WORKING PAPERS

2023-08

An α-MaxMin Utility Representation for Close and Distant Future Preferences with Temporal Biases

Jean-Pierre Drugeon, Thai Ha-Hui

CEPS
Centre for Economics at Paris-Saclay

An α-MaxMin Utility Representation for Close and Distant Future Preferences with Temporal Biases*

Jean-Pierre Drugeon[†] and Thai Ha-Huy[‡]

October 27, 2023

*The authors would like to thank Gaetano Bloise, Alain Chateauneuf, Rose-Anne Dana, and Joanna Franaszek for their insightful comments. Jean-Pierre Drugeon would like to thank PGSE and the support of the *Ambiguity in Dynamic Environments* ANR (co-funded with Queen Mary University, Bielefeld University and Kyoto University) research project. Thai Ha-Huy would like to thank the Labex MME-DII (ANR-11-LBX-0023-01) and Paris School of Economics (campus Jourdan) for support during the writing of this article.

†Paris School of Economics and Centre National de la Recherche Scientifique. Email: jean-pierre.drugeon@cnrs.fr

‡Université Paris-Saclay, Univ Evry, EPEE, 91025, Evry-Courcouronnes, France. Email: thai.hahuy@univ-evry.fr

ABSTRACT

This paper provides a framework for understanding preferences over utility streams across different time periods. We analyze preferences for the close future, for the distant future, and a synthesis of both, establishing a representation involving weights over time periods. Examining scenarios where two utility streams cannot be robustly compared to each other, we introduce notions in which one has more "potential" to be preferred over another, which lead to MaxMin, MaxMax, and α-MaxMin representations. Finally, we consider temporal bias in the form of violations of stationarity. For close future preferences, we obtain a generalization of quasi-hyperbolic discounting. For distant future preferences, we obtain Banach limits and discuss the relationship with exponential discounting.

KEYWORDS: Axiomatisation, Myopia, Multiple Discounts, α-MaxMin Citeria, Temporal Biases, Banach Limits, Infinite Dimensional Topologies.

JEL CLASSIFICATION: D11, D15, D90.

CONTENTS

1. Introduction

1.1 Motivation and results

Imagine a scenario where a government is required to assess very long-term policies, such as environmental ones. One of its goals is to strike a balance between the welfare of the present generations and that of far remote future generations. The government may rely on the advice of a group of experts to evaluate the close and distant future values of the projects. When evaluating very long-term projects, a scientific committee of this kind can include economics, politicians, environmentalists, and even philosophers. Therefore, pointing out that opinions would significantly differ is anything from surprising. Based on these recommendations, it is assumed that the government establishes two orders $\succeq_c$ and $\succeq_d$ representing its preferences for the near and the far remote future.

In Section 2, we characterize fundamental properties of temporal orders. In the evaluations under the close future order $\succeq_c$, the far remote future becomes negligible. Following a substantial body of preferences in the literature, the utility levels of a finite number of generations end up mostly defining the value of the close future. The distant future order, $\succeq_d$, as for itself, displays a radically distinct behavior: the preferences would not change if the utility levels were changed for a finite number of generations. Finally, the government's total order $\succeq$, which incorporates both the close and the distant futures, synthesizes these two classes of preferences.

Under standard conditions of axiomatic intertemporal literature, namely translation invariance and positive homogeneity, Lemma 2.1 states that such orders can be represented by index functions that are constantly additive and positively homogeneous. By adding the natural Pareto condition that the options preferred by both $\succeq_c$ and $\succeq_d$ are also preferred by the total order $\succeq$, in Proposition 2.1, the index function of $\succeq$ can be represented by a convex combination of the two other ones. Interestingly, the parameter of the convex combination is not a constant but depends on the utility stream at stake. Two configurations emerge: a first where the economic agent desires to smooth the difference between the close and the distant futures and a second that corresponds to the opposite behavior.

In Section 3, we present the robust preorders $\succeq_c^*$ and $\succeq_d^*$ to further investigate the potential for an expert comparison that is unanimous. A given utility stream is robustly superior to another under an order belonging to $\{\succeq_c, \succeq_d\}$ if and only if this comparison does not depend on the reference stream, in the sense that combining two streams with a third one does not affect the comparison between them. As a result, in Propositions 3.1 and 3.2, each order can be characterized by a set of probabilities such that the robust comparison is equivalent to the agreement of every evaluation under these probabilities. In other words, a robust comparison is possible if and only if every expert agrees to it.

These robust pre-orders are generally incomplete and can be represented by a set of weights over time periods, which reflects the diversity of experts' opinions. For the close future pre-order $\succeq_c^*$, this result is presented in Proposition 3.1. The set of weights is a subset of countably additive probabilities, representing different discount rate systems.[1] For the distant future pre-order $\succeq_d^*$, by Proposition 3.2, the corresponding set builds from a set purely finitely additive probabilities.[2]

For each order $\succeq_c$ and $\succeq_d$, the difference in opinions of experts naturally leads to situations where two utility streams are not robustly comparable, but one has more potential to be preferred than the other one. In Section 3.2, we present two categories of *potentially better* properties. In the first category, a utility stream is thought to have a greater potential to be preferred if it is robustly better to every constant stream that is robustly dominated by the other one. Additionally, in the second category, if a stream is not robustly worse than every constant stream that is not robustly better than the other, it has a greater chance of being preferred.

Proposition 3.3 presents some important representations of preferences. We obtain a MaxMin criterion, where the value of a utility stream is determined by the worst

[1] A countably additive probability can be presented as a real sequence $(\omega_0, \omega_1, \omega_2, \dots)$ such that $\omega_s \geq 0$ for every s and $\sum_{s=0}^{\infty} \omega_s = 1$.

[2] They are also known under the name *charges*, see Bhaskara Rao and Bhaskara Rao (1983). The evaluation of a utility stream under a purely finitely additive probability does not change if we change only a finite number of values in this stream. A detailed definition is given in Appendix **A**.

evaluation, under the assumption that having more potential in the first category implies being favored. We also obtain a MaxMax criterion, where only the best evaluation is taken into account, if a stream that has greater potential in the second category is preferred. Under a more demanding condition than the aforementioned ones, requiring that having more potential in both categories implies being better, we obtain an α-MaxMin representation that encompasses the MaxMin and MaxMax criteria as special cases.

In Section 4, we assume a weakened version of *stationarity* to address the potential for present biases.[3] The evaluation of each expert satisfies a *delay*-stationary property in the sense that, the comparison between two sequences does not depend upon the beginning date if this date is sufficiently far into the future. Therefore, a delayed equivalence assumption, axiom **A2**, is taken into consideration in the case of a close future order. According to this, for every stream, there exists a delayed stream that preserves its robustly improving capacity while mixing with another delayed one. A generalization of the quasi-hyperbolic discounting representation is established in Proposition 4.1. The set that characterizes the robust order is a convex hull of discount rates systems satisfying the following property: from a date in the future, the rate of trade-off between a date and its subsequence one becomes constant.

Under the same delay-stationarity assumption applied to the close future, in the case of purely finitely additive measures that characterize the distant future robust preorder, Proposition 4.2 states that these measures belong to the set of *Banach limits*, which are linear functions on the set ℓ_∞[4] with a special property that the evaluation of a utility stream under a Banach limit does not change if it is shifted one (or many) period(s) to the future.[5] This property echoes the close future evaluation under exponential discounting for which the comparison between two streams does

[3]In inter-temporal axiomatic literature, stationarity is usually understood as the comparison between two streams is the same if we move them to any date in the future.

[4]The set of bounded real sequences $(x_0, x_1, x_2, \dots)$ such that $\sup_{s\geq 0} |x_s| < \infty$.

[5]For every $b \in \mathbb{R}$ and $x, y \in \ell_\infty$, using a Banach limit, the comparison between $(b, x_0, x_1, x_2, \dots)$ and $(b, y_0, y_1, y_2, \dots)$ is the same as between x and y. For intuition about these objects, one can have in mind the infimum limit *liminf* and the supremum limit *limsup* of utility streams. These functions satisfy every property of Banach limits, minus the linearity. For a rigorous definition, see page 55 in Becker and Boyd (1997).

not change if we shift them to the same date in the future. This property makes Banach limits the counterpart of exponential discounting in the evaluation of the close future.

1.2 Contribution and related literature

The pioneering work by Hurwicz (1951) maked the inception of the α-MaxMin expected utility,[6] a criterion that takes the form of a convex combination of the best and worst evaluations, as the choice of a statistician facing a class of possible probabilities. In an article that is considered as a "Big Bang in decision theory after Savage",[7] Gilboa and Schmeidler (1989) introduced an approach to taking decisions under uncertainty and the associated MaxMin criteria. By using two distinct approaches, Kopylov (2003) and Ghirardato et al. (2004) established the α-MaxMin representation, where α represents the ambiguous attitude.[8] The recursive version for α-MaxMin preference in a continuous time configuration was explored by Beißner et al. (2020) as an application in optimization problems, and the differentiability aspects were examined by Beißner and Werner (2021).

Chambers and Echenique (2018) analyzed how regular discounting criteria could reconcile conflicting viewpoints held by different experts; however, another stream of the literature focuses on phenomena related to temporal biases. This led to a growing interest in the quasi-hyperbolic discounting representation, initially introduced by Phelps and Pollack (1968) and more recently analyzed by Laibson (1997) and Montiel Olea and Strzalecki (2014).

This study's main objective is to examine the applicability of such representation while dealing with discounted infinite utility streams. We develop the analysis in three regards.

[6]For experiments finding support for Hurwicz's predictions, see the recent contribution by Bleichrodt et al. (2023).

[7]See Karni et al. (2022).

[8]While Kopylov (2003) assumes that the decision-maker conforms to the expected utility on a set called *unambiguous acts*, Ghirardato et al. (2004), using the *unambiguous preference*, derived the set of possible probabilities and the degree of ambiguity attitude. For a survey, see Trautmann and van de Kuilen (2015).

First, we consider the arbitrarily remote components of the utility streams. The purpose is to provide alternative representations that would complement the close future, which is largely used in most researches. In this sense, we add an additional concept to the one that is well described in Brown and Lewis (1981) and Sawyer (1988) and is referred to as *myopic economic agent*.[9] While in Gilles (1989), purely finitely additive measures indicate the presence of asset bubbles in exchange and production economies, in our article, they characterize the distant future order and its index function being presented in Drugeon and Ha-Huy (2022).

Our second concern is about the fact that, usually, the decision is based on a set of different weight systems.[10] Then, we present an approach that aggregates such differences and extends the usual MaxMin criterion to the more general class of α-MaxMin criteria. This effort echoes contributions of α-MaxMin in literature on multiple probabilities, such as Frick et al. (2022) and Chateauneuf et al. (2021). Our "potentially better" categories are similar to the notions of "potential" and "security-potential dominance" presented in Frick et al. (2022). We define our notion using the robust order whereas Frick et al. (2022) based their definition on a fundamental being called *objective rational preference*.

The third concern of this article is to encompass general temporal biases within a multiple discounting configuration and present a generalized version of quasi-hyperbolic representation, passing through the robust order and the delay stationarity axiom. In a recent contribution, Bach et al. (2023) generalized the system of axioms in Chambers and Echenique (2018) to the scope for one-step present bias and quasi-hyperbolic discounting, with properties named *delay-Invariance to stationary relabeling* and *Existence of a future reference plan* as key axioms.[11] The

[9] A myopic agent attributes very small values for utilities in the distant future. A very strong version of myopia, where the agent cares only about what happens before some fixed date but is indifferent afterward, is studied in de Andrade et al. (2021).

[10] For example, in situations with the lack of available information, the ambiguity about the system of appropriate discount rate, or the difference in opinions of the experts that the economic agent consults, etc.

