## DOCUMENT DE RECHERCHE EPEE

centre d'Etude des Politiques Economiques de l'universite d'Evry

## Discrete Time Macroeconomic Dynamics

Stefano BOSI
01-07R

# Discrete Time Macroeconomic Dynamics 

Stefano Bosi<br>EPEE, University of Evry

February 3, 2003

## Contents

0.1 Introduction ..... 7
1 Elements of Dynamics ..... 9
1.1 Ordinary Difference Equations ..... 9
1.2 Autonomous Difference Equations ..... 10
1.3 Linear Dynamics ..... 12
1.4 Non-Linear Dynamics ..... 14
I Demand Functions ..... 23
2 The Consumption Function ..... 25
2.1 A Keynesian Language ..... 25
2.2 The Relative Income Theory ..... 29
2.3 The Life-Cycle Theory ..... 29
2.4 The Permanent Income Theory ..... 31
2.5 Two-Period Utility Maximization ..... 34
2.6 Intertemporal Utility Maximization ..... 37
2.7 Two-Period Stochastic Consumption ..... 42
2.8 Three-Period Stochastic Consumption ..... 47
2.9 The Random Walk of Consumption ..... 49
3 The Investment Function ..... 63
3.1 The Investment Function ..... 64
3.2 Duality ..... 65
3.3 Static Profit Maximization ..... 69
3.4 Dynamic Behavior ..... 71
3.5 Tobin's $q$ ..... 73
3.6 The Neutrality Theorem of Modigliani-Miller ..... 82
II General Equilibrium ..... 87
4 Exogenous Saving ..... 89
4.1 Growth Accounting ..... 89
4.1.1 Statistics on growth ..... 90
4.1.2 Facts ..... 90
4.1.3 Comments ..... 91
4.2 The Solow Model ..... 92
4.3 Exogenous Technical Progress ..... 99
4.4 Endogenous Growth in a Solow Framework ..... 104
4.5 Open Economy ..... 105
4.5.1 Human Capital ..... 105
4.5.2 Taxes and Absolute Convergence ..... 107
4.5.3 Transition Dynamics in a Closed Economy ..... 107
4.5.4 Transition Dynamics in an Open Economy ..... 109
5 Endogenous Saving ..... 113
5.1 Two-Period Equilibrium Model ..... 113
5.1.1 Decentralized Equilibrium ..... 113
5.1.2 Planner's Problem ..... 115
5.2 Infinite-Lived Agents ..... 116
5.2.1 Decentralized Equilibrium ..... 116
5.2.2 Central Planner ..... 122
5.2.3 Open Economy ..... 127
5.2.4 $A k$ Model ..... 128
5.2.5 Transaction Costs and Indeterminacy ..... 130
5.2.6 More on Endogenous Growth ..... 144
5.2.7 More on Indeterminacy ..... 147
5.3 Overlapping Generations Models and Bubbles ..... 151
5.3.1 Rational Bubbles ..... 151
5.3.2 Rational Bubbles and Growth ..... 162
III Exercises ..... 171
6 Exercises ..... 173
6.1 Elements of Dynamics ..... 173
6.1.1 Autonomous Difference Equations ..... 173
6.1.2 Autonomous Difference Equations ..... 174
6.1.3 Planar Systems ..... 175
6.1.4 Planar Systems ..... 176
6.1.5 Planar Systems ..... 176
6.1.6 Planar Systems ..... 177
6.2 The Consumption Function ..... 178
6.2.1 The Life-Cycle Hypothesis ..... 178
6.2.2 The Life-Cycle Hypothesis ..... 179
6.2.3 The Life-Cycle Hypothesis ..... 180
6.2.4 The Permanent Income Hypothesis ..... 181
6.2.5 The Permanent Income Hypothesis ..... 186
6.2.6 The Permanent Income Hypothesis ..... 187
6.2.7 The Permanent Income Hypothesis ..... 189
6.2.8 The Permanent Income Hypothesis ..... 191
6.2.9 Three-Period Utility Maximization ..... 193
6.2.10 Infinite Horizon Utility Maximization ..... 193
6.2.11 Constant Elasticity of Intertemporal Substitution ..... 196
6.2.12 Infinite Horizon Utility Maximization ..... 197
6.2.13 Two-Period Stochastic Maximization ..... 201
6.2.14 Three-Period Stochastic Maximization ..... 203
6.2.15 Infinite Horizon Stochastic Optimization ..... 208
6.3 The Investment Function ..... 212
6.3.1 Static Behavior ..... 212
6.3.2 Static Behavior ..... 216
6.3.3 Dynamic Behavior ..... 216
6.4 Exogenous Saving ..... 219
6.4.1 A Static Linear $I S-L M$ Model ..... 219
6.4.2 Time to Double ..... 220
6.4.3 The Solow Model with a Cobb-Douglas Production Function ..... 220
6.4.4 $C E S$ Case ..... 222
6.5 Endogenous Saving ..... 228
6.5.1 The Clower Constraint ..... 228
6.5.2 Barro Model ..... 231
6.5.3 The Diamond Model with Central Planner ..... 235
6.5.4 The Diamond Model with Market Economy ..... 237
6.5.5 The Decentralized Equilibrium in an Overlapping Gen- eration Model ..... 241

[^0]
### 0.1 Introduction

This work is the fruit of several years of teaching to undergraduate students in the University of Evry and in the ENSAI of Rennes.

The aim of the book is to equip students with basic tools to understand the intertemporal macroeconomics in terms of general equilibrium dynamics. Equilibrium simply means equality between aggregate demand and supply in the market of a good. An equilibrium is said to be general if all the markets are in equilibrium. A general equilibrium is said to be dynamic if all the markets are in equilibrium at each period.

More in details we are interested in studying the rational behavior of a price-taker agent who chooses the intertemporal profiles of his relevant economic variables such as the consumption or investment, to maximize an individual objective such as an intertemporal utility function or a profit function under a set of constraints such as an intertemporal budget constraint. The solutions of individual programs are the intertemporal demand functions which can be aggregated to set up the market clearing conditions and to compute the dynamic general equilibrium prices and quantities. The micro-foundation of a macroeconomic system is this construction of individual demands before aggregation. The evolution of equilibrium prices and quantities across the time will be the very center of our intellectual effort. More in details we will focus on the occurrence of economic cycles in the short run and growth in the long run.

The first chapter presents a short overview about the ordinary difference equations and provides the set of mathematical instruments required during the course. All the models will be set up in discrete time to make easier the access to less skilled students. Only the knowledge of partial derivatives as well as some basic notion of linear algebra are needed. Moreover the discrete time approach will turn out to be more adapted to construct the monetary versions of the benchmark models.

The first part focuses on the individual behaviors and demand functions. In the second chapter we study the consumer's behavior and we derive his intertemporal demand function. The third chapter is centered around the study of the optimal investment decision and around the computation of the intertemporal investment function.

The second part treats general equilibrium dynamics. Within this part the fourth chapter introduces the most elementary models of growth characterized by an exogenous saving rate. Chapter 5 generalizes the fourth one
by revising preferences, i.e. assuming an endogenous saving in economies populated by infinite-lived agents or overlapping generations.

The third part is constituted by five sections corresponding to the five chapters of the previous parts.

The real models presented in the course are often integrated by their monetary versions.

Evry, February 1, 2003.

## Chapter 1

## Elements of Dynamics

The economic dynamics are modeled in either (i) discrete time or (ii) continuous time. The agents optimize their objectives under a system of constraints, i.e. they maximize either ( $i$ ) Lagrangian functionals or (ii) Hamiltonian functionals, and they obtain as reduced forms respectively $(i)$ systems of difference equation or (ii) systems of differential equations. If time $t$ variable we choose at time 0 , does not affect the state of dynamic system and returns before $t$, a discrete time problem can be shaped in the Bellman (1957) recursive form.

Before we enter the economic aspects, an elementary overview is provided about the discrete time dynamics. All our economic applications will be made with a discrete time approach. Therefore we will not deal with systems of differential equations in continuous time but only with systems of difference equation in discrete time.

### 1.1 Ordinary Difference Equations

We capture in general the dynamic relation between more lagged variables by an implicit function: $F\left(t, y_{t}, y_{t+1}, \ldots, y_{t+m}\right)=0$. By applying the implicit function theorem, the higher ordered vector $y_{t+m}$ can be made locally explicit: $y_{t+m}=G\left(t, y_{t}, y_{t+1}, \ldots, y_{t+m-1}\right)$. By simplicity let us assume the we have a globally explicit form. By operating a substitution of variables we obtain a first order system: $x_{t+1}=f\left(t, x_{t}\right)$ under the usual initial condition $x_{0}$.

$$
\begin{aligned}
x_{1, t} & \equiv y_{t} \\
x_{2, t} & \equiv y_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
& \vdots \\
x_{i, t} \equiv & y_{t+i-1} \\
& \vdots \\
x_{m, t} \equiv & y_{t+m-1}
\end{aligned}
$$

Thereby

$$
\begin{aligned}
x_{1, t+1}= & x_{2, t}, \\
x_{2, t+1}= & x_{3, t} \\
& \vdots \\
x_{i, t+1}= & x_{i+1, t}, \\
& \vdots \\
x_{m-1, t+1}= & x_{m, t}, \\
x_{m, t+1} \equiv & y_{t+m}=G\left(t, y_{t}, y_{t+1}, \ldots, y_{t+m-1}\right)=G\left[t, x_{1, t}, x_{2, t}, \ldots, x_{m, t}\right] .
\end{aligned}
$$

This system gets the compact form $x_{t+1}=f\left(t, x_{t}\right)$, where $x_{t} \equiv\left(x_{1, t}, \ldots, x_{m, t}\right)^{t}$ and $x_{0}$ is given.

### 1.2 Autonomous Difference Equations

In the economic models we shall consider, this system of difference equations will be autonomous with respect to the time:

$$
x_{t+1}=f\left(x_{t}\right) .
$$

Eigenvalues and Eigenvectors. Before we introduce the notion of eigenvalue, the definition of complex number is required. One meets the complex numbers in the solution of second degree algebraic equations: $a x^{2}+$ $b x+c=0$. The roots are $x=-[b /(2 a)] \pm \sqrt{[b /(2 a)]^{2}-c / a}$. If $b^{2}<4 a c$ two conjugate complex numbers appear with form $\alpha \pm \beta i$ where $\alpha=-[b /(2 a)]$ denotes the real part of the complex number and $\beta=\sqrt{c / a-[b /(2 a)]^{2}}$ denotes the coefficient of the imaginary part. By definition $i \equiv \sqrt{-1}$. Every
complex number can be represented in a Gaussian plane.


Figure 1. The Gaussian plane.
Conjugate complex numbers are symmetric with respect to the axis of abscissas. In particular real numbers are complex numbers with a nul coefficient of the imaginary part. They lie on the axis of abscissas in the Gaussian plane. The modulus of a complex number is the Euclidean length of the corresponding vector in the Gaussian plane: $\sqrt{\alpha^{2}+\beta^{2}}$. Observe that the product of two conjugate complex numbers is just the square of their modulus: $(\alpha+\beta i)(\alpha-\beta i)=\alpha^{2}+\beta^{2}$. A unit circle is plotted in the Gaussian plane: $\alpha^{2}+\beta^{2}=1$.

Now we can define the eigenvalues $\lambda$ 's and the eigenvectors $v$ 's of a square matrix $J$. An eigenvalue and its associated eigenvector constitute a solution for the algebraic equation:

$$
J v=\lambda v .
$$

Computation of eigenvalues is performed as follows. An equivalent equation is $(J-\lambda I) v=0$, where $I$ is the identity matrix having the same dimension of $J$. Let me observe that $(J-\lambda I) v$ must be a non-trivial linear combination of the columns of matrix $J-\lambda I$, i.e. the determinant of this matrix must be zero:

$$
|J-\lambda I|=0 .
$$

The solution is detailed in the case of square matrices of order 2 .

$$
J=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] .
$$

Hence

$$
|J-\lambda I|=\left|\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]-\lambda\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right|=\left|\left[\begin{array}{cc}
a-\lambda & b \\
c & d-\lambda
\end{array}\right]\right|=0
$$

i.e. $(a-\lambda)(d-\lambda)-b c=0$. One needs to solve the second degree equation with respect to $\lambda: \lambda^{2}-(a+d) \lambda+a d-b c=0$. The trace of matrix $J$ is the sum of elements on the principal diagonal: $\operatorname{tr} J=a+d$. The determinant is constituted by the following expression: $\operatorname{det} J=a d-b c$. Eventually the characteristic polynomial is given by $\lambda^{2}-\lambda \operatorname{tr} J+\operatorname{det} J$. The two solutions $\lambda_{1}$ and $\lambda_{2}$ allow a decomposition: $\lambda^{2}-(a+d) \lambda+a d-b c=\left(\lambda-\lambda_{1}\right)\left(\lambda-\lambda_{2}\right)=0$ with $a+d=\operatorname{tr} J=\lambda_{1}+\lambda_{2}$ and $a d-b c=\operatorname{det} J=\lambda_{1} \lambda_{2}$. After we have computed the two eigenvalues $\lambda_{1}$ and $\lambda_{2}$, we solve the algebraic linear system $J v_{i}=\lambda_{i} v_{i}$ to find for each eigenvalue $\lambda_{i}$ the related eigenvector $v_{i}$ (see among the others Hale and Koçak (1991, p. 228)). A square matrix of dimension $n$ has $n$ eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$, which are possibly multiple. Note that the eigenvalues may be complex. In this case, they are pairwise conjugated. Assume for the sake of simplicity that the eigenvalues are all real. We write: $J v_{1}=\lambda_{1} v_{1}, \ldots, J v_{n}=\lambda_{n} v_{n}$, and we compact this list of column vectors in two matrices: $\left[J v_{1}, \ldots, J v_{n}\right]=\left[\lambda_{1} v_{1}, \ldots, \lambda_{n} v_{n}\right]$, i.e.

$$
J\left[v_{1}, \ldots, v_{n}\right]=\left[v_{1}, \ldots, v_{n}\right]\left[\begin{array}{ccc}
\lambda_{1} & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & \lambda_{n}
\end{array}\right]
$$

In the simplest case of eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$ all real, the matrix

$$
\Lambda \equiv\left[\begin{array}{ccc}
\lambda_{1} & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & \lambda_{n}
\end{array}\right]
$$

is a diagonal matrix. Let the matrix of eigenvectors be denoted by $V$ : $V \equiv\left[v_{1}, \ldots, v_{n}\right]$. Hence $J V=V \Lambda$, i.e. $\Lambda=V^{-1} J V$. As we will see the transformation matrix $V$ implements a basis change in the vector space. It diagonalizes $J$, i.e. it reduces the original matrix to a Jordan canonical form.

### 1.3 Linear Dynamics

We consider a general linear system of difference equations.

$$
\begin{equation*}
x_{t+1}=J x_{t}+c . \tag{1.1}
\end{equation*}
$$

This system is equivalent to the following.

$$
\begin{equation*}
x_{t+1}-x=J\left(x_{t}-x\right), \tag{1.2}
\end{equation*}
$$

where $x$ is the fixed point of system (1.1), i.e. the solution of $x=J x+c$ :

$$
x=(I-J)^{-1} c .
$$

The proof is simple.

$$
\begin{aligned}
x_{t+1}-x & =J\left(x_{t}-x\right), \\
x_{t+1}-(I-J)^{-1} c & =J\left[x_{t}-(I-J)^{-1} c\right], \\
x_{t+1} & =J x_{t}+(I-J)^{-1} c-J(I-J)^{-1} c \\
& =J x_{t}+(I-J)(I-J)^{-1} c \\
& =J x_{t}+c .
\end{aligned}
$$

If the original system is already linear, the global dynamics can be made explicit. From (1.2) we obtain

$$
x_{t+1}-x=V \Lambda V^{-1}\left(x_{t}-x\right),
$$

where $\Lambda$ diagonalizes $J$. Hence

$$
x_{t}-x=V \Lambda^{t} V^{-1}\left(x_{0}-x\right)
$$

where $x_{0}$ is the starting point of trajectory, because

$$
\left(V \Lambda V^{-1}\right)\left(V \Lambda V^{-1}\right)=V \Lambda^{2} V^{-1}
$$

In particular whenever all the eigenvalues are real and distinct, the matrix power becomes

$$
\Lambda^{t}=\left[\begin{array}{ccc}
\lambda_{1}^{t} & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & \lambda_{n}^{t}
\end{array}\right]
$$

If $V=I$ and $x^{*}=0: x_{t}=\Lambda^{t} x_{0}$, i.e.

$$
x_{i t}=\lambda_{i}^{t} x_{i 0}
$$

for every $i=1, \ldots, n$. Otherwise the dynamics represented by $x_{i t}=\lambda_{i}^{t} x_{i 0}$, just refer to the equivalent space whose basis is obtained by implementing the change of basis in the original space via the transformation matrix $V$.

