On Ramsey equilibrium with dependent preferences*

Jean-Paul Barinci†

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Abstract

This paper introduces consumption externalities in a one-sector Ramsey model with heterogeneous households and borrowing constraints. Externalities are taken into account by writing that the felicity functions depend on the consumption of all the households of the economy. Focusing on the class of equilibria in which the most patient household owns the whole capital stock, it is proved that there exist non-convergent Ramsey equilibria even though the Maximum Income Monotonicity (MIM) condition holds.

Key words: Consumption externalities; borrowing constraints; heterogeneous households; local bifurcation.

JEL Classification: E22; E23; O40.

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†EPEE, University of Evry Val d'Essonne, jeanpaul.barinci@univ-evry.fr

1 Introduction

The present paper consider the issue of how consumption externalities, that is nonmarket interdependence between households, affect the dynamics exhibited by the one-sector Ramsey model with heterogeneous households and borrowing constraints.

Dynamic general equilibrium models seek knowledge about the time paths of prices and wealth distribution of decentralized market economies. For instance, the one sector Ramsey model with heterogenous households, each a distinct individual with different tastes and endowments, focuses on the interaction between the households' time-preference rates, limitations on their choices due to borrowing constraints, and the technological possibilities for capital accumulation.

Whenever the forward markets structure is complete, the most patient household would emerge as the dominant household: the consumption of relatively more impatient households is driven towards zero as their incomes are entirely devoted to debt service; in the long run only the most patient household has positive wealth, consumes the entire output on the economy, and determines prices (impatient households are "unimportant"). This result, known as *Ramsey's conjecture*, has been proved by Bewley (1982) and Coles (1986).[1]

Becker (1980) put forward a solution in order to circumvent the eventual disappearance of impatient households from the economy demand side: the incompleteness of the forward markets structure. A non-negativity constraint on the capital holdings would prevent all consumption ultimately going to the most patient; households would always have a wage income available for consumption. The model of Becker (1980), nowadays known as the *Ramsey model*, then describes a competitive one-sector economy with heterogeneous households that are subject to no-borrowing constraints. In a comprehensive survey, Becker (2006) points out that in the context of the Ramsey model, Ramsey's conjecture does not hold in general. The only major result about the dynamics of Ramsey equilibria that can be proved under standard assumptions is the so-called *recurrence property*: every households other than the most patient one must attain the zero-capital state infinitely often. Thus, the recurrence property does not confirm Ramsey's conjecture

[1] The accuracy of the Ramsey conjecture is obtained provided each household's tastes are represented by a time additive separable utility function with a fixed rate of time-preference. Indeed, this outcome occurs either *asymptotically* (at every finite date the households have positive but small and shrinking consumption) or *eventually* (in finite time). The latter result is due to the assumption that marginal utility is bounded, even at zero consumption.

about the eventual distribution of wealth.[2] Furthermore, Becker and Foias (1987)(1994) and Sorger (1994) demonstrated that Ramsey equilibria can display non-convergent behavior, even if the *turnpike property* holds, *i.e.*, even if eventually the most patient household owns the entire capital stock. Becker and Foias (1987) came up with the first set of sufficient condition for the convergence of the capital stock known as Capital Income Monotonicity (CIM). If the production function is such that the capital income is monotone increasing in the capital stock, then the turnpike property holds (the wealth distribution becomes degenerate in finite time), and additionally all variables converge asymptotically towards their steady state values. Becker, Dubey and Mitra (2014) have established that a weaker condition, Maximum Income Monotonicity (MIM), is indeed sufficient.[3] [4]

In the Ramsey model social interactions are exclusively mediated by markets. It provided a suitable framework to investigate the idea that the capital market functions as a powerful mechanism generating and maintaining a highly skewed distribution of wealth. Yet, social relations and interactions, although hardly insignificant for the welfare of individuals and the allocation of resources, are largely beyond the scope of the competitive market. Widespread externalities are an appropriate device to account for non-market interactions within competitive market economies. As a matter of fact, widespread externalities are those created by and simultaneously affecting large numbers of individuals. Unlike local externalities, they are related to the entire society, and cannot be removed by negotiations between individuals. Widespread *consumption* externalities are thus a device to formalize out of markets dependencies among individuals within large societies.

McKenzie (1955) was the first to prove explicitly the existence of competitive equilibrium in a finite, convex economy where each consumer's preferences depend on the allocation of resources among others consumers (see also Arrow and Hahn (1971)). The smallness of theoretical results currently avail-

[2]Indeed, Becker, Dubey and Mitra (2014) recently provide an example of a Ramsey equilibrium in which the most patient household reaches a no capital position infinitely often.