[11] In a different approach, Drugeon and Ha-Huy (2023) focused on recursive time-dependent orders as Wakai (2007), where the discount rate is chosen in *each* period by a comparison between the utility value of the present and that of the future.

similarity between this article and Bach et al. (2023) is that, in order to establish quasi-hyperbolic representation or its generalization, they both require a stationarity property that starts at some fixed date in the future.

1.3 Contents

Section 2 describes the Pareto aggregation of close future and distant future preferences. Section 3 presents robust pre-orders and conditions for MaxMin, MaxMax, and α-MaxMin representations. Section 4 strengthens these results by incorporating them in a temporally-biased multiple discounts configuration. Section 5 concludes the article. The mathematical preparations and the proofs are given in the Appendix.

2. Pareto aggregation of close future and distant future preferences

This paper adopts an axiomatic approach to the evaluation of bounded utility streams in a discrete-time configuration. Letters such as x, y, z will be used for streams of utilities with values in $\mathbb{R}$. Denote by ℓ_∞ the set of bounded real sequences.

Notation $\mathbf{1}$ will denote the constant stream $(1, 1, \ldots)$. Similarly, $b\mathbf{1}$ and $b^*\mathbf{1}$ will be used for constant streams $(b, b, \ldots)$ and $(b^*, b^*, \ldots)$. The notations λ, μ, and χ will be used for constant scalars.

For every $x \in \ell_\infty$ and $T \geq 0$, let $x_{[0,T]} = (x_0, x_1, \ldots, x_T)$ be its *head* $T+1$ first components and $x_{[T+1,\infty)} = (x_{T+1}, x_{T+2}, \ldots)$ its *tail* starting from date $T+1$. Given sequences x and y, the sequence $(y_{[0,T]}, x_{[T+1,+\infty)})$ denotes $(y_0, y_1, \ldots, y_T, x_{T+1}, x_{T+2}, \ldots)$. The sequence $(y_{[0,T]}, x)$ denotes $(y_0, y_1, \ldots, y_T, x_0, x_1, x_2, \ldots)$. By convention, if $T = -1$, let $(y_{[0,T]}, x_{[T+1,+\infty)})$ be the sequence $x = (x_0, x_1, x_2, \ldots)$.

An economic agent evaluates utility streams that belong to ℓ_∞, while attempting to strike a balance between the welfare of the close and the distant futures. She has two original preferences, which are represented by the close future order $\succeq_c$ and the distant future order $\succeq_d$. The distant future order has an opposite tendency and ignores the close future, whereas the close future order ignores the distant future. The economic agent establishes her main preference, $\succeq$, based on these two preferences, synthesizing the comparisons made by $\succeq_c$ and $\succeq_d$.

As a constructive example, imagine a situation where a government has to evaluate very long-run policies. The government may rely on the advice of a group of economists, environmentalists, politicians, or even philosophers to evaluate the values of the close and distant future of projects. Each expert shares her opinion on an appropriate discount rate system for the close future as well as some rules for the calculus of the distant future. The evaluation of the close and distant future will be determined based on the experts' advice.

Any expert will recognize that the rule must satisfy two basic properties: positive homogeneity of degree one and additivity. These guarantee that the properties *homogeneity* and *additivity* are satisfied. According to homogeneity, an expert's comparison of two streams after multiplying a positive scalar by both of them remains unchanged. Additivity ensures that by adding a third stream, the expert's comparison of two streams remains intact. In other words, the preferences of an expert do not depend on the reference stream. After having consulted the experts, the preferences are described by two completed orders: $\succeq_c$ and $\succeq_d$.

The opinions of such experts obviously differ due to their diverse experiences, and the purpose of this article is to propose an axiomatic approach that aggregates their disparate preferences.

We now return to the basic properties concerning preferences. The intuition behind the orders in the above illustration is presented in Axiom **F1**. For any order $\widehat{\succeq} \in \{\succeq, \succeq_c, \succeq_d\}$, let $x \widehat{\sim} y$ denote $x \widehat{\succeq} y$ and $y \widehat{\succeq} x$. The order $\widehat{\succeq}$ is nontrivial if there is x and y such that $x \succeq y$ and $y \not\succeq x$. Let the notation be presented as $x \widehat{\succ} y$.

Axiom **F 1.** Every order $\widehat{\succeq}$ belonging to $\{\succeq_c, \succeq_d, \succeq\}$ satisfies the following properties.[12]

[12] These properties are well-known in the literature, and an interested reader may find a detailed discussion about their significance in Chambers and Echenique (2018), Bach et al. (2023), or Drugeon and Ha-Huy (2022).

(i) *Completeness*, *transitivity* and *monotonicity*: For every $x, y \in \ell_\infty$, either $x \widehat{\succsim} y$ or $y \widehat{\succsim} x$. If $x \widehat{\succsim} y$ and $y \widehat{\succsim} z$, then $x \widehat{\succsim} z$. If $x_s \geq y_s$ for every $s \in \mathbb{N}$, then $x \widehat{\succsim} y$.

(ii) *Archimedeanity*: For $x \in \ell_\infty$ and real values b, b' satisfying $b\mathbf{1} \,\widehat{\succ}\, x \,\widehat{\succ}\, b'\mathbf{1}$, there are $\lambda, \mu \in (0,1)$ such that $(1-\lambda)b\mathbf{1} + \lambda b'\mathbf{1} \,\widehat{\succ}\, x \,\widehat{\succ}\, (1-\mu)b\mathbf{1} + \mu b'\mathbf{1}$.

(iii) *Constant additivity*: For every stream x, y, constants b and scalar $\lambda \in [0,1)$, $x \widehat{\succsim} y$ if and only if $(1-\lambda)x + \lambda b\mathbf{1} \widehat{\succsim} (1-\lambda)y + \lambda b\mathbf{1}$.

(iv) *Nontriviality of the main order*: There exist $x, y \in \ell_\infty$ such that $x \succ y$.

The completeness, transitivity, and monotonicity properties are standard in the literature. Archimedeanity ensures continuity with respect to the sup-norm topology in ℓ_∞.

The constant additivity property is admittedly less immediate. It guarantees that constant streams are comparison-neutral: mixing two sequences with a constant stream does not alter their comparison. In decision theory literature, this property is known as *certainty independence*. In the context of intertemporal preferences, this condition states that the utility level of different dates is given in the same unit. As in Chambers and Echenique (2018), this allows us to perform interpersonal (or in the case of this article, intertemporal) comparison of utility.[13]

These conditions, combined with the nontriviality property, have as a consequence that the order $\widehat{\succsim}$ can be represented by an index function $\widehat{I}$: for every x and y in ℓ_∞, $x \widehat{\succsim} y$ if and only if $\widehat{I}(x) \geq \widehat{I}(y)$. The function $\widehat{I}$ satisfies constant additivity and homogeneity of degree 1 properties. Specifically, this function is defined as:

$$\widehat{I}(x) = \sup\Big\{ b \text{ such that } x \widehat{\succsim} b\mathbf{1} \Big\}.$$

For every $x \in \ell_\infty$, $\lambda \geq 0$ and $b \in \mathbb{R}$, we have:[14]

(i) $\widehat{I}(\lambda x) = \lambda \widehat{I}(x)$,

(ii) $\widehat{I}(x + b\mathbf{1}) = \widehat{I}(x) + b$.

[13] See Chambers and Echenique (2018), pages 1331-1332 for a detailed discussion.

[14] The proof of this property can be found in Drugeon and Ha-Huy (2022), Lemma 2.1.

By convention, if the order $\widehat{\succeq}$ is trivial, we let $\widehat{I}(x) = 0$ for every $x \in \ell_\infty$. We then have three index functions $I_c(x), I_d(x)$ and $I(x)$ corresponding to the three orders $\succeq_c, \succeq_d$ and $\succeq$.

The following axiom details the precise properties of these three orders. The close future order satisfies a myopia property, as described in Brown and Lewis (1981): it attributes a very small value for utilities in distant dates. Otherwise, the distant future order attributes no value for the close future. The main order satisfies a version of the Pareto property in aggregating close future and distant future orders.

Axiom **F 2.** The close future, distant future, and main orders respectively satisfy:

(i) *Distant future insensitivity*: Consider the close future order $\succeq_c$. For every $x, y, z \in \ell_\infty$, $\epsilon > 0$, there exists T_0 such that for $T \geq T_0$,

$$\Big(x_{[0,T]}, y_{[T+1,\infty)}\Big) \succeq_c \Big(x_{[0,T]}, z_{[T+1,\infty)}\Big) - \epsilon\mathbf{1}.$$

(ii) *Close future insensitivity*: Consider the distant future order $\succeq_d$. For every $x, y, z \in \ell_\infty$,

$$\Big(y_{[0,T]}, x_{[T+1,\infty)}\Big) \sim_d \Big(z_{[0,T]}, x_{[T+1,\infty)}\Big),$$

for every $T \geq 0$.

(iii) *Consistency*: Consider the main order $\succeq$. For $x, y \in \ell_\infty$, if $x \succeq_c y$ and $x \succeq_d y$, then $x \succeq y$.

It is easy to verify that, under axioms **F1** and **F2**, for every $x, y \in \ell_\infty$:

$$I_c(x) = \lim_{T\to\infty} I_c\Big(x_{[0,T]}, y_{[T+1,\infty)}\Big),$$
$$I_d(x) = I_d\Big(y_{[0,T]}, x_{[T+1,\infty)}\Big), \text{for every } T \geq 0.$$

From now on, we always impose **F1** and **F2** on the three orders $\succeq_c$, $\succeq_d$ and $\succeq$.

The consistency condition requires that the main order never contradicts the close future and the distant future orders when these two orders are in agreement with each other. With the nontriviality of the main order, a direct consequence of consistency is that at least one of the two close future and distant future orders is nontrivial.

More specifically, if the distant future order $\succeq_d$ is trivial, the two orders $\succeq$ and $\succeq_c$ are equivalent. The economic agent cares only about the close future. This is the usual situation of literature where the remote future is negligible. Similarly, if the close future order $\succeq_c$ is trivial, then $\succeq$ and $\succeq_d$ coincide.

We now focus on the relation between the index functions. The evaluation of a utility stream is a convex combination of its close future and distant future values. The parameter of the convex combination is characterized by the two following parameters. Let

$$\chi_g = \lim_{T\to\infty} I(0\mathbf{1}_{[0,T]}, \mathbf{1}_{[T+1,\infty)}),$$
$$\chi_\ell = -\lim_{T\to\infty} I(0\mathbf{1}_{[0,T]}, -\mathbf{1}_{[T+1,\infty)}).$$

These two values can be interpreted as the perception of the economic agent about the importance of constant gains and losses in the distant future. They both belong to the closed interval $[0, 1]$. The case $\chi_g = \chi_\ell = 1$ corresponds to the configuration where the close future order is trivial and $\succeq = \succeq_d$. The opposite case, $\chi_g = \chi_\ell = 0$ implies that the distant future order is trivial and $\succeq = \succeq_c$, a well-known configuration of the literature.

Lemma 2.1 is crucial in the establishment of the formula linking the close and the distant future values of the utility streams. The value of a stream is a convex combination of its close and distant values. The parameter of this convex combination in use is χ_g if the close future value is smaller than the other one. In the reverse situation, it is χ_ℓ that is selected to determine the weight in the convex combination.

Lemma **2.1.** *Consider a stream x.*

(i) *If $I_c(x) \leq I_d(x)$, then*

$$I(x) = (1 - \chi_g)I_c(x) + \chi_g I_d(x).$$

(ii) *If $I_c(x) \geq I_d(x)$, then*

$$I(x) = (1 - \chi_\ell)I_c(x) + \chi_\ell I_d(x).$$

We can establish two different behaviors based on Lemma 2.1. The first one corresponds to the situation where the economic agent desires to smooth the difference between the close future and the distant future values. The second one exhibits the opposite behavior.