Note that the product of the eigenvalues is given by

$$
\operatorname{det} J=\operatorname{det}\left(V \Lambda V^{-1}\right)=\operatorname{det} V \operatorname{det} \Lambda(\operatorname{det} V)^{-1}=\operatorname{det} \Lambda
$$

and the trace of $J$ equals the trace of $\Lambda$ i.e. the sum of eigenvalues.
There are three topological classes of stationary states.
(i) If all the eigenvalues lie in the unit circle in the Gaussian plane, the stationary state is stable in all directions and is said to be a sink (see figure $2)$.
(ii) If all the eigenvalues are outside the unit circle, the stationary state is instable in all directions and is said to be a source (figure 3).
(iii) If eventually some eigenvalues lie inside and some others outside the unit circle, then the stationary state is stable according to some particular directions and unstable with respect the other ones, and it is said to be a saddle point (figure 4).


Figure 2. Sink.


Figure 3. Source.


Figure 4. Saddle.

### 1.4 Non-Linear Dynamics

We consider a general non-linear system of ordinary and autonomous difference equations:

$$
x_{t+1}=f\left(x_{t}\right) .
$$

By definition the steady state is the fixed point of the dynamic system. It is obtained as solution of the algebraic equation

$$
x=f(x) .
$$

Let the function $f \in C^{\infty}$ be differentiable infinitely many times. It can be represented by a Taylor series and approximated by a polynomial.

$$
f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}\left(x_{0}\right)}{n!}\left(x-x_{0}\right)^{n} .
$$

A linear approximation of $f$ is obtained by canceling out the terms of a Taylor series after the first order:

$$
f(x) \approx f\left(x_{0}\right)+f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)
$$

In general if we consider real vector functions of more variables we can linearize the original function around the point $x_{0}$ by considering only the first order term in the Taylor series:

$$
f(x)=f\left(x_{0}\right)+D_{x} f\left(x_{0}\right)\left(x-x_{0}\right)+\ldots,
$$

i.e.

$$
f(x)-f\left(x_{0}\right) \approx D_{x} f\left(x_{0}\right)\left(x-x_{0}\right)
$$

Let now $x_{0}=x$ be the stationary state of dynamic system, i.e.: $x=f(x)$. We know that $x_{t+1}=f\left(x_{t}\right)$. We obtain

$$
f\left(x_{t+1}\right)-f(x) \approx D_{x} f(x)\left(x_{t}-x\right)
$$

and eventually

$$
x_{t+1}-x \approx D_{x_{t}} f(x)\left(x_{t}-x\right),
$$

where $D_{x_{t}} f(x)$ is the Jacobian matrix of our dynamic system computed at the steady state. For instance the two-dimensional dynamic system

$$
\left[\begin{array}{l}
x_{1 t+1} \\
x_{2 t+1}
\end{array}\right]=f\left[\begin{array}{l}
x_{1 t} \\
x_{2 t}
\end{array}\right]
$$

is approximated by

$$
\left[\begin{array}{c}
x_{1 t+1} \\
x_{2 t+1}
\end{array}\right]-\left[\begin{array}{c}
x_{1} \\
x_{2}
\end{array}\right] \approx\left[\begin{array}{cc}
\partial f_{1} / \partial x_{1 t} & \partial f_{1} / \partial x_{2 t} \\
\partial f_{2} / \partial x_{1 t} & \partial f_{2} / \partial x_{2 t}
\end{array}\right]^{*}\left(\left[\begin{array}{l}
x_{1 t} \\
x_{2 t}
\end{array}\right]-\left[\begin{array}{c}
x_{1} \\
x_{2}
\end{array}\right]\right)
$$

where the asterisk denotes the evaluation at the steady state.
The Großman-Hartman ${ }^{1}$ theorem ensures that the local dynamics generated by the Jacobian matrix $J$, which linearizes a dynamic system around its

[^1]stationary state, are topologically equivalent to the local dynamics generated by the Jordan canonical form $\Lambda$ one derives from $J$.

More explicitly a stable steady state remains stable after the linear transformation operated by $V$, an unstable steady state remains unstable, a saddle point is transformed in a saddle point. The Jacobian for the new equivalent dynamics computed in the steady state, is exactly $\Lambda$.

If the system is non-linear, we perform the linearization around the steady state and the reduction to the Jordan canonical form.

The Jacobian matrix

$$
J=\left[\begin{array}{ll}
\partial f_{1} / \partial x_{1 t} & \partial f_{1} / \partial x_{2 t} \\
\partial f_{2} / \partial x_{1 t} & \partial f_{2} / \partial x_{2 t}
\end{array}\right]^{*}
$$

evaluated at the steady state allows us to characterize the equilibrium stability. Conditions for stability look like those we encounter in continuous time, but now what really matters is just the position of each eigenvalue with respect to the unit circle in the Gaussian plane instead of the sign of the eigenvalue real part (see figure 1).

The set of steady states is partitioned in three classes of topological equivalence. Necessary and sufficient conditions are provided to know the equivalence class for each steady state. (i) If the eigenvalues of $J(x)=D_{x} f(x)$ lie all in the interior of unit circle, the steady state is locally stable in all the directions. (ii) If the eigenvalues of $J(x)=D_{x} f(x)$ locate all outside the unit circle, the steady state is locally unstable in all the directions.(iii) If at least one eigenvalue of $J(x)=D_{x} f(x)$ lies into the unit circle and at least one outside, the stationary state is a saddle point. For the latter case, a finer partition of the topological class will depend on the exact number of eigenvalues into the unit circle.

Bifurcations. The dynamic system may capture either a numerical relation or a parametric relation between the variables. In the latter case the change of one parameter may transform the steady state from a qualitative point of view: the stationary state enters a new equivalence class of stability. For instance a saddle point of planar systems may become a sink or a source. The critical value of parameter allowing for a stability change is said to be a bifurcation value.

Indeterminacy and Endogenous Fluctuations. A variable, which has been determined prior to time $t$, is said to be predetermined at time $t$.

For instance the stock of capital $k_{t}$ is a predetermined variable, because it depends on the investment decisions, which has been taken in the previous period $t-1$. Whenever the stable manifold, i.e. the union of all the converging trajectories to the same attractor, has a dimension greater than the number of predetermined variables figuring in the dynamic system, there is a multiplicity and more precisely a continuum of equilibrium paths. This kind of multiplicity is said to be local indeterminacy. If the equilibria are indeterminate, the agents may individually saturate this degree of freedom by relating their choices to exogenous random signals, which do not affect the fundamentals (technology, preferences and endowments) and are said to be sunspots. Usually the probability distribution of a sunspot is assumed to be common knowledge and it is inferred from past realizations of signal. In other words the sunspot shocks the believes instead of fundamentals. If the way of relating the economic future to this distribution is the same for all the agents, the believes are shared. If the choices of the agents and shared believes satisfy the stochastic version of our dynamic system, the shared believes become self-fulfilling prophecies. Local indeterminacy is the necessary condition to observe stochastic (sunspot) equilibria, i.e. stochastic endogenous fluctuations.

One-Dimensional Dynamics. If $f$ is real valued and $x_{t}$ is a scalar, the eigenvalue is always real and the stability condition becomes: $\left|f^{\prime}(x)\right|<1$. If $x$ is a non predetermined variable, the converging sequence of point is indeterminate (see figure 30).

Two-Dimensional Dynamics. Dynamic economic model often present two-dimensional systems as reduced form, i.e. two difference equations of first order. Hence the rest of the chapter is dedicated to planar systems and their local characterization. We shall apply a simple geometrical method to study the steady state stability in a trace-determinant plane. The theorem of Großman-Hartman ensures that hyperbolic non-linear dynamics are equivalent to the linearized dynamics represented by $\Lambda$, the Jordan canonical form. We know that $\operatorname{det} J=\operatorname{det} \Lambda$. In the planar case one easy checks that $\operatorname{tr} J=\operatorname{tr} \Lambda$.

Theorem $1 A$ partition of the $(T, D)$-plane is defined by the straight lines $D=1, D=T-1$ and $D=-T-1$. The stationary state $x$ is a sink, a source
or a saddle point according to the position of the pair $(T, D)=(\operatorname{tr} J, \operatorname{det} J)$ in the $(T, D)$-plane as detailed in figure 6 .


Figure 5. Trace and determinant.

Proof. (Hint). The characteristic polynomial is defined as follows: $\lambda^{2}-$ $\lambda \operatorname{tr} J+\operatorname{det} J=0$, where $\operatorname{tr} J=\lambda_{1}+\lambda_{2}$ and $\operatorname{det} J=\lambda_{1} \lambda_{2}$. First we consider a stable stationary state, i.e. the case of both the two eigenvalues inside the unit circle in the Gaussian plane (their modulus is less than one, i.e. $\alpha^{2}+\beta^{2}<1$ ). This condition is necessary and sufficient to observe a local contraction of dynamics to the steady state: $x_{t}-x=V \Lambda^{t} V^{-1}\left(x_{0}-x\right)$. We must prove that the trace and determinant of the Jacobian matrix lie in the interior of triangle represented by the shaded area in figure 6, i.e. $D<1, D>T-1$, and $D>-T-1$. Consider the characteristic polynomial $\lambda^{2}-(\operatorname{tr} J) \lambda+\operatorname{det} J=\lambda^{2}-\left(\lambda_{1}+\lambda_{2}\right) \lambda+\left(\lambda_{1} \lambda_{2}\right)=\left(\lambda-\lambda_{1}\right)\left(\lambda-\lambda_{2}\right)=0$. Two cases matter.
(i) The two eigenvalues are complex, i.e. conjugated. In this case either both of them are inside the unit circle, or both are outside. Then it is sufficient to compute their product: $\operatorname{det} J=(a+b i)(a-b i)=a^{2}+b^{2} \gtrless 1$. If both are outside $(D>1)$, we observe a source. If both the eigenvalues lie inside the unit circle, we observe a $\operatorname{sink}(D<1)$. The saddle configuration is ruled out. The condition to obtain two complex roots of characteristic polynomial $\lambda_{ \pm}=\left(\operatorname{tr} J \pm \sqrt{\operatorname{tr} J^{2}-4 \operatorname{det} J}\right) / 2=\operatorname{tr} J / 2 \pm \sqrt{(\operatorname{tr} J / 2)^{2}-\operatorname{det} J}$
is $(\operatorname{tr} J / 2)^{2}<\operatorname{det} J$.


Figure 6.
(ii) The two eigenvalues are real ${ }^{2}\left((\operatorname{tr} J / 2)^{2}>\operatorname{det} J\right)$. In this case $T^{2}-$ $4 D>0$. Thus $-1<\lambda_{i}<1$, with $i=1,2$. The condition to observe a sink is $-1<\operatorname{tr} J / 2 \pm \sqrt{(\operatorname{tr} J / 2)^{2}-\operatorname{det} J}<1$. More explicitly the system of inequalities becomes: $-1<T / 2+\sqrt{(T / 2)^{2}-D}, T / 2+\sqrt{(T / 2)^{2}-D}<1$, $-1<T / 2-\sqrt{(T / 2)^{2}-D}$ and $T / 2-\sqrt{(T / 2)^{2}-D}<1$; or equivalently $\sqrt{(T / 2)^{2}-D}>-1-T / 2, \sqrt{(T / 2)^{2}-D}<1-T / 2, \sqrt{(T / 2)^{2}-D}<T / 2+1$, and $\sqrt{(T / 2)^{2}-D}>T / 2-1$. Hence four regions are defined in the $(T, D)-$ plane.

$$
\text { (1) }\left\{\begin{array}{l}
(T / 2)^{2}-D>1+T+(T / 2)^{2} \text { and }-1-T / 2>0, \\
D<-T-1 \text { or } T>-2 .
\end{array}\right. \text {, }
$$

[^2](2) $\left\{\begin{array}{l}(T / 2)^{2}-D>1-T+(T / 2)^{2} \text { and } 1-T / 2>0 \\ D>T-1 \text { or } T<2 .\end{array}\right.$,
(3) $\left\{\begin{array}{l}(T / 2)^{2}-D>1+T+(T / 2)^{2} \text { and } T>-2 \\ D>-T-1 .\end{array}\right.$,
(4) $\left\{\begin{array}{l}(T / 2)^{2}-D>1-T+(T / 2)^{2} \text { or } T<2 \\ D<T-1 .\end{array}\right.$

The four regions are represented as follows.


Figure 7.


Figure 9.


Figure 8.


Figure 10.

The intersection of the four regions constitutes the set of solutions for the inequality system.


Figure 11.
We are still in the real case: $(T / 2)^{2}-D>0$, i.e. $D<(T / 2)^{2}$. This case is illustrated by figure 12 .


Figure 12.
In the complex case, the part of the plane corresponding to sinks is the complement of the shaded area in figure 12, with respect to the triangle. Hence the entire region for stability is the union of the two shaded areas represented in figures 6 and 12, i.e. just the interior of the triangle. In a very similar manner the other regions of partition in figure 5 are characterized.

## Part I

## Demand Functions

## Chapter 2

## The Consumption Function

### 2.1 A Keynesian Language

The production of goods and services makes it possible to satisfy human needs. A undesired good is not produced. In this respect the demand determines the production and becomes effective only whether the consumers have an income to actually transform their consumption desires in purchases.

Thereby the income plays an explicative role for consumption. Often the break of the sound mechanism between consumption and revenue is at the origin of economic crises.

The global demand is the aggregate value of what firms and households (with the State and the foreign sector) want to purchase in terms of good and services for each level of income. The basic Keynesian terminology specifies the demand components as demands for consumption, investment, public spending and net exports. Among them the consumption demand is the main component.

The consumption of a good is a destruction process which allows the agents to satisfy directly their needs without affecting the production. We must not confuse it with the consumption of intermediary goods during the production activity.

In the Sixties the households' consumption represented $60 \%$ of $G D P$. In addition the consumption of private administrations and that of the public administration must be taken into account because the households enjoy this kind of collective consumption. In the last three decades this consumption has constituted $20 \%$ of the global consumption.

The disposable income is the revenue households receive from the firms augmented by the transfers from the government and diminished by the direct taxes and social contributions. In other words it is the income households have at disposal to consume or save.

The consumption function is the relation between consumption and disposable income. More precisely the consumption function represents the consumption level the economic agents desire for each level of disposable income.

Let $D$ be the disposable income. The average propensity to consume is given by

$$
c \equiv C / D
$$

The marginal propensity to consume is the change in consumption entailed by an additional money unit of revenue.

$$
\partial C / \partial D
$$

Similarly we define the average and marginal propensities to save:

$$
\begin{aligned}
& S / D \\
& \partial S / \partial D .
\end{aligned}
$$

We notice that

$$
\begin{aligned}
C / D+S / D & =1 \\
\partial C / \partial D+\partial S / \partial D & =1
\end{aligned}
$$

The average propensity to consume with respect to the disposable income is about $86 \%$.

The Keynesian theory of the propensity to consume tried to explain some stylized facts: the consumption depends on the real revenue, the marginal propensity to consume is less than 1 , there exists an incompressible threshold $C_{0}$, the average propensity to consume is decreasing.

All these informations are properly interpreted by a linear consumption function

$$
C=C_{0}+c D
$$

with $c<1$. The figure 14 illustrates the concept of propensity to consume.


Figure 13. Consumption function.
The Keynesian consumption function is the simplest representation of the observed facts. All the successive theory refines the Keynesian view and tries to provide a micro-foundation for the aggregate consumption demand.

The saving is defined as the difference between the disposable income and consumption.

$$
S \equiv D-C=-C_{0}+s D,
$$

where $s=1-c$ is the marginal propensity to save.
By simplicity we assume taxes and transfers from the Government to be zero $(D=Y)$ as well as the public spending and the import-export.

The core of the Keynesian theory focuses on the relation between saving (or equivalently consumption) and investment. At equilibrium

$$
\begin{equation*}
Y=C+I \tag{2.1}
\end{equation*}
$$

The supply $Y$ equals demand which is specified as a demand for consumption $C$ and for investment $I$. Keynes (1936) introduces the propensity to consumption $c$ as a constant ratio between aggregate consumption and income.

$$
\begin{equation*}
C=c Y . \tag{2.2}
\end{equation*}
$$

Combining formulas (2.1) and (2.2) we obtain

$$
Y=\frac{1}{1-c} I
$$

and in differential terms

$$
d Y=\frac{1}{1-c} d I .
$$

This is the well-known Keynesian multiplier.
A larger definition of aggregate demand includes the public spending $G$ and the net export $E$, i.e. the difference between export and import.

$$
Y=C+I+G+E .
$$

The Keynesian multipliers becomes

$$
d Y=\frac{1}{1-c}(d I+d G+d E) .
$$

In particular when $d I=d E=0$, the public spending becomes the chief tool to stimulate the economy:

$$
d Y=\frac{1}{1-c} d G
$$

To clarify the mechanism let us consider the following example. The higher is the propensity to consumption, the higher is the impact of public spending on product.

| propensity $c$ | $50 \%$ | $75 \%$ |
| :--- | :--- | :--- |
| multiplier | 2 | 4 |

Roughly speaking the saver turns out to be a public enemy in a caricatured Keynesian view.

