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Abstract

In this article, we consider a multidimensional economy where the standard supermodularity property fails. We generalize the notion of net gain of investment, introduced by Kamihigashi and Roy [7] and applied to one-sector growth models, to the case of multiple capital stocks. We prove the convergence to the set of steady states without relying on the monotonicity of optimal path. Our approach differs from the standard dynamic programming based on convexity or supermodularity. We find that preferences are key to shape the economy in the long run.

Keywords: net gain of investment, multidimensional economy, nonconvexities.

JEL codes: C61, D51.

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1 Introduction

Nice properties of dynamic models often rest on convenient but restrictive assumptions such as convexity or supermodularity.[1] The seminal book by Stokey et al. [17] is an introduction to dynamic programming with economic applications under nice convexity properties. Le Van and Dana [8] show how strict convexities in technology and preferences ensure not only the uniqueness of solutions but also their convergence (sections 2.4.3 and 2.4.4). Dechert and Nishimura [4] consider a convex-concave production function and find multiple solutions only for a zero-measure set of initial states. In the lack of a suitable convexity structure, the supermodularity à la Amir [1, 2] entails a monotonicity property and unambiguous long-run dynamics.

These classical approaches face a common difficulty. To study the economy in the long run, scholars use monotonicity based on convexity or supermodularity. However, these condition may fail when a facet can not be treated in one-sector models, such as the externalities between different sectors. For instance, investing in Internet or other communication facilities can speed up the transfer of information and enhance, by this, the productivity. In a different context, an increase in industrial activities may harm other sectors such as agriculture and tourism, or curb the regeneration of renewable resources such as forests or oceans. These cross effects can rule out convexity or, even, supermodularity properties, with two main consequences: (1) the uniqueness of the optimal path is no longer ensured; (2) its monotonicity may also fail. The second issue is more complicated to tackle. For these reasons, scholars refrain from a global analysis and focus on a neighborhood of the steady state. Nevertheless and precisely a global approach is needed to overcome these difficulties.

In this article, we consider a small open economy where the capital prices are supposed to be given. We tackle the lack of supermodularity from a different viewpoint by considering the innovative concept of *net gain of investment* in the

[1] For a definition of supermodularity, its main properties and applications, see Amir [1], [2].

spirit of Majumdar and Nermuth [11], and Majumdar and Mitra [10]. We remain closer to Kamihigashi and Roy [6, 7], a growth model with a nonsmooth and nonconvex technology.

The net gain of investment is a function with nice properties despite the monotonicity failure. We find that, even when it does not increase over time, it converges to the supremum. In addition, the discounted sum of future net gains always exceeds the current value and increases over time. The limit of the discounted sums of net gains is also informative about the economy in the long run. We obtain convergence of the economy to a set of steady states, generically finite and rather easy to study. Last but not least, the case where optimal paths converge to infinity is also characterized. These results are very general: we do not need any restriction on technology but only the concavity of preferences. In other words, we prove that preferences are key to shape the economy in the long run.

Dechert and Nishimura [4] paved the way to generalize the optimal growth theory to the nonconvex case. In their article, under a convex-concave production function, the optimal path is monotonic and converges either to a positive steady state or to zero, depending whether the initial stock is above or below a critical level, the poverty trap.[2]

Kamihigashi and Roy [7] extend the analysis with more general production functions, not only nonconvex, but also, possibly, non-differentiable and discontinuous. To overcome these difficulties, they develop new tools such as the net gain function or the Euler inequalities. These authors provide different conditions for the optimal path to converge to a non-trivial steady state or zero, or to diverge to infinity, and for the turnpike property to hold, that is the convergence of optimal capital stock to the *golden rule* when the discount factor tends to one.

Because the *net gain of investment* is key, Kamihigashi and Roy [7] is the closest work to ours. In their contribution, as in Dechert and Nishimura [4], the monotonicity of optimal path is a direct consequence of supermodularity.[3] When the

[2] For details, see Le Van and Dana [8].

[3] See their Lemma 3.1.

optimal path is bounded, it converges either to a positive steady state or to zero. Hence, the main interest of Kamihigashi and Roy [7] rests on the long-run values instead of on the convergence *per se*.

Hung et al. [5] study a model of optimal growth where the convexity fails by aggregation of two concave production functions, representing two technologies. One is more costly but also more productive than the other (for a high level of capital stock). These authors also study the net gain of investment. In their framework, supermodularity ensures the monotonicity of the optimal path and the Euler equation allows them to find candidates for the steady states. Hung et al. [5] provide conditions to determine these steady states. As in Kamihigashi and Roy [7] and, as we will see, here, they find that the maximizer of net gain of investment is a steady state.[4]

While the results in Decher and Nishimura [4], Kamihigashi and Roy [7], and Hung et al. [5] are grounded on the monotonicity of the optimal path, our approach no longer rests on this property and, hence, as a general viewpoint, it unifies these different results.

The rest of the article is organized as follows. Section 2 introduces the fundamentals and sufficient conditions to ensure the existence of an optimal solution. Section 3 defines the *net gain of investment*, while section 4 proves the existence and the structure of the set of steady states. Section 5 addresses the stability issue in terms of convergence to the set of steady states. Some examples are given in Section 6. Section 7 concludes. All proofs are gathered in the Appendix.

2 Fundamentals

Capital and resource stocks are represented by a d-dimensional vector. The production function $f : \mathbb{R}^d_+ \to \mathbb{R}_+$ transforms these inputs in a single output which is consumed or invested.

We denote by $\theta \in \mathbb{R}^d_{++}$ the vector of input prices expressed in units of output

[4] See Lemmas 1, 2 and Proposition 4 in Hung et al. [5].

(numeraire). The economic agent is a price-taker and prices are positive and constant over time: $\theta_i > 0$ with $i = 1, \ldots, d$. The assumption of a constant price vector to a large extent simplifies the dynamic analysis and allows us to focus on the optimization aspects we are interested in, without caring about general equilibrium feedbacks and price adjustments.

At date t, the capital stock x_t yields $f(x_t)$. The agent consumes a part c_t of this aggregate product and invests in the new capital stock I_{t+1} according to her budget constraint $c_t + \theta \cdot I_{t+1} \leq f(x_t)$, where $\cdot$ denotes the scalar product between θ and I_{t+1}. Let $e_i \in [0, 1]$ be the depreciation rate of capital i. The stock of capital i at date $t + 1$ is defined as

$$x_{i,t+1} = I_{i,t+1} + (1 - e_i)\, x_{i,t}.$$

The initial stock of capital x_0 and agent's preferences (the discount rate $\delta \in (0, 1)$ and the utility function u) are given. She solves the following dynamic program:

$$v(x_0) = \max \sum_{t=0}^{\infty} \delta^t u(c_t), \tag{2.1}$$

where v denotes the value function, subject to

$$\begin{aligned} c_t + \theta \cdot I_{t+1} &\leq f(x_t), \\ x_{i,t+1} &= I_{i,t+1} + (1 - e_i)\, x_{i,t}, \\ c_t &\geq 0,\ x_{i,t} \geq 0 \text{ and } I_{i,t} \geq 0, \end{aligned}$$

for any $t \geq 0$ and $0 \leq i \leq d$.