[3]Attempts have been made to seek alternative conditions which guarantee the convergence of the equilibrium capital sequence. For instance, Borissov and Dubey (2015) relax the no borrowing condition by letting the households to be able to borrow against their next period wage income and show that irrespective of production function, the capital stock sequence converges; see also Becker, Borissov and Dubey (2015).

[4]One notices that the properties of *continuous*-time formulation of the Ramsey model stand in stark contrast to the ones of the discrete-time version. As a matter of fact, Mitra and Sorger (2013) prove that in the continuous-time Ramsey economy (i) the unique steady state equilibrium is globally asymptotically stable and (ii) along *every* Ramsey equilibrium the most patient household eventually owns the whole stock of capital.

able has recently motivated the study of the general equilibrium exchange model with consumption externalities (see, *e.g.*, Geanakoplos and Polemarchakis (2008), Bonnisseau and del Mercato (2010), Dufwenberg, Heidhues, Kirchsteiger, Riedel and Sobel (2011)).

The current paper introduces non-market interactions among households in the standard Ramsey model. It will be assumed that, beside their own consumption, households are concerned with the consumption of others. More specifically, each household's *felicity* function will depend at any date on the state of the economy which is specified by the overall consumption distribution.[5] Our purpose here is to characterize a special class of equilibrium allocations, in which the *turnpike property* holds. The latter, which is actually satisfied by the stationary equilibrium, will be ensured by a myopia argument. The interest of this class of equilibria is that they are easily characterized by making use of the *dynamical systems approach*, initiated by Becker and Foias (1990) and extensively used since then. The main result is that in presence of consumption externalities, the Maximum Income Monotonicity condition is not sufficient to ensure the convergence of the capital stock towards its steady state value. It provides an original illustration that Ramsey equilibria can display non-convergent behavior.

This paper is organized as follows. Section 2 is devoted to the model and basic assumptions. Section 3 defines a Ramsey equilibrium with consumption externalities. Section 4 establishes the existence of non-convergent equilibria, even though the turnpike property applies and the MIM condition holds. Section 5 concludes.

2 The model

This section describes the economy under consideration. Except for the assumption that individual tastes are dependent, this is the standard competitive Ramsey model with borrowing constraints comprehensively surveyed by Becker (2006).

Time is discrete; period are indexed by $t \geq 0$. The production sector consists of a set of identical competitive firms, which transform labor and capital into a homogeneous output good. The set of firms has unit measure. Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$. The common technology is described

[5]Other types of households' dependencies could have been considered. Consumption externalities could impact the time preference rather than the felicity function; households could be concerned with the distribution of wealth rather than consumption, along the line of Balasko (2015). But it should be noted that the analysis would have been much more intricate.

by the linearly homogeneous production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$. At the beginning of period t, every firm hires L_t unit of labor and K_t unit of capital in order to produce the amount of output $F(K_t, L_t)$. Let denotes the rental rates for labor and capital in period t by w_t and r_t, respectively. In every period $t \geq 0$, firms solve the static problem:

$$\mathrm{P}^f = \max_{(K_t,L_t)} F(K_t, L_t) - r_t K_t - w_t L_t \tag{1}$$

In order to state assumptions, it will be useful to define a reduced production function written only in terms of capital. Define the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(K) = F(K, \ell)$, where ℓ is the total labor endowment of the economy; see below. It is assumed that

Assumption 1. The reduced production function f is continuous on $\mathbb{R}_+$ and $\mathcal{C}^2$ on $\mathbb{R}^2_{++}$ with $f(0) = 0$, $f'(K) > 0$, and $f''(K) < 0$ for all $K \in \mathbb{R}_{++}$. In addition, it holds that $\lim_{K\to 0} f'(K) = +\infty$ and $\lim_{K\to+\infty} f'(K) = 0$ (Inada conditions).

There is a finite number H of households labeled by $h \in \mathcal{H} := \{1, ..., H\}$. The lifetime preferences of households $h \in H$ are describe by an additively separable utility function characterized by (u^h, δ^h), where u^h is the felicity or one-period utility function and δ^h is the discount factor. In order to consider a simple setting in which there are consumption externalities, suppose that each household cares about her own consumption and also about the consumption by all the other households in the economy. The consumption of others may matter because individuals are altruistic, envious, nonconformist, or even malevolent. In this setting,

$$u^h : \mathbb{R}_+ \times \mathbb{R}^{H-1}_+ \to \mathbb{R},$$

so that $u^h(c^h, \mathbf{c}^{-h})$ represents household h's felicity associated with the consumption c^h and the consumption by the other households $\mathbf{c}^{-h} := (c^i)_{i\in\mathcal{H}\setminus\{h\}}$.