When we talk about the consumption function or about the investment function, we refer to the determinants of the aggregate consumption or investment. We could explain also the key variables which determine the public spending and the net export.

For now we have provided the simplest consumption function as in the General Theory of Keynes:

$$
C=c Y .
$$

This assumption has been confirmed by the data analyzed by Kuznets (1946) for the U.S. economy in the last hundred years.

All the successive consumption theories tried to provide microeconomic foundations of the aggregate relation

$$
C=C(D) .
$$

### 2.2 The Relative Income Theory

A first step towards finer interpretations of consumer's behavior is the theory of the relative income by Duesemberry (1949).

The individuals are sensitive to the consumption of the others and to their past consumption experiences. More precisely on the one hand they observe the habits of their reference social group and imitate them. On the other hand the consumption of a given period depends more on the highest past income than on the current one.

During the recession the disposable income decreases, the individuals do not reduce enough the consumption and by consequence the saving falls down more than the income. This lack of saving entails a contraction of the investments and a deeper recession.

### 2.3 The Life-Cycle Theory

Modigliani and Brumberg (1954) provide the seminal contribution of the class of models better known as "life-cycle-permanent income" literature. During the life the consumer faces income cycles and prefers to have a constant consumption.

Let us assume that the consumer lives three periods: a non-active youth $(n)$, an active life $(a)$ and the final retirement period $(r)$.

The active life income finances a constant consumption over the three stages.

To be more explicit we consider an example.
The length of the non-active life is $T_{n}=20$ years. The active life measures $T_{a}=40$ years. The retirement period goes on $T_{r}=20$ years. The income is $y$ per year during the active life. We assume a free access to the credit market to finance the consumption during the youth with zero interest rate. We could conceive a system of transfers between generations in alternative.

Therefore the consumption will be given by

$$
c=\frac{y T_{a}}{T_{n}+T_{a}+T_{r}}=\frac{y 40}{20+40+20}=\frac{y}{2} .
$$

In this example the consumer will consume half of the annual income of the active phase each years of the whole life.

We can design a more complex consumption problem by assuming that the current consumption does not depend only on the life wealth ( $y 40$ ) but
also directly on the transitory income. In other word we can introduce a consumption sensitivity to income fluctuations.

To be more precise we set a linear relation between consumption and wealth $w$, between consumption and transitory income $y$ :

$$
c=a w+b y .
$$

This linear function can be specified as a weighted average between a permanent income $w / T$ and a transitory income

$$
c=a w+b y
$$

with

$$
a=(1-b) / T,
$$

where $T \equiv T_{n}+T_{a}+T_{r}$ is the entire length of the life. We obtain

$$
\begin{aligned}
& c_{n}=(1-b) y T_{a} / T+b 0, \\
& c_{a}=(1-b) y T_{a} / T+b y, \\
& c_{r}=(1-b) y T_{a} / T+b 0 .
\end{aligned}
$$

We verify that the intertemporal budget constraint is respected.

$$
\begin{aligned}
& c_{n} T_{n}+c_{a} T_{a}+c_{r} T_{r} \\
= & {\left[(1-b) y \frac{T_{a}}{T}\right] T_{n}+\left[(1-b) y \frac{T_{a}}{T}\right] T_{a}+\left[(1-b) y \frac{T_{a}}{T}\right] T_{r} } \\
= & {\left[(1-b) y \frac{T_{a}}{T}\right]\left(T_{n}+T_{a}+T_{r}\right)+b y T_{a} } \\
= & {\left[(1-b) y \frac{T_{a}}{T}\right] T+b y T_{a} } \\
= & y T_{a}=W .
\end{aligned}
$$

This model augments the simple life-cycle model by taking into account an excess sensitivity of consumption to the current income.

As we shall see later on this theory will be progressively generalized by the permanent income theory and the intertemporal utility maximization.

### 2.4 The Permanent Income Theory

The permanent income theory (Friedman, 1957) generalizes the life-cycle hypothesis by introducing a positive interest rate on the credit market and thereby a discount factor for consumption and income less than one.

The period between the instant $t-1$ and $t$ is said to be the period $t$. Individuals live $T$ periods and receive at period $t$ a capital income $y_{t-1}^{c}$ and the labor income $y_{t}^{l}$. The discounted value of the future income is

$$
\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t}\left(y_{t}^{k}+y_{t}^{l}\right)
$$

where $r$ is the market interest rate (individuals can borrow and lend).
The non-human wealth is the discounted value of capital income.

$$
N \equiv \sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t} y_{t}^{k}
$$

The human wealth is the discounted value of labor income.

$$
H \equiv \sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t} y_{t}^{l}
$$

The total wealth is the sum of human and non-human wealth:

$$
W \equiv N+H
$$

Individuals want to consume a constant amount of good at each period. As there are no consumption and no satisfaction after the death, they want to spend the entire wealth to maximize the permanent (stationary) consumption $y^{p}$.

$$
\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t} y^{p}=W
$$

$y^{p}$ is said by Friedman to be a permanent income. As in the life-cycle theory the permanent income is a smoothed consumption.

We can solve now for the permanent consumption:

$$
\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t} y^{p}=\frac{y^{p}}{1+r} \sum_{t=0}^{T-1}\left(\frac{1}{1+r}\right)^{t}
$$

$$
\begin{aligned}
& =\frac{y^{p}}{1+r} \frac{1-(1+r)^{-T}}{1-(1+r)^{-1}} \\
& =\frac{y^{p}}{r}\left[1-(1+r)^{-T}\right] .
\end{aligned}
$$

Hence

$$
\frac{y^{p}}{r}\left[1-(1+r)^{-T}\right]=W=\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t}\left(y_{t}^{k}+y_{t}^{l}\right) .
$$

We obtain

$$
\begin{equation*}
y^{p}=\frac{r}{1-(1+r)^{-T}} W \tag{2.3}
\end{equation*}
$$

Notice that

$$
\lim _{T \rightarrow \infty}\left[1-(1+r)^{-T}\right]=1
$$

If the consumer's life is long enough, for instance 80 years, we can approximate

$$
T=80 \approx \infty
$$

and formula (2.3) becomes

$$
y^{p}=r W
$$

under the assumption of positive interest rate $(r>0)$.
In other words the permanent consumption (income) is a constant fraction of the total wealth according to the interest rate.

The saving at period $t$ is equal to

$$
s_{t}=y_{t}-y^{p} .
$$

We observe that it could be negative, i.e. the consumer could dissave or borrow in the credit market.

It is possible to think a slightly different framework where the consumer decides an increasing consumption across the time. Let $\gamma$ be the constant growth rate he wants. The question is to find the initial consumption to make binding the wealth constraint.

We can write

$$
c_{t}=c_{1}(1+\gamma)^{t-1}
$$

and substituting in

$$
\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t} c_{t}=W
$$

we obtain

$$
\begin{align*}
\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t}\left[c_{1}(1+\gamma)^{t-1}\right] & =\frac{c_{1}}{1+\gamma} \sum_{t=1}^{T}\left(\frac{1+\gamma}{1+r}\right)^{t} \\
& =\frac{c_{1}}{1+\gamma} \frac{1+\gamma}{1+r} \sum_{t=0}^{T-1}\left(\frac{1+\gamma}{1+r}\right)^{t} \\
& =\frac{c_{1}}{1+r} \frac{1-[(1+\gamma) /(1+r)]^{T}}{1-(1+\gamma) /(1+r)} \\
& =\frac{c_{1}}{r-\gamma}\left[1-\left(\frac{1+\gamma}{1+r}\right)^{T}\right] \\
& =W . \tag{2.4}
\end{align*}
$$

Let $r>\gamma$. In this case

$$
\lim _{T \rightarrow \infty}\left[1-\left(\frac{1+\gamma}{1+r}\right)^{T}\right]=1
$$

For an infinite-lived agent ( $T=\infty$ ) equation (2.4) becomes

$$
c_{1}=(r-\gamma) W
$$

Thereby the higher the preferred consumption growth rate, the lower the initial consumption.

A more realistic formulation assumes consumption sensitivity to the current income as well as a smoothing behavior:

$$
c_{t}=c\left(y_{t}, y\right)
$$

where $c_{t}, y_{t}$ and $y$ denote respectively the current consumption, the current income and the permanent one. For an infinite-lived consumer we know that

$$
y=r W
$$

where $r$ denotes the interest rate and $W$ the wealth. A similar version of this permanent income hypothesis consists of formulating a new function

$$
c_{t}=d\left(y_{t}, W\right) \equiv c\left(y_{t}, r W\right)
$$

where the dependence on the current (fluctuating) income and on the wealth is stressed.

The simplest specification is the following

$$
c_{t}=c\left(y_{t}, y\right)=\alpha y_{t}+(1-\alpha) y
$$

If we set $y=r W$, the intertemporal budget constraint is respected:

$$
\begin{aligned}
\sum_{t=1}^{\infty} \frac{c_{t}}{(1+r)^{t}} & =\sum_{t=1}^{\infty} \frac{\alpha y_{t}+(1-\alpha) y}{(1+r)^{t}} \\
& =\alpha \sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}}+(1-\alpha) \sum_{t=1}^{\infty} \frac{r W}{(1+r)^{t}} \\
& =\alpha W+(1-\alpha) r W \sum_{t=1}^{\infty} \frac{1}{(1+r)^{t}} \\
& =\alpha W+(1-\alpha) W \frac{r}{1+r} \sum_{t=0}^{\infty} \frac{1}{(1+r)^{t}} \\
& =\alpha W+(1-\alpha) W \frac{r}{1+r} \frac{1}{1-(1+r)^{-1}} \\
& =\alpha W+(1-\alpha) W=W \\
& =\sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}}
\end{aligned}
$$

### 2.5 Two-Period Utility Maximization

This model is due to Fisher (1930).
We introduce now a true intertemporal utility functional. The individual lives two periods 0 and 1 . He receives the revenue $y_{0}$ in the first period and $y_{1}$ in the second. He consumes an amount $c_{0}$ during the first period and $c_{1}$ during the last period.

The consumption today and tomorrow are interpreted as two distinct goods which enter a simple intertemporal utility function

$$
U\left(c_{0}, c_{1}\right)
$$

which is assumed to be quasi-concave (i.e. preferences are convex). For instance we can specify $U$ as follows:

$$
U\left(c_{0}, c_{1}\right) \equiv u\left(c_{0}\right)+\beta u\left(c_{1}\right)
$$

where $u$ is the per-period utility function and $\beta$ is the subjective discount factor. Utility is usually assumed to be increasing ( $u^{\prime}>0$ ) and strictly concave $\left(u^{\prime \prime}<0\right)$. The consumer faces an intertemporal budget constraint:

$$
c_{0}+\frac{c_{1}}{1+r} \leq y_{0}+\frac{y_{1}}{1+r} .
$$

The consumer's program becomes

$$
\begin{aligned}
& \max U\left(c_{0}, c_{1}\right) \\
& c_{0}+c_{1} /(1+r) \leq y_{0}+y_{1} /(1+r)
\end{aligned}
$$

The solution gives the consumption demand functions for $t=0,1$. This program can be interpreted as an usual microeconomic consumer's program where the price of $c_{0}$ has been normalized to one, while the price of $c_{1}$ in terms of $c_{0}$ is equal to $1 /(1+r)$.

The Lagrangian function is the following

$$
\Lambda=U\left(c_{0}, c_{1}\right)+\lambda\left[y_{0}+\frac{y_{1}}{1+r}-c_{0}-\frac{c_{1}}{1+r}\right] .
$$

Solving with respect to $c_{0}$ and $c_{1}$, we get

$$
\begin{aligned}
& \frac{\partial \Lambda}{\partial c_{0}}=\frac{\partial U}{\partial c_{0}}-\lambda=0 \\
& \frac{\partial \Lambda}{\partial c_{1}}=\frac{\partial U}{\partial c_{1}}-\frac{\lambda}{1+r}=0
\end{aligned}
$$

After the multiplier elimination, we obtain the usual first order condition: the marginal rate of substitution equals the negative of the "prices" ratio.

$$
M R S=-\frac{\partial U / \partial c_{0}}{\partial U / \partial c_{1}}=-\frac{1}{(1+r)^{-1}}=-(1+r)
$$

This equation is said to be Euler equation. The second order condition for maximization is respected because of the quasi-concavity of utility. $-(1+r)$ is the slope of the budget constraint in the consumption space. The budget constraint is tangent to an indifference curve. The tangency point gives the optimal consumption point. The solution is a usual demand function depending on the relative price $(1+r)$ and the income. Notice that the interest rate $r$ is the credit price.

As the utility function is assumed to be strictly increasing, then the budget constraint holds with equality. The solution satisfies the following system:

$$
\begin{aligned}
M R S & =-(1+r), \\
c_{0}+\frac{c_{1}}{1+r} & =y_{0}+\frac{y_{1}}{1+r} .
\end{aligned}
$$

The budget constraint can be rewritten as follows:

$$
c_{1}=y_{1}+(1+r)\left(y_{0}-c_{0}\right) .
$$

If for instance the consumer saves in the first period a positive amount $y_{0}-c_{0}$, then the second period consumption is given by the second period income $y_{1}$ and the gross fruit of the saving $(1+r)\left(y_{0}-c_{0}\right)$.

Example. Assume that the utility function is shaped as follows

$$
U\left(c_{0}, c_{1}\right)=\ln c_{0}+\frac{1}{1+\theta} \ln c_{1}
$$

where $\theta$ is measure of time preference. We obtain

$$
M R S=-\frac{1 / c_{0}}{1 /\left[(1+\theta) c_{1}\right]}=-(1+r)
$$

Therefore

$$
c_{1}=\frac{1+r}{1+\theta} c_{0} .
$$

Notice that the consumption growth rate is equal to

$$
\frac{1+r}{1+\theta}-1
$$

The consumption is stationary $\left(c_{0}=c_{1}\right)$ if and only if

$$
r=\theta .
$$

In a general equilibrium framework with $r=f^{\prime}(k)$ that this equality is interpreted as a modified golden rule.

$$
f^{\prime}(k)=\theta .
$$

We can now solve the consumer's system.

$$
\begin{aligned}
c_{0}+\frac{1}{1+r}\left(\frac{1+r}{1+\theta} c_{0}\right) & =y_{0}+\frac{y_{1}}{1+r} \\
c_{0}^{*} & =\frac{1+\theta}{2+\theta}\left(y_{0}+\frac{y_{1}}{1+r}\right) \\
c_{1}^{*} & =\frac{1}{2+\theta}\left[(1+r) y_{0}+y_{1}\right] .
\end{aligned}
$$

The consumption is no longer constant as in the life-cycle permanent income theory. The saving of the first period is given by

$$
\begin{aligned}
s_{0}^{*} & \equiv y_{0}-c_{0}^{*} \\
& =y_{0}-\frac{1+\theta}{2+\theta}\left(y_{0}+\frac{y_{1}}{1+r}\right) \\
& =\frac{y_{0}}{2+\theta}-\frac{1+\theta}{2+\theta} \frac{y_{1}}{1+r} .
\end{aligned}
$$

It can be negative. The comparative statics are obtained by deriving $c_{0}$ and $c_{1}$ with respect to $\theta, r, y_{0}, y_{1}$. For instance

$$
\frac{\partial c_{0}^{*}}{\partial \theta}>0
$$

The more impatient the consumer, the higher the initial consumption.

$$
\begin{aligned}
& \frac{\partial c_{0}^{*}}{\partial y_{0}}>0 \\
& \frac{\partial c_{0}^{*}}{\partial y_{1}}>0
\end{aligned}
$$

Higher incomes has a positive impact on consumption.

$$
\frac{\partial c_{0}^{*}}{\partial r}<0 .
$$

If the interest rate is higher, then the consumer saves more in the first period.