Proposition **2.1.** *Consider* χ_g *and* χ_ℓ.

(i) *If* $\chi_g \leq \chi_\ell$, *then, for every stream* x,

$$I(x) = \min_{\chi_g \leq \lambda \leq \chi_\ell} \Big[(1-\lambda) I_c(x) + \lambda I_d(x)\Big].$$

(ii) *If* $\chi_g \geq \chi_l$, *then, for every stream* x,

$$I(x) = \max_{\chi_\ell \leq \lambda \leq \chi_g} \Big[(1-\lambda) I_c(x) + \lambda I_d(x)\Big].$$

As an example, consider the order $\succeq$ that is represented as follows, with $0 < \chi_g < \chi_\ell < 1$ and D a compact subset of $(0,1)$:

$$I(x) = \min_{\chi_g \leq \lambda \leq \chi_\ell} \left[(1-\lambda) \min_{\delta \in D} \left((1-\delta) \sum_{s=0}^{\infty} \delta^s x_s\right) + \lambda \liminf_{s \to \infty} x_s\right].$$

In this example, the initial order $\succeq$ can be decomposed into two suborders $\succeq_c$ and $\succeq_d$ with two associated index functions that are available as:

$$I_c(x) = \min_{\delta \in D} \left((1-\delta) \sum_{s=0}^{\infty} \delta^s x_s\right),$$

$$I_d(x) = \liminf_{s \to \infty} x_s.$$

3. Axiomatization of α-MaxMin representations

We want to study a preorder that represents the unanimous comparison, in the close as well as in the distant future, after the orders $\succeq_c$ and $\succeq_d$ have been established. An important question is: what should this order's principal properties be? A simple answer that comes to mind is that the preorder must preserve the homogeneity and additivity properties, since every expert provides her evaluation rule that satisfies them. Such a line of reflection intuitively leads to the well-known definition of a robust order in decision theory: one stream is robustly better than another one if mixing them with a third one does not alter the comparison.

Definition **3.1.** *Consider an order* $\widehat{\succeq} \in \{\succeq_c, \succeq_d\}$. *Let the pre-order* $\widehat{\succeq}^*$ *be defined by*

$$x\widehat{\succeq}^* y \quad \textit{iff for every } \lambda \in (0,1) \textit{ and } z \in \ell_\infty, \textit{ we have } (1-\lambda)x + \lambda z \widehat{\succeq} (1-\lambda)y + \lambda z.$$

We may also interpret robust dominance as being preferred independently from the reference stream. If we begin from z, it is always better to mix it with x than with y. Generally, this preorder is not complete. This brings to mind Bewley (2002), whose contribution establishes a set of probability distributions rather than a single one, due to a lack of completeness. A choice is robustly better than another one if, calculated by every probability of this set, its expected utility is always better than the other one.

Similarly, for each order $\widehat{\succeq} \in \{\succeq_c, \succeq_d\}$, there exists a set of probabilities (finitely additive or infinitely additive) such that a utility stream x is robustly $\widehat{\succeq}$-better than a utility stream y if and only that is confirmed by every evaluation using these probabilities. In other words, these probabilities characterize the difference between the opinions of the experts.

Section 3.1 will present in detail their characterization.

3.1 Representation of the robust preorders

This approach leads to a characterization of the order $\widehat{\succeq}$ by a set of continuous linear functions on ℓ_∞, similar to Gilboa and Schmeidler (1989) and Bewley (2002). These can be considered as finitely additive measures on the set of natural numbers $\{0, 1, 2, \ldots\}$. Interestingly, they are countably additive in the case of the close future order $\succeq_c$, and purely finitely additive in the case of the distant future order $\succeq_d$. A rigorous definition of these notions is given in Appendix **A**.

3.1.1 Representation of close future order

Axiom **A1** states a strong version of myopia: though there are disagreements between experts about discount rate system to evaluate the close future, their close future evaluations all attribute very small values for sufficiently remote dates. The well-known axiom continuity at infinity in Chambers and Echenique (2018) might be viewed as the close future equivalent of this axiom.

Axiom A 1. For every $b \in (0, 1)$, there exists T_0 such that, for every $T \geq T_0$,

$$\left(\mathbf{1}_{[0,T]}, 0\mathbf{1}_{[T+1,\infty)}\right) \succeq_c^* b\mathbf{1}.$$

Under axiom **A1**, the robust preorder $\succeq_c^*$ is represented by a set of weights Ω_c that builds from countably additive probabilities. To be more precised, Ω_c is a set of sequences $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$ such that $\omega_s \geq 0$ for every s, and $\sum_{s=0}^{\infty} \omega_s = 1$. A probability in this set can be considered a possible system of discount rates that is used to evaluate the close future.

Proposition **3.1.** *Assume that the close future order $\succeq_c$ is nontrivial. Under axiom* **A1**, *there exists a set of discount rate systems $\Omega_c \subset \ell_1$*[15] *that is compact with respect to the weak topology in ℓ_1 and satisfies the two following properties:*

(i) *For every $\omega \in \Omega_c$, $\omega_s \geq 0$, $\forall s$, and $\sum_{s=0}^{\infty} \omega_s = 1$.*

(ii) *For every streams x and y, we have $x \succeq_c^* y$ if and only if*

$$\sum_{s=0}^{\infty} \omega_s x_s \geq \sum_{s=0}^{\infty} \omega_s y_s,$$

for every $\omega \in \Omega_c$.

As a remark, observe that for every stream x,

$$\inf_{\omega \in \Omega_c} \sum_{s=0}^{\infty} \omega_s x_s \leq I_c(x) \leq \sup_{\omega \in \Omega_c} \sum_{s=0}^{\infty} \omega_s x_s.$$

The economic agent attributes to the close future of x a value lying between the worst and the best evaluations of experts.

3.1.2 Representation of distant future order

Since the distant future order $\succeq_d$ does not take into account the close future, the robust preorder $\succeq_d^*$ satisfies that same property and depends only on the distant

[15]The set of absolutely convergent real sequences $(\omega_0, \omega_1, \omega_2, \ldots)$ such that $\sum_{s=0}^{\infty} |\omega_s| < \infty$.

future of the utility streams. It is characterized by a set Ω_d that builds from purely finitely additive properties.

Under the evaluation of a continuous linear function $\phi \in \Omega_d$, the value of a stream x, denoted $\phi \cdot x$, depends only on the distant behavior of x and does not change if there are only a finite number of changes in the values x_s. More precisely, for every stream y and $T \geq 0$, we have

$$\phi \cdot \left(y_{[0,T]}, x_{[T+1,\infty)}\right) = \phi \cdot x.$$

Hence, the value $\phi \cdot x$ can be interpreted as the evaluation under ϕ of the stream x in the distant future.

Proposition **3.2.** *Assume that the distant future order $\succeq_d$ is nontrivial. There exists a set of purely finitely additive probabilities Ω_d such that $x \succeq_d^* y$ if and only if $\phi \cdot x \geq \phi \cdot y$ for every $\phi \in \Omega_d$.*

Observe that for every stream x,

$$\inf_{\phi \in \Omega_d} \phi \cdot x \leq I_d(x) \leq \sup_{\phi \in \Omega_d} \phi \cdot x.$$

As in the case of close future value, the economic agent attributes to the distant future of x a value lying between the worst and the best evaluations of experts.

3.2 MaxMin, MaxMax and α-MaxMin representations

Once the robust preorders $\succeq_c^*$ and $\succeq_d^*$ have been established, a natural interest arises about situations where the two utility streams are not robustly comparable, but one has more potential to be preferred than the other one. We will define two categories of being potentially better for either of the two orders $\succeq_c$ or $\succeq_d$.

The following defines the first category of potentially better. If x is robustly better than every constant stream that is robustly dominated by y, then it is considered as having more potential to be preferred. The second category is defined as x having more potential to be preferred if it is not robustly worse than a constant stream that is not robustly better than y.

Definition **3.2.** *Consider an order $\widehat{\succeq} \in \{\succeq_c, \succeq_d\}$.*

(i) *Under the order* $\widehat{\succeq}$, *x is potentially better than y in the first category if, for every constant b,*

$$y\widehat{\succeq}^* b\mathbf{1} \text{ implies } x\widehat{\succeq}^* b\mathbf{1}.$$

(ii) *Under the order* $\widehat{\succeq}$, *x is potentially better than y in the second category if, for every constant b,*

$$b\mathbf{1}\widehat{\not\succeq}^* y \text{ implies } \widehat{b}\mathbf{1}\widehat{\not\succeq}^* x.$$

Under the axiom ensuring that having more potential in the first category implies to be preferred, we obtain a MaxMin criterion, where the value of a utility stream is determined by the worst evaluation. Similarly, if a stream that is potentially better in the second category is the preferred one, we obtain a MaxMax criterion, where only the best evaluation is taken into account. Under a more demanding condition than the aforementioned ones, requiring that having more potential in both categories implies to be preferred, we obtain an α-MaxMin representation that encompases the MaxMin and the MaxMax criteria as special cases.

Proposition **3.3.** *Consider an order* $\widehat{\succeq} \in \{\succeq_c, \succeq_d\}$ *and its corresponding preorder* $\widehat{\succeq}^*$. *Assume that this order is nontrivial.*

(i) *If for every streams x and y, x being potentially better than y in the first category implies* $x\widehat{\succeq}y$, *then the order* $\widehat{\succeq}$ *has a MaxMin representation:*

$$\widehat{I}(x) = \inf_{P\in\widehat{\Omega}} P\cdot x$$

for every stream x.

(ii) *If for every streams x and y, x being potentially better than y in the second category implies* $x\widehat{\succeq}y$, *then the order* $\widehat{\succeq}$ *has a MaxMax representation:*

$$\widehat{I}(x) = \sup_{P\in\widehat{\Omega}} P\cdot x$$

for every stream x.

(iii) *If for every streams x and y, x being potentially better than y in* both *categories implies* $x\widehat{\succeq}y$, *then the order* $\widehat{\succeq}$ *has an* α-MaxMin *representation: there exists unique* $0 \le \alpha \le 1$ *such that for every stream x,*

$$\widehat{I}(x) = \alpha \sup_{P\in\widehat{\Omega}} P\cdot x + (1-\alpha)\inf_{P\in\widehat{\Omega}} P\cdot x.$$

The MaxMin representation can be considered as a unanimous choice. The economic agent determines that a stream x has at least value b if and only if every expert agrees with that evaluation. We can also consider that the agent is very prudent in following the experts' advice. The MaxMax exhibits the exact opposite behavior in this aspect. The opinion of one expert that x has at least value b is sufficient for the agent to have the same conclusion. The α-MaxMin is a way to balance the two. The agent considers not only the worst but also the best evaluation.

Following two distinct approaches, Chateauneuf et al. (2021) and Frick et al. (2022) are interested in the uniqueness of the α-MaxMin representation. In this regard, the "potentially better" categories in this article echoes in some way the security and potential dominance properties in Frick et al. (2022).

4. Specification of multiple discount sets by stationarity-type axioms

4.1 Temporal bias axiom

In order to properly characterize the sets Ω_c and Ω_d, consider the definition 4.1, which characterizes the impatience and stationary properties of the robust preorders. Fix the order $\widehat{\succeq} \in \{\succeq_c, \succeq_d\}$ and its robust preorder $\widehat{\succeq}^*$. The idea expressed in definition 4.1 represents the intuition that beginning from a sufficiently remote date in the future, the evaluation of every expert becomes stationary. After a certain time, her comparison between two streams no longer depends on the starting date. The temporal bias phenomenon, one of the main causes for the violation of the stationary property, has only a finite range of influence on the experts. In fact, there exists a date such that their evaluations all become stationary afterward.