Assumption 2. For each $h \in \mathcal{H}$, $0 < \delta^h < 1$, and $1 > \delta^1, \delta^2, ..., \delta^H > 0$.

The households have been indexed from the most patient to the least patient according to the magnitude of their discount factors.

Assumption 3. For each $h \in \mathcal{H}$ the function $u^h : \mathbb{R}_+ \times \mathbb{R}^{H-1}_+ \to \mathbb{R}$ is continuous and $\mathcal{C}^2$ on $\mathbb{R}_{++} \times \mathbb{R}^{H-1}_{++}$. Furthermore, for each $\mathbf{c}^{-h} \in \mathbb{R}^{H-1}_{++}$, the function $u^h(\cdot, \mathbf{c}^{-h})$ is strictly increasing and strictly concave on $\mathbb{R}_{++}$.

Assumption 4. For each $h \in \mathcal{H}$, the felicity function u^h is non-separable in externalities:

$$\frac{\partial}{\partial c^i}\left(\frac{\partial u^h}{\partial c^h}\right) \neq 0, \qquad \forall c^i \in \mathbf{c}^{-h}$$

Assumption 4 implies that externalities do influence not only the felicity levels but also the marginal rate of substitution. In other words, an agent's evaluation of a trade is allowed to depend on the trades engaged by other members of the economy. More precisely, the marginal rate of substitution between any pair of adjacent dates for household $h \in \mathcal{H}$ depends on the consumption of all other households at those dates:

$$MRS^h_{t,t+1} = \frac{\frac{\partial u^h}{\partial c^h}(c^h_t, \mathbf{c}^{-h}_t)}{\delta_h \frac{\partial u^h}{\partial c^h}(c^h_{t+1}, \mathbf{c}^{-h}_{t+1})} \tag{2}$$

This feature is critical. As a matter of fact, the mere dependence of u^h on $\mathbf{c}^{-h}_t$ does not mean that the economic behavior of a household will depend upon the consumption of the others. If, for instance, each household's felicity function is additive separable in the consumption of the rest of households, the presence of externalities would have welfare effects, but it would not affect the behavior of any household.[6] One expects consumption externalities to affect the outcome competitive markets if and only if they have an effect on the marginal rates of substitution. Sole non-additively separable externalities introduce intricate interdependencies.

Household $h \in \mathcal{H}$ is endowed with $k^h > 0$ units of capital at time $t = 0$ and $\ell^h > 0$ units of labor at all dates $t \geq 0$.[7] Let x^h_t denotes the capital stock held by household h at the beginning of period t. The characteristic of competitive environment is that every household behaves as though he were unable to influence the market prices or the actions of other households. Given the prices sequences $\{w_t\}^{\infty}_{t=0}$ and $\{r_t\}^{\infty}_{t=0}$, and the sequence of consumption patterns of other households $\{\mathbf{c}^{-h}_t\}^{\infty}_{t=0}$, each household $h \in \mathcal{H}$ solves[8] :

$$\mathrm{P}^h = \max_{\{c^h_t, x^h_{t+1}\}} \sum_{t=0}^{+\infty} \delta^t_h u^h\left(c^h_t, \mathbf{c}^{-h}_t\right) \tag{3}$$

[6] In a pure exchange economy Dufwenberg, Heidhues, Kirchsteiger, Riedel and Sobel (2011), showed that with additively separable utility functions, equilibrium prices and allocations are those of the economy without externalities.

[7] Notice that, because the utility is derived solely from consumption goods, the competitive household will offer its entire endowment of labor services to the market in each time period.

[8] For simplicity, we assume that capital fully depreciate within the period.

subject to

$$c_t^h + x_{t+1}^h = (1 + r_t)x_t^h + w_t\ell^h \tag{4}$$
$$c_t^h \geq 0, x_{t+1}^h \geq 0, x_0^h = k^h, \{\mathbf{c}_t^{-h}\}_{t=0}^{\infty} \text{ given} \tag{5}$$

where the constraint (5) states that households must have non-negative wealth at each time; they are not allowed to finance present consumption by borrowing against future income.

A collection $\mathcal{E} = \left(f, \{u_h, \delta_h, k^h, \ell_h\}_{h\in\mathcal{H}}\right)$ satisfying Assumptions 1-4, $k^h \geq 0$, $\ell_h > 0$, is said to be a *dependent household felicities Ramsey economy*, or simply an economy.