### 2.6 Intertemporal Utility Maximization

The household maximizes an intertemporal utility function. By simplicity we assume that his life goes on forever and that the utility functional is
additively separable.

$$
\begin{equation*}
U\left(c_{1}, c_{2}, \ldots\right) \equiv \sum_{t=1}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right) \tag{2.5}
\end{equation*}
$$

The utility function $u$ is assumed to be increasing and strictly concave. The consumer has a free access to credit market as lender or borrower, so he faces an intertemporal budget constraint.

$$
\sum_{t=1}^{\infty} \frac{1}{(1+r)^{t}} c_{t} \leq \sum_{t=1}^{\infty} \frac{1}{(1+r)^{t}} y_{t}
$$

The revenue at period $t$ is given by the capital and labor income.

$$
y_{t} \equiv y_{t}^{k}+y_{t}^{l}
$$

The Lagrangian for the program is given by

$$
\Lambda=\sum_{t=1}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right)+\lambda\left[\sum_{t=1}^{\infty} \frac{1}{(1+r)^{t}} y_{t}-\sum_{t=1}^{\infty} \frac{1}{(1+r)^{t}} c_{t}\right] .
$$

Notice that $\lambda$ is independent on time. Deriving with respect to the generic choice variable $c_{t}$, we get the corresponding first order condition

$$
\frac{\partial \Lambda}{\partial c_{t}}=0
$$

i.e.

$$
(1+\theta)^{-t} u^{\prime}\left(c_{t}\right)=\lambda(1+r)^{-t}
$$

To eliminate the multiplier we compute the intertemporal marginal rate of substitution:

$$
I M R S_{t+1}=\frac{(1+\theta)^{-t} u^{\prime}\left(c_{t}\right)}{(1+\theta)^{-t-1} u^{\prime}\left(c_{t+1}\right)}=\frac{(1+r)^{t+1}}{(1+r)^{t}}
$$

Notice that the right-hand side is just the price ratio. We obtain

$$
\begin{equation*}
u^{\prime}\left(c_{t}\right)=\frac{1+r}{1+\theta} u^{\prime}\left(c_{t+1}\right) \tag{2.6}
\end{equation*}
$$

This is the non-stochastic Euler equation.

$$
\sum_{t=1}^{\infty} \frac{c_{t}}{(1+r)^{t}}=\sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}}
$$

The constraint is now binding because the utility function is monotonic.
To provide an explicit solution we consider a particular class of utility functions.

As usual in macrodynamics an utility function with a constant elasticity of intertemporal substitution ( $C E S$ ) is used. The elasticity of substitution between the consumption at time $s$ and consumption at time $t$ is given by

$$
\sigma \equiv-\frac{u^{\prime}\left(c_{s}\right) / u^{\prime}\left(c_{t}\right)}{c_{s} / c_{t}} \frac{d\left(c_{s} / c_{t}\right)}{d\left[u^{\prime}\left(c_{s}\right) / u^{\prime}\left(c_{t}\right)\right]} .
$$

Taking the limit for $s$ converging to $t$, one obtains in continuous time

$$
\sigma\left(c_{t}\right)=-\frac{u^{\prime}\left(c_{t}\right)}{u^{\prime \prime}\left(c_{t}\right) c_{t}}=-\left[\frac{u^{\prime \prime}\left(c_{t}\right)}{u^{\prime}\left(c_{t}\right) / c_{t}}\right]^{-1}
$$

that is the negative inverse of the elasticity of marginal utility (for more details see Blanchard and Fischer (1989), chapter 2). In discrete time we adopt the latter formula as a definition. An isoelastic function with elasticity $\sigma$ has the form

$$
u\left(c_{t}\right)=C_{1} \frac{c_{t}^{1-1 / \sigma}}{1-1 / \sigma}+C_{2}
$$

where $C_{1}$ and $C_{2}$ are integration constants. To see that, reconsider the definition of elasticity:

$$
-\frac{u^{\prime}\left(c_{t}\right)}{u^{\prime \prime}\left(c_{t}\right) c_{t}}=\sigma
$$

Hence $-u^{\prime \prime}\left(c_{t}\right) / u^{\prime}\left(c_{t}\right)=1 /\left(\sigma c_{t}\right)$. We can write

$$
-\frac{d}{d c_{t}} \ln u^{\prime}\left(c_{t}\right)=\frac{1}{\sigma} \frac{d}{d c_{t}} \ln c_{t} .
$$

The indefinite integral is

$$
-\int \frac{d}{d c_{t}} \ln u^{\prime}\left(c_{t}\right) d c_{t}=\frac{1}{\sigma} \int \frac{d}{d c_{t}} \ln c_{t} d c_{t}
$$

Thereby $-\ln u^{\prime}\left(c_{t}\right)=\left(\ln c_{t}\right) / \sigma+c$, where $c$ is an indefinite integration constant. Taking the power with base $e$ we obtain $e^{-\ln u^{\prime}\left(c_{t}\right)}=e^{\left(\ln c_{t}\right) / \sigma+c}$ and $e^{\ln \left[u^{\prime}\left(c_{t}\right)\right]^{-1}}=e^{c} e^{\ln c_{t}^{1 / \sigma}}$, i.e. $\left[u^{\prime}\left(c_{t}\right)\right]^{-1}=e^{c} c_{t}^{1 / \sigma}$ and $u^{\prime}\left(c_{t}\right)=e^{-c} c_{t}^{-1 / \sigma}$. The integral is now defined between 0 and $c_{t}$ :

$$
\int_{c_{0}}^{c_{t}} u^{\prime}\left(x_{t}\right) d x_{t}=e^{-c} \int_{c_{0}}^{c_{t}} x_{t}^{-1 / \sigma} d x_{t}
$$

Finally $\left[u\left(x_{t}\right)\right]_{c_{0}}^{c_{t}}=e^{-c}\left[x_{t}^{1-1 / \sigma} /(1-1 / \sigma)\right]_{c_{0}}^{c_{t}}$ and

$$
u\left(c_{t}\right)-u\left(c_{0}\right)=e^{-c}\left[c_{t}^{1-1 / \sigma} /(1-1 / \sigma)-c_{0}^{1-1 / \sigma} /(1-1 / \sigma)\right] .
$$

Hence $u\left(c_{t}\right)=e^{-c} c_{t}^{1-1 / \sigma} /(1-1 / \sigma)-e^{-c} c_{0}^{1-1 / \sigma} /(1-1 / \sigma)+u\left(c_{0}\right)$. We can specify the two integration constants as $c=0$ and $u\left(c_{0}\right)=\left(c_{0}^{1-1 / \sigma}-1\right)$ $/(1-1 / \sigma)$, to obtain the standard $C E S$ function

$$
u\left(c_{t}\right)=\frac{c_{t}^{1-1 / \sigma}-1}{1-1 / \sigma}
$$

By applying the definition, it is possible to check that the elasticity of intertemporal substitution is actually $\sigma$. For $\sigma=1$, this isoelastic function is replaced by the logarithm:

$$
\frac{c_{t}^{1-1 / \sigma}-1}{1-1 / \sigma} \rightarrow \ln c_{t}
$$

Check that the logarithm function has a constant elasticity of intertemporal substitution just equal to one.

Coming back to equation (2.6), we write

$$
\frac{c_{t}^{-1 / \sigma}}{c_{t+1}^{-1 / \sigma}}=\frac{1+r}{1+\theta} .
$$

The consumption growth rate is given by

$$
\frac{c_{t+1}}{c_{t}}=\left(\frac{1+r}{1+\theta}\right)^{\sigma}
$$

Therefore

$$
c_{t}=c_{1}\left(\frac{1+r}{1+\theta}\right)^{\sigma(t-1)}
$$

and

$$
\begin{aligned}
\sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} & =\sum_{t=1}^{\infty} \frac{c_{1}[(1+r) /(1+\theta)]^{\sigma(t-1)}}{(1+r)^{t}} \\
& =\frac{c_{1}}{1+r} \sum_{t=1}^{\infty} \frac{[(1+r) /(1+\theta)]^{\sigma(t-1)}}{(1+r)^{t-1}} \\
& =\frac{c_{1}}{1+r} \sum_{t=1}^{\infty}\left[\frac{(1+r)^{\sigma-1}}{(1+\theta)^{\sigma}}\right]^{t-1} \\
& =\frac{c_{1}}{1+r} \sum_{t=0}^{\infty}\left[\frac{(1+r)^{\sigma-1}}{(1+\theta)^{\sigma}}\right]^{t}
\end{aligned}
$$

The series converges if and only if

$$
\begin{align*}
\frac{(1+r)^{\sigma-1}}{(1+\theta)^{\sigma}} & <1 \\
(1+r)^{\sigma-1} & <(1+\theta)^{\sigma}, \\
(\sigma-1) \ln (1+r) & <\sigma \ln (1+\theta) \\
\frac{\sigma-1}{\sigma} & <\frac{\ln (1+\theta)}{\ln (1+r)} . \tag{2.7}
\end{align*}
$$

We assume that $r>0$. Then

$$
\frac{\ln (1+\theta)}{\ln (1+r)}>0 .
$$

The inequality (2.7) is for instance respected if $\sigma<1$ (weak elasticity of intertemporal substitution).

Under inequality (2.7) we obtain

$$
\begin{aligned}
\sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} & =\frac{c_{1}}{1+r} \sum_{t=0}^{\infty}\left[\frac{(1+r)^{\sigma-1}}{(1+\theta)^{\sigma}}\right]^{t} \\
& =\frac{c_{1}}{1+r} \frac{1}{1-(1+r)^{\sigma-1} /(1+\theta)^{\sigma}} \\
& =\frac{c_{1}}{1+r-[(1+r) /(1+\theta)]^{\sigma}}
\end{aligned}
$$

We are able now to determine the initial consumption and then the entire path.

$$
\begin{align*}
c_{1} & =\left[1+r-\left(\frac{1+r}{1+\theta}\right)^{\sigma}\right] \sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} \\
c_{t} & =\left(\frac{1+r}{1+\theta}\right)^{\sigma(t-1)} c_{1} \\
& =\left(\frac{1+r}{1+\theta}\right)^{\sigma(t-1)}\left[1+r-\left(\frac{1+r}{1+\theta}\right)^{\sigma}\right] \sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} \\
& =\left[(1+r)\left(\frac{1+r}{1+\theta}\right)^{\sigma(t-1)}-\left(\frac{1+r}{1+\theta}\right)^{\sigma t}\right] \sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} . \tag{2.8}
\end{align*}
$$

It is possible to perform the comparative statics by evaluating the impact of $r$ and $\theta$ on the path $\left\{c_{t}\right\}_{t=1}^{\infty}$.

The saving at each period is given by

$$
\begin{aligned}
s_{t} & =y_{t}-c_{t} \\
& =y_{t}-\left(\frac{1+r}{1+\theta}\right)^{\sigma(t-1)}\left[1+r-\left(\frac{1+r}{1+\theta}\right)^{\sigma}\right] \sum_{\tau=1}^{\infty} \frac{y_{\tau}}{(1+r)^{\tau}} .
\end{aligned}
$$

We observe that if for instance we set $r=\theta$, equation (2.8) becomes

$$
c_{t}=r \sum_{\tau=1}^{\infty} \frac{y_{t}}{(1+r)^{t}}
$$

If moreover we assume a constant revenue across the time, we have

$$
c_{t}=r \frac{y}{1+r} \sum_{\tau=0}^{\infty} \frac{1}{(1+r)^{t}}=y \frac{r}{1+r} \frac{1+r}{r}=y
$$

This clarifies the sense of a smoothed consumption as a permanent income.

### 2.7 Two-Period Stochastic Consumption

Before entering the economic application, let us introduce the meaning of the expectation operator.

Let $x$ be a random variable, i.e. a probability distribution which associated to each element of the $\sigma$-algebra structuring the domain, a probability measure.

To be less abstract we could consider a discrete domain (finite or countable) $\left\{x_{i}\right\}$ and we introduces a probability function $\pi_{i}=\pi\left(x_{i}\right)$, which respects the axioms of probability. A random variable is perfectly identified by this probability distribution.

The average of the random variable is

$$
\mu=\sum_{i=1}^{n} x_{i} \pi_{i}
$$

with

$$
\begin{aligned}
\sum_{i=1}^{n} \pi_{i} & =1 \\
\pi_{i} & \geq 0
\end{aligned}
$$

for every $i$. If the domain is infinite $n=\infty$.
If we take into account a domain which is a continuum, the probability function becomes a probability density $\pi$ which associates to each point $x$ of the domain $X$ a density value $\pi(x)$. Under the usual integrability assumptions, the average of the random variable is given by

$$
\int_{x \in X} x \pi(x) d x
$$

where

$$
\begin{array}{r}
\int_{x \in X} \pi(x) d x=1 \\
\pi(x) \geq 0
\end{array}
$$

for every $x$. More complex $\sigma$-algebra could be taken into account.
We assume now that a rational economic agent forecasts the future realization of the random variable $x$ by computing the average. This average is called expectation $E$. Let us be more precise. Assume that the individual wants to forecast at period $t$ the future realization $x_{t+1}$. The information set $I_{t}$ at his disposal at period $t$ is used by him for computing the subjective
probability distribution $\pi$ for the random variable $x_{t+1}$. If he uses all the information in the information set $I_{t}$, the expectation is said to be rational. A simple example of information set is the sequence of all past realizations of the variable: $I_{t}=\left\{\ldots, x_{t-1}, x_{t}\right\}$. In this case the individual can use the information to construct the probability distribution of $x_{t+1}$. In the real life the information set is much more rich.

We formalize these assumptions as follows. The rational expectation at time $t$ (conditional to the disposable information set $I_{t}$ ) of the random variable $x_{t+1}$ is given by

$$
E_{t} x_{t+1}=E\left[x_{t+1} \mid I_{t}\right] .
$$

If the domain of the random variable is discrete, we obtain

$$
E_{t} x_{t+1}=\sum_{i=1}^{n} x_{t+1, i} \pi_{t, i}
$$

where the probability function $\pi_{t} \equiv\left(\pi_{t, 1}, \ldots, \pi_{t, n}\right)$ is computed from $I_{t}$.
If the domain of the random variable is continuous, we get

$$
\int_{x_{t+1} \in X_{t+1}} x_{t+1} \pi_{t}\left(x_{t+1}\right) d x_{t+1}
$$

where the probability density function $\pi_{t} \equiv \pi_{t}\left(x_{t+1}\right)$ is computed from $I_{t}$. As $\sum$ and $\int$ are linear operators the expectation operator is linear as well. To see that consider the following example.

$$
\begin{aligned}
E[f(x)+g(y)] & =\int_{x} \int_{y}[f(x)+g(y)] \pi(x, y) d y d x \\
& =\int_{x} \int_{y} f(x) \pi(x, y) d y d x+\int_{x} \int_{y} g(y) \pi(x, y) d y d x \\
& =\int_{x} f(x) \int_{y} \pi(x, y) d y d x+\int_{y} g(y) \int_{x} \pi(x, y) d x d y \\
& =\int_{x} f(x) \pi(x) d x+\int_{y} g(y) \pi(y) d y \\
& =E f(x)+E f(y) .
\end{aligned}
$$

More generally

$$
E_{t} f\left(x_{t+1}\right)=E\left[f\left(x_{t+1}\right) \mid I_{t}\right]
$$

$$
\begin{aligned}
& =\sum_{i=1}^{n} f\left(x_{t+1, i}\right) \pi_{t, i}, \\
\text { or } & =\int_{x_{t+1} \in X_{t+1}} f\left(x_{t+1}\right) \pi_{t}\left(x_{t+1}\right) d x_{t+1},
\end{aligned}
$$

where $f$ is a function.
We enter now the computations of a stochastic consumption function.
The consumer maximizes an expected utility function

$$
E_{t}\left[u\left(c_{t}\right)+\frac{1}{1+\theta} u\left(c_{t+1}\right)\right] .
$$

He consumes $c_{t}$ at the beginning of the first period and $c_{t+1}$ at the beginning of the second period. He receives a stochastic revenue $y_{t}$ at the end of the first period. He can borrow and lend on the credit market at the constant interest rate $r$. There are $S_{t}$ states of nature for the income: $y_{t}\left(s_{t}\right), s_{t}=1, \ldots, S_{t}$, and $S_{t}$ associated probabilities: $\pi_{t}\left(s_{t}\right)$. Clearly

$$
\sum_{s_{t}} \pi_{t}\left(s_{t}\right)=1
$$

The expected intertemporal utility is computed as follows:

$$
\begin{aligned}
E_{t}\left[u\left(c_{t}\right)+\frac{1}{1+\theta} u\left(c_{t+1}\right)\right] & =\sum_{s_{t}}\left[u\left(c_{t}\right)+\frac{1}{1+\theta} u\left(c_{t+1}\left(s_{t}\right)\right)\right] \pi_{t}\left(s_{t}\right) \\
& =u\left(c_{t}\right)+\frac{1}{1+\theta} \sum_{s_{t}} u\left(c_{t+1}\left(s_{t}\right)\right) \pi_{t}\left(s_{t}\right)
\end{aligned}
$$

We observe that the first period consumption is certain because it precedes the revenue realization, while the second period consumption is stochastic, because it depends on the stochastic revenue realization.