Definition **4.1.** T^*-delay stationarity*: Let $T^* \geq 0$. The order $\widehat{\succeq}$ satisfies* T^*-delay stationarity *if for every $x \in \ell_\infty$ and a constant b such that*

$$(b\mathbf{1}_{[0,T^*-1]}, x)\widehat{\succeq}^* b\mathbf{1},$$

we have

$$(b\mathbf{1}_{[0,T^*-1]}, x)\widehat{\succeq}^*(b\mathbf{1}_{[0,T^*]}, x)\widehat{\succeq}^* b\mathbf{1}.$$

To be more precise, for an order $\widehat{\succeq}$ satisfying this definition:

(i) Case $T^* = 0$ corresponds to the Stationarity property:

$$x\widehat{\succeq}^* b\mathbf{1} \text{ implies } x\widehat{\succeq}^*(b, x)\widehat{\succeq}^* b\mathbf{1}.$$

(ii) Case $T^* = 1$ corresponds to the Quasi-hyperbolic discounting property:

$$(b, x)\widehat{\succeq}^* b\mathbf{1} \text{ implies } (b, x)\widehat{\succeq}^*(b, b, x)\widehat{\succeq}^* b\mathbf{1}.$$

(iii) Case $T^* \geq 1$ can be considered as a T^*-steps quasi-hyperbolic discounting property:

$$(\underbrace{b, b, \ldots, b}_{T^* \text{ times}}, x)\widehat{\succeq}^* b\mathbf{1} \text{ implies } (\underbrace{b, b, \ldots, b}_{T^* \text{ times}}, x)\widehat{\succeq}^*(\underbrace{b, b, \ldots, b}_{T^*+1 \text{ times}}, x)\widehat{\succeq}^* b\mathbf{1}.$$

The choice to build the impatience and stationarity properties from the comparison of a stream with another constant one is based on the purpose of practicability. It is indeed simpler to ask an economic agent, or to observe her behavior, about whether she values a utility stream of at least b.

In definition 4.1(iii), the comparison $(b\mathbf{1}_{[0,T^*-1]}, x)\widehat{\succeq}^*(b\mathbf{1}_{[0,T^*]}, x)$ characterizes impatience whereas $(b\mathbf{1}_{[0,T^*]}, x)\widehat{\succeq}^* b\mathbf{1}$ features T^*-delay stationarity. In other words, even while the effect according to preorder $\widehat{\succeq}^*$ weakens over time, if a combination is robustly better than a constant sequence, it stays robustly better if it is advanced into the future.

4.2 Temporal bias representation of close future preorder

If the close order $\succeq_c$ satisfies T^*-delay stationary, one can obtain a characterization for the sets discount rate systems Ω. For a stream x, let $\mathcal{C}(x)$ be the supremum of the values $b \in \mathbb{R}$ such that $x \succeq_c^* b\mathbf{1}$. We can interpret $\mathcal{C}(x)$ as the lowest evaluation of the close future of x by experts. A utility stream y may be robustly improved in the close future by mixing with x if we have

$$\mathcal{C}\left(\frac{1}{2}x+\frac{1}{2}y\right)>\mathcal{C}(y).$$

In other words, every expert agrees mixing y with x is a better choice than staying with y. Axiom **A2** states that, there exists a delayed-equivalence of x that keeps its robustly improving capacity on delayed sequences.

Axiom A 2. Let the close future order $\succeq_c$ satisfy the condition in definition 4.1. For a utility stream x, let $b=\mathcal{C}(x)$. There exists a utility stream y such that

(i) $\mathcal{C}\left(\left(b\mathbf{1}_{[0,T^*-1]},y\right)\right)=b,$

(ii) for every stream $\widehat{y}$ satisfying $\mathcal{C}\left(\left(b\mathbf{1}_{[0,T^*-1]},\widehat{y}\right)\right)=b$, one has

$$\mathcal{C}\left(\frac{1}{2}x+\frac{1}{2}\left(b\mathbf{1}_{[0,T^*-1]},\widehat{y}\right)\right)>b \text{ if and only if } \mathcal{C}\left(\frac{1}{2}\left(b\mathbf{1}_{[0,T^*-1]},y\right)+\frac{1}{2}\left(b\mathbf{1}_{[0,T^*-1]},\widehat{y}\right)\right)>b.$$

Let us discuss some intuitions for this property. Stationarity is typically described as the independence of comparisons with the beginning date. In the most common form, stationarity can be expressed as follows: the comparison between x and y is equivalent to the comparison between $\left(b\mathbf{1}_{[0,T]},x\right)$ and $\left(b\mathbf{1}_{[0,T]},y\right)$, with every utility level b and date T. This property plays a pivotal role in establishing exponential discounting, where the assessment of utility streams is conducted using a discount rate system denoted as $(\omega_s)_{s=0}^{\infty}$, with $\omega_s=(1-\delta)\delta^s$, where $\delta\in(0,1)$.

Now, consider a weaker form of stationarity known as the "Invariance to stationary relabeling" property (ISTAT), introduced by Chambers and Echenique (2018). This property assumes that for any x that is equivalent to a constant sequence $b\mathbf{1}$, not only x is equivalent to $\left(b\mathbf{1}_{[0,T]},x\right)$, but this stream is equivalent to any convex combination between x and $\left(b\mathbf{1}_{[0,T]},x\right)$. Formally, in the context of our close future order, $x\sim_c b\mathbf{1}$ implies

$$x\sim_c \frac{1}{2}x+\frac{1}{2}\left(b\mathbf{1}_{[0,T]},x\right),\ \text{for every } T\geq 0.$$

This property plays a central role in establishing the following close future index function:

$$I_c(x)=\inf_{\delta\in\mathcal{D}}\left[(1-\delta)\sum_{s=0}^{\infty}\delta^s x_s\right].$$

Here, the set of possible discount factors $\mathcal{D}$ is a closed subset of $(0, 1)$. The exponential discount rate is no longer unique. In this scenario, each expert's criterion is stationary, but they may differ in the degree to which they discount the (close) future. The economic agent adopts a MaxMin approach by maximizing the worst-case evaluation. Consequently, the condition described in Chambers and Echenique (2018) can be seen as a specific form of stationarity.

In the context of this article, this property implies the following claim:[16] for every $T \geq 0$, if $\mathcal{C}\left(\left(b\mathbf{1}_{[0,T-1]}, \widehat{y}\right)\right) = \mathcal{C}(x) = b$, then

$$\mathcal{C}\left(\frac{1}{2}x + \frac{1}{2}\left(b\mathbf{1}_{[0,T-1]}, \widehat{y}\right)\right) > b \text{ iff } \mathcal{C}\left(\frac{1}{2}\left(b\mathbf{1}_{[0,T-1]}, x\right) + \frac{1}{2}\left(b\mathbf{1}_{[0,T-1]}, \widehat{y}\right)\right) > b. \tag{1}$$

The robustly improving capacity using the utility stream x remains consistent regardless of the starting date. If we can robustly enhance a delayed sequence $\left(b\mathbf{1}_{[0,T-1]}, \widehat{y}\right)$ by mixing it with x, the result is the same when using a delayed version of x, namely, the sequence $\left(b\mathbf{1}_{[0,T-1]}, x\right)$. In other words, the delayed version of x maintains its robustly improving capacity.

In cases where it is assumed that each expert may be subject to temporal bias phenomena, the full stationarity property is not upheld. Consequently, we have an interest in exploring a more lenient version of it. Axiom **A2** captures this bias phenomena, by weakening the property in (1). There are two primary adjustments. First, it delays the point from which stationarity can occur to a fixed date, denoted as T^*. Second, instead of using directly a delayed version of x, this axiom assumes the existence of a T^*-delayed equivalence of x, namely $\left(b\mathbf{1}_{[0,T^*-1]}, y\right)$. This equivalence preserves the robustly improving capacity of x when combined with a delayed stream. If we can robustly improve the delayed sequence $\left(b\mathbf{1}_{[0,T^*-1]}, \widehat{y}\right)$ through mixing with x, the same outcome can be achieved using $\left(b\mathbf{1}_{[0,T^*-1]}, y\right)$.

We then can interpret the results in Proposition 4.1 below as follows: starting from date T^* in the future, the capacity for robust improvement is independent of the stream's initial date. From this date, the evaluation made by each expert become stationary.

[16] In fact, the two properties are equivalent. This equivalence can be easily verified after referring to Proposition 4.1.

Proposition **4.1.** *Assume axiom* **A1**. *Assume also that the order $\succeq_c$ is nontrivial and satisfies the* delay stationarity *property.*

(i) Stationarity: *If $T^* = 0$, then there exists $\mathcal{D} \subset (0,1)$ such that Ω_c is the convex hull of probabilities*

$$\left\{\left(1-\delta, (1-\delta)\delta, \ldots, (1-\delta)\delta^s, \ldots\right)\right\}_{\delta\in\mathcal{D}}.$$

(ii) Quasi-hyperbolic discounting: *Consider the case $T^* = 1$. By adding axiom* **A2**, *there exists $\mathcal{D} \in (0,1)^2$ such that Ω_c is the convex hull of the probabilities*

$$\left\{\left(1-\delta_0, \delta_0(1-\delta), \delta_0\delta(1-\delta), \delta_0\delta^2(1-\delta), \ldots, \delta_0(1-\delta)\delta^s, \ldots\right)\right\}_{(\delta_0,\delta)\in\mathcal{D}}.$$

(iii) T^*-steps quasi hyperbolic discounting: *Consider the case $T^* \geq 1$. By adding axiom* **A2**, *there exists $\mathcal{D} \in (0,1)^{T^*+1}$ such that Ω_c is the convex hull of the set of discount rate systems:*

$$\Big\{\Big(1-\delta_0, \delta_0(1-\delta_1), \delta_0\delta_1(1-\delta_2), \ldots, \delta_0\delta_1\cdots\delta_{T^*-1}(1-\delta), \ldots$$
$$\ldots, \delta_0\delta_1\ldots\delta_{T^*-1}\delta^s(1-\delta), \ldots\Big)\Big\}_{(\delta_0,\delta_1,\ldots\delta_{T^*-1},\delta)\in\mathcal{D}}.$$

In recent research dealing with multiple temporal biased discount rates and operating within a similar axiomatic system configuration of Chambers and Echenique (2018), the authors Bach et al. (2023) established a representation for multiple quasi-hyperbolic discounting. This generalization is founded on two key axioms: "Delayed-Invariance to Stationary Relabeling" (Delayed-ISTAT) and "Existence of a Future Reference Plan" (EFREP).

The delayed-ISTAT can be considered a modification of the ISTAT condition presented in Chambers and Echenique (2018). It asserts that if $(b,x) \sim_c b\mathbf{1}$ holds, then not only should $(b,b,x) \sim_c b\mathbf{1}$ be valid, but also mixing (b,x) with every delayed version $\left(b\mathbf{1}_{[0,T]}, x\right)$ of x does not improve it. Supplementing this property is the EFREP condition, which shares a conceptual alignment with our notion of delayed equivalence. It supposes the existence of a "reference plan" for x in the future, denoted as y, in a way that if the combination with x enhances a 1-period delayed sequence, the same result is achieved by using the 1-period delayed version of y.

The combination of Delayed-ISTAT and EFREP allows the authors to generalize the findings of Chambers and Echenique (2018).

These two conditions, together with T^*-delay stationarity and axiom **A2**, exhibit a common trait. They introduce a more lenient form of stationarity when compared to the characteristics typically associated with exponential discounting. Beginning from a specific date in the future (1 in the case of quasi-hyperbolic discounting, T^* in the case of our article), the comparison of the economic agent satisfies a weak version of stationarity.