For $x_t \in \mathbb{R}^d_+$, let $F(x_t) \equiv f(x_t) + \sum_{i=1}^{d} (1 - e_i)\, \theta_i x_{i,t}$. The initial program (2.1) can be rewritten as:

$$v(x_0) = \max \sum_{t=0}^{\infty} \delta^t u(c_t), \tag{2.2}$$

subject to

$$c_t + \theta \cdot x_{t+1} \leq F(x_t),$$

$$x_{i,t+1} \geq (1 - e_i)\, x_{i,t},$$

$$c_t \geq 0 \text{ and } x_{i,t} \geq 0,$$

for any $t \geq 0$ and $0 \leq i \leq d$.

In the particular case of reversible capital investment, that is $e_i = 1$ for every $i = 1, 2, \ldots, d$, (2.2) coincides with (2.1) and $F(x) = f(x)$.

In the general case, given the price system θ and the capital stock x_t, we define the set of affordable inputs:

$$\Gamma\left(x_t\right) \equiv \left\{x_{t+1} \in \mathbb{R}_+^d \text{ such that } 0 \leq \theta \cdot x_{t+1} \leq F\left(x_t\right) \text{ and } x_{i,t+1} \geq \left(1 - e_i\right) x_{i,t} \text{ for any } i\right\}.$$

Let $\Pi\left(x_0\right)$ be the set of feasible paths $\left(x_t\right)_{t=0}^{\infty}$ such that $x_{t+1} \in \Gamma\left(x_t\right)$ for any $t \geq 0$.

We introduce some conditions that ensure the existence of a solution to program (2.2). For the sake of simplicity, we assume from the outset the continuity of the maximization with respect to the product topology. Readers interested in technical details are referred to Le Van and Morhaim [9] (conditions H1 to H8 and Theorem 1).

Assumption F1. 1. *The production function F is continuous and increasing.*

2. *The utility function is strictly concave and strictly increasing.*

3. *For any x_0 with at least one strictly positive component, there exists a sequence $\left(x_t\right)_{t=0}^{\infty} \in \Pi\left(x_0\right)$ such that $\sum_{t=0}^{\infty} \delta^t u\left(F\left(x_t\right) - \theta \cdot x_{t+1}\right) > -\infty$ and the value function is well-defined: $v\left(x_0\right) < \infty$.*

4. *The function $\sum_{t=0}^{\infty} \delta^t u\left(F\left(x_t\right) - \theta \cdot x_{t+1}\right)$ is upper semi-continuous with respect to the product topology.*

5. *The value function v is upper semi-continuous in $\mathbb{R}_+^d$.*

6. *For any feasible path starting from x_0, we have $\lim_{T\to\infty} \delta^T \theta \cdot x_T = 0$.*

7. *For any feasible path starting from x_0, we have $\sum_{t=0}^{\infty} \delta^t \left[F\left(x_t\right) - \theta \cdot x_{t+1}\right] < \infty$.*

Le Van and Morhaim [9] provide conditions ensuring Assumption F1. In their Lemma 2, H1 and H2 imply the compactness of $\Pi\left(x_0\right)$ with respect to the product topology. In their Theorem 1, H1, H2, H4, H5 and H6 imply conditions 3, 4, 5 in F1. The most important condition, which ensures the continuity property of the value function v, is *tail-insensitivity* (H6 in Le Van and Morhaim [9]).

Conditions 6 and 7 in Assumption F1 are new. Let us give the intuition behind them. Given a feasible path $(x_t)_{t=0}^{\infty}$, condition 6 states that the growth rate never overcomes the discount factor. In other words, the growth rate is dominated by the discount factor. In condition 7, the consumption is given at date t by $c_t = F(x_t) - \theta \cdot x_{t+1}$. This condition simply means $\sum_{t=0}^{\infty} \delta^t c_t < \infty$: the economy can diverge to infinity, but, from any initial state, the *discounted sum* of consumptions is finite.

It is worth to remark that, these conditions are always satisfied if the economy is bounded, for example under condition H2 in Le Van and Morhaim [9] with $0 < \gamma < 1$, according to their Lemma 1.

Under F1, as shown among others by Le Van and Morhaim [9] (Theorem 1), an optimal path exists. The value function is a solution of the Bellman functional equation:

$$v(x) = \max_{\substack{\theta \cdot y \leq F(x) \\ y_i \geq (1-e_i) x_i \forall i}} \left[u(F(x) - \theta \cdot y) + \delta v(y) \right].$$

Let σ denote the optimal policy correspondence

$$\sigma(x) \equiv \arg \max_{\substack{\theta \cdot y \leq F(x) \\ y_i \geq (1-e_i) x_i \forall i}} \left[u(F(x) - \theta \cdot y) + \delta v(y) \right].$$

This correspondence allows for optimal paths and multiple steady states. By the Maximum Theorem, this correspondence is upper hemi-continuous (Theorem 3.6, chapter 3 in Stokey et al. [17]).

A feasible sequence $(x_t)_{t=0}^{\infty}$ is an optimal path beginning from x_0, if and only if for every t, we have $x_{t+1} \in \sigma(x_t)$ or, equivalently,

$$v(x_t) = u(F(x_t) - \theta \cdot x_{t+1}) + \delta v(x_{t+1}) = \max_{\substack{\theta \cdot y \leq F(x_t) \\ y_i \geq (1-e_i) x_i \forall i}} \left[u(F(x_t) - \theta \cdot y) + \delta v(y) \right].$$

In terms of capital stock, a steady state is a fixed point $x^* \in \sigma(x^*)$: the constant sequence $(x^*, x^*, \dots)$ is an optimal path starting from x^*.

In the next section, we study the behavior of the economy in the long run. First, we introduce the notion of *net gain of investment*, then we focus on the existence of steady states. Finally, we consider the convergence in the long run and a more general technology correspondence.

3 The net gain of investment

In the spirit of Kamihigashi and Roy [6, 7], we define the *net gain of investment* function as follows:

$$\psi\left(x_t\right) \equiv \delta F\left(x_t\right) - \theta \cdot x_t,$$

for any t.

Interestingly, for any feasible path $(x_t)_{t=0}^{\infty} \in \Pi\left(x_0\right)$, the discounted sum of consumption is equal to the discounted sum of net gains generated by the initial capital stock x_0. More precisely, we observe that the discounted sum of consumption is equal to the sum of $F(x_0)$ and the discounted sum of net gain:

$$\begin{aligned}\sum_{t=0}^{\infty} \delta^t c_t &= \sum_{t=0}^{\infty} \delta^t \underbrace{\left[F\left(x_t\right) - \theta \cdot x_{t+1}\right]}_{=c_t} = \sum_{t=0}^{T-1} \delta^t \left[F\left(x_t\right) - \theta \cdot x_{t+1}\right] + \delta^T \sum_{s=0}^{\infty} \delta^s c_{T+s} \\ &= F(x_0) + \sum_{t=0}^{T-2} \delta^t \left[\delta F(x_{t+1}) - \theta \cdot x_{t+1}\right] - \delta^{T-1}\theta \cdot x_T + \delta^T \sum_{s=0}^{\infty} \delta^s c_{T+s} \\ &= F\left(x_0\right) + \sum_{t=1}^{\infty} \delta^{t-1} \underbrace{\left[\delta F(x_t) - \theta \cdot x_t\right]}_{=\psi(x_t)} - \lim_{T\to\infty} \delta^T \theta \cdot x_{T+1},\end{aligned}$$

because of part 7 of Assumption F1. By part 6 of Assumption F1, we obtain:

$$\sum_{t=0}^{\infty} \delta^t c_t = F\left(x_0\right) + \sum_{t=0}^{\infty} \delta^t \psi\left(x_{t+1}\right). \tag{3.1}$$

The following lemma is an indispensable step to prove that, if the sequence $(x_t)_{t=0}^{\infty}$ is optimal, then the discounted sum of net gains of investment increases over time.