There are $S_{t}$ possible budget constraints as many as the states of nature (the consumer must always repay his debt).

$$
\begin{aligned}
c_{t}+\frac{1}{1+r} c_{t+1}\left(s_{t}\right) & =y_{t}\left(s_{t}\right) \\
s_{t} & =1, \ldots, S_{t} .
\end{aligned}
$$

The choice variables are: $c_{t}, c_{t+1}(1), \ldots, c_{t+1}\left(S_{t}\right)$. We set the Lagrangian

$$
\begin{aligned}
\Lambda_{t}= & u\left(c_{t}\right)+\frac{1}{1+\theta} \sum_{s_{t}} u\left(c_{t+1}\left(s_{t}\right)\right) \pi_{t}\left(s_{t}\right) \\
& +\sum_{s_{t}} \lambda\left(s_{t}\right)\left[y_{t}\left(s_{t}\right)-c_{t}-\frac{1}{1+r} c_{t+1}\left(s_{t}\right)\right] .
\end{aligned}
$$

We compute the first order conditions:

$$
\begin{align*}
\frac{\partial \Lambda_{t}}{\partial c_{t}} & =u^{\prime}\left(c_{t}\right)-\sum_{s_{t}} \lambda\left(s_{t}\right)=0  \tag{2.9}\\
\frac{\partial \Lambda_{t}}{\partial c_{t+1}\left(s_{t}\right)} & =\frac{u^{\prime}\left(c_{t+1}\left(s_{t}\right)\right) \pi_{t}\left(s_{t}\right)}{1+\theta}-\frac{\lambda\left(s_{t}\right)}{1+r}=0  \tag{2.10}\\
s_{t} & =1, \ldots, S_{t}
\end{align*}
$$

Therefore aggregating equations (2.9-2.10), we get

$$
\begin{aligned}
u^{\prime}\left(c_{t}\right) & =\sum_{s_{t}} \lambda\left(s_{t}\right) \\
\sum_{s_{t}} \frac{u^{\prime}\left(c_{t+1}\left(s_{t}\right)\right) \pi_{t}\left(s_{t}\right)}{1+\theta} & =\sum_{s_{t}} \frac{\lambda\left(s_{t}\right)}{1+r} .
\end{aligned}
$$

Therefore

$$
\frac{1+r}{1+\theta} \sum_{s_{t}} u^{\prime}\left(c_{t+1}\left(s_{t}\right)\right) \pi_{t}\left(s_{t}\right)=\sum_{s_{t}} \lambda\left(s_{t}\right)=u^{\prime}\left(c_{t}\right)
$$

Using the expectation operator we obtain a stochastic Euler equation:

$$
u^{\prime}\left(c_{t}\right)=\frac{1+r}{1+\theta} E_{t} u^{\prime}\left(c_{t+1}\right)
$$

To explicitly solve the problem, we require the knowledge of the fundamentals. In the third part of the handbook we treat an explicit functional form and we completely characterize the solution (see the exercise "TwoPeriod Stochastic Consumption").

### 2.8 Three-Period Stochastic Consumption

We focus now on a slightly more complex framework. The consumer maximizes the intertemporal utility function over three periods.

$$
E_{t}\left[u\left(c_{t}\right)+\frac{1}{1+\theta} u\left(c_{t+1}\right)+\left(\frac{1}{1+\theta}\right)^{2} u\left(c_{t+2}\right)\right] .
$$

He receives a stochastic revenue $y_{t}\left(s_{t}\right)$ at the end of the first period and a stochastic revenue $y_{t+1}\left(s_{t+1}\right)$ at the end of the second period. The time sequence of economic actions is the following:

$$
c_{t}, y_{t}\left(s_{t}\right), c_{t+1}\left(s_{t}\right), y_{t+1}\left(s_{t+1}\right), c_{t+2}\left(s_{t}, s_{t+1}\right)
$$

with

$$
\begin{aligned}
s_{t} & =1, \ldots, S_{t} \\
s_{t+1} & =1, \ldots, S_{t+1}
\end{aligned}
$$

We notice that the consumption of third period $\left(c_{t+2}\left(s_{t}, s_{t+1}\right)\right)$ depends on the entire history $\left(s_{t}, s_{t+1}\right)$ of the states of nature.

We assume that the probability distribution of $y_{t+1}\left(s_{t+1}\right)$ is independent on the realization $s_{t}$. In other words the probability distributions $\pi_{t}\left(s_{t}\right)$ and $\pi_{t+1}\left(s_{t+1}\right)$ are independent. Clearly

$$
\begin{aligned}
\sum_{s_{\tau}} \pi_{\tau}\left(s_{\tau}\right) & =1 \\
\tau & =t, t+1
\end{aligned}
$$

The expected intertemporal utility is computed as follows:

$$
\begin{aligned}
& E_{t}\left[u\left(c_{t}\right)+\frac{1}{1+\theta} u\left(c_{t+1}\right)+\left(\frac{1}{1+\theta}\right)^{2} u\left(c_{t+2}\right)\right] \\
= & \sum_{\substack{s_{t}, s_{t+1}}}\left[u\left(c_{t}\right)+\frac{1}{1+\theta} u\left(c_{t+1}\left(s_{t}\right)\right)+\left(\frac{1}{1+\theta}\right)^{2} u\left(c_{t+2}\left(s_{t}, s_{t+1}\right)\right)\right] \\
& * \pi\left(s_{t}, s_{t+1}\right) \\
= & u\left(c_{t}\right)+\frac{1}{1+\theta} \sum_{s_{t}} u\left(c_{t+1}\left(s_{t}\right)\right) \pi_{t}\left(s_{t}\right) \\
& +\left(\frac{1}{1+\theta}\right)^{2} \sum_{s_{t}, s_{t+1}} u\left(c_{t+2}\left(s_{t}, s_{t+1}\right)\right) \pi\left(s_{t}, s_{t+1}\right)
\end{aligned}
$$

where $\pi_{t}\left(s_{t}, s_{t+1}\right)$ is the joint probability that the state of nature $s_{t}$ happens at the end of the first period and the state of nature $s_{t+1}$ happens at the end of the second. We observe that

$$
\pi_{t}\left(s_{t}\right)=\sum_{s_{t+1}} \pi\left(s_{t}, s_{t+1}\right)
$$

If the probability distributions $\pi_{t}$ and $\pi_{t+1}$ are independent

$$
\pi\left(s_{t}, s_{t+1}\right)=\pi_{t}\left(s_{t}\right) \pi_{t+1}\left(s_{t+1}\right) .
$$

The number of possible histories (paths) $\left(s_{t}, s_{t+1}\right)$ for the revenues is equal to

$$
S_{t} S_{t+1}
$$

This is exactly the number of intertemporal budget constraints the consumer must respect:

$$
\begin{aligned}
c_{t}+\frac{1}{1+r} c_{t+1}\left(s_{t}\right)+\left(\frac{1}{1+r}\right)^{2} c_{t+2}\left(s_{t}, s_{t+1}\right) & =y_{t}\left(s_{t}\right)+\frac{1}{1+r} y_{t+1}\left(s_{t+1}\right), \\
s_{t} & =1, \ldots, S_{t} \\
s_{t+1} & =1, \ldots, S_{t+1}
\end{aligned}
$$

Notice that the probability distribution of $c_{t+2}$ depends on those of $y_{t}$ and $y_{t+1}$.

The Lagrangian gets the following form

$$
\begin{aligned}
\Lambda_{t} \equiv & u\left(c_{t}\right)+\frac{1}{1+\theta} \sum_{s_{t}} u\left(c_{t+1}\left(s_{t}\right)\right) \pi_{t}\left(s_{t}\right) \\
& +\left(\frac{1}{1+\theta}\right)^{2} \sum_{s_{t}, s_{t+1}} u\left(c_{t+2}\left(s_{t}, s_{t+1}\right)\right) \pi\left(s_{t}, s_{t+1}\right)+\sum_{s_{t}, s_{t+1}} \lambda\left(s_{t}, s_{t+1}\right) \\
& *\left[y_{t}\left(s_{t}\right)+\frac{1}{1+r} y_{t+1}\left(s_{t+1}\right)-c_{t}-\frac{1}{1+r} c_{t+1}\left(s_{t}\right)-\left(\frac{1}{1+r}\right)^{2} c_{t+2}\left(s_{t}, s_{t+1}\right)\right] .
\end{aligned}
$$

Notice that the number of multipliers is $S_{t} S_{t+1}$.
We compute the first order conditions.

$$
\begin{equation*}
\frac{\partial \Lambda_{t}}{\partial c_{t}}=u^{\prime}\left(c_{t}\right)-\sum_{s_{t}, s_{t+1}} \lambda\left(s_{t}, s_{t+1}\right)=0 \tag{2.11}
\end{equation*}
$$

$$
\begin{align*}
\frac{\partial \Lambda_{t}}{\partial c_{t+1}\left(s_{t}\right)} & =\frac{u^{\prime}\left(c_{t+1}\left(s_{t}\right)\right) \pi_{t}\left(s_{t}\right)}{1+\theta}-\sum_{s_{t+1}} \frac{\lambda\left(s_{t}, s_{t+1}\right)}{1+r}=0,  \tag{2.12}\\
\frac{\partial \Lambda_{t}}{\partial c_{t+2}\left(s_{t}, s_{t+1}\right)} & =\frac{u^{\prime}\left(c_{t+2}\left(s_{t}, s_{t+1}\right)\right) \pi\left(s_{t}, s_{t+1}\right)}{(1+\theta)^{2}}-\frac{\lambda\left(s_{t}, s_{t+1}\right)}{(1+r)^{2}}=0 .
\end{align*}
$$

Aggregating (2.11) over $s_{t}$ and (2.12) over $s_{t}, s_{t+1}$, we obtain

$$
\begin{aligned}
u^{\prime}\left(c_{t}\right) & =\sum_{s_{t}, s_{t+1}} \lambda\left(s_{t}, s_{t+1}\right) \\
\frac{1+r}{1+\theta} \sum_{s_{t}} u^{\prime}\left(c_{t+1}\left(s_{t}\right)\right) \pi_{t}\left(s_{t}\right) & =\sum_{s_{t}, s_{t+1}} \lambda\left(s_{t}, s_{t+1}\right) \\
\left(\frac{1+r}{1+\theta}\right)^{2} \sum_{s_{t}, s_{t+1}} u^{\prime}\left(c_{t+2}\left(s_{t}, s_{t+1}\right)\right) \pi\left(s_{t}, s_{t+1}\right) & =\sum_{s_{t}, s_{t+1}} \lambda\left(s_{t}, s_{t+1}\right)
\end{aligned}
$$

Dividing the first equation by the second and the second by the third we have

$$
\begin{aligned}
u^{\prime}\left(c_{t}\right) & =\frac{1+r}{1+\theta} \sum_{s_{t}} u^{\prime}\left(c_{t+1}\left(s_{t}\right)\right) \pi_{t}\left(s_{t}\right) \\
\sum_{s_{t}} u^{\prime}\left(c_{t+1}\left(s_{t}\right)\right) \pi_{t}\left(s_{t}\right) & =\frac{1+r}{1+\theta} \sum_{s_{t}, s_{t+1}} u^{\prime}\left(c_{t+2}\left(s_{t}, s_{t+1}\right)\right) \pi\left(s_{t}, s_{t+1}\right)
\end{aligned}
$$

that is more compactly

$$
\begin{aligned}
u^{\prime}\left(c_{t}\right) & =\frac{1+r}{1+\theta} E_{t} u^{\prime}\left(c_{t+1}\right) \\
E_{t} u^{\prime}\left(c_{t+1}\right) & =\frac{1+r}{1+\theta} E_{t} u^{\prime}\left(c_{t+2}\right)
\end{aligned}
$$

The first equation is the same one of the two-period model, while the second constitutes a general form for the stochastic Euler equation.

### 2.9 The Random Walk of Consumption

A more sophisticated foundation of the consumption function has been provided by Hall in 1978. The consumer is at period $t$ and from this period on
solves a stochastic version of program (2.5):

$$
\begin{aligned}
& \max _{c_{t}, \ldots, c_{T}} E_{t} \sum_{\tau=t}^{T}(1+\theta)^{-(\tau-t)} u\left(c_{\tau}\right), \\
& b_{\tau+1} \leq(1+r) b_{\tau}+y_{\tau}-c_{\tau}
\end{aligned}
$$

where $T$ is the end of the consumer's life (he still lives $T-t$ periods), $\theta$ measures his impatience, $u$ is the utility of a period, $c_{\tau}$ the random consumption at period $\tau, b_{\tau}$ are the random bonds at period $\tau$ providing during the period a non-random constant return of $r . y_{t}$ is the random revenue.

The expectation operator is linear and the objective is rewritten as follows

$$
\sum_{\tau=t}^{T}(1+\theta)^{-(\tau-t)} E_{t} u\left(c_{\tau}\right)
$$

The random constraint of wealth accumulation is now correctly specified.

$$
\begin{equation*}
b_{\tau+1}\left(s_{t}, \ldots, s_{\tau}\right) \leq(1+r) b_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right)+y_{\tau}\left(s_{\tau}\right)-c_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right), \tag{2.13}
\end{equation*}
$$

where $s_{\tau} \in\left\{1, \ldots, S_{\tau}\right\}$ is the state of nature observed at period $\tau$ with probability $\pi_{\tau}\left(s_{\tau}\right)$. The corresponding realization of the random revenue is $y_{\tau}\left(s_{\tau}\right)$.

Let

$$
\pi\left(s_{t}, \ldots, s_{\tau}\right)
$$

be the joint probability of the history of states of nature ${ }^{1}$

$$
\left(s_{t}, \ldots, s_{\tau}\right)
$$

We notice that the consumption choice at time $\tau$ depends on the entire history of revenues, i.e. of states of nature. The same holds for the bonds. The optimization is performed in $t$, when the individual has at his disposal the information set $I_{t}$.

[^3]There are as many multipliers as constraints. The multipliers of the generic constraint (2.13) is

$$
\lambda\left(s_{t}, \ldots, s_{\tau}\right)
$$

Notice that the first constraint is given by

$$
b_{t+1}\left(s_{t}\right) \leq(1+r) b_{t}+y_{t}\left(s_{t}\right)-c_{t},
$$

because the wealth $b_{t}$ is inherited from the previous period and consumption choice $c_{t}$ happens before the realization of the state of nature $s_{t}$.

The general timing is

$$
b_{t}, c_{t}, y_{t}, b_{t+1}, c_{t+1}, y_{t+1}, \ldots
$$

To have an idea of problem complexity we compute the number of consumption variables until $\tau$ as

$$
1+\sum_{j=t}^{\tau-1} \prod_{i=t}^{j} S_{i}
$$

(the sum of the number of histories).
The expected intertemporal utility is written more explicitly

$$
\begin{aligned}
& \sum_{\tau=t}^{T}\left(\frac{1}{1+\theta}\right)^{\tau-t} E_{t} u\left(c_{\tau}\right) \\
= & u\left(c_{t}\right)+\sum_{\tau=t+1}^{T} \sum_{s_{t}, \ldots, s_{\tau-1}}\left(\frac{1}{1+\theta}\right)^{\tau-t} u\left(c_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right)\right) \pi\left(s_{t}, \ldots, s_{\tau-1}\right) .
\end{aligned}
$$

We write down the Lagrangian:

$$
\begin{aligned}
\Lambda_{t} \equiv & u\left(c_{t}\right)+\sum_{\tau=t+1}^{T} \sum_{s_{t}, \ldots, s_{\tau-1}}\left(\frac{1}{1+\theta}\right)^{\tau-t} u\left(c_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right)\right) \pi\left(s_{t}, \ldots, s_{\tau-1}\right) \\
& +\sum_{s_{t}} \lambda_{t}\left(s_{t}\right)\left[(1+r) b_{t}+y_{t}\left(s_{t}\right)-c_{t}-b_{t+1}\left(s_{t}\right)\right] \\
& +\sum_{\tau=t+1}^{T} \sum_{s_{t}, \ldots, s_{\tau-1}} \lambda_{\tau}\left(s_{t}, \ldots, s_{\tau}\right) \\
& *\left[(1+r) b_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right)+y_{\tau}\left(s_{\tau}\right)-c_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right)-b_{\tau+1}\left(s_{t}, \ldots, s_{\tau}\right)\right]
\end{aligned}
$$

We can choose $c_{\tau}$ as a control variable and $b_{\tau+1}$ as a state variable. $y_{\tau}$ is not a choice variable.

We want to derive the generic Euler equation between the period $\tau$ and $\tau+1$.