3 Competitive equilibrium with externalities

The assumed competitive organization of markets along the widespread nature of externalities justify using a noncooperative perfect foresight equilibrium. More specifically, a competitive equilibrium in a Ramsey economy with widespread externalties is defined so that (i) agents (households and firms) maximize their goals by perfectly anticipating and taking as given both the sequences of prices and levels of externalities; (ii) the induced demands and supplies balance at every point of time; and (iii) the resulting levels of externalities coincide at every date with expected levels. Formally:

Definition 1. The sequences of rental rates $\{r_t^\star, w_t^\star\}_{t=0}^{\infty}$ and allocations $\{K_t^\star, L_t^\star, (c_t^{h\star}, x_t^{h\star})_{h\in\mathcal{H}}\}_{t=0}^{\infty}$ constitute an equilibrium for an economy $\mathcal{E}$ provided:

1. for all $h \in \mathcal{H}$, $\{c_t^{h\star}, x_t^{h\star}\}_{t=0}^{\infty}$ solve P^h given $\{r_t^\star, w_t^\star\}_{t=0}^{\infty}$ and $\left\{\mathbf{c}^{-h\star}\right\}_{t=0}^{\infty}$.
2. for each $t \geq 0$, $(K_t^\star, L_t^\star)$ solves P^f given $(r_t^\star, w_t^\star)$.
3. The capital market clears: $K_0^\star = \sum_{h\in\mathcal{H}} k^h$ and, for all $t \geq 1$, $K_t^\star = \sum_{h\in\mathcal{H}} x_t^{h\star}$.
4. The labor market clears: for each $t \geq 0$, $L_t^\star = \sum_{h\in\mathcal{H}} \ell^h := \ell$.

Walras law ensuring balance on the output market, *i.e.*, $\sum_{h\in\mathcal{H}} \left(c_t^{h\star} + x_{t+1}^{h\star}\right) = f\left(K_t^\star\right)$, for all $t \geq 1$, $\sum_{h\in\mathcal{H}} \left(c_0^{h\star} + k^h\right) = f\left(K_0^\star\right)$.

Our purpose is to characterize a special class of equilibria which allows the use of the dynamical systems approach initiated by Becker and Foias (1990).

This special class of equilibria arise when the *turnpike property* applies.[9] The equilibrium path has the turnpike property when the capital stocks held by relatively impatient households (the ones of whom discount factors are below the highest discount factor in the economy), *i.e.*, $\mathcal{H}^\imath := \{h \in \mathcal{H} \backslash \{1\}\}$, are zero for all time. In order for the turnpike property to hold from the model's start, it must be the case that the initial capital endowment of relatively impatient households be zero, *i.e.*, $k^h = 0$, for $h \in \mathcal{H}^\imath$. Yet, a household always earns a wage payment at each time; it always has the option of saving, hence acquiring capital. The equilibrium path must be constructed in such way that only the most patient household holds capital. The following proposition states the necessary and sufficient conditions for the turnpike property to hold for Ramsey economies where $k^h = 0$, $\forall h \in \mathcal{H}^\imath$.

Proposition 1. *Make Assumptions 1-4.* $\{\{x_t^{1\star}\}, \{(x_t^{h\star} = 0)_{h\in\mathcal{H}^\imath}\}\}$ *is an equilibrium capital sequence for each household with* $r_t^\star = f'(x_t^{1\star})$, $\ell w_t^\star = f(x_t^{1\star}) - x_t^{1\star} f'(x_t^{1\star})$, *if and only if* $x_0^{1\star} = k^1$, $x_t^{1\star} > 0$, $t \geq 1$*:*

$$\frac{\partial u_1}{\partial c_t^1}\left(c_t^{1\star}, (w_t^\star \ell^h)_{h\in\mathcal{H}^\imath}\right) = \delta_1 r_{t+1}^\star \frac{\partial u_1}{\partial c_{t+1}^1}\left(c_{t+1}^{1\star}, (w_{t+1}^\star \ell^h)_{h\in\mathcal{H}^\imath}\right), \quad t \geq 1 \tag{6}$$

for $h \in \mathcal{H}^\imath$

$$\frac{\partial u_h}{\partial c_t^h}\left(w_t^\star \ell^h, \left(c_t^{1\star}, (w_t^\star \ell^i)_{\{i\in\mathcal{H}^\imath\backslash\{h\}\}}\right)\right) \geq$$
$$\delta_h r_{t+1}^\star \frac{\partial u_h}{\partial c_{t+1}^h}\left(w_{t+1}^\star \ell^h, \left(c_{t+1}^{1\star}, (w_{t+1}^\star \ell^i)_{\{i\in\mathcal{H}^\imath\backslash\{h\}\}}\right)\right) \tag{7}$$

and for each $h \in \mathcal{H}$*,*

$$\lim_{t\to\infty} \delta_h^t \frac{\partial u_h}{\partial c_t^h}\left(c_t^{h\star}, \mathbf{c}_t^{-h\star}\right) x_{t+1}^{h\star} = 0. \tag{8}$$