Lemma 3.1. *For any initial state x_0 such that $\theta \cdot x_0 \leq F(x_0)$,*

$$u\left(F\left(x_0\right) - \theta \cdot x_0\right) \leq (1-\delta)\, v\left(x_0\right). \tag{3.2}$$

The intuition for Lemma 3.1 is that, given x_0 such that $0 \leq \theta \cdot x_0 \leq F(x_0)$, $(x_0, x_0, \ldots)$ is feasible. It is clear that

$$v(x_0) \geq \sum_{s=0}^{\infty} \delta^s u\left(F(x_0) - \theta \cdot x_0\right) = \frac{u\left(F(x_0) - \theta \cdot x_0\right)}{1-\delta},$$

which implies the inequality in (3.2).

Proposition 3.1 below states that either the economy is at a steady state, or the *net gain of investment* increases at some date in the future. This result echoes Kamihigashi and Roy [6, 7].

PROPOSITION 3.1. *Consider an initial capital stock x_0 and an optimal path $(x_t)_{t=0}^{\infty}$ starting from x_0.*

1. *If for every $t \geq 0$, we have $\psi(x_t) \leq \psi(x_0)$, then the constant sequence $(x_0, x_0, \ldots)$ is also an optimal path starting from x_0.*[5] *Moreover, for every $t \geq 0$, we have $F\left(x_t\right) = F\left(x_0\right)$, $\theta \cdot x_t = \theta \cdot x_0$ and $\psi\left(x_t\right) = \psi\left(x_0\right)$.*

2. *If the constant path $(x_0, x_0, \ldots)$ is not an optimal path, then there exists some $t \geq 1$ such that $\psi\left(x_t\right) > \psi\left(x_0\right)$.*

The following proposition establishes that the *net gain of investment* is always smaller than the *discounted sum of future net gains of investment* and that this sum increases over time. In other words, along the optimal path, the future gain always exceeds the present one. These general results draws a picture of the dynamic behavior of the economy.

PROPOSITION 3.2. *Consider an optimal path $(x_t)_{t=0}^{\infty}$.*

1. *For any $t \geq 0$,*

$$\psi\left(x_t\right) \leq (1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s+1}\right).$$

[5] This sequence may differ from $(x_0, x_1, x_2, \ldots)$ since the uniqueness of optimal path is no longer guaranteed.

2. *For any $t \geq 0$,*

$$(1-\delta)\sum_{s=0}^{\infty}\delta^s\psi\left(x_{t+s}\right) \leq (1-\delta)\sum_{s=0}^{\infty}\delta^s\psi\left(x_{t+s+1}\right),$$

where the system of weights $\left((1-\delta)\,\delta^s\right)_{s=0}^{\infty}$ well defines an infinite-dimensional average of net gains of investment.

4 The existence of steady states

Echoing Proposition 3.2 in Kamihigashi and Roy [7], Proposition 3.1 entails that every input vector x^M with the largest net investment is a steady state. Indeed, since the value of *net gain of investment* can no longer increase in the future if the starting point is x^M, part 1 of Proposition 3.1 states that it is a steady state.

Corollary 4.1. *If* $\arg\max\left[\delta F\left(x\right)-\theta\cdot x\right]$ *is a nonempty set, then every x^M belonging to this set is a steady state of the economy.*

In the literature, standard compactness properties imply the existence of x^M and, hence, the existence of a steady state. The next proposition characterizes the set of steady states without any assumption of convexity in technology.

Let S^* be the set of $x \in \mathbb{R}_+^d$ such that an optimal path $(x_t)_{t=0}^{\infty}$ exists with the following properties:

1. $\psi\left(x\right)=\sup_{t\geq 0}\psi\left(x_t\right)$.

2. There exists a subsequence $(x_{t_n})_{n=0}^{\infty}$ converging to x, that is $\lim_{n\to\infty}x_{t_n}=x$.

Evidently, the existence of a bounded optimal path $(x_t)_{t=0}^{\infty}$ implies the nonemptiness of S^*. Indeed, picking the subsequence $(x_{t_n})_{n=0}^{\infty}$ such that $\lim_{n\to\infty}\psi\left(x_{t_n}\right)=\sup_{t\geq 0}\psi\left(x_t\right)$, we observe that, by the boundedness hypothesis, the sequence $(x_{t_n})_{n=0}^{\infty}$ belongs to a compact set. Hence, there exists a subsequence $\left(x_{t_{n_k}}\right)_{k=0}^{\infty}$ of $(x_{t_n})_{n=0}^{\infty}$ that converges to some $x \in \mathbb{R}_+^d$. By the definition of S^* , we have $x \in S^*$. Hence, S^* is nonempty. Clearly, every steady state belongs to S^*. The next proposition states that S^* coincides with the whole set of steady states.

Proposition 4.1. *Assume that there exists a bounded optimal path. Then S^* is nonempty and:*

1. *Every capital stock x^* is a steady state if and only if $x^* \in S^*$.*
2. *Consider a bounded optimal path $(x_t)_{t=0}^{\infty}$. If, for some T, $\psi(x_T) = \sup_{t \geq 0} \psi(x_t)$, then, for any $t \geq T$, $\psi(x_t) = \psi(x_T)$ and x_t is a steady state.*

One of the main concerns is to know whether the economy grows to infinity, converges to a steady state, or collapses to zero. On the one hand, standard assumptions such as Inada conditions, ensure that an optimal path is bounded away from zero. On the other hand, low productivity for high capital levels implies that the economy is bounded.[6] By Proposition 4.1, the existence of such an optimal path implies the existence of a non-trivial steady state. More precisely, we get the following characterization.

Corollary 4.2. *There exists a non-trivial steady state if and only if there exists a bounded optimal path which is bounded away from zero.*

5 Stability issue

We now establish our main result.

By definition of the set S^*, a bounded optimal path $(x_t)_{t=0}^{\infty}$ visits the neighborhood of S^* infinitely many times. The next theorem establishes a stronger property of the set S^*. Its proof rests on Proposition 3.2, which implies the existence of the limit $\left((1-\delta)\sum_{s=0}^{\infty} \delta^s \psi(x_{t+s})\right)_{t=0}^{\infty}$. Moreover, the value of this limit is precisely $\sup_{t \geq 0} \psi(x_t)$.

According to Proposition 4.1, the optimal path converges to S^*, which is the set of steady states. Here, convergence means that, for any $\varepsilon > 0$, there exists a critical

[6] Condition H2 in Le Van and Morhaim [9], with $0 < \gamma < 1$, is weaker but still sufficient. Proposition 3.3 in Kamihigashi and Roy [7] achieves a similar results in the case of a one-dimensional economy.

date T such that $x_t \in S^* + B(0, \varepsilon)$ for any $t \geq T$, with $B(0, \varepsilon)$ denoting the sphere of radius ε centered in the origin 0^d.