The fundamental difference between the two previously mentioned papers and the present one lies in the nature of the order under consideration. While Bach et al. (2023) worked on a complete order $\succeq$, this study focuses on the analysis of a robust preorder $\succeq_c^*$. If the results in Bach et al. (2023) relies on two core axioms, delayed-ISTAT and EFREP, our version of T^*-steps quasi-hypecbolic discounting is based on T^*-delay stationarity and the existence of delayed equivalence. However, in contrast to Bach et al. (2023), who established a MaxMin representation of the index function, this study explores a broader spectrum of possible orders and index functions, including the α-MaxMin representation. Unsurprisingly, these two distinct approaches are underpinned by rather different systems of axioms.

4.3 Temporal bias representation of distant future preorder

Assuming that $\succeq_d$ satisfies the T^*-delay stability property, we can deduce crucial characteristics of the set of purely finitely additive measures that characterize the preorders $\succeq_d^*$. The set Ω_d builds from Banach limits. This reflects an exponential discounting property where the comparison between two sequences does not depend on the beginning date chosen.

Proposition 4.2. *Assume that the order $\succeq_d$ is not trivial and satisfies T^*-*delay stability *with $T^* \geq 1$. Then every purely finitely additive probability $\phi \in \Omega_d$ is a Banach limit: for every $x \in \ell_\infty$,*

$$\phi \cdot x = \phi \cdot (0, x).$$

From this result, we have $\phi \cdot x \geq \phi \cdot y$ if and only if $\phi \cdot (0, x) \geq \phi \cdot (0, y)$. This property echoes the stationarity of exponential discounting. The comparison between two utility streams under exponential discounting remains the same if we shift them to a future date, similar to the case of Banach limits. As a result, the Banach limits in the evaluation of the distant future can be considered as a counterpart of exponential discounting in the evaluation of the close future.

Axiom **A3** establishes that a utility stream dominates (or is dominated) by a constant one if and only if its values are all greater (or worse) in distant future.

Axiom A 3. For any $x \in \ell_\infty$ and $b \in \mathbb{R}$,

(i) If there exist $\epsilon > 0$ and an infinite number of times s such that $b > x_s + \epsilon$, then $x \not\succeq_d^* b\mathbf{1}$.

(ii) If there exist $\epsilon > 0$ and an infinite number of times s such that $x_s > b + \epsilon$, then $b\mathbf{1} \not\succeq_d^* x$.

Proposition **4.3.** *Assume that under the order $\succeq_d$, for every streams x and y, x being potetially better than y implies $x \succeq_d y$. Adding axiom* **A3***, there exists unique $0 \leq \alpha_d \leq 1$ such that the distant future index function can be represented as:*

$$I_d(x) = \alpha_d \limsup_{s \to \infty} x_s + (1 - \alpha_d) \liminf_{s \to \infty} x_s.$$

5. Conclusion

This article deviates from the classical method of intertemporal preferences analysis.[17] We offer a new approach to the literature using utility streams rather than choice under uncertainty over a range of states of the world. This approach enables us to characterize preferences by sets of finitely additive probabilities, using techniques well-developed in decision theory. This work makes three contributions: it considers the distant future, it identifies circumstances that guarantee α-MaxMin

[17]To name some contributions in the axiomatic of discounted utilities literature, see the seminal articles by Koopmans (1960) and Koopmans (1972), or more recent ones, Fishburn and Rubinstein (1982) and Dolmas (1995). For a review, see Bleichrodt et al. (2008).

representations, and provides a generalized form of multiple quasi-hyperbolic discounting. The method used, which involves robust comparison, may complement the approaches followed by Chambers and Echenique (2018) and Bach et al. (2023) in the studies of temporal preferences.

The results can be used to depict how rational economic agents would behave if they were given advice by a group of experts with a finite range of temporal biases. By observing the choice between pairs of streams that are comparable under notions of "potentially better", we can determine whether our agent has a MaxMin, MaxMax or α-MaxMin criterion. The parameter α could be used as a degree of prudence while considering experts' opinions. The case $\alpha = 0$ corresponds to a very high prudence level. The agent exclusively evaluates x values at least b if and only if experts unanimously agree with that. Case $\alpha = 1$ represents an opposite behavior, where the evaluation of at least one expert is sufficient.

The issue of how to incorporate these preferences in an optimization context with long-term policies, for example, optimal growth model with natural resources, especially the renewable ones, could be of interest. However, in our opinion, this question is complicated, and that should be the subject of another research, hopefully in the close future.

A. Mathematical preparations

A.1 Topologies

The set of bounded utility streams ℓ_∞ is equipped the sup −norm topology. A sequence of streams $\{x_n\}_{n=0}^{\infty} \subset \ell_\infty$ converges to x in this topology if

$$\lim_{n\to\infty} \left[\sup_{s\geq 0} |x_{n,s} - x_s|\right] = 0.$$

The set of continuous linear functions defined on ℓ_∞, namely its *dual*, is denoted by $(\ell_\infty)^*$. This set can be decomposed as $(\ell_\infty)^* = \ell_1 \oplus \ell_1^d$. The set ℓ_1 is constituted by real number sequences $\omega = (\omega_0, \omega_1, \omega_2, \dots)$ such that

$$\sum_{s=0}^{\infty} |\omega_s| < \infty.$$

A sequence of $\{\omega^n\}_{n=0}^{\infty} \subset \ell_1$ converges to ω in *weak*-topology if for every $x \in \ell_\infty$, we have

$$\lim_{n\to\infty} \sum_{s=0}^{\infty} \omega_s^n x_s = \sum_{s=0}^{\infty} \omega_s x_s.$$

Sometimes, instead of $\sum_{s=0}^{\infty} \omega_s x_s$, we can write simply $\omega \cdot x$. If $\omega_s \geq 0$ for every s and $\sum_{s=0}^{\infty} \omega_s = 1$, we call ω a countably additive probability.

The set ℓ_1^d is constituted by purely finitely signed measures.[18] In this article, we will focus only on a special subset of it, the set of purely finitely additive probabilities, with a characterization that will be presented in the second part of this section.

A.2 Characterization set of the robust preorder and probability decomposition

Fix an order $\widehat{\succeq} \in \{\succeq_c, \succeq_d\}$. Suppose that this order is not trivial. Define $\widehat{\Omega}$ as:

$$\widehat{\Omega} = \Big\{P \in (\ell_\infty)^* \text{ such that } P \cdot x \geq 0 \text{ for every } x \widehat{\succeq}^* 0\mathbf{1} \text{ and } P \cdot \mathbf{1} = 1\Big\}.$$

It is obvious the set $\widehat{\Omega}$ is convex. Moreover, if a stream x satisfies $x_s \geq 0$ for every x, then $P \cdot x \geq 0$ for every $P \in \widehat{\Omega}$. This means a $P \in \widehat{\Omega}$ can be considered a measure on the set of natural numbers $\{0, 1, 2, \ldots\}$, in the sense that for every subset $S \subset \{0, 1, 2, \ldots\}$, we may define $P(S)$ as $P \cdot x$, where $x_s = 1$ if $s \in S$ and $x_s = 0$ if $s \notin S$. Using Theorems 1.23 and 1.24 in Yosida and Hewitt (1952), each P belonging to $\widehat{\Omega}$ is a finitely additive probability on $\{0, 1, 2, \ldots\}$ and can be decomposed as

$$P = (1 - \lambda)\omega + \lambda\phi,$$

where $0 \leq \lambda \leq 1$, ω is a countably additive probability belonging to ℓ_1, and ϕ is a purely finitely additive probability belonging to ℓ_1^d.

To be precise, ϕ satisfies the following property: if $\tilde{\omega} \in \ell_1$ such that $\tilde{\omega}_s \geq 0$ $\forall s$ and for every subset $S \subset \{0, 1, 2, \ldots\}$, we have $\sum_{s\in S} \tilde{\omega}_s \leq \phi(S)$, then $\tilde{\omega}_s = 0$ for every s. In other words, the evaluation of $x \in \ell_\infty$ under ϕ, the value $\phi \cdot x$, is not affected if we change only a finite numbers of values x_s. For every $x, y \in \ell_\infty$ and $T \geq 0$, we obtain $\phi \cdot (y_{[0,T]}, x_{[T+1,\infty)}) = \phi \cdot x$. Hence, we can consider ϕ an evaluation of the distant future of utility streams. Function ϕ is called a Banach limit if for every $x \in \ell_\infty$, we have $\phi \cdot x = \phi \cdot (0, x)$.

[18] See Dunford and Schwartz (1966).

This decomposition will be used in the establishment of finitely additive probabilities that characterize the robust order $\widehat{\succsim}$, Propositions 3.1 and 3.2.

For a given convex subset $\Omega_c \subset \ell_1$, ω is an exposed point of Ω_c if there exists $x \in \ell_\infty$ such that $\tilde{\omega} \cdot x > \omega \cdot x$ for every $\tilde{\omega} \in \Omega_c \setminus \{\omega\}$. Though in this article, we consider only the exposed points of a convex set in ℓ_1, curious readers may refer to the note of Radner (1965) on the same objects in the space of ℓ_∞. Recently, Becker (2022) gave an interesting discussion on the subject, concerning not only exposed points, but also Banach limits that appear in Proposition 4.2.

B. Proof of Lemma 2.1

Fix $x \in \ell_\infty$. To simplify the exposition, let $a = I_c(x)$ and $b = I_d(x)$. We will prove that the evaluation of the whole stream $I(x)$ is a convex combination of a and b and that the parameter of the convex combination depends on these two values.

According to the monotonicity property, the parameters χ_g and χ_ℓ are limits of decreasing sequences. Hence, they are well defined.

Now, we prove the following assertion

$$I(x) = \lim_{T\to\infty} I(a\mathbf{1}_{[0,T]}, b\mathbf{1}_{[T+1,\infty)}).$$

This is obviously true if the close future order $\succeq_c$ is trivial, with a consequence that $\succeq = \succeq_d$. Consider the case that $\succeq_c$ is nontrivial. Fix any $\epsilon > 0$. From the distant future insensitivity property, we have

$$\begin{aligned} I_c(x) &> I_c\Big((a-\epsilon)\mathbf{1}\Big) \\ &= \lim_{T\to\infty} I_c\Big((a-\epsilon)\mathbf{1}_{[0,T]}, (b-\epsilon)\mathbf{1}_{[T+1,\infty)}\Big). \end{aligned}$$

Hence, for T that is sufficiently large, $x \succeq_c \Big((a-\epsilon)\mathbf{1}_{[0,T]}, (b-\epsilon)\mathbf{1}_{[T+1,\infty)}\Big)$.

From the close future insensitivity property,

$$\begin{aligned} I_d(x) &\geq I_d\Big((b-\epsilon)\mathbf{1}\Big) \\ &= I_d\Big((a-\epsilon)\mathbf{1}_{[0,T]}, (b-\epsilon)\mathbf{1}_{[T+1,\infty)}\Big), \end{aligned}$$

for every T. Hence, $x \succeq_d \Big((a-\epsilon)\mathbf{1}_{[0,T]}, (b-\epsilon)\mathbf{1}_{[T+1,\infty)}\Big)$.