Proof. Follows primarily from the marginal conditions of the agents maximization problems P^f and P^h. As regards the production sector, if $0 < r_t^\star < \infty$, under Assumption 1 there is a unique positive stock $K_t^\star$ which solves P^f at each t:

$$f'(K_t^\star) = r_t^\star$$

; in addition, the corresponding wage $w_t^\star$ is positive and defined by

$$w_t^\star = \left[f(K_t^\star) - K_t^\star f'(K_t^\star)\right] \frac{1}{L_t^\star}.$$

[9]Many papers examine Ramsey equilibria when the turnpike property obtains; see among others Becker and Foias (1994), Becker and Tsyganov (2002), and Sorger (1994).

The statement about prices follows from the market clearing conditions $x_t^{1\star} = K_t^\star$ and $\ell := \sum_{h\in\mathcal{H}} \ell^h = L_t^\star$. As for households, the no arbitrage or Euler condition for $\{c_t^{h\star}, x_t^{h\star}\}$ to solve P^h are $c_t^{h\star} > 0$ and:

$$\frac{\partial u_h}{\partial c_t^h}\left(c_t^{h\star}, \mathbf{c}_t^{-h\star}\right) \geq \delta_h r_{t+1} \frac{\partial u_h}{\partial c_{t+1}^h}\left(c_{t+1}^{h\star}, \mathbf{c}_{t+1}^{-h\star}\right) \quad \text{with "=" if} \quad x_{t+1}^{h\star} > 0 \tag{9}$$

The euler condition is sufficient provided that $\lim_{t\to\infty} \delta_h^t \frac{\partial u_h}{\partial c_t^h}\left(c_t^{h\star}, \mathbf{c}_t^{-h\star}\right) = 0$.
□

Clearly, whenever the turnpike property holds, the resulting properties of the model are deduced by examining this special case where the aggregate capital stock and the most patient household's stock are the same. The resulting path of aggregate stocks and consumptions for the most patient household, together with the assignment of wage income to the relatively more impatient households always expresses an equilibrium for some economy. That is, the felicity functions of the other households and their discount factors can always be chosen to support the specially constructed path as an equilibrium.

Following Becker and Tsyganov (2002), the turnpike property obtains whenever relatively impatient households are sufficiently *myopic* in comparison the the dominant household's time preference. It could be emphasized that the myopia argument is particularly suitable here inasmuch as, by assumption, discount factors are exogenous, *i.e.*, not related to consumption externalities.

Formally, fixing the felicity functions, consider the sequence $\{c_t^{1\star}, x_t^{1\star}\}_{t=0}^\infty$, with $x_0^{1\star} = k^1 > 0$.

$$\frac{\partial u_1}{\partial c_t^1}\left(c_t^{1\star}, \left(w(x_t^{1\star})\ell^h\right)_{h\in\mathcal{H}^\imath}\right) = \delta_1 r(x_{t+1}^{1\star}) \frac{\partial u_1}{\partial c_{t+1}^1}\left(c_{t+1}^{1\star}, \left(w(x_{t+1}^{1\star})\ell^h\right)_{h\in\mathcal{H}^\imath}\right) \tag{10}$$

$$c_t^{1\star} + x_{t+1}^{1\star} = r(x_t^{1\star})x_t^{1\star} + w(x_t^{1\star})\ell^1 \tag{11}$$

Let

$$\underline{\delta_h} := \inf_{t,t+1} \frac{\frac{\partial u_h}{\partial c_t^h}\left(w_t^\star \ell^h, \left(c_t^{1\star}, (w_t^\star \ell^i)_{i\in\mathcal{H}^\imath\setminus\{h\}}\right)\right)}{r_{t+1}^\star \frac{\partial u_h}{\partial c_{t+1}^h}\left(w_{t+1}^\star \ell^h, \left(c_{t+1}^{1\star}, (w_{t+1}^\star \ell^i)_{i\in\mathcal{H}^\imath\setminus\{h\}}\right)\right)}. \tag{12}$$

Clearly, if $\delta_h \leq \underline{\delta_h}$ for each $h \in \mathcal{H}^\imath$, then the no arbitrage conditions (7) remain slack along the considered path. It follows that the sequence $\{c_t^{1\star}, x_t^{1\star}\}_{t=0}^\infty$

is an equilibrium. Assuming $\delta_h \leq \underline{\delta_h}$, entails that household h perfectly anticipating the prices $\{(w_t^\star, r_t^\star)\} = \{(w(x_t^{1\star}, r(x_t^{1\star})\}$ has no incentive to acquire capital.