THEOREM 5.1. *1. Either any optimal path converges to the set of steady states S^* or it is unbounded.*

2. For any bounded optimal path $(x_t)_{t=0}^{\infty}$,

$$\lim_{t\to\infty} \psi(x_t) = \sup_{t\geq 0} \psi(x_t).$$

In sections 4 and 5, we have found that, if there is a bounded optimal path, then the set of steady states is nonempty, and any bounded optimal path converges to it. Now, to study the possibility of unbounded growth, we replace the assumption of boundedness with the following conditions.

Assumption A1. *1. Unbounded net gains of investment:* $\lim_{\|x\|\to\infty} \psi(x) = \infty$.

2. For any compact set C:

$$\sup_{x_0 \in C} \sup_{\chi_0 \in \Pi(x_0)} \sum_{t=0}^{\infty} \delta^t \psi(x_{t+1}) < \infty.$$

The first condition is less restrictive than the condition (4.8) of Proposition 4.6 in Kamihigashi and Roy [7] considering marginal productivities larger than $1/\delta$ for higher levels of capital.

Since, for any feasible path $(x_t)_{t=0}^{\infty}$, $\sum_{t=0}^{\infty} \delta^t c_t = F(x_0) + \sum_{t=0}^{\infty} \delta^t \psi(x_{t+1})$, the second condition rules out any discounted sum of intertemporal consumptions taking an infinite value. The economy can grow forever, but the rate of growth never overtakes the discount factor δ. This condition is similar, but not identical, to *tail-insensitivity* (Assumption H6 in Le Van and Morhaim [9]).

PROPOSITION 5.1. *Assume A1. Any optimal path either converges to the set of steady states or diverges to infinity.*

To conclude, let us evoke the possible extension of our results to the case where capital prices are no longer constant but depend on resources. More precisely, let us replace the value of capital $\theta \cdot x_t$, where the price vector θ is constant, by a continuous and increasing cost function $\Theta(x_t)$. In this case, under the additional assumption of capital reversibility, that is full capital depreciation ($e_i = 1$ for $i = 1, \ldots, d$), all the results of Sections 3, 4 and 5 remain valid. One can check it, by replacing the scalar product $\theta \cdot x_t$ by the cost function $\Theta(x_t)$ in each proof of these sections.

6 Applications

6.1 An economy with convex-concave production function

Focus on a one-sector economy with a convex-concave production function, where a single good is consumed and invested. Agent's preferences are the same:

$$\max \sum_{t=0}^{\infty} \delta^t u(c_t),$$

$$c_t + k_{t+1} \leq F(k_t),$$

$$c_t, k_t \geq 0$$

for any $t \geq 0$.

This model is considered by Le Van and Dana [8] and by Dechert and Nishimura [4] in a more general context where the production is no longer convex-concave. Using the concept of *net gain of investment*, we can address the issue from a different viewpoint and quickly recover the main results of the third chapter in Le Van and Dana [8].

For simplicity, assume that F is a continuous S-shaped function with $F(0) = 0$: strictly convex for $k_t < k_I$ and strictly concave for $k_t > k_I$ where k_I denotes the inflection point. Moreover, suppose $F'(k_t) < 1$ for any k_t large enough. This condition implies a bounded economy. Hence, as in Theorem 2 by Dechert and Nishimura [4] or in Proposition 3.4.5 by Le Van and Dana [8], applying our Theorem 5.1, we find that any optimal path converges to a steady state. We require also

$\delta F'(k_I) > 1$. Otherwise, $F'(k_I)$ being the highest productivity level, $\delta F'(k) < 1$ for every k and the economy converges to zero: the unique steady state is the trivial one.

Dechert and Nishimura [4] and Le Van and Dana [8] show that, under *mild discounting*, any optimal path with $k_0 > 0$ converges to a strictly positive steady state. Let us explain why.

The function of *net gain of investment* is defined as $\psi(k_t) \equiv \delta F(k_t) - k_t$. Under mild discounting ($\delta > 1/F'(0)$), we have $\psi(0) = 0 < \psi(k_t)$ for every $0 < k_t \leq k_I$. Then, starting from $0 < k_0 \leq k_I$, the optimal path does not converges to zero. Therefore, the limit is strictly positive. Starting from $k_0 \geq k^*$, every optimal path is bounded from below by this limit.

PROPOSITION 6.1. *Assume that $\delta > 1/F'(0)$ (mild discounting). Let $k_0 > 0$. The economy converges to a strictly positive steady state, which is the unique solution to $\delta F'(k) = 1$.*

For a detailed proof of Proposition 6.1, see the Appendix.

Now, consider the problem with $\delta < 1/F'(0)$. As above, we assume $1/F'(k_I) < \delta$ and $F'(\infty) < 1 < 1/\delta$. We consider two subcases: (1) intermediate discounting and (2) strong discounting. Let us provide a formal definition.

Since $F'(0) < 1/\delta$ and $F'(\infty) < 1/\delta < F'(k_I)$, there exist k_s and k^s with $0 < k_s < k_I < k^s$ solutions to the equation $F'(k) = 1/\delta$.

The intermediate and strong discounting are defined as follows:[7]

LEMMA 6.1. *1. Intermediate discounting: $k^s/F(k^s) < \delta < 1/F'(0)$.*

2. Strong discounting: $\delta < k^s/F(k^s)$.

The following lemma is key to obtain the main results.

LEMMA 6.2. *Let $1/F'(k_I) < \delta < 1/F'(0)$. Then:*

[7] Since $\psi(k^s) = \max_{k \geq k_s} \psi(k)$, our notions are equivalent to the ones in Le Van and Dana [8], pages 50-51.

1. $k_s = \arg\min_{0 \leq k \leq k^s} \psi(k)$.

2. *If* $k^s / F(k^s) < \delta < 1/F'(0)$ *(intermediate discounting),* $k^s = \arg\max_{k \geq 0} \psi(k)$. *Moreover,* k^s *is a steady state.*

3. *If* $1/F'(k_I) < \delta < k^s / F(k^s)$ *(strong discounting),* $\psi(k) < \psi(0) = 0$ *for any* $k > 0$.

0 is always a (trivial) steady state. In the case of intermediate discounting, since $k^s = \arg\max_{k \geq 0} \psi(k)$, by Corollary 4.1, k^s is also a steady state.

Since $k_s = \arg\min_{0 \leq k \leq k^s} \psi(k)$, starting from $k_0 \neq k_s$, by part (2) of Theorem 5.1, every optimal path converges either to 0 or k^s. Thus, if k_s is a steady state, the interval $[0, k_s]$ is a poverty trap with $\lim_{t \to \infty} k_t = 0$ if $k_0 < k_s$, and $k_t = k_s$ for any $t \geq 0$ if $k_0 = k_s$.[8]

Proposition 6.2. *If* k_s *is a steady state, then every optimal path starting from* $k_0 < k_s$ *converges to* 0 *and every optimal path starting from* $k_0 > k_s$ *converges to* k^s.

What happens if k_s is not a steady state?[9]

Let us show that an upper bound, say $\underline{k}$, exist for the poverty trap. From point 2 of Lemma 6.2, we know that, under intermediate discounting, k^s is a steady state. We also have $\psi(k_0) > 0$ for every $\underline{k} < k_0 \leq k^s$. Hence, starting from k_0, the economy converges to k^s.