According to the consistency property, for every T that is sufficiently large, we have

$$x \succeq \Big((a-\epsilon)\mathbf{1}_{[0,T]}, (b-\epsilon)\mathbf{1}_{[T+1,\infty)}\Big).$$

This implies

$$I(x) \geq \limsup_{T\to\infty} I\Big(a\mathbf{1}_{[0,T]}, b\mathbf{1}_{[T+1,\infty)}\Big) - \epsilon.$$

Using the same arguments, we get

$$I(x) \leq \liminf_{T\to\infty} I\Big(a\mathbf{1}_{[0,T]}, b\mathbf{1}_{[T+1,\infty)}\Big) + \epsilon.$$

Since ϵ is chosen arbitrarily,

$$I(x) = \lim_{T\to\infty} I\Big(a\mathbf{1}_{[0,T]}, b\mathbf{1}_{[T+1,\infty)}\Big).$$

The assertion has been proven. Now, assume that $I_c(x) \leq I_d(x)$. Then

$$\begin{aligned} I(x) &= \lim_{T\to\infty} I(a\mathbf{1}_{[0,T]}, b\mathbf{1}_{[T+1,\infty)}) \\ &= \lim_{T\to\infty} I\Big(0\mathbf{1}_{[0,T]}, (b-a)\mathbf{1}_{[T+1,\infty)}\Big) + a \\ &= (b-a)\lim_{T\to\infty} I\Big(0\mathbf{1}_{[0,T]}, \mathbf{1}_{[T+1,\infty)}\Big) + a \\ &= (b-a)\chi_g + a \\ &= (1-\chi_g)I_c(x) + \chi_g I_d(x). \end{aligned}$$

Consider the case $I_c(x) \geq I_d(x)$. Observe that

$$\chi_\ell = 1 - \lim_{T\to\infty} I(\mathbf{1}_{[0,T]}, 0\mathbf{1}_{[T+1,\infty)}).$$

Using the same arguments as in the first part of the proof, we have

$$I(x) = (1-\chi_\ell)I_c(x) + \chi_\ell I_d(x).$$

C. Proof of Proposition 2.1

Consider the case $\chi_g \leq \chi_\ell$. For a stream x, if $I_c(x) \leq I_d(x)$, we have

$$\begin{aligned} I(x) &= (1-\chi_g)I_c(x) + \chi_g I_d(x) \\ &= \min_{\chi_g \leq \lambda \leq \chi_\ell} \Big[(1-\lambda)I_c(x) + \lambda I_d(x)\Big]. \end{aligned}$$

The last equality comes from $I_c(x) \le I_d(x)$ and $\chi_g \le \chi_\ell$.

If $I_c(x) \ge I_d(x)$, we have

$$\begin{aligned} I(x) &= (1-\chi_\ell)I_c(x) + \chi_\ell I_d(x) \\ &= \min_{\chi_g \le \lambda \le \chi_\ell} \Big[(1-\lambda)I_c(x) + \lambda I_d(x)\Big]. \end{aligned}$$

The last equality comes from $I_c(x) \ge I_d(x)$ and $\chi_g \le \chi_\ell$.

Using the same arguments, in the case $\chi_g \ge \chi_\ell$, for every stream x,

$$I(x) = \max_{\chi_\ell \le \lambda \le \chi_g} \Big[(1-\lambda)I_c(x) + \lambda I_d(x)\Big].$$

D. Proof of Proposition 3.1

Relying upon the same arguments as in Section **A**, there exists a set of finitely additive probabilities $\Omega_c \subset (\ell_\infty)^*$ such that

$$x \succeq_c^* y \Leftrightarrow P \cdot x \ge P \cdot y,$$

for every $P \in \Omega_c$. By Yosida and Hewitt (1952), every $P \in \Omega_c$ can be decomposed as $P = (1-\lambda)\omega + \lambda\phi$, with $\omega \in \ell_1$ is a countable additivity probability and ϕ is a purely finitely additive probability.

Suppose that $\lambda\phi \neq 0$, or equivalently $\lambda\phi \cdot \mathbf{1} > 0$. Fix b such that $1-\lambda < b < 1$. By axiom **A1**, there exists a large enough T_0 such that for $T \ge T_0$,

$$\Big(\mathbf{1}_{[0,T]}, 0\mathbf{1}_{[T+1,\infty)}\Big) \succeq_c^* b\mathbf{1}.$$

Hence,

$$\Big((1-\lambda)\omega, \lambda\phi\Big) \cdot \Big(\mathbf{1}_{[0,T]}, 0\mathbf{1}_{[T+1,\infty)}\Big) \ge b,$$

with a direct consequence that $1-\lambda \ge b$, a contradiction.

To sum up, for every $\Big((1-\lambda), \lambda\phi\Big) \in \Omega_c$, $\lambda\phi = 0$. Hence, Ω_c can be considered a subset of probabilities that is included in ℓ_1. Moreover, for every $\epsilon > 0$, there exists T such that $\sum_{s=0}^{T} \omega_s > 1-\epsilon$ for every $\omega \in \Omega_c$. By the Dunford-Petit critetion in Dunford and Schwartz (1966), this implies the weak compactness of Ω_c in ℓ_1. This property will be used in the proof of Proposition 4.1.

E. Proof of Proposition 3.2

Let Ω_d be the set of finitely additive probabilities being defined as $P \in \Omega_d$ if and only if $P \cdot \mathbf{1} = 1$ and $P \cdot x \geq 0$ for every x such that $x \succeq_d^* 0\mathbf{1}$. As presented in Section **A**, by Yosida and Hewitt (1952), a probability $P \in \Omega_d$ can be decomposed as $P = (1-\lambda)\omega + \lambda\phi$, with $\omega \in \ell_1$ is a countably additive probability and ϕ is a purely finitely additive probability belonging to ℓ_d^1.

We prove that $(1-\lambda)\omega = 0$. Indeed, suppose the contrary. Then $\lambda < 1$ and there exists T such that $\omega_T > 0$. Take a constant $c > 0$ such that $(1-\lambda)\omega_T c > \lambda$ and let $x = \left(-c\mathbf{1}_{[0,T]}, \mathbf{1}\right)$. For every $z \in \ell_\infty$ one has $I_d(x+z) = I_d(\mathbf{1}+z) \geq I_d(z)$. Hence, $x \succeq_d^* 0\mathbf{1}$. Then

$$(1-\lambda)\omega \cdot x + \lambda\phi \cdot x \geq 0,$$

which implies $-(1-\lambda)\omega_T c + \lambda \geq 0$, a contradiction. This contradiction implies that $(1-\lambda)\omega = 0$, which also implies $\lambda = 1$. The weights set Ω_d can be considered a subset of purely finitely additive probabilities belonging to ℓ_d^1.

F. Proof of Proposition 3.3

Fix an order $\widehat{\succeq}$ belonging to $\{\succeq_c, \succeq_d\}$. Assume that being potentially better in the first category implies to be preferred. Consider two streams x and y, we will prove that x is potentially better than y in the first category if and only if:

$$\inf_{P \in \widehat{\Omega}} P \cdot x \geq \inf_{P \in \widehat{\Omega}} P \cdot y.$$

Assume that x is potentially better than y under $\widehat{\succeq}$. Stream y robustly dominates a constant one $b\mathbf{1}$ if and only if $\inf_{P \in \widehat{\Omega}} P \cdot y \geq b$. Let $b = \inf_{P \in \widehat{\Omega}} P \cdot y$. By the very definition of the first category of the potentially better property, we have $x \widehat{\succeq}^* b\mathbf{1}$, with a direct consequence that $\inf_{P \in \widehat{\Omega}} P \cdot x \geq \inf_{P \in \widehat{\Omega}} P \cdot y$.

Assume that $\inf_{P \in \widehat{\Omega}} P \cdot x \geq \inf_{P \in \widehat{\Omega}} P \cdot y$. If $y \widehat{\succeq}^* b\mathbf{1}$, then $\inf_{P \in \widehat{\Omega}} P \cdot x \geq \inf_{P \in \widehat{\Omega}} P \cdot y \geq b$. This implies that $x \widehat{\succeq}^* b\mathbf{1}$.

The condition of Proposition 3.3(i) is thus equivalent to: if $\inf_{P \in \widehat{\Omega}} P \cdot x \geq \inf_{P \in \widehat{\Omega}} P \cdot y$, then $x \widehat{\succeq} y$. For every stream x, let $b = \inf_{P \in \widehat{\Omega}} P \cdot x$ and $y = b\mathbf{1}$. Since $\inf_{P \in \widehat{\Omega}} P \cdot x = \inf_{P \in \widehat{\Omega}} P \cdot y$, we have $x \widehat{\sim} y$, with $\widehat{I}(x) = b$ as a direct consequence.

As for part (ii), using the same arguments, we can prove that if having more potential in the second category implies to be preferred, $\widehat{I}(x) = \sup_{P\in\widehat{\Omega}} P \cdot x$.

Consider the most interesting part, namely (iii). Using the same arguments as in the proof of part (i), the condition in part (iii) can be rewritten as: if $\inf_{P\in\widehat{\Omega}} P \cdot x \geq \inf_{P\in\widehat{\Omega}} P \cdot y$ and $\sup_{P\in\widehat{\Omega}} P \cdot x \geq \sup_{P\in\widehat{\Omega}} P \cdot y$, then $x \widehat{\succeq} y$.

For every x, it is obvious that $\sup_{P\in\widehat{\Omega}} P \cdot x \widehat{\succeq} \widehat{I}(x) \widehat{\succeq} \inf_{P\in\widehat{\Omega}} P \cdot x$. Hence, there exists $0 \leq \alpha_x \leq 1$ such that

$$\widehat{I}(x) = \alpha_x \sup_{P\in\widehat{\Omega}} P \cdot x + (1 - \alpha_x) \inf_{P\in\widehat{\Omega}} P \cdot x.$$

If $\inf_{P\in\widehat{\Omega}} P \cdot x < \sup_{P\in\widehat{\Omega}} P \cdot x$, the value α_x is unique.

To end the proof and establish an α-MaxMin representation, we prove that for every streams x and y such that $\inf_{P\in\widehat{\Omega}} P\cdot x < \sup_{P\in\widehat{\Omega}} P\cdot x$ and $\inf_{P\in\widehat{\Omega}} P\cdot y < \sup_{P\in\widehat{\Omega}} P\cdot y$, we obtain $\alpha_x = \alpha_y$.

First, observe that we can find $\lambda > 0$ and a constant b such that

$$\lambda \sup_{P\in\widehat{\Omega}} P \cdot y + b = \sup_{P\in\widehat{\Omega}} P \cdot x,$$

$$\lambda \inf_{P\in\widehat{\Omega}} P \cdot y + b = \inf_{P\in\widehat{\Omega}} P \cdot x.$$

Let $\tilde{x} = \lambda y + b$. We have

$$\widehat{I}(\tilde{x}) = \lambda \widehat{I}(y) + b,$$

$$\sup_{P\in\widehat{\Omega}} P \cdot \tilde{x} = \lambda \sup_{P\in\widehat{\Omega}} P \cdot y + b,$$

$$\inf_{P\in\widehat{\Omega}} P \cdot \tilde{x} = \lambda \inf_{P\in\widehat{\Omega}} P \cdot y + b.$$

Hence, $\alpha_{\tilde{x}} = \alpha_y$. Observe that

$$\sup_{P\in\widehat{\Omega}} P \cdot \tilde{x} = \sup_{P\in\widehat{\Omega}} P \cdot x,$$

$$\inf_{P\in\widehat{\Omega}} P \cdot \tilde{x} = \inf_{P\in\widehat{\Omega}} P \cdot x.$$

Hence, $\tilde{x} \widehat{\sim} x$. This implies $\widehat{I}(\tilde{x}) = \widehat{I}(x)$, and $\alpha_x = \alpha_y$. Let the common value be α. For every stream x, we have

$$\widehat{I}(x) = \alpha \sup_{P\in\widehat{\Omega}} P \cdot x + (1 - \alpha) \inf_{P\in\widehat{\Omega}} P \cdot x.$$

G. Proof of Proposition 4.1

The proof of this proposition begins by a preparative Lemma G.1. Under the hypothesis that the close future order $\succeq_c$ satisfies definition 4.1, for each stream x, the value of the worst-case scenario corresponding to $(b^*\mathbf{1}_{[0,T^*-1]}, x)$, evaluated under order $\succeq_c$, neither change with a shift of the stream to the future nor with a convex combination with this shift. In other words, beginning from T^*, the robust order satisfies a version of stationarity.