Assumption 5. Let $\underline{\delta_h}$ defined by (12). For each $h \in \mathcal{H}^\imath$, $\delta_h \leq \underline{\delta_h}$.

Observe that, as $\underline{\delta_h}$ depends, among other things, upon δ_1, Assumption 5 is equivalent to: $\delta_h << \delta_1$, that is impatient households are strongly myopic in comparison to $h = 1$.

Characterization of the class of competitive equilibria satisfying the turnpike property summarized in the following lemma stated without proof (in the sequel, the "$\star$" denoting a Ramsey equilibrium are dropped to simplify the notation as meaning is clear).

Lemma 1. *Make Assumptions (1)-(5). Let $\delta := \delta_1$, $c := c^1$, and $x := x^1$. A Ramsey equilibrium is a sequence $\{c_t, x_t\}_{t=0}^{\infty}$, with $x_0 = k^1$, such that*

$$\frac{\partial u_1}{\partial c_t^1}\left(c_t, \left(w(x_t)\ell^h\right)_{h\in\mathcal{H}^\imath}\right) = \delta r'(x_{t+1})\frac{\partial u_1}{\partial c_{t+1}^1}\left(c_{t+1}, \left(w(x_{t+1})\ell^h\right)_{h\in\mathcal{H}^\imath}\right) \tag{13}$$

$$c_t + x_{t+1} = r(x_t)x_t + w(x_t)\ell^1 := g(x_t) \tag{14}$$

where g is called the dominant household's income function.

It is straightforward to see that the stationary equilibrium, that is the equilibrium in which prices and allocations remain constant over time, belongs to the class of equilibria satisfying the turnpike property.[10] Indeed,

$$r := f'(x) = \delta^{-1} \tag{15}$$

$$c + x = f'(x)x + w(x)\ell^1 \tag{16}$$

The no arbitrage conditions (9) ensuring that $x^h = 0$, $\forall h \in \mathcal{H}^\imath$, being satisfied, only the most patient household holds capital.

When the turnpike property holds, as this is the case for the stationary one, non-stationary equilibria are the trajectories of the dynamical system in the plane defined by (13)-(14).

[10] The existence and uniqueness of the stationary equilibrium are ensured by Assumption 1; the positivity of the equilibrium rental rate r follows from Assumption 2.

4 Maximum income monotonicity and non-convergent Ramsey equilibria

Becker, Dubey and Mitra (2014) established that if the maximal income monotonicity (MIM) condition is satisfied, the capital stock sequence along *every* Ramsey equilibrium path is convergent and consequently the turnpike property on the capital ownership pattern holds. In this subsection, it is shown that there exists dependent felicities Ramsey equilibria satisfying (by assumption) the turnpike property that fail to converge even when the MIM holds.

The existence of non-convergent equilibria will be proved by a local analysis of the equilibrium conditions around the stationary equilibrium. The properties of non-stationary equilibria close to the stationary one are indeed described by the difference equations:

$$\begin{pmatrix} dx_{t+1} \\ dc_{t+1} \end{pmatrix} = \begin{pmatrix} g'(x) & -1 \\ \frac{1}{\eta}\frac{c}{x}\left(\theta g'(x) - \frac{w'(x)x}{w(x)}\varepsilon\right) & 1 - \frac{1}{\eta}\frac{c}{x}\theta \end{pmatrix} \begin{pmatrix} dx_t \\ dc_t \end{pmatrix} \tag{17}$$

for

$$\begin{aligned} \eta &:= -\frac{\partial^2 u_1}{\partial c_1 \partial c_1}\left(c, \left(w(x)\ell^h\right)_{h\in\mathcal{H}^\imath}\right) \\ \varepsilon &:= \sum_{h\in\mathcal{H}^\imath} \frac{\partial^2 u_1}{\partial c_1 \partial c_h}\left(c, \left(w(x)\ell^h\right)_{h\in\mathcal{H}^\imath}\right) \\ \theta &:= \frac{r'(x)x}{r(x)} + \frac{w'(x)x}{w(x)}\varepsilon \end{aligned}$$