The first order conditions are:

$$
\begin{aligned}
& \frac{\partial E_{t} U}{\partial c_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right)} \\
= & \left(\frac{1}{1+\theta}\right)^{\tau-t} u^{\prime}\left(c_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right)\right) \pi\left(s_{t}, \ldots, s_{\tau-1}\right)-\sum_{s_{\tau}} \lambda_{\tau}\left(s_{t}, \ldots, s_{\tau}\right) \\
= & 0 \\
& \frac{\partial E_{t} U}{\partial b_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right)} \\
= & \sum_{s_{\tau}} \lambda_{\tau}\left(s_{t}, \ldots, s_{\tau}\right)(1+r)-\lambda_{\tau-1}\left(s_{t}, \ldots, s_{\tau-1}\right)=0 .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_{s_{\tau}} \lambda_{\tau}\left(s_{t}, \ldots, s_{\tau}\right) & =\left(\frac{1}{1+\theta}\right)^{\tau-t} u^{\prime}\left(c_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right)\right) \pi\left(s_{t}, \ldots, s_{\tau-1}\right) \\
\lambda_{\tau}\left(s_{t}, \ldots, s_{\tau}\right) & =(1+r) \sum_{s_{\tau+1}} \lambda_{\tau+1}\left(s_{t}, \ldots, s_{\tau+1}\right)
\end{aligned}
$$

Aggregating across the histories $\left(s_{t}, \ldots, s_{\tau}\right)$ we obtain

$$
\begin{aligned}
& \sum_{s_{t}, \ldots, s_{\tau-1}}\left(\frac{1}{1+\theta}\right)^{\tau-t} u^{\prime}\left(c_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right)\right) \pi\left(s_{t}, \ldots, s_{\tau-1}\right) \\
= & \sum_{s_{t}, \ldots, s_{\tau-1}} \sum_{s_{\tau}} \lambda_{\tau}\left(s_{t}, \ldots, s_{\tau}\right) \\
= & \sum_{s_{t}, \ldots, s_{\tau}} \lambda_{\tau}\left(s_{t}, \ldots, s_{\tau}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{s_{t}, \ldots, s_{\tau}} \lambda_{\tau}\left(s_{t}, \ldots, s_{\tau}\right) \\
= & (1+r) \sum_{s_{t}, \ldots, s_{\tau}} \sum_{s_{\tau+1}} \lambda_{\tau+1}\left(s_{t}, \ldots, s_{\tau+1}\right) \\
= & (1+r) \sum_{s_{t}, \ldots, s_{\tau+1}} \lambda_{\tau+1}\left(s_{t}, \ldots, s_{\tau+1}\right) .
\end{aligned}
$$

After eliminating the multipliers we get

$$
\begin{aligned}
& \sum_{s_{t}, \ldots, s_{\tau-1}}\left(\frac{1}{1+\theta}\right)^{\tau-t} u^{\prime}\left(c_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right)\right) \pi\left(s_{t}, \ldots, s_{\tau-1}\right) \\
= & (1+r) \sum_{s_{t}, \ldots, s_{\tau}}\left(\frac{1}{1+\theta}\right)^{\tau+1-t} u^{\prime}\left(c_{\tau+1}\left(s_{t}, \ldots, s_{\tau}\right)\right) \pi\left(s_{t}, \ldots, s_{\tau}\right),
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
& \sum_{s_{t}, \ldots, s_{\tau-1}} u^{\prime}\left(c_{\tau}\left(s_{t}, \ldots, s_{\tau-1}\right)\right) \pi\left(s_{t}, \ldots, s_{\tau-1}\right) \\
= & \frac{1+r}{1+\theta} \sum_{s_{t}, \ldots, s_{\tau}} u^{\prime}\left(c_{\tau+1}\left(s_{t}, \ldots, s_{\tau}\right)\right) \pi\left(s_{t}, \ldots, s_{\tau}\right) .
\end{aligned}
$$

Finally we obtain

$$
\begin{equation*}
E_{t} u^{\prime}\left(c_{\tau}\right)=\frac{1+r}{1+\theta} E_{t} u^{\prime}\left(c_{\tau+1}\right) \tag{2.14}
\end{equation*}
$$

that is the stochastic Euler equation we found in the three-period model.
In other words we obtain the equality between the stochastic marginal rate of intertemporal substitution and the price ratio $1+r$ (which could be viewed as a marginal rate of transformation with $r=f^{\prime}\left(k_{\tau}\right)$ for every $\left.\tau\right)$.

$$
(1+\theta) \frac{E_{t} u^{\prime}\left(c_{\tau}\right)}{E_{t} u^{\prime}\left(c_{\tau+1}\right)}=1+r .
$$

We notice that

$$
\begin{equation*}
E_{t} c_{t}=c_{t} \tag{2.15}
\end{equation*}
$$

because $c_{t} \in I_{t}$. Setting $\tau=t$ in (2.14), we obtain

$$
\begin{align*}
E_{t} u^{\prime}\left(c_{t}\right) & =\frac{1+r}{1+\theta} E_{t} u^{\prime}\left(c_{t+1}\right), \\
u^{\prime}\left(c_{t}\right) & =\frac{1+r}{1+\theta} E_{t} u^{\prime}\left(c_{t+1}\right), \tag{2.16}
\end{align*}
$$

because of (2.15).
There is a simple economic interpretation of (2.16). If the consumer renounces to one unit of consumption in $t$, he reduces the utility of $u^{\prime}\left(c_{t}\right)$ and
increases in $t+1$ the utility of the expected gain $(1+r) E_{t} u^{\prime}\left(c_{t+1}\right)$. However the latter expression must be discounted according to the time preference

$$
\frac{1+r}{1+\theta} E_{t} u^{\prime}\left(c_{t+1}\right)
$$

Subjective cost and benefit are equal at the optimum.
If we set by simplicity $r=\theta$, we get

$$
u^{\prime}\left(c_{t}\right)=E_{t} u^{\prime}\left(c_{t+1}\right) .
$$

In the seminal paper of 1978 Hall assumes a quadratic utility function

$$
\begin{aligned}
u\left(c_{t}\right) & =a c_{t}-\frac{b}{2} c_{t}^{2} \\
u^{\prime}\left(c_{t}\right) & =a-b c_{t}
\end{aligned}
$$

We observe that a quadratic function allows for satiation and further negative effects of consumption. Equation (2.16) becomes

$$
a-b c_{t}=a-b E_{t} c_{t+1},
$$

because of the linearity of the marginal utility. Eventually

$$
c_{t}=E_{t} c_{t+1}
$$

This equation is crucial in the Hall's construction. Today consumption constitutes the best forecast for tomorrow consumption. Notice also that

$$
\begin{aligned}
E_{t} c_{t+2} & =E_{t}\left[E_{t+1} c_{t+2}\right] \\
& =E_{t} c_{t+1} \\
& =c_{t}
\end{aligned}
$$

because $I_{t} \subset I_{t+1}$. By induction we obtain

$$
\begin{equation*}
E_{t} c_{t+\tau}=c_{t} \tag{2.17}
\end{equation*}
$$

The best forecast at time $t$ of consumption at date $t+\tau$ is still the consumption at date $t$.

We want now to write the intertemporal budget constraint under perfect foresight.

$$
\begin{aligned}
b_{t+1}= & (1+r) b_{t}+y_{t}-c_{t}, \\
b_{t+2}= & (1+r)\left[(1+r) b_{t}+y_{t}-c_{t}\right]+y_{t+1}-c_{t+1}, \\
= & (1+r)^{2} b_{t}+(1+r) y_{t}-(1+r) c_{t}+y_{t+1}-c_{t+1} \\
b_{t+3}= & (1+r)\left[(1+r)^{2} b_{t}+(1+r) y_{t}-(1+r) c_{t}+y_{t+1}-c_{t+1}\right]+y_{t+2} \\
& -c_{t+2}, \\
= & (1+r)^{3} b_{t}+(1+r)^{2} y_{t}-(1+r)^{2} c_{t}+(1+r) y_{t+1}-(1+r) c_{t+1} \\
& +y_{t+2}-c_{t+2} \\
= & (1+r)^{3} b_{t}+\left[(1+r)^{2} y_{t}+(1+r) y_{t+1}+y_{t+2}\right] \\
& -\left[(1+r)^{2} c_{t}+(1+r) c_{t+1}+c_{t+2}\right], \\
b_{T+1}= & (1+r)^{T+1-t} b_{t}+\sum_{\tau=t}^{T}(1+r)^{T-\tau} y_{\tau}-\sum_{\tau=t}^{T}(1+r)^{T-\tau} c_{\tau} .
\end{aligned}
$$

The rational consumer does not die with bound. He is forced to not die with debts.

$$
b_{T+1}=0 .
$$

Therefore

$$
\begin{aligned}
b_{t}+\sum_{\tau=t}^{T} \frac{(1+r)^{T-\tau}}{(1+r)^{T+1-t}} y_{\tau} & =\sum_{\tau=t}^{T} \frac{(1+r)^{T-\tau}}{(1+r)^{T+1-t}} c_{\tau} \\
b_{t}+\sum_{\tau=t}^{T}(1+r)^{-(\tau+1-t)} y_{\tau} & =\sum_{\tau=t}^{T}(1+r)^{-(\tau+1-t)} c_{\tau} .
\end{aligned}
$$

Under uncertainty conditional to information disposable at time $t$, we obtain the stochastic intertemporal budget constraint.

$$
b_{t}+E_{t} \sum_{\tau=t}^{T}(1+r)^{-(\tau+1-t)} y_{\tau}=E_{t} \sum_{\tau=t}^{T}(1+r)^{-(\tau+1-t)} c_{\tau}
$$

where

$$
b_{t}=E_{t} b_{t},
$$

because $b_{t} \in I_{t}$.

The expectation operator is linear. We know that $E_{t} c_{t+\tau}=c_{t}$ for every $\tau \geq 0$. Hence we obtain, by substituting the first order condition (2.17) in the intertemporal budget constraint, the microfounded consumption demand function.

$$
\begin{aligned}
b_{t}+E_{t} \sum_{\tau=t}^{T}(1+r)^{-(\tau+1-t)} y_{\tau} & =\sum_{\tau=t}^{T}(1+r)^{-(\tau+1-t)} E_{t} c_{\tau} \\
& =\sum_{\tau=t}^{T}(1+r)^{-(\tau+1-t)} c_{t} \\
& =c_{t} \sum_{\tau=t}^{T}(1+r)^{-(\tau+1-t)} \\
& =c_{t} \sum_{\tau=0}^{T-t}(1+r)^{-(\tau+1)} \\
& =c_{t} \frac{1}{1+r} \sum_{\tau=0}^{T-t}\left(\frac{1}{1+r}\right)^{-\tau} \\
& =c_{t} \frac{1}{1+r} \frac{1-(1+r)^{-(T-t+1)}}{1-(1+r)^{-1}} \\
& =c_{t} \frac{1-(1+r)^{-(T-t+1)}}{r} .
\end{aligned}
$$

The consumption demand function becomes

$$
c_{t}=\frac{r}{1-(1+r)^{-(T-t+1)}}\left[b_{t}+E_{t} \sum_{\tau=t}^{T}(1+r)^{-(\tau+1-t)} y_{\tau}\right],
$$

which depends on the price system $(r)$, the expected revenues $\left(y_{\tau}\right)$ and the initial wealth $\left(b_{t}\right)$, as in the usual consumer's theory.

We assume now by simplicity an infinite-lived consumer $(T=\infty)$. Thereby as

$$
\lim _{T \rightarrow \infty}(1+r)^{-(T-t+1)}=0
$$

the consumption demand gets a straightforward form

$$
c_{t}=r\left[b_{t}+E_{t} \sum_{\tau=t}^{\infty}(1+r)^{-(\tau+1-t)} y_{\tau}\right] .
$$

If $b_{t}$ is the initial non-human wealth and $y_{\tau}$ 's are labor incomes, the expression into the square brackets is the sum of non-human and expected human wealth, i.e. it is the total wealth. Thereby the current consumption is just given by the interests on the expected total wealth.

This is the stochastic version of the life-cycle-permanent income theory.
To better understand as the consumption reacts to income changes we evaluate $c_{t}-c_{t-1}$.

As

$$
c_{t-1}=E_{t-1} c_{t}
$$

the consumption updating is the following

$$
\begin{aligned}
c_{t}-c_{t-1}= & c_{t}-E_{t-1} c_{t} \\
= & r\left[b_{t}+E_{t} \sum_{\tau=t}^{\infty}(1+r)^{-(\tau+1-t)} y_{\tau}\right] \\
& -E_{t-1}\left[r\left[b_{t}+E_{t} \sum_{\tau=t}^{\infty}(1+r)^{-(\tau+1-t)} y_{\tau}\right]\right] \\
= & r \sum_{\tau=t}^{\infty}(1+r)^{-(\tau+1-t)}\left(E_{t} y_{\tau}-E_{t-1} y_{\tau}\right) .
\end{aligned}
$$

Then if $I_{t-1}$ is strictly included in $I_{t}$ and new relevant information is arrived the expected income sequence may change and may affect the consumption. If for instance

$$
E_{t} y_{\tau}>E_{t-1} y_{\tau}
$$

for every $\tau$, then

$$
c_{t}>c_{t-1}
$$

To conclude this section we consider an explicit income process.

$$
\begin{equation*}
y_{t}=\rho y_{t-1}+(1-\rho) \bar{y}+\varepsilon_{t} \tag{2.18}
\end{equation*}
$$

with $\rho \leq 1$. The revenue $y_{t}$ is a weighted average of $y_{t-1}$ and the average of the process $\bar{y}$, plus the innovation $\varepsilon_{t}$, a random variable we assume for instance i.i.d. and with a zero mean.

We want now to compute

$$
E_{t} y_{\tau}-E_{t-1} y_{\tau}
$$

with $\tau>t-1$.
From (2.18) we get

$$
\begin{aligned}
y_{\tau}= & \rho y_{\tau-1}+(1-\rho) \bar{y}+\varepsilon_{\tau} \\
= & \rho\left[\rho y_{\tau-2}+(1-\rho) \bar{y}+\varepsilon_{\tau-1}\right]+(1-\rho) \bar{y}+\varepsilon_{\tau} \\
= & \rho^{2} y_{\tau-2}+\rho(1-\rho) \bar{y}+\rho \varepsilon_{\tau-1}+(1-\rho) \bar{y}+\varepsilon_{\tau} \\
= & \rho^{2}\left[\rho y_{\tau-3}+(1-\rho) \bar{y}+\varepsilon_{\tau-2}\right]+\rho(1-\rho) \bar{y}+\rho \varepsilon_{\tau-1}+(1-\rho) \bar{y}+\varepsilon_{\tau} \\
= & \rho^{3} y_{\tau-3}+\rho^{2}(1-\rho) \bar{y}+\rho^{2} \varepsilon_{\tau-2}+\rho(1-\rho) \bar{y}+\rho \varepsilon_{\tau-1}+(1-\rho) \bar{y}+\varepsilon_{\tau} \\
= & \rho^{\tau-(t-2)} y_{t-2}+(1-\rho) \bar{y}\left[\rho^{\tau-t+1}+\ldots+\rho+1\right] \\
& +\left[\rho^{\tau-t+1} \varepsilon_{t-1}+\ldots+\rho \varepsilon_{\tau-1}+\varepsilon_{\tau}\right] \\
= & \rho^{\tau-(t-2)} y_{t-2}+(1-\rho) \bar{y} \sum_{i=0}^{\tau-t+1} \rho^{i}+\sum_{i=t-1}^{\tau} \rho^{\tau-i} \varepsilon_{i} \\
= & \rho^{\tau-(t-2)} y_{t-2}+(1-\rho) \bar{y} \frac{1-\rho^{\tau-(t-2)}}{1-\rho}+\sum_{i=t-1}^{\tau} \rho^{\tau-i} \varepsilon_{i} \\
= & \rho^{\tau-(t-2)} y_{t-2}+\left[1-\rho^{\tau-(t-2)}\right] \bar{y}+\sum_{i=t-1}^{\tau} \rho^{\tau-i} \varepsilon_{i} .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
E_{t} y_{\tau}-E_{t-1} y_{\tau}= & \rho^{\tau-(t-2)} y_{t-2}+\left[1-\rho^{\tau-(t-2)}\right] \bar{y}+\sum_{i=t-1}^{\tau} \rho^{\tau-i} E_{t} \varepsilon_{i} \\
& -\left[\rho^{\tau-(t-2)} y_{t-2}+\left[1-\rho^{\tau-(t-1)}\right] \bar{y}+\sum_{i=t-1}^{\tau} \rho^{\tau-i} E_{t-1} \varepsilon_{i}\right] \\
= & \sum_{i=t-1}^{\tau} \rho^{\tau-i} E_{t} \varepsilon_{i}-\sum_{i=t-1}^{\tau} \rho^{\tau-i} E_{t-1} \varepsilon_{i} .
\end{aligned}
$$