Lemma **G.1.** *Assume that the order $\succeq_c$ is not trivial and satisfies the T^*-delay stationarity property.*

(i) *For any constant b, $(b\mathbf{1}_{[0,T^*-1]}, x) \succeq_c^* b\mathbf{1}$ implies:*

$$(b\mathbf{1}_{[0,T^*-1]}, x) \succeq_c^* (b\mathbf{1}_{[0,T^*]}, x) \succeq_c^* (b\mathbf{1}_{[0,T^*+1]}, x) \succeq_c^* \ldots \succeq_c^* b\mathbf{1}.$$

(ii) *If $b^* = \mathcal{C}\Big((b^*\mathbf{1}_{[0,T^*-1]}, x)\Big)$, then for any $T \geq T^*$,*

$$\mathcal{C}\Big(b^*\mathbf{1}_{[0,T]}, x\Big) = b^*.$$

(iii) *If $b^* = \mathcal{C}((b^*\mathbf{1}_{[0,T^*-1]}, x))$, then for any $T \geq T^*$,*

$$\mathcal{C}\left(\frac{1}{2}(b^*\mathbf{1}_{[0,T^*-1]}, x) + \frac{1}{2}\left(b^*\mathbf{1}_{[0,T]}, x\right)\right) = b^*.$$

Proof. The proof of part (i) is obvious, using delay stationary property.

(ii) It is obvious that if $x \succeq_c^* y$, then $\mathcal{C}(x) \geq \mathcal{C}(y)$. Let $b^* = \mathcal{C}\Big(b^*\mathbf{1}_{[0,T^*-1]}, x\Big)$. From part (i), for $T \geq T^*$,

$$(b^*\mathbf{1}_{[0,T^*-1]}, x) \succeq_c^* (b^*\mathbf{1}_{[0,T]}, x) \succeq_c^* b^*\mathbf{1}.$$

This implies $b^* = \mathcal{C}\left(b^*\mathbf{1}_{[0,T^*-1]}, x\right) \geq \mathcal{C}\left(b^*\mathbf{1}_{[0,T]}, x\right) \geq b^*$.

(iii) Since $(b^*\mathbf{1}_{[0,T^*-1]}, x) \succeq_c^* (b^*\mathbf{1}_{[0,T]}, x)$,

$$\begin{aligned} b^* &= \mathcal{C}\left((b^*\mathbf{1}_{[0,T^*-1]}, x)\right) \\ &\geq \mathcal{C}\left(\frac{1}{2}(b^*\mathbf{1}_{[0,T^*-1]}, x) + \frac{1}{2}(b^*\mathbf{1}_{[0,T]}, x)\right) \\ &\geq b^*. \end{aligned}$$

QED

Now, return to the main part of the proof. For each probability $\omega = (\omega_0, \omega_1, \dots) \in \ell_1$ and $T \geq 0$, let ω^T be the probability defined as

$$\omega_s^T = \frac{\omega_{T+s}}{\sum_{s'=0}^{\infty} \omega_{T+s'}}.$$

It is worth noting that, for $x \in \ell_\infty$ and a constant b, $\omega \cdot \left(b\mathbf{1}_{[0,T-1]}, x\right) = b$ if and only if $\omega^T \cdot x = b$.

Let $\Omega_c^{T^*} = \left\{\omega^{T^*} \text{ such that } \omega \in \Omega_c\right\}$. First, observe that from axiom **A1**, we have $\Omega_c^{T^*}$ is a weak compact subset of ℓ_1. Take $\omega \in \Omega_c$ such that ω^{T^*} is an exposed point of $\Omega_c^{T^*}$.

We will prove that $\omega^{T^*} = (\omega^{T^*})^T$ for all $T \geq 0$.

From the definition of ω, there exists $x \in \ell_\infty$ such that $\omega^{T^*} \cdot x < \tilde{\omega}^{T^*} \cdot x$ for every $\tilde{\omega} \in \Omega_c \setminus \{\omega\}$. Let $b^* = \omega^{T^*} \cdot x$. It is obvious that the following inequality is verified:

$$b^* = \omega \cdot \left(b^*\mathbf{1}_{[0,T^*-1]}, x\right) < \tilde{\omega} \cdot \left(b^*\mathbf{1}_{[0,T^*-1]}, x\right).$$

This implies that $\mathcal{C}\left(\left(b^*\mathbf{1}_{[0,T^*-1]}, x\right)\right) = b^*$. Obviously, $\left(b^*\mathbf{1}_{[0,T^*-1]}, x\right) \succeq_c^* b^*\mathbf{1}$. Fix $T \geq 0$ and from Lemma G.1,

$$\mathcal{C}\left(\frac{1}{2}\left(b^*\mathbf{1}_{[0,T^*-1]}, x\right) + \frac{1}{2}\left(b^*\mathbf{1}_{[0,T^*+T]}, x\right)\right) = b^*.$$

This implies that there exists ω' such that

$$\begin{aligned} b^* &= \omega' \cdot \left(\frac{1}{2}\left(b^*\mathbf{1}_{[0,T^*-1]}, x\right) + \frac{1}{2}\left(b^*\mathbf{1}_{[0,T^*+T]}, x\right)\right) \\ &= \min_{\omega \in \Omega_c} \omega \cdot \left(\frac{1}{2}\left(b^*\mathbf{1}_{[0,T^*-1]}, x\right) + \frac{1}{2}\left(b^*\mathbf{1}_{[0,T^*+T]}, x\right)\right). \end{aligned}$$

From (i), $\omega' \cdot \left(b^*\mathbf{1}_{[0,T^*-1]}, x\right) \geq b^*$ and $\omega' \cdot \left(b^*\mathbf{1}_{[0,T^*+T]}, x^*)\right) \geq b^*$. It follows that

$$\omega' \cdot \left(b^*\mathbf{1}_{[0,T^*-1]}, x\right) = \omega' \cdot \left(b^*\mathbf{1}_{[0,T^*+T]}, x^*\right) = b^*.$$

Hence:

$$\begin{aligned} (\omega')^{T^*} \cdot x &= b^*, \\ (\omega')^{T^*} \cdot \left(b^*\mathbf{1}_{[0,T^*+T]}, x\right) &= b^*. \end{aligned}$$

Since ω^{T^*} is an exposed point of $\Omega_c^{T^*}$, the first equality implies that $(\omega')^{T^*} = \omega^{T^*}$.

Observe that $\omega \cdot \Big(b^*\mathbf{1}_{[0,T^*+T]}, x\Big) = b^*$ is equivalent to $(\omega^{T*})^T \cdot x = b^*$. Moreover, $(\omega^{T*})^T$ belongs to $\Omega_c^{T^*}$. Indeed, suppose the contrary: from the weak compactness of $\Omega_c^{T^*}$, there exists $\epsilon > 0$ such that the intersection between $\Omega_c^{T^*}$ and the open set $\Big\{\tilde{\omega}$ such that $\Big\|\tilde{\omega} - (\omega^{T*})^T\Big\|_{\ell_1} < \epsilon\Big\}$ is empty. From Hahn-Banach theorem, there exist x' and a constant b such that $\tilde{\omega}^{T^*} \cdot x' > b > \omega^{T*} \cdot x'$ for every $\tilde{\omega} \in \Omega_c$. This implies that $\Big(b\mathbf{1}_{[0,T^*-1]}, x'\Big) \succeq_c^* b\mathbf{1}$ and therefore that $\Big(b\mathbf{1}_{[0,T^*-1]}, x'\Big) \succeq_c^* \Big(b\mathbf{1}_{[0,T^*+T]}, x'\Big) \succeq_c^* b\mathbf{1}$, hence $\omega \cdot \Big(b\mathbf{1}_{[0,T^*+T]}, x'\Big) \geq b$, which is equivalent to $(\omega^{T*})^T \cdot x' \geq b$, a contradiction.

The probability $(\omega^{T*})^T$ belongs to $\Omega_c^{T^*}$, and satisfies $(\omega^{T*})^T \cdot x = b^*$. From the definition of ω^{T^*} and x, $\omega^{T^*} = (\omega^{T*})^T$, for every $T \geq 0$. It follows that

$$\omega_s^{T^*} = \frac{\omega_{T^*+T+s}}{\sum_{s'=0}^{\infty} \omega_{T^*+T+s'}} \text{ and } \omega_{s+1}^{T^*} = \frac{\omega_{T^*+T+s+1}}{\sum_{s'=0}^{\infty} \omega_{T^*+T+s'}}.$$

This implies that for every T, s:

$$\frac{\omega_{s+1}^{T^*}}{\omega_s^{T^*}} = \frac{\omega_{T^*+T+s+1}}{\omega_{T^*+T+s}}.$$

This is equivalent, for some $\delta > 0$ and for every $s \geq 0$, to

$$\frac{\omega_{s+1}^{T^*}}{\omega_s^{T^*}} = \delta,$$

or to $\omega_s^{T^*} = \delta^s \omega_0^{T^*}$ for every $s \geq 0$. Since $\sum_{s=0}^{\infty} \omega_s^{T^*} = 1$, we have $0 < \delta < 1$ and $\omega_s = \Big(1 - \delta\Big)\delta^s$ for $s \geq 0$.

To sum up, every exposed point of $\Omega_c^{T^*}$ has an exponential representation. The set $\Omega_c^{T^*}$ is weakly compact, according to Theorem 4 in Amir and Lindentrauss (1968), $\Omega_c^{T^*}$ is the convex hull of its exposed points. This implies the existence of a subset $D^* \subset (0, 1)$ such that

$$\Omega_c^{T^*} = \text{convex}\Big\{(1 - \delta, (1 - \delta)\delta, \ldots, (1 - \delta)\delta^s, \ldots)\Big\}_{\delta \in D^*}.$$

Part (i), where $T^* = 0$ is proven.

Consider the case $T^* \geq 1$. Observe that if ω^{T^*} is an exposed point of $\Omega_c^{T^*}$, then ω is an exposed point of Ω_c. Indeed, in that case, there exists $x \in \ell_\infty$ such that $\omega^{T^*} \cdot x < \tilde{\omega}^{T^*} \cdot x$ for every $\tilde{\omega} \in \Omega_c \setminus \{\omega\}$. Let $b = \omega^{T^*} \cdot x$. It is easy to verify that $b = \omega \cdot \Big(b\mathbf{1}_{[0,T^*-1]}, x)\Big)$ and $b < \tilde{\omega} \cdot \Big(b\mathbf{1}_{[0,T^*-1]}, x)$. Hence, ω is an exposed point of Ω_c.

Consider an exposed point ω of Ω_c. We will prove that ω^{T^*} is an exposed point of $\Omega_c^{T^*}$. In this stage of the proof, we need axiom **A2**, to prove that the T^*-delay equivalence of an exposed point of Ω_c is an exposed point of $\Omega_c^{T^*}$.

By the choice of ω, there exists $x \in \ell_\infty$ such that $\omega \cdot x < \tilde{\omega} \cdot x$, for every $\tilde{\omega} \in \Omega_c \setminus \{\omega\}$. Let $b = \mathcal{C}(x) = \omega \cdot x$. Consider the utility stream y, which is a T^*-delay equivalence of x, being defined in the statement of axiom **A2**. Taking $\widehat{y} = y$, from the obvious property $\mathcal{C}\left(\frac{1}{2}\Big(b\mathbf{1}_{[0,T^*-1]}, y\Big) + \frac{1}{2}\Big(b\mathbf{1}_{[0,T^*-1]}, y\Big)\right) = b$, one has

$$\mathcal{C}\left(\frac{1}{2}x + \frac{1}{2}\Big(b\mathbf{1}_{[0,T^*-1]}, y\Big)\right) = b.$$

Using the same arguments as those used in the proof of Lemma G.1, we obtain

$$\omega \cdot x = \omega \cdot \Big(b\mathbf{1}_{[0,T^*-1]}, y\Big) = b.$$

Since $\mathcal{C}\left(\Big(c\mathbf{1}_{[0,T^*-1]}, y\Big)\right) = b$, for every $\tilde{\omega} \in \Omega_c$, $\tilde{\omega} \cdot \Big(b\mathbf{1}_{[0,T^*-1]}, y\Big) \geq b$, which is equivalent to $\tilde{\omega}^{T^*} \cdot y \geq b$. We prove that for every exposed point $\widehat{\omega}^{T^*}$ of $\Omega_c^{T^*}$ that differs to ω^{T^*}, $\widehat{\omega}^{T^*} \cdot y > b$.