The characteristic polynomial of the Jacobian matrix is $P(\lambda) = \lambda^2 - T\lambda + D$, where

$$T = g'(x) + 1 - \frac{1}{\eta}\frac{c}{x}\theta, \tag{18}$$

$$D = g'(x) - \frac{1}{\eta}\frac{c}{x}\frac{w'(x)x}{w(x)}\varepsilon. \tag{19}$$

Consider

$$P(1) = 1 - T + D = \frac{1}{\eta}\frac{c}{x}\frac{r'(x)x}{r(x)}. \tag{20}$$

Under Assumptions 1 and 3, $P(1) < 0$. It follows that the characteristic roots are real; one (at least) being unstable, i.e., has a norm greater than one. In other terms, the stationary equilibrium is either a saddle-point or (locally) unstable. Yet, the existence of sustained cycles is not precluded. Indeed, whenever it happens that one of the eigenvalues equals -1, the flip bifurcation theorem teaches us that, generically, there exists a periodic orbit of period two. In order for an eigenvalue to be equal to -1, it must be the case that $P(-1) = 1 + T + D = 0$, that is

$$P(-1) = 2(1 + g'(x)) - \frac{1}{\eta}\frac{c}{x}\left[\frac{r'(x)x}{r(x)} + 2\frac{w'(x)x}{w(x)}\varepsilon\right] = 0. \tag{21}$$

Consider first the case without externalities, $\varepsilon = 0$. From (21), flip cycles require $g'(x) < 0$. Conversely, whenever $g'(x) > 0$, the stationary equilibrium is a saddle-point. This is the result of Becker and Foias (1994). Observe that $g(\cdot)$ boils down to the *maximun income* emphasized by Becker, Dubey and Mitra (2014) whenever $\ell^h = \ell/H$, $\forall h \in \mathcal{H}$. Clearly, the maximum income is monotone increasing (MIM) ensures the convergence of the capital stock along a (local) Ramsey equilibrium.

Now consider the case where consumption externalities are present and non vanishing at the stationary equilibrium, $\varepsilon \neq 0$. Provided that $\varepsilon > 0$, the holding of MIM is no longer sufficient to rule non-convergent equilibria. More precisely, as long as the term is square brackets is positive, even though $g'(x) > 0$, $1+T+D$ may vanish for a proper degree of the patient household consumption substitutability measured by $1/\eta$.

Given this result, one can seek restrictions on the primitives of the model, *i.e.*, the specification of felicity and production functions, and the patient household time preference, under which a flip bifurcation is allowed to occur. We postulate $g'(x) > 0$ for the remainder of this paper without further mention. First, notice that $g'(x)$ can be written

$$g'(x) = r(x)\left(1 + \frac{r'(x)x}{r(x)} + \frac{w'(x)x}{w(x)}\frac{w(x)}{r(x)}\frac{\ell^1}{x}\right),$$

and, from (16),

$$\frac{c}{x} = r(x)\left(1 + \frac{w(x)}{r(x)}\frac{\ell^1}{x}\right) - 1.$$

Restrictions on the production function can be expressed conveniently in terms of the capital share and the elasticity of substitution of F evaluated at the stationary equilibrium, respectively:

$$
\begin{aligned}
s &:= s(x) = \frac{f'(x)x}{f(x)} \\
\sigma &:= \sigma(x) = \frac{f'(x)\left(f(x) - xf'(x)\right)}{xf(x)f''(x)}
\end{aligned}
$$

Remembering that $r(x) = f'(x)$ and $w(x) = f(x) - xf'(x)$, one shows that:

$$
\begin{aligned}
\frac{w'(x)x}{w(x)} &= \frac{s}{\sigma} \\
\frac{r'(x)x}{r(x)} &= -\frac{1-s}{\sigma} \\
\frac{w(x)}{r(x)} &= \frac{1-s}{s}\frac{x}{\ell}
\end{aligned}
$$

Making use of the relations above and $f'(x) = \delta^{-1}$, one finally gets:

$$
\begin{aligned}
g'(x) &= \delta^{-1}\left(1 - \frac{1-s}{\sigma} + \frac{1-s}{\sigma}\frac{\ell^1}{\ell}\right), \\
\frac{c}{x} &= \delta^{-1}\left(1 + \frac{1-s}{s}\frac{\ell^1}{\ell}\right) - 1.
\end{aligned}
$$