Lemma 6.3. *If* $k^s / F(k^s) < \delta < 1/F'(0)$ *(intermediate discounting), then there exists* $\underline{k} > 0$ *solution to* $\delta = k/F(k)$ *such that any optimal path starting from* $k_0 \geq \underline{k}$ *converges to* k^s.

We can prove the existence of a poverty trap $\left[0, \hat{k}\right) \subset [0, \underline{k}]$.

[8] See also Corollary 3.4.3 and Proposition 3.4.6 in Le Van and Dana [8].

[9] k_s is no longer a steady state under some sufficient conditions considered in Proposition 3.4.7 in Le Van and Dana [8].

Proposition 6.3. *Under intermediate discounting, there exists* $0 < \hat{k} \leq \underline{k}$ *such that the economy converges to* 0 *if* $k_0 < \hat{k}$ *and to* k^s *if* $k_0 > \hat{k}$.

Under strong discounting, k^s plays the role of $\underline{k}$.

Proposition 6.4. *Let* $1/F'(k_I) < \delta < k^s/F(k^s)$ *(strong discounting).*

1. *If* k^s *is not a steady state, then the economy converges to* 0 *for any* $k_0 \geq 0$.

2. *If* k^s *is a steady state, then there exists* $0 < \hat{k} \leq k^s$ *such that the economy converges to* 0 *if* $k_0 < \hat{k}$ *and to* k^s *if* $k_0 > \hat{k}$.

6.2 An economy with renewable resources

Consider the economy à la Dam *et al.* [3]. In this article, the authors study an economy with physical capital k_t and renewable resources y_t. The regeneration capacity of the renewable resources is a function of the existing stock and the industrial activity, which is an increasing function of capital stock k_t.

Precisely, given the stocks of resources and capital (k_t, y_t) at time t, the natural resources at time $t+1$ are given by a regeneration function $\eta(k_t, y_t)$ which is decreasing and convex in k_t, and increasing and concave in y_t. The production function f and the regeneration function η are supposed to satisfy the Inada conditions: $f'(0) = \infty$ and $\eta'_y(k, 0) = \infty$ for any $k > 0$.

The capital good can be consumed or invested at the unit price, while the natural resources can be only invested at the price $p > 0$. Thus, $\theta = (1, p)$ with $x_t = (k_t, y_t) \in \mathbb{R}^2_+$. For simplicity, the physical capital fully depreciates ($e_k = 1$), while the natural resources do not ($e_y = 0$). The consumer faces the following program:

$$\max \sum_{t=0}^{\infty} \delta^t u(c_t),$$

subject to

$$c_t + k_{t+1} + p y_{t+1} \leq F(k_t, y_t) \equiv f(k_t) + p\eta(k_t, y_t),$$

$$0 \leq y_{t+1} \leq \eta(k_t, y_t).$$

Given the stock $(k_t, y_t) \in \mathbb{R}^2_+$, the set of affordable outputs is

$$\Gamma(k_t, y_t) \equiv \left\{(k_{t+1}, y_{t+1}) \in \mathbb{R}^2_+ \text{ such that } k_{t+1} + py_{t+1} \leq F(k_t, y_t) \text{ and } y_{t+1} \leq \eta(k_t, y_t)\right\}.$$

The additional constraint $y_{t+1} \leq \eta(k_t, y_t)$ simply means that we can sell the renewable resources to buy the physical capital but not the converse. The set affordable outputs is smaller that the budget set

$$B(k_t, y_t) \equiv \left\{(k_{t+1}, y_{t+1}) \in \mathbb{R}^2_+ \text{ such that } k_{t+1} + py_{t+1} \leq F(k_t, y_t)\right\}$$

since this set may contain (k_{t+1}, y_{t+1}) such that $y_{t+1} > \eta(k_t, y_t)$. The results of the previous sections no longer apply directly.

Let us adapt the proof of Lemma 3.1 to this new context. Given the initial condition (k_0, y_0), the inequality 3.2 becomes

$$u(F(k_0, y_0) - k_0 - py_0) \leq (1 - \delta) v(k_0, y_0). \tag{6.1}$$

Indeed, recall that inequality 3.2 is satisfied if either $k_0 + py_0 > F(k_0, y_0)$ or $(k_0, y_0) \in \Gamma(k_0, y_0)$. Now, focus on the remaining case, $k_0 + py_0 \leq F(k_0, y_0)$ and $y_0 > \eta(k_0, y_0)$. By the concavity of η with respect to y, this inequality implies that $y_0 > \overline{y}$, the solution to $\eta(k_0, y) = y$. Moreover, we have $y < \eta(k_0, y)$ for $0 < y < \overline{y}$ and $y > \eta(k_0, y)$ for $y > \overline{y}$.

Choose $\tilde{y}$ such that $0 < \tilde{y} < \overline{y}$ and let $\tilde{x} \equiv (k_0, \tilde{y})$. Then, $\tilde{x}$ satisfies the following properties:

1. $\theta \cdot \tilde{x} < \theta \cdot x_0$,
2. $F(\tilde{x}) - \theta \cdot \tilde{x} = F(x_0) - \theta \cdot x_0$.

The first property implies that $\tilde{x} \in \Gamma(x_0)$. The second one jointly with $\tilde{y} < \eta(k_0, \tilde{y})$ implies that the stock $\tilde{x}$ belongs to $\Gamma(\tilde{x})$. In other words, this stock can replicate itself. Then, the sequence $(x_0, \tilde{x}, \tilde{x}, \dots)$ is feasible. Combining with $\theta \cdot x_0 > \theta \cdot \tilde{x}$, we get

$$
\begin{aligned}
(1-\delta)\, v\,(x_0) &\geq (1-\delta)\, u\,(F\,(x_0)-\theta\cdot\tilde{x})+(1-\delta)\sum_{t=1}^{\infty}\delta^t u\,(F\,(\tilde{x})-\theta\cdot\tilde{x}) \\
&> (1-\delta)\sum_{t=0}^{\infty}\delta^t u\,(F\,(x_0)-\theta\cdot x_0)=u\,(F\,(x_0)-\theta\cdot x_0)\,.
\end{aligned}
$$

Hence, inequality 6.1 is proven for any (k_0, y_0). Following the same arguments in the proofs of Propositions 3.1 and 3.2, and in Theorem 5.1, we are able to conclude that the economy always converges to its set of steady states (see also Dam *et al.* [3]).

7 Concluding comments

One-dimensional nonconvex economies have been studied in the literature. Readers are referred to Mitra and Ray [12], Dechert and Nishimura [4] and Kamihigashi and Roy [7] for a deterministic setting, and to Nishimura and Stachurski [15], and Nishimura *et al.* [16] for a stochastic approach. The supermodular property of their models is key. When supermodularity fails, the economic dynamics may exhibit complex behavior. Under a sufficiently low discount factor, cycles and chaos may arise, even in one-dimensional economies. Montrucchio and Sorger [13], and Nishimura and Sorger [14] provide excellent surveys of this literature.

In our model, we have considered a multi-dimensional economy à la Ramsey, where income is shared into consumption and investment, and proven its convergence to the set of steady states. Therefore, the study of the long-run behavior of the economy comes down to the analysis of this set, which, in literature, is often finite. The Euler equations allow us to compute the set of steady states and to study their properties.