We observe that

$$
\begin{aligned}
E_{t-1} \varepsilon_{i} & =0, \forall i \geq t-1, \\
E_{t} \varepsilon_{i} & =0, \forall i \geq t \\
E_{t} \varepsilon_{t-1} & =\varepsilon_{t-1},
\end{aligned}
$$

because

$$
\begin{aligned}
\varepsilon_{t} & \notin I_{t} \\
\varepsilon_{t-1} & \in I_{t} .
\end{aligned}
$$

Hence

$$
E_{t} y_{\tau}-E_{t-1} y_{\tau}=\rho^{\tau-t+1} \varepsilon_{t-1} .
$$

for $\tau>t-1$. The consumption updating becomes

$$
\begin{aligned}
c_{t}-c_{t-1} & =r\left(b_{t}-E_{t-1} b_{t}\right)+r \sum_{\tau=t}^{\infty}(1+r)^{-(\tau+1-t)}\left(E_{t} y_{\tau}-E_{t-1} y_{\tau}\right) \\
& =r\left(b_{t}-E_{t-1} b_{t}\right)+r \sum_{\tau=t}^{\infty}(1+r)^{-(\tau+1-t)} \rho^{\tau-t+1} \varepsilon_{t-1} \\
& =r\left(b_{t}-E_{t-1} b_{t}\right)+\frac{r}{1+r} \rho \varepsilon_{t-1} \sum_{\tau=t}^{\infty}(1+r)^{-(\tau-t)} \rho^{\tau-t} \\
& =r\left(b_{t}-E_{t-1} b_{t}\right)+\frac{r}{1+r} \rho \varepsilon_{t-1} \sum_{i=0}^{\infty}\left(\frac{\rho}{1+r}\right)^{i} \\
& =r\left(b_{t}-E_{t-1} b_{t}\right)+\frac{r}{1+r} \rho \varepsilon_{t-1} \frac{1}{1-\rho /(1+r)} \\
& =r\left(b_{t}-E_{t-1} b_{t}\right)+\frac{r \rho}{1+r-\rho} \varepsilon_{t-1}
\end{aligned}
$$

because

$$
\frac{\rho}{1+r}<1 .
$$

Thereby

$$
c_{t}-c_{t-1}=r\left(b_{t}-E_{t-1} b_{t}\right)+\frac{r \rho}{1+r-\rho} \varepsilon_{t-1} .
$$

We observe that the change $\varepsilon_{t}$ is a unexpected change in revenue, while the term

$$
\frac{r \rho}{1+r-\rho}<1
$$

(because $\rho<1$ ) measures the proportional impact of the change on consumption.

We want now to compute the explicit expression for $c_{t}$.

$$
\begin{aligned}
c_{t} & =r\left[b_{t}+E_{t} \sum_{\tau=t}^{\infty}(1+r)^{-(\tau+1-t)} y_{\tau}\right] \\
& =r\left\{b_{t}+E_{t} \sum_{\tau=t}^{\infty}(1+r)^{-(\tau+1-t)}\left\{\rho^{\tau-(t-2)} y_{t-2}+\left[1-\rho^{\tau-(t-2)}\right] \bar{y}+\sum_{i=t-1}^{\tau} \rho^{\tau-i} \varepsilon_{i}\right\}\right\}
\end{aligned}
$$

$$
\begin{aligned}
= & r\left\{b_{t}+\sum_{\tau=t}^{\infty}(1+r)^{-(\tau+1-t)}\left\{\rho^{\tau-(t-2)} y_{t-2}+\left[1-\rho^{\tau-(t-2)}\right] \bar{y}+\sum_{i=t-1}^{\tau} \rho^{\tau-i} E_{t} \varepsilon_{i}\right\}\right\} \\
= & r\left\{b_{t}+\sum_{\tau=t}^{\infty}(1+r)^{-(\tau+1-t)}\left\{\rho^{\tau-(t-2)} y_{t-2}+\left[1-\rho^{\tau-(t-2)}\right] \bar{y}+\rho^{\tau-t+1} \varepsilon_{t-1}\right\}\right\} \\
= & r b_{t}+r \rho y_{t-2} \sum_{\tau=t}^{\infty}\left(\frac{\rho}{1+r}\right)^{\tau-(t-1)} \\
& +r \bar{y}\left[\sum_{\tau=t}^{\infty}\left(\frac{1}{1+r}\right)^{\tau-(t-1)}-\rho \sum_{\tau=t}^{\infty}\left(\frac{\rho}{1+r}\right)^{\tau-(t-1)}\right] \\
& +\varepsilon_{t-1} \frac{r \rho}{1+r} \sum_{\tau=t}^{\infty}\left(\frac{\rho}{1+r}\right)^{\tau-t} \\
= & r b_{t}+r \rho y_{t-2} \frac{\rho}{1+r} \sum_{i=0}^{\infty}\left(\frac{\rho}{1+r}\right)^{i} \\
& +r \bar{y}\left[\frac{1}{1+r} \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}-\frac{\rho^{2}}{1+r} \sum_{i=0}^{\infty}\left(\frac{\rho}{1+r}\right)^{i}\right] \\
& +\varepsilon_{t-1} \frac{r \rho}{1+r} \sum_{i=0}^{\infty}\left(\frac{\rho}{1+r}\right)^{i} .
\end{aligned}
$$

Solving the series we obtain

$$
\begin{aligned}
c_{t}= & r b_{t}+r y_{t-2} \frac{\rho^{2}}{1+r} \frac{1}{1-\rho /(1+r)} \\
& +r \bar{y}\left[\frac{1}{1+r} \frac{1}{1-1 /(1+r)}-\frac{\rho^{2}}{1+r} \frac{1}{1-\rho /(1+r)}\right] \\
& +\varepsilon_{t-1} \frac{r \rho}{1+r} \frac{1}{1-\rho /(1+r)} \\
= & r b_{t}+r y_{t-2} \frac{\rho^{2}}{1+r-\rho}+r \bar{y}\left[\frac{1}{r}-\frac{\rho^{2}}{1+r-\rho}\right]+\frac{r \rho}{1+r-\rho} \varepsilon_{t-1} \\
= & r b_{t}+r y_{t-2} \frac{\rho^{2}}{1+r-\rho}+r \bar{y}\left[\frac{1}{r}-\frac{\rho^{2}}{1+r-\rho}\right] \\
& +\frac{r \rho}{1+r-\rho}\left[y_{t-1}-\rho y_{t-2}-(1-\rho) \bar{y}\right] \\
= & r b_{t}+r y_{t-2} \frac{\rho^{2}}{1+r-\rho}+\bar{y}-\frac{r \rho^{2}}{1+r-\rho} \bar{y}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{r \rho}{1+r-\rho} y_{t-1}-\frac{r \rho^{2}}{1+r-\rho} y_{t-2}-\frac{r \rho}{1+r-\rho} \bar{y}+\frac{r \rho^{2}}{1+r-\rho} \bar{y} \\
= & r b_{t}+\bar{y}-\frac{r \rho}{1+r-\rho} \bar{y}+\frac{r \rho}{1+r-\rho} y_{t-1} \\
= & r b_{t}+\left(1-\frac{r \rho}{1+r-\rho}\right) \bar{y}+\frac{r \rho}{1+r-\rho} y_{t-1} \\
= & r b_{t}+\alpha \bar{y}+(1-\alpha) y_{t-1}
\end{aligned}
$$

where

$$
\alpha \equiv \frac{r \rho}{1+r-\rho} .
$$

## Chapter 3

## The Investment Function

Consumption and investment are the chief determinants of the aggregate demand. The purpose of this chapter is to investigate the investment behavior and to provide a micro-foundation, i.e. an explanation based on the individual behavior, which is largely shared by the macroeconomists.

The demand for investment is the desired variation of the physical capital level. This level is the sum of the long run capital such as machines, and short run capital (goods for production or sale). Broadly speaking, the underlying idea is that of immobilized input.

The demand for investment depends above all on the expectations about the future demand. There is no direct link between the current level of production and the forecast on the evolution of demand and production.

A restrictive definition of investment is often adopted in national accounting. The investment is said to be gross formation of fixed capital. The $G F F C$ represents the value of the durable goods which have been bought by national firms to be used at least during one year.

The GFFC includes ( $i$ ) the buildings which have been purchased by the institutional sectors (companies, administrations, financial institutions,...), (ii) the furniture, the machines and similar equipments which have been bought by such institutional sectors.

An investment rate is defined in the national accounting as the ratio $G F F C$ over the gross domestic product (GDP).

$$
\frac{G F F C}{G D P} .
$$

In western countries this ratio fluctuates around $20 \%$.

Alternative classifications for the investment are possible such as that based on the institutional sectors, branches (agriculture, industry, tertiary sector) and so on.

A Formal Definition. The net investment is the variation of the stock of capital with respect to time:

$$
I_{n t}=K_{t+1}-K_{t}
$$

The gross investment is the net investment plus the capital depreciation:

$$
\begin{aligned}
I_{t} & =K_{t+1}-K_{t}+\delta K_{t} \\
& =K_{t+1}-(1-\delta) K_{t},
\end{aligned}
$$

where $\delta \in[0,1]$ is the depreciation rate of capital.

### 3.1 The Investment Function

Criterion of the Discounted Value. Firms can finance their investment by either using their internal founds or loaning outside for instance from households or financial institutions.

We assume that the initial investment cost is equal $C_{0}$ and that the investment will provide during $n$ period a non-stochastic return. Let the return of period $t$ be $R_{t}$. The producer compares the discounted sum of future revenues with the initial cost. In other words the entrepreneur will invest if and only if the intertemporal profit of a project is strictly positive:

$$
V \equiv \sum_{t=1}^{n} \frac{R_{t}}{(1+i)^{t}}>C_{0}
$$

where $V$ is the discounted value of the future revenues provided by the productive combination we say to be an investment. $i$ is the market interest rate, which measures the opportunity cost of investment (the agent could lend the amount $C_{0}$ to the credit market and have $i C_{0}$ as net return).

Criterion of the Internal Rate of Return. How could we compute the rate of return which is specific to each investment?

Let us assume that the initial cost $C_{0}$ and the sequence of revenues $\left\{R_{t}\right\}_{t=1}^{n}$ are known. The internal rate of return $r$ of the productive process simply solves the following algebraic equation:

$$
\sum_{t=1}^{n} \frac{R_{t}}{(1+r)^{t}}=C_{0}
$$

The criterion of the internal rate of return just consists in comparing the market interest rate $i$ which measures an opportunity cost with $r$.

If $r>i$ the agent will perform the investment, if $r<i$ the agent will supply the monetary amount on the credit market.

If the future revenues are constant, the internal rate of return can be easily computed:

$$
\begin{aligned}
C_{0} & =\sum_{t=1}^{n} \frac{R}{(1+r)^{t}}=\frac{R}{1+r} \sum_{t=0}^{n} \frac{1}{(1+r)^{t}}=\frac{R}{1+r} \frac{1+r}{r} \\
r & =R / C_{0}
\end{aligned}
$$

The Investment Function. If the market interest rate $i$ is too high, few investment projects will display an internal rate of return $r$ greater than the opportunity cost $i$, and will be implemented. If $i$ is lower, more projects will be performed. Then there exists a negative relationship between the interest rate $i$ on the credit market and the number of realized investments.

The investment function describes the relation between the explicative variables such as the interest rate and the aggregate investment.

$$
\begin{aligned}
I & =I(i) \\
I^{\prime} & <0 .
\end{aligned}
$$

In the following section we will consider another explicative variable as argument: the aggregate demand variation.

### 3.2 Duality

Production Maximization. A price-taker producer wants to maximize the production under a constraint of cost. Let $f(x)$ be a production function, $x$ be the vector of inputs and $c$ be the maximal cost the agent
can pay. If $w$ is the given input price vector, the maximization program is formalized as follows.

$$
\begin{aligned}
& \max _{x} f(x), \\
& w x \leq c .
\end{aligned}
$$

We consider without loss of generality the case of a function $f$ of two inputs $x_{1}$ and $x_{2}$. The Lagrangian is

$$
f\left(x_{1}, x_{2}\right)+\lambda\left[c-w_{1} x_{1}-w_{2} x_{2}\right]
$$

We obtain the first order condition

$$
M R S=-w_{1} / w_{2}
$$

where $M R S$ denotes the marginal rate of substitution, jointly with the budget constraint.

The solution provides the demands $x_{1}(w, c)$ and $x_{2}(w, c)$ as a function of the factor price vector and the given cost. Moreover we know by substitution the product supply

$$
y(w, c)=f\left(x_{1}(w, c), x_{2}(w, c)\right)
$$

Cost Minimization. The agent must produce at least an amount $y$ of output and wants to minimize the cost.

The program is

$$
\begin{gathered}
\min _{x} w x, \\
f(x) \geq y .
\end{gathered}
$$

The first order conditions are

$$
w=\lambda D_{x} f
$$

where $w$ is the input price vector and $D_{x} f$ is the gradient of the production function. As usual $\lambda$ is a Lagrangian multiplier. After eliminating the Lagrangian multiplier and using the production constraint we obtain the factor demand as function of input prices and the minimally required production level:

$$
x^{*}=x^{*}(w, y) .
$$

We notice that $x, w$ are vectors, while $y$ is a scalar.

By substitution we get the cost function that is the minimal cost:

$$
c(w, y)=w x^{*}
$$

This function is also said to be the total cost.
The average cost is defined as follows

$$
\frac{c(w, y)}{y}
$$

while the marginal cost is given by a partial derivative

$$
\frac{\partial c(w, y)}{\partial y}
$$

It is possible to show that the cost function: $(i)$ is non-decreasing in $w$, (ii) it is homogeneous of degree one, (iii) concave in $w,(i v)$ continuous in $w$ (see among others Varian, 1992).

The Shephard's lemma allows us to compute the optimal demand functions from a total cost:

$$
x_{i}(w, y)=\frac{\partial C(w, y)}{\partial w_{i}}
$$

The proof is just an application of the envelope theorem ${ }^{1}$.

Duality. The production maximization

$$
\begin{gathered}
\max _{x} f(x) \\
w x \leq c
\end{gathered}
$$

${ }^{1}$ Proof. Let $y$ be given. We can maximize the negative of the cost:

$$
M(w) \equiv-C(w, y) .
$$

Therefore

$$
\frac{\partial M}{\partial w_{i}}=\frac{\partial(-w x)}{\partial w_{i}}=-x_{i}
$$

and finally

$$
x_{i}(w, y)=\frac{\partial C(w, y)}{\partial w_{i}} .
$$

and the cost minimization are equivalent programs under mild assumptions. In other words the solutions (optimal factor purchase) are the same.

$$
\begin{gathered}
\min _{x} w x \\
f(x) \geq y
\end{gathered}
$$

The main condition to have the equivalence is the convexity of the input requirement set, i.e. the quasi-concavity of the production function.

In words the duality theory provides the conditions to link the informations concerning the production function with the informations about the cost function.

If the input requirement set for every production level is convex, then we can reconstruct exactly the production function from the cost function. In this case we obtain

$$
c^{*}(w, y)=\min w x
$$

with $x \in V(y)$, where

$$
V(y) \equiv\{x: f(x) \geq y\}
$$

The converse is always possible: from a production function we can derive the factor demands to be substituted in the cost to obtain the minimal cost, i.e. the cost function $c(w, y)=w x^{*}(p, w)$.

The duality relation can be easily viewed in the case of one input. We can show the inverse relationship between the marginal cost and productivity. It is always possible to find the cost function because an increasing production function $y=f(x)$ is always quasi-concave. Monotonicity of $f$ implies $x=f^{-1}(y)$. The cost function becomes $c(w, y)=\min w x=w f^{-1}(y)$. The marginal cost is given by

$$
\partial c / \partial y=w\left[1 / f^{\prime}(x)\right]=w / f^{\prime}(x)
$$

The inverse relationship between the marginal cost and productivity is now clear.

The following table illustrates the dual links between production and costs
for a function of two factors.


The two figures below describe the duality.


Figure 14. $\operatorname{Max} f$.


Figure 15. Min $c$.

### 3.3 Static Profit Maximization

Which is the right level of capital a firm needs? Let the production function depend on capital and labor (for instance the number of workers):

$$
Y=F(K, L) .
$$

We assume $F$ to be concave (production divisibility rules out convex functions even if quasi-concave). The profit is usually defined

$$
p F(K, L)-p_{k} K-w L,
$$

where $p_{t}$ is the price of the product, $p_{k}$ is the usage cost of capital and $w$ is the wage for the workers.

The unconstrained profit maximization gives

$$
\begin{align*}
\frac{\partial F}{\partial K} & =\frac{p_{k}}{p}  \tag{3.1}\\
\frac{\partial F}{\partial L} & =\frac{w}{p} \tag{3.2}
\end{align*}
$$

In words capital productivity must equal the real cost of capital and labor productivity must equal the real wage.