Now, assume that both ε and the term in square brackets in relation (21) are positive. Solving in η delivers the critical value, labeled η^c:

$$
\eta^c = \alpha + \beta\varepsilon \tag{22}
$$

for

$$
\begin{aligned}
\alpha &:= \frac{1}{2}\frac{\frac{c}{x}\frac{r'(x)x}{r(x)}}{1 + g'(x)} = -\frac{1}{2}\frac{\left(\delta^{-1}\left(1 + \frac{1-s}{s}\frac{\ell^1}{\ell}\right) - 1\right)\frac{1-s}{\sigma}}{1 + \delta^{-1}\left(1 - \frac{1-s}{\sigma} + \frac{1-s}{\sigma}\frac{\ell^1}{\ell}\right)} \\
\beta &:= \frac{\frac{c}{x}\frac{w'(x)x}{w(x)}}{1 + g'(x)} = \frac{1}{2}\frac{\left(\delta^{-1}\left(1 + \frac{1-s}{s}\frac{\ell^1}{\ell}\right) - 1\right)\frac{s}{\sigma}}{1 + \delta^{-1}\left(1 - \frac{1-s}{\sigma} + \frac{1-s}{\sigma}\frac{\ell^1}{\ell}\right)}
\end{aligned}
$$

As η goes through η^c a flip bifurcation occurs, which implies that, for an open interval of value of η close to η^c there exit periodic competitive equilibria of period 2.

At that stage it could be noted that, beside the assumed MIM condition on the production function, the only restriction on the primitives of the model required for the existence of period two cycles is the positivity of the term in square brackets in relation (21), *i.e.*,:

$$\varepsilon > \frac{1}{2}\frac{1-s}{s}.$$

The sign of $\eta^c - \varepsilon$ is however of interest. A negative value would mean that, along the stationary equilibrium, the effect of changes in the consumption $(\mathbf{c}^{-1})$ of impatient households on the patient household's marginal felicity - ε - dominates the effect of change in the consumption c^1 of the patient household supporting a flip bifurcation - η^c -. In other words, at the steady state, the patient household's marginal felicity should be *more sensitive* to the consumption of the other (impatient) households than to his own consumption in order for a two period Ramsey equilibria to exist. In such a case, the existence of non-convergent equilibria would rest upon *strong* externalties.[11] We shall now establish that non-convergent equilibria exist even though external effects on the impatient household felicity are dominated by the effect of his own consumption.

Lemma 2. *Let* η^c *be given by* (22), *then*
if $\beta < 1$, $\forall\varepsilon > 0$, $\eta^c < \varepsilon$;
if $\beta > 1$, $\exists\, \overline{\varepsilon} > 0$ *such that* $\eta^c > \varepsilon$, $\forall\varepsilon > \overline{\varepsilon}$.

Proof. The statement straightforwardly follows from $\alpha < 0$. □

Now, make

Assumption 5.

$$\sigma < \overline{\sigma} := \frac{1-\delta s}{1+\delta}.$$

Summing up the above discussions, we have:

[11] In the case of finite pure exchange economies with consumption externalities, Bonnisseau and del Mercato (2010) established that the standard assumptions do not suffice to guarantee the generic regularity of the competitive equilibrium. An additional assumption on the second order external effects on utility, that ensures that the external effect on one consumer's marginal utilities is dominated by the effect of his own consumption, is required. In the absence of this assumption, they provided an example where equilibria are indeterminate for all initial endowments.

Proposition 2. *Make Assumptions (1)-(5). It exists $\overline{\varepsilon} > 0$, actually given by*

$$\overline{\varepsilon} := \frac{1}{2}\frac{\frac{c}{x}\frac{1-s}{\sigma}}{\frac{c}{x}\frac{s}{\sigma} - (1+g'(x))}, \tag{23}$$

such that for all $\varepsilon > \overline{\varepsilon}$, there is η^c given by (22), satisfying $\eta^c > \varepsilon$, such that for values of η close to η^c, there generically exist period two cycles.

Proof. Assumption 5 implies that $\beta > 1$. The statement follows from Lemma 2 and the flip bifurcation theorem. □

5 Conclusion

This paper has introduced consumption externalities in a standard Ramsey model with heterogeneous agents and borrowing constraints. It has been shown that the Maximum Income Monotonocity (MIM) assumption is no longer sufficient to rule out non-convergent Ramsey equilibria, even if the turnpike property applies. Furthermore, the existence of such equilibria is compatible with the second order external effects on felicity functions. This clearly establish that nonmarket interdependences may have noticeable positive (as opposite to normative) influence on competitive market mechanisms.

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