8 Appendix

8.1 Proof of Lemma 3.1

The inequality $\theta \cdot x_0 \leq F(x_0)$ implies that the constant sequence $(x_0, x_0, \dots)$ is feasible. Hence,

$$v(x_0) \geq \sum_{t=0}^{\infty} \delta^t u(F(x_0) - \theta \cdot x_0) = \frac{u(F(x_0) - \theta \cdot x_0)}{1-\delta}$$

and we obtain (2.2).

8.2 Proof of Proposition 3.1

Fix any x_0 and an optimal path $(x_t)_{t=0}^{\infty}$ starting from x_0.

(1) Suppose that $\psi(x_t) \leq \psi(x_0)$ for every $t \geq 1$. According to (3.1), we have

$$\begin{aligned}\sum_{t=0}^{\infty} \delta^t c_t &= F(x_0) + \sum_{t=0}^{\infty} \delta^t \psi(x_{t+1}) \leq F(x_0) + \sum_{t=0}^{\infty} \delta^t \psi(x_0) \\ &= F(x_0) + \frac{\psi(x_0)}{1-\delta} = \frac{F(x_0) - \theta \cdot x_0}{1-\delta}.\end{aligned} \tag{8.1}$$

First, observe that this implies $F(x_0) \geq \theta \cdot x_0$. Hence, $x_0 \in \Gamma(x_0)$ and the sequence $(x_0, x_0, \dots)$ is feasible. Moreover,

$$u(F(x_0) - \theta \cdot x_0) \geq u\left((1-\delta)\sum_{t=0}^{\infty} \delta^t c_t\right).$$

Noticing that $(1-\delta)\sum_{t=0}^{\infty} \delta^t = 1$ and, using Jensen inequality, we obtain

$$u(F(x_0) - \theta \cdot x_0) \geq u\left((1-\delta)\sum_{t=0}^{\infty} \delta^t c_t\right) \geq (1-\delta)\sum_{t=0}^{\infty} \delta^t u(c_t) = (1-\delta) v(x_0). \tag{8.2}$$

Inequalities (8.2) entail that $(x_0, x_0, \dots)$ is also an optimal path. Moreover, inequalities in (8.2) become equalities. This implies $\psi(x_t) = \psi(x_0)$ for every

$t \geq 0$ because, if $\psi(x_t) < \psi(x_0)$ for some t, then (8.1) writes $(1-\delta)\sum_{t=0}^{\infty} \delta^t c_t < F(x_0) - \theta \cdot x_0$ and, by the Jensen inequality and (8.2) taken with equalities,

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t u(c_t) \leq u\left(\sum_{t=0}^{\infty} (1-\delta)\delta^t c_t\right) < u(F(x_0) - \theta \cdot x_0) = (1-\delta) v(x_0),$$

leading to a contradiction with the optimality of $(c_t)_{t=0}^{\infty}$.

Since $u\left((1-\delta)\sum_{t=0}^{\infty} \delta^t c_t\right) = (1-\delta)\sum_{t=0}^{\infty} \delta^t u(c_t)$, the strict concavity of u jointly with the Jensen inequality implies $c_t = c_0$ and, therefore, $F(x_t) - \theta \cdot x_{t+1} = F(x_0) - \theta \cdot x_1$ for any $t \geq 0$. Combining this with $\psi(x_t) = \psi(x_0)$, we get

$$\delta F(x_t) - \theta \cdot x_t = \delta F(x_0) - \theta \cdot x_0, \tag{8.3}$$

$$\delta[F(x_t) - \theta \cdot x_{t+1}] = \delta[F(x_0) - \theta \cdot x_1]. \tag{8.4}$$

Subtracting (8.3) from (8.4), we find $\theta \cdot x_t = \delta\theta \cdot x_{t+1} + (\theta \cdot x_0 - \delta\theta \cdot x_1)$ for any t and, then,

$$\begin{aligned}\theta \cdot x_t &= \delta^2\theta \cdot x_{t+2} + \delta(\theta \cdot x_0 - \delta\theta \cdot x_1) + (\theta \cdot x_0 - \delta\theta \cdot x_1) \\ &= \delta^T\theta \cdot x_{t+T} + \frac{1-\delta^T}{1-\delta}(\theta \cdot x_0 - \delta\theta \cdot x_1) = \frac{\theta \cdot x_0 - \delta\theta \cdot x_1}{1-\delta},\end{aligned} \tag{8.5}$$

where we have obtained the last equality by letting T converge to infinity and using condition 6 in Assumption F1.

Setting $t = 1$ in equality (8.5), we get $\theta \cdot x_0 = \theta \cdot x_1$. Hence, for any $t \geq 0$, we have $\theta \cdot x_t = \theta \cdot x_0$. As $\psi(x_t) = \psi(x_0)$, this also implies that $F(x_t) = F(x_0)$.

(2) Assume the contrary, that is that, for every $t \geq 0$, we have $\psi(x_t) \leq \psi(x_0)$. Repeating the arguments of part 1, we obtain that the constant path $(x_0, x_0, \ldots)$ is optimal, a contradiction.

8.3 Proof of Proposition 3.2

(1) We observe that (3.2) holds with x_0, but also with x_t. Consider the case $\theta \cdot x_t \leq F(x_t)$. Using (3.1), we obtain along the optimal path:

$$
\begin{aligned}
&u\left(F\left(x_t\right)-\theta \cdot x_t\right) \leq (1-\delta)\, v\left(x_t\right) = (1-\delta) \sum_{s=0}^{\infty} \delta^s u\left(c_{t+s}\right) \\
&\leq u\left((1-\delta) \sum_{s=0}^{\infty} \delta^s c_{t+s}\right) = u\left((1-\delta)\, F\left(x_t\right) + (1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s+1}\right)\right).
\end{aligned}
$$

This implies

$$
F\left(x_t\right)-\theta \cdot x_t \leq (1-\delta)\, F\left(x_t\right) + (1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s+1}\right), \tag{8.6}
$$

which is equivalent to

$$
\psi\left(x_t\right) \leq (1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s+1}\right). \tag{8.7}
$$

In the case where $\theta \cdot x_t > F(x_t)$, it is clear that (8.6) holds, and we obtain (8.7).

(2) Using the result in part 1, we get

$$
\begin{aligned}
&(1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s}\right) \\
&= (1-\delta)\, \psi\left(x_t\right) + \delta\,(1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s+1}\right) \\
&\leq (1-\delta)\left[(1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s+1}\right)\right] + \delta\left[(1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s+1}\right)\right] \\
&= (1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s+1}\right).
\end{aligned}
$$

8.4 Proof of Corollary 4.1

By the definition of x^M, for any optimal path $(x_t)_{t=0}^{\infty}$, we have $\psi\left(x_t\right) \leq \psi\left(x^M\right)$ for any t. If $x_0 = x^M$, then $\psi\left(x_t\right) \leq \psi\left(x_0\right)$ for any t. Proposition 3.1 holds, that is $x_0 = x^M$ is a steady state.

8.5 Proof of Proposition 4.1

Let a bounded optimal path exist.

(1) First, we prove that if $x^* \in S^*$, then x^* is a steady state.