Assume the contrary, and consider a point $\widehat{\omega}^{T^*}$, which is an exposed point and $\widehat{\omega}^{T^*} \cdot y = b$. There exists y' such that $\widehat{\omega}^{T^*} \cdot y' < \tilde{\omega}^{T^*} \cdot y'$, for every $\tilde{\omega}^{T^*} \in \Omega^{T^*} \setminus \{\widehat{\omega}^{T^*}\}$, including ω^{T*}. Let $\widehat{y} = y' + \Big(b - \widehat{\omega}^{T^*} \cdot y'\Big)\mathbf{1}$. The stream $\widehat{y}$ satisfies

$$b = \widehat{\omega}^{T^*} \cdot \widehat{y} < \tilde{\omega}^{T^*} \cdot \widehat{y},$$

for every $\tilde{\omega}^{T^*} \in \Omega^{T^*} \setminus \{\widehat{\omega}^{T^*}\}$, including ω^{T^*}. Moreover, for every $\tilde{\omega}^{T^*} \in \Omega_c^{T^*}$,

$$\tilde{\omega}^{T*} \cdot \left(\frac{1}{2}y + \frac{1}{2}\widehat{y}\right) \geq b,$$

with the equality being obtained at $\tilde{\omega}^{T^*} = \widehat{\omega}^{T^*}$. One has

$$\mathcal{C}\left(\frac{1}{2}\Big(b\mathbf{1}_{[0,T^*-1]}, y\Big) + \frac{1}{2}\Big(b\mathbf{1}_{[0,T^*-1]}, \widehat{y}\Big)\right) = b.$$

Contrary to this, inequality $\omega^{T^*} \cdot y' > b$ implies $\omega \cdot \Big(b\mathbf{1}_{[0,T^*-1]}, y'\Big) > b$, and

$$\omega \cdot \left(\frac{1}{2}x + \frac{1}{2}\Big(b\mathbf{1}_{[0,T^*-1]}, \widehat{y}\Big)\right) > b.$$

For any $\tilde{\omega} \in \Omega_c \setminus \{\omega\}$, $\tilde{\omega} \cdot x > b$. Hence, the satisfaction of the strict inequality

$$\begin{aligned}\tilde{\omega}\left(\frac{1}{2}x + \frac{1}{2}\Big(b\mathbf{1}_{[0,T^*-1]}, \widehat{y}\Big)\right) &= \frac{1}{2}\tilde{\omega} \cdot x + \frac{1}{2}\tilde{\omega} \cdot \Big(b\mathbf{1}_{[0,T^*-1]}, \widehat{y}\Big) \\ &> b.\end{aligned}$$

The compactness of Ω_c implies that $\mathcal{C}\left(\frac{1}{2}x + \frac{1}{2}\Big(b\mathbf{1}_{[0,T^*-1]}, \widehat{y}\Big)\right) > b$, a contradiction. Hence, for every $\widehat{\omega}^{T^*} \in \Omega_c^{T^*} \setminus \{\omega^{T^*}\}$, one has $\widehat{\omega}^{T^*} \cdot y > b$. This implies ω^{T^*} is an exposed point of $\Omega_c^{T^*}$, and has an exponential representation with some discount rate δ. It is easy to find $\delta_0, \delta_1, \ldots, \delta_{T^*-1} \in (0,1)$ such that $\omega_0 = 1 - \delta_0, \omega_1 = \delta_0(1-\delta_1), \ldots, \omega_{T^*-1} = \delta_0\delta_1 \ldots \delta^{T^*-1}(1-\delta)$ and $\omega_{T+s} = \delta_0\delta_1 \ldots \delta_{T^*-1} \times \delta^s(1-\delta)$, for $s \geq 0$.

The set Ω_c being the convex hull of its exposed points, the proof is completed.

H. Proof of Proposition 4.2

Fix $b \leq \inf_{s\geq 0} x_s$. Obviously, for every $T \geq T^*$, $(b\mathbf{1}_{[0,T]}, x_{[T+1,\infty)}) \succeq_d^* b\mathbf{1}$. It follows that

$$\Big(b\mathbf{1}_{[0,T]}, x_{[T+1,\infty)}\Big) \succeq^* \Big(b\mathbf{1}_{[0,T+1]}, x_{[T+1,\infty)}\Big).$$

From the head-insensitivity property of the distant future order $\succeq_d$,

$$x \succeq_d^* \Big(0, x\Big).$$

Hence, for every purely finitely additive probability ϕ belonging to Ω_d,

$$\phi \cdot x \geq \phi \cdot \Big(0, x\Big).$$

By applying the same arguments with $-x$ in the place of x, and $b \leq -\sup_{s\geq 0} x_s$, it follows that $\phi \cdot (-x) \geq \phi \cdot \Big(0, -x\Big)$. From the linearity of ϕ, we obtain

$$\phi \cdot x = \phi \cdot \Big(0, x\Big).$$

I. Proof of Proposition 4.3

First, observe that for every purely finitely additive probability $\phi \in \Omega_d$, $x \in \ell_\infty$, one has

$$\liminf_{s\to\infty} x_s \leq \phi \cdot x \leq \limsup_{s\to\infty} x_s.$$

Axiom **A3** implies that

$$\inf_{\phi\in\Omega_d} \phi \cdot x = \liminf_{s\to\infty} x_s,$$

$$\sup_{\phi\in\Omega_d} \phi \cdot x = \limsup_{s\to\infty} x_s.$$

Indeed, assume the contrary. Consider the case $\liminf_{s\to\infty} x_s < \inf_{\phi\in\Omega_d} \phi \cdot x$. This implies the existence of $b \in \mathbb{R}$ and $\epsilon > 0$ such that $b > \liminf_{s\to\infty} x_s + \epsilon$ and $x \succeq_d^* b\mathbf{1}$, a contradiction with part (i) of axiom **A3**. For the case in which $\sup_{\phi\in\Omega_d} \phi \cdot x < \limsup_{s\to\infty} x_s$, using part (ii), similar arguments lead us to a contradiction.

Therefore, the decomposition is a direct consequence of Proposition 3.3.

References

Amir, D. and J. Lindentrauss (1968): The Structure of Weakly Compact Sets in Banach Spaces, *Annals of Mathematics* 88, 35-46.

Bach D. X., Bich, P. and B. Wigniolle (2023): Prudent aggregation of quasi-hyperbolic experts. *Working paper.*

Beissner, P., Q. Lin and F. Riedel (2020): Dynamically Consistent α-MaxMin Expected Utility, *Mathematical Finance* 30, 1073-1102.

Beissner, P. and J. Werner (2021): Optimal Allocations with α-MEU, CEU and CPT, *Working Paper.*

Becker, R. A. (2022): Maximal points of convex set in ℓ_∞ revisited, *Pure and Applied Functional Analysis* 7, 2003-2039.

Becker, R.A. and J. H. Boyd (1997): Capital Theory, Equilibrium Analysis an Recursive Utility, *Blackwell.*

Bewley, T.F. (2002): Knightian Decision Theory. Part I. *Decisions in Economics and Finance* 25, 79-110.

Bhaskara Rao, K.P.S. and M. Bhaskara Rao (1983): Theory of Charges. *Academic Press.*

Bleichrodt, H., S. Grant and J. Yang (2023): Testing Hurwicz expected utility. *Econometrica* 91, 1393-1416.

Bleichrodt, H., K. Rohde and P. Wakker (2008): Koopmans' constant discounting for intertemporal choice: a simplification and a generalisation, *Journal of Mathematical Psychology* 52, 341-347.

Brown, D. and L. Lewis (1981): Myopic Economic Agents, *Econometrica* 49, 369-268.

Chateauneuf, A., X. Qu, C. Ventura and V. Vergopoulos (2021): On the Falsifiability of α-MaxMin, *Working Paper*.

Chambers, C. and F. Echenique (2018): On multiple discount rates, *Econometrica* 86, 1325-1346.

de Andrade, M., L. Bastianello and J. Orrillo (2021): Future Blindness, *Working Paper*.

Dolmas, J. (1995): Time-additive Representations of Preferences when Consumption grows without Bound, *Economic Letters* 47, 317-325.

Drugeon, J.-P. and T. Ha-Huy (2022): A not so Myopic Axiomatisation of Discounting, *Economic Theory* 73, 349-376.

Drugeon, J.-P. and T. Ha-Huy (2023): Time-Varying Discount Rates with Recursive Time-Dependent Orders, *Working Paper*.

Dunford N. and J. T. Schwartz (1966): Linear Operators, Part I : general Theory, *3rd ed., Interscience*, New York, 1966.

Fishburn, P. and A. Rubinstein (1982): Time Preference, *International Economic Review* 23, 677-694.

Frick, M., R. Iijima and Y. Le Yaouanq (2022): Objective Rationality Foundations for (Dynamic) α-MEU, *Journal of Economic Theory* 200, https://doi.org/10.1016/j.jet.2021.105394.

Ghirardato, P., F. Maccheroni and M. Marinacci (2004): Differentiating Ambiguity and Ambiguity Attitude, *Journal of Economic Theory* 118, 133-173.

Gilboa, I. and D. Schmeidler (1989): MaxMin expected utility with non-unique prior, *Journal of Mathematical Economics* 18, 141-153.

Gilles, C. (1989): Charges as Equilibrium prices and Asset Bubbles, *Journal of Mathematical Economics* 18, 155-167.

HURWICZ, L. (1951): Some Specification Problems and Applications to Econometric Models, *Econometrica* 19, 343-344.

KARNI, E., F. MACCHERONI and M. MARINACCI (2022): David Schemeidler's contributions to decision theory, *Theory and Decision* **93**: 219-235.

KOOPMANS, T. J. (1960): Stationary Ordinal Utility and Impatience, *Econometrica* 28, 287-309.

KOOPMANS, T. J. (1972): T.J. Representation of Preference Orderings over Time. In: *Decision and Organisation.* McGuire, C. and R. Radner, eds. Amsterdam: North-Holland.

KOPYLOV, I. (2003): α-MaxMin Expected Utility, *Working Paper.*

LAIBSON, D. (1997): Golden Eggs and Hyperbolic Discounting, *Quarterly Journal of Economics* 112, 443-478.

MONTIEL OLEA J. L. and T. STRZALECKI (2014): Axiomatisation and Measurement of Quasi-Hyperbolic discounting, *Quartely Journal of Economics*, 1449-1499.

PHELPS, P. and R. POLLACK (1968): On Second-best National Saving and Game-Equilibrium Growth, *Review of Economics Studies* 35, 185-199.

RADNER, R. (1965): A note on maximal points of convex sets in ℓ_∞, *Proceedings of the Fifth Berkeley Symposium on Probability and Statistics*, University of California Press, Berkeley.

SAWYER, C. (1988): Stationary Recursive Preferences: Myopia and Impatience Reconsidered—the Many Goods Case, *Working Paper.*

TRAUTMAN, S. T., and G. VAN DE KUILEN (2015): Ambiguity Attitudes, *The Wiley Blackwell Handbook of Judgment and Decision Making* 2, 89-116.

WAKAI, T. (2007): A Model of Utility Smoothing, *Econometrica* 73, 157-63, 2007.

YOSIDA K. and E. HEWITT (1952): Finitely additive measures. *Transactions of the American Mathematical Society* 72, 46-66.