Suppose the contrary: x^* is not a steady state. Take any optimal path $(x_t)_{t=0}^{\infty}$ and the subsequence $(x_{t_n})_{n=0}^{\infty}$ such that $\lim_{n\to\infty} x_{t_n} = x^*$. For each n, consider the sequence $\chi_{t_n} \equiv (x_{t_n}, x_{t_n+1}, x_{t_n+2}, \dots)$. By the compactness of the economy with respect to the product topology, there exists a subsequence $\left(\chi_{t_{n_k}}\right)_{k=0}^{\infty}$ of $(\chi_{t_n})_{n=0}^{\infty}$ that converges to some sequence χ^* in this topology. Let $\chi^* \equiv (x_0^*, x_1^*, \dots)$. The convergence in product topology means that for any $t \geq 0$, $\lim_{n\to\infty} x_{t_{n_k}+t} = x_t^*$. By the upper hemi-continuity property of the optimal policy correspondence σ, the sequence χ^* is an optimal path starting from $x_0^* = \lim_{n\to\infty} x_{t_n} = x^*$.

Because of Proposition 3.1 and the assumption that x^* is not a steady state, there exists some T such that $\psi\left(x_T^*\right) > \psi\left(x^*\right)$. By the convergence in the product topology, $\lim_{n\to\infty} x_{t_{n_k}+T} = x_T^*$. Hence, there exists a sufficiently large k such that $\psi\left(x_{t_{n_k}+T}\right) > \psi\left(x^*\right) = \sup_{t\geq 0} \psi\left(x_t\right)$, a contradiction.

Consider x^*, that is a steady state. Since $(x^*, x^*, \dots)$ is a bounded optimal path beginning from x^*, by the definition of S^*, $x^* \in S^*$.

(2) Assume that a critical date T exists such that $\psi\left(x_T\right) = \sup_{t\geq 0} \psi\left(x_t\right)$. The statement is a direct consequence of Proposition 3.1.

8.6 Proof of Theorem 5.1

(1) Fix an optimal path $(x_t)_{t=0}^{\infty}$ and assume that it is bounded. Let $s^* \equiv \sup_{t\geq 0} \psi\left(x_t\right)$.

We will prove that the sequence $(x_t)_{t=0}^{\infty}$ converges to S^*, in the sense that, for any $\varepsilon > 0$, there exists T such that, for $t \geq T$, $x_t \in S^* + B\left(0, \varepsilon\right)$, where $B\left(0, \varepsilon\right)$ is the sphere of radius ε centered in 0^d.

Suppose the contrary. Then there exists some $\varepsilon > 0$ and a subsequence $(x_{t_n})_{n=0}^{\infty}$ such that, for any n , $x_{t_n} \notin S^* + B\left(0, \varepsilon\right)$. Let $s_* = \sup_{n\geq 0} \psi(x_{t_n})$.

We claim that $s_* < s^*$. Indeed, assume the contrary, $s_* = s^*$. From the boundedness of the optimal sequence, there exists a subsequence $\left(x_{t_{n_k}}\right)_{k=0}^{\infty}$ which converges to some $\hat{x}$. By definition of $(x_{t_n})_{n=0}^{\infty}$, we have $\hat{x} \notin S^*$, while, if $s_* = s^*$, then $\hat{x}$ meets the two conditions for $\hat{x} \in S^*$, a contradiction. Hence, $s_* < s^*$.

Let us prove that

$$\lim_{t\to\infty} \left[(1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s}\right) \right] = s^*.$$

Indeed, by part 2 of Proposition 3.2, the sequence $\left((1-\delta)\sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s}\right)\right)_{t=0}^{\infty}$ is increasing. The following limit exists and it is not larger than s^*:

$$\lim_{t\to\infty} \left[(1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s}\right) \right] \leq s^*. \tag{8.8}$$

Either there is T such that $\psi\left(x_T\right) = s^*$ and, according to part 2 of Proposition 4.1, $\psi\left(x_t\right) = \psi\left(x_T\right)$ for any $t \geq T$, or there is a subsequence $(x_{\tau_n})_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \psi\left(x_{\tau_n}\right) = s^*$.

In the first case,

$$s^* = \psi\left(x_T\right) = (1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{T+s}\right) = \lim_{t\to\infty} \left[(1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s}\right) \right].$$

In the second case, according to parts 1 and 2 (monotonicity property) of Proposition 3.2, we get

$$s^* = \lim_{n\to\infty} \psi\left(x_{\tau_n}\right) \leq \lim_{n\to\infty} \left[(1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{\tau_n+s+1}\right) \right] = \lim_{t\to\infty} \left[(1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s}\right) \right],$$

and, according to (8.8), again $\lim_{t\to\infty} \left[(1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t+s}\right)\right] = s^*$.

For every n, we have

$$\begin{aligned} (1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t_n+s}\right) &= (1-\delta)\, \psi\left(x_{t_n}\right) + \delta\, (1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t_n+s+1}\right) \\ &\leq (1-\delta)\psi(x_{t_n}) + \delta s^*, \end{aligned}$$

which implies

$$\lim_{n\to\infty} \left[(1-\delta) \sum_{s=0}^{\infty} \delta^s \psi\left(x_{t_n+s}\right) \right] \leq \lim_{n\to\infty} \left[(1-\delta)\, \psi\left(x_{t_n}\right) + \delta s^* \right] \leq (1-\delta) s_* + \delta s^* < s^*,$$

a contradiction. Thus, any bounded optimal path $(x_t)_{t=0}^{\infty}$ converges to S^*.

(2) Let us prove that, for any bounded optimal path $(x_t)_{t=0}^{\infty}$, $\lim_{t\to\infty} \psi(x_t) = \sup_{t\geq 0} \psi(x_t)$.

Assume that there exists a subsequence $(x_{t_n})_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \psi(x_{t_n}) = \underline{s} < s^*$.

Using the same arguments as in the proof of the first part of the Theorem, for every n, we have

$$\begin{aligned}(1-\delta)\sum_{s=0}^{\infty}\delta^s\psi\left(x_{t_n+s}\right) &= (1-\delta)\,\psi\left(x_{t_n}\right)+\delta\left(1-\delta\right)\sum_{s=0}^{\infty}\delta^s\psi\left(x_{t_n+s+1}\right)\\ &\leq (1-\delta)\psi(x_{t_n})+\delta s^*.\end{aligned}$$

Hence, $\lim_{t\to\infty}(1-\delta)\sum_{s=0}^{\infty}\delta^s\psi\left(x_{t_n+s}\right)\leq(1-\delta)\underline{s}+\delta s^* < s^*$, a contradiction.

8.7 Proof of Proposition 5.1

Fix an optimal path $(x_t)_{t=0}^{\infty}$. By Theorem 5.1, if this sequence is bounded, then it converges to the set of steady states.

Consider the case of an unbounded optimal path $(x_t)_{t=0}^{\infty}$, with $\sup_{t\geq 0}\|x_t\| = \infty$. We will prove that $\lim_{t\to\infty}\|x_t\| = \infty$.

By part 1 of Assumption A1, we have $\sup_{t\geq 0}\psi(x_t) = \infty$. Using Proposition 3.2,

$$\limsup_{t\to\infty}\left[(1-\delta)\sum_{s=0}^{\infty}\delta^s\psi\left(x_{t+s}\right)\right]\geq \limsup_{t\to\infty}\psi(x_t) = \infty.$$

Applying again Proposition 3.2, we obtain

$$\lim_{t\to\infty}\left[(1-\delta)\sum_{s=0}^{\infty}\delta^s\psi\left(x_{t+s}\right)\right] = \limsup_{t\to\infty}\left[(1-\delta)\sum_{s=0}^{\infty}\delta^s\psi\left(x_{t+s}\right)\right] = \infty.$$

By part 2 of Assumption A1, the existence of a subsequence $(x_{t_n})_{n=0}^{\infty}$ that is bounded in a compact set implies

$$\liminf_{n\to\infty}\left[(1-\delta)\sum_{s=0}^{\infty}\delta^s\psi\left(x_{t_n+s}\right)\right] < \infty,$$

a contradiction. The sequence $(x_t)_{t=0}^{\infty}$ diverges to infinity, with $\lim_{t\to\infty}\|x_t\| = \infty$.

8.8 Proof of Proposition 6.1

By Euler equation and the differentiability of F, a strictly positive steady state is solution to $\delta F'(k) = 1$. Since $\delta F'(k) > 1$ for $0 < k < k_I$, and F is strictly concave in $[k_I, \infty)$, the positive solution to this equation is unique.

If $\delta F'(0) \geq 1$, then $\delta F'(k) > 1$ for any $0 < k \leq k_I$, where k_I denotes the inflection point. Hence, the function $\psi(k) = \delta F(k) - k$ is strictly increasing in $[0, k_I]$ and we have $\psi(k) > \psi(0)$ for any $0 < k \leq k_I$. By the second part of Theorem 5.1, any optimal path with initial condition $0 < k_0 \leq k_I$ does not converge to 0, but to a strictly positive steady state k^*.

Consider now an optimal path starting from $k_0 > k_I$. Using Theorem 1 and Corollary 1 in Dechert and Nishimura [4], we have that this path is bounded from below by k^*.

8.9 Proof of Lemma 6.2

(1) ψ is strictly convex in $[0, k_I]$ because F is strictly convex in the same interval. Hence, $\psi'(k_s) = 0$ implies that $k_s = \arg\min_{0 \leq k \leq k_I} \psi(k)$. The property that $\psi(k)$ is increasing in completes the proof.

(2) An intermediate discounting implies $\psi(k^s) > \psi(0)$. From the strict concavity of ψ on $[k_I, \infty)$ and $\psi'(k^s) = 0$, we have $\psi(k^s) = \max_{k_I \leq k} \psi(k)$. Then, $\psi(k^s) > \psi(k_I)$ and $\psi(k^s) > \max\{\psi(k_I), \psi(0)\} = \max_{0 \leq k \leq k_I} \psi(k)$. According to Corollary 4.1, k^s is also a steady state.

(3) A strong discounting implies $\psi(k^s) < \psi(0) = 0$. From the strict concavity of ψ on $[k_I, \infty)$ and $\psi'(k^s) = 0$, we have $\max_{k_I \leq k} \psi(k) = \psi(k^s) < 0$. Then, $\psi(k_I) < \max_{k_I \leq k} \psi(k) < 0$. The strict convexity of ψ on $[0, k_I]$ jointly with $\psi(k_I) < 0 = \psi(0)$ ensures that $\psi(k) < 0$ for any $k \in (0, k_I]$.

8.10 Proof of Proposition 6.2

Let k_s be a steady state. Consider an optimal path $(k_t)_{t=0}^{\infty}$ starting from k_0. If $k_0 < k_s$, by Theorem 1 in Dechert and Nishimura [4], $k_t < k_s$ for any t. This path does not converges to k_s since, according to point 1 of Lemma 6.2, $\psi(k_t) > \psi(k_s)$ for any t and, according to point 2 of our Theorem 5.1, $\lim_{t\to\infty} \psi(k_t) = \sup_{t\geq 0} \psi(k_t)$.

If this path converges to some $0 < k < k_s$, then k is a strictly positive steady state. Euler equation implies that $F'(k) = 1/\delta$ and $\psi'(k) = 0$. But k_s is the lower solution to the equation $F'(k) = 1/\delta$, a contradiction. Hence, $\lim_{t\to\infty} k_t = 0$.

If $k_0 > k_s$, by Theorem 1 in Dechert and Nishimura [4], we have $k_t > k_s$ for any t.

This path does not converges to k_s since, according to point 1 of Lemma 6.2, $\psi(k_t) > \psi(k_s)$ for any t and, according to point 2 of our Theorem 5.1, $\lim_{t\to\infty} \psi(k_t) = \sup_{t\geq 0} \psi(k_t)$. Thus, this optimal sequence converges to a steady state $k > k_s$. According to the Euler equation, since k is a strictly positive steady state, $F'(k) = 1/\delta$. But k^s is the higher solution to the equation $F'(k) = 1/\delta$, a contradiction. Hence, $\lim_{t\to\infty} k_t = k^s$.

8.11 Proof of Lemma 6.3

Under the intermediate discounting, $\delta < 1/F'(0)$, that is $\psi'(0) < 0$, and $\psi(k^s) > \psi(0) = 0$. Thus, there exists a strictly positive solution $\underline{k} > k_s$ to the equation $\psi(k) = 0$. Let $(k_t)_{t=0}^{\infty}$ be an optimal path starting from $\underline{k}$.

Recall that there are three candidates for the limit point of this sequence: 0, k_s and k^s.

Assume that this path converges either to 0 or k_s. By point 2 of Proposition 3.1, point 2 of Theorem 5.1 and the fact that $\underline{k}$ is not a steady state, the sequence $(\psi(k_t))_{t=0}^{\infty}$ increases strictly to its supremum. We then obtain $\psi(\underline{k}) < \lim_{t\to\infty} \psi(k_t) \leq 0$, a contradiction. Hence, $\lim_{t\to\infty} k_t = k^s$.

Starting from $k_0 \geq \underline{k}$, by Theorem 1 in Dechert and Nishimura [4], the optimal path is bounded away from k_s and, therefore, it converges to k^s.

8.12 Proof of Proposition 6.3

If $k^s/F(k^s) < \delta < 1/F'(0)$ (intermediate discounting), then $F'(0) < 1/\delta < F(k^s)/k^s \leq \max_{k>0}[F(k)/k]$ and Theorem 3.4.2 in Le Van and Dana [8] applies.

8.13 Proof of Proposition 6.4

(1) Fix $0 \leq k_0 \leq k^s$ and $k_0 \neq k_s$. Since $\psi(k_0) > \psi(k_s)$, because of point 2 of Theorem 5.1, any optimal path starting from k_0 converges either to 0 or to k^s. If k^s is not a steady state, then 0 is the unique candidate.

According to Corollary 1 in Dechert and Nishimura [4], every optimal path is monotonic. Consider $k_0 \in (k_s, k^s)$. The convergence of any optimal path starting from k_0 to 0 implies that any optimal path starting from k_s converges also to 0 by Theorem 1 in Dechert and Nishimura [4]. Since neither k^s nor k_s are steady states, any optimal path converges to 0.

(2) If $1/F'(k_I) < \delta < k^s/F(k^s)$ (strong discounting), then $F(k^s)/k^s < 1/\delta < F'(k_I)$. Then, $\max_{k>0}[F(k)/k] < 1/\delta$ and part 2 of Theorem 3.4.3 in Le Van and Dana [8] applies.

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