These necessary first order conditions turns out to be sufficient for profit maximization under the assumption of concavity of $F$.

Profit Function. As above we assume that the producer is price-taker and has at disposal a concave production function. He wants to maximize the profit, i.e. the difference between takings $p y$ and production costs $w x$. The technological constraint is respected

$$
\max _{(y, x)} p y-w x
$$

with $y=f(x)$. If for instance $f$ is a function of two factors $x_{1}, x_{2}$ the profit is given by

$$
\pi\left(x_{1}, x_{2}\right) \equiv p f\left(x_{1}, x_{2}\right)-w_{1} x_{1}-w_{2} x_{2}
$$

The maximal profit is said to be the profit function. This function depends only on the price vector.

The first order condition is

$$
p D_{x} f=w,
$$

where $D_{x} f$ is the gradient of $f$ and $w$ is a vector. If $f$ is a function of only one input the first order condition becomes

$$
\begin{equation*}
f^{\prime}(x)=\frac{w}{p} \tag{3.3}
\end{equation*}
$$

(compare with (3.2)). In words the marginal productivity must equal the real remuneration of factors. The second order condition we require, is the production function to be concave (if $f \in C^{2}$ the Hessian matrix is required
to be negative definite). As usual first and second order conditions ensure a correct maximization. In the case of functions of one factor $y=f(x)$ we get (3.3) and

$$
f^{\prime \prime}(x) \leq 0
$$

Profit maximization provides the optimal product supply and factor demands:

$$
\begin{aligned}
x^{*} & =x^{*}(p, w) \\
y^{*} & =f\left(x^{*}(p, w)\right)=y^{*}(p, w)
\end{aligned}
$$

To know the product supply and factor demands from the profit function a useful proposition is the Hotelling's lemma.

$$
\begin{aligned}
y(p, w) & =\partial \pi^{*} / \partial p \\
x_{i}(p, w) & =-\partial \pi^{*} / \partial w_{i}
\end{aligned}
$$

where $\pi^{*}$ is the profit function (maximal profit). The proof is an application of the envelope theorem ${ }^{2}$.

The profit function has the following properties. (i) it is non-decreasing in $p$ and non-increasing in $w$. (ii) It is homogeneous of degree one in $(p, w)$, (iii) convex in $(p, w)$, (iv) continuous in $(p, w)$. See among others Varian (1992).

### 3.4 Dynamic Behavior

The firm value $V_{t}$ is a sum of discounted future profits.

$$
\begin{aligned}
V_{t} & =\sum_{\tau=0}^{\infty}\left(\frac{1}{1+i}\right)^{\tau+1}\left[p_{t+\tau} Y_{t+\tau}-w_{t+\tau} L_{t+\tau}-p_{k, t+\tau} I_{t+\tau}\right] \\
I_{t+\tau} & \leq K_{t+1+\tau}-(1-\delta) K_{t+\tau}
\end{aligned}
$$

where $i$ is the market interest rate, $Y_{t+\tau}, L_{t+\tau}, I_{t+\tau}$ and $K_{t+\tau}$ are respectively the product, the labor services, the investment and the capital of the period $t+\tau . p_{t+\tau}, w_{t+\tau}, p_{k, t+\tau}$ are respectively the product price, the wage and the investment price. Eventually $\delta$ is the depreciation rate of capital.

[^4]We set the infinite horizon Lagrangian

$$
\begin{aligned}
\Lambda_{t} \equiv & \sum_{\tau=0}^{\infty}\left(\frac{1}{1+i}\right)^{\tau+1}\left[p_{t+\tau} Y_{t+\tau}-w_{t+\tau} L_{t+\tau}-p_{k, t+\tau} I_{t+\tau}\right] \\
& +\sum_{\tau=0}^{\infty} \lambda_{t+\tau}\left[K_{t+1+\tau}-(1-\delta) K_{t+\tau}-I_{t+\tau}\right]
\end{aligned}
$$

and we obtain the first order conditions

$$
\begin{aligned}
\frac{\partial \Lambda_{t}}{\partial K_{t+\tau}} & =\left(\frac{1}{1+i}\right)^{\tau+1} p_{t+\tau} \frac{\partial Y_{t+\tau}}{\partial K_{t+\tau}}+\lambda_{t+\tau-1}-\lambda_{t+\tau}(1-\delta)=0 \\
\frac{\partial \Lambda_{t}}{\partial I_{t+\tau}} & =-\left(\frac{1}{1+i}\right)^{\tau+1} p_{k, t+\tau}-\lambda_{t+\tau}=0 \\
\tau & =0, \ldots, \infty
\end{aligned}
$$

We eliminate the multipliers:

$$
\begin{aligned}
\lambda_{t+\tau} & =-\left(\frac{1}{1+i}\right)^{\tau+1} p_{k, t+\tau} \\
\left(\frac{1}{1+i}\right)^{\tau+1} p_{t+\tau} \frac{\partial Y_{t+\tau}}{\partial K_{t+\tau}} & =\left(\frac{1}{1+i}\right)^{\tau} p_{k, t+\tau-1}-(1-\delta)\left(\frac{1}{1+i}\right)^{\tau+1} p_{k, t+\tau}, \\
p_{t+\tau} \frac{\partial Y_{t+\tau}}{\partial K_{t+\tau}} & =(1+i) p_{k, t+\tau-1}-(1-\delta) p_{k, t+\tau}, \\
p_{t+\tau} \frac{\partial Y_{t+\tau}}{\partial K_{t+\tau}} & =\left[(1+i) \frac{p_{k, t+\tau-1}}{p_{k, t+\tau}}-(1-\delta)\right] p_{k, t+\tau}
\end{aligned}
$$

For every $t$ we get

$$
\begin{equation*}
p_{t} \frac{\partial Y_{t}}{\partial K_{t}}=\left[(1+i) \frac{p_{k, t-1}}{p_{k t}}-(1-\delta)\right] p_{k t} . \tag{3.4}
\end{equation*}
$$

The inflation factor on the capital market is

$$
1+\pi_{k t} \equiv \frac{p_{k t}}{p_{k, t-1}}
$$

We recall to mind the Fisher's formula

$$
(1+i)=(1+\pi)(1+r)
$$

where $i$ and $r$ are respectively the nominal and the real interest rate.
Equation (3.4) becomes

$$
\begin{align*}
p_{t} \frac{\partial Y_{t}}{\partial K_{t}} & =\left[\frac{1+i}{1+\pi_{k t}}-(1-\delta)\right] p_{k t} \\
& =\left[1+r_{t}-(1-\delta)\right] p_{k t} \\
& =\left(r_{t}+\delta\right) p_{k t} . \tag{3.5}
\end{align*}
$$

This formula generalizes (3.1) to a dynamic context.
In words firm value maximization (intertemporal profit maximization) requires that the marginal nominal productivity ( $L H S$ ) equal the nominal usage cost of capital (RHS).

### 3.5 Tobin's $q$

The previous model is augmented to take into account the adjustment costs, i.e. the further costs the firm sustains to adapt the production organization to new machines and in general long run inputs ${ }^{3}$.

We assume these costs to depend on the net investment and the capital level:

$$
A\left(I_{n t}, K_{t}\right)
$$

where

$$
I_{n t} \equiv I_{t}-\delta K_{t} .
$$

More precisely they are assumed to be homogenous of degree one.

$$
A\left(I_{n t}, K_{t}\right)=C\left(I_{n t} / K_{t}\right) K_{t},
$$

where the intensive function $C$ is specified as follows:

$$
\begin{aligned}
C(0) & =0 \\
C^{\prime}(0) & =0 \\
C^{\prime}(x) & >0 \text { for every } x>0, \\
C^{\prime \prime}(x) & >0 \text { for every } x \geq 0
\end{aligned}
$$

( $C$ is increasing and convex).

[^5]For example we choose

$$
\begin{aligned}
A\left(I_{t}, K_{t}\right) & =I_{n t}^{\alpha} K_{t}^{(1-\alpha)} \\
\alpha & >1
\end{aligned}
$$

Then

$$
C\left(I_{n t} / K_{t}\right)=\left(I_{n t} / K_{t}\right)^{\alpha} .
$$

The production is reduced by these costs:

$$
Y_{t}-C\left(I_{n t} / K_{t}\right) K_{t}
$$

where

$$
Y_{t} \equiv F\left(K_{t}, L_{t}\right)
$$

For the sake of simplicity we assume that the capital input and the output are the same good and therefore have the same price which is normalized to one.

$$
\begin{aligned}
p_{t} & =p_{k t} \equiv 1, \\
i & =r
\end{aligned}
$$

(no inflation).
In real terms the firm value becomes

$$
\begin{aligned}
V_{t} & =\sum_{\tau=0}^{\infty}\left(\frac{1}{1+r}\right)^{\tau+1}\left[Y_{t+\tau}-C\left(I_{n, t+\tau} / K_{t+\tau}\right) K_{t+\tau}-w_{t+\tau} L_{t+\tau}-I_{n, t+\tau}-\delta K_{t+\tau}\right] \\
I_{n, t+\tau} & \leq K_{t+1+\tau}-K_{t+\tau}, \\
\tau & =0,1, \ldots,
\end{aligned}
$$

because the gross investment is equal to $I_{t}=I_{n t}+\delta K_{t}$.
The infinite horizon Lagrangian is

$$
\begin{aligned}
\Lambda_{t} \equiv & \sum_{\tau=0}^{\infty}\left(\frac{1}{1+r}\right)^{\tau+1}\left[Y_{t+\tau}-C\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right) K_{t+\tau}-w_{t+\tau} L_{t+\tau}-I_{n t}-\delta K_{t}\right] \\
& +\sum_{\tau=0}^{\infty} \lambda_{t+\tau}\left[K_{t+1+\tau}-K_{t+\tau}-I_{n, t+\tau}\right] .
\end{aligned}
$$

We find the first order conditions.

$$
\begin{aligned}
0= & \frac{\partial \Lambda_{t}}{\partial K_{t+\tau}}=\left(\frac{1}{1+r}\right)^{\tau+1}\left[\frac{\partial Y_{t+\tau}}{\partial K_{t+\tau}}+C^{\prime}\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right) \frac{I_{n, t+\tau}}{K_{t+\tau}}-C\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right)-\delta\right] \\
& +\lambda_{t+\tau-1}-\lambda_{t+\tau}, \\
0= & \frac{\partial \Lambda_{t}}{\partial I_{n, t+\tau}}=-\left(\frac{1}{1+r}\right)^{\tau+1}\left[C^{\prime}\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right)+1\right]-\lambda_{t+\tau}, \\
\tau= & 0, \ldots, \infty .
\end{aligned}
$$

Hence

$$
\begin{align*}
\lambda_{t+\tau}= & -\left(\frac{1}{1+r}\right)^{\tau+1}\left[C^{\prime}\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right)+1\right],  \tag{3.6}\\
\lambda_{t+\tau-1}= & \lambda_{t+\tau}-\left(\frac{1}{1+r}\right)^{\tau+1} \\
& {\left[\frac{\partial Y_{t+\tau}}{\partial K_{t+\tau}}+C^{\prime}\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right) \frac{I_{n, t+\tau}}{K_{t+\tau}}-C\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right)-\delta\right] . } \tag{3.7}
\end{align*}
$$

Tobin's $q$ is defined now as follows

$$
q_{t+\tau} \equiv-(1+r)^{\tau+1} \lambda_{t+\tau} .
$$

Proposition 2 The economic meaning of this multiplier redefinition is the following

$$
q_{t}=\sum_{\tau=1}^{\infty}\left(\frac{1}{1+r}\right)^{\tau}\left[\frac{\partial Y_{t+\tau}}{\partial K_{t+\tau}}-\delta+C^{\prime}\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right) \frac{I_{n, t+\tau}}{K_{t+\tau}}-C\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right)\right] .
$$

Tobin's $q$ is the marginal value of the firm with respect to the capital, i.e. the marginal value of a further unit of capital invested in the firm. In other words it is the discounted sum of all the future marginal net productivity.

## Proof.

By recurrence from (3.7)

$$
\begin{aligned}
\lambda_{t}= & \lambda_{t+1} \\
& -\left(\frac{1}{1+r}\right)^{2}\left[\frac{\partial Y_{t+1}}{\partial K_{t+1}}+C^{\prime}\left(\frac{I_{n, t+1}}{K_{t+1}}\right) \frac{I_{n, t+1}}{K_{t+1}}-C\left(\frac{I_{n, t+1}}{K_{t+1}}\right)-\delta\right]
\end{aligned}
$$

$$
\begin{aligned}
= & \lambda_{t+2} \\
& -\left(\frac{1}{1+r}\right)^{3}\left[\frac{\partial Y_{t+2}}{\partial K_{t+2}}+C^{\prime}\left(\frac{I_{n, t+2}}{K_{t+2}}\right) \frac{I_{n, t+2}}{K_{t+2}}-C\left(\frac{I_{n, t+2}}{K_{t+2}}\right)-\delta\right] \\
& -\left(\frac{1}{1+r}\right)^{2}\left[\frac{\partial Y_{t+1}}{\partial K_{t+1}}+C^{\prime}\left(\frac{I_{n, t+1}}{K_{t+1}}\right) \frac{I_{n, t+1}}{K_{t+1}}-C\left(\frac{I_{n, t+1}}{K_{t+1}}\right)-\delta\right] \\
= & \lambda_{t+3} \\
& -\left(\frac{1}{1+r}\right)^{4}\left[\frac{\partial Y_{t+3}}{\partial K_{t+3}}+C^{\prime}\left(\frac{I_{n, t+3}}{K_{t+3}}\right) \frac{I_{n, t+3}}{K_{t+3}}-C\left(\frac{I_{n, t+3}}{K_{t+3}}\right)-\delta\right] \\
& -\left(\frac{1}{1+r}\right)^{3}\left[\frac{\partial Y_{t+2}}{\partial K_{t+2}}+C^{\prime}\left(\frac{I_{n, t+2}}{K_{t+2}}\right) \frac{I_{n, t+2}}{K_{t+2}}-C\left(\frac{I_{n, t+2}}{K_{t+2}}\right)-\delta\right] \\
& -\left(\frac{1}{1+r}\right)^{2}\left[\frac{\partial Y_{t+1}}{\partial K_{t+1}}+C^{\prime}\left(\frac{I_{n, t+1}}{K_{t+1}}\right) \frac{I_{n, t+1}}{K_{t+1}}-C\left(\frac{I_{n, t+1}}{K_{t+1}}\right)-\delta\right] .
\end{aligned}
$$

We get

$$
\begin{aligned}
\lambda_{t}= & -\sum_{\tau=1}^{\infty}\left(\frac{1}{1+r}\right)^{\tau+1}\left[\frac{\partial Y_{t+\tau}}{\partial K_{t+\tau}}+C^{\prime}\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right) \frac{I_{n, t+\tau}}{K_{t+\tau}}-C\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right)-\delta\right] \\
& +\lim _{\tau \rightarrow \infty} \lambda_{t+\tau} .
\end{aligned}
$$

The definition of Tobin's $q$ entails:

$$
\begin{aligned}
q_{t}= & -(1+r) \lambda_{t} \\
= & \sum_{\tau=1}^{\infty}\left(\frac{1}{1+r}\right)^{\tau}\left[\frac{\partial Y_{t+\tau}}{\partial K_{t+\tau}}+C^{\prime}\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right) \frac{I_{n, t+\tau}}{K_{t+\tau}}-C\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right)-\delta\right] \\
& -(1+r) \lim _{T \rightarrow \infty} \lambda_{t+T .} .
\end{aligned}
$$

By definition we observe that $\lambda_{t+T} \equiv-q_{t+T} /(1+r)^{T+1}$. Thereby

$$
-(1+r) \lim _{T \rightarrow \infty} \lambda_{t+T}=\lim _{T \rightarrow \infty}(1+r)^{-T} q_{t+T}
$$

and

$$
\begin{aligned}
q_{t}= & \sum_{\tau=1}^{\infty}\left(\frac{1}{1+r}\right)^{\tau}\left[\frac{\partial Y_{t+\tau}}{\partial K_{t+\tau}}-\delta+C^{\prime}\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right) \frac{I_{n, t+\tau}}{K_{t+\tau}}-C\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right)\right] \\
& +\lim _{T \rightarrow \infty}(1+r)^{-T} q_{t+T} .
\end{aligned}
$$

The (necessary) transversality condition for optimization (absence of bubbles) requires

$$
\lim _{T \rightarrow \infty}(1+r)^{-T} q_{t+T}=0
$$

The formula of proposition follows.
From equation (3.6) we obtain

$$
q_{t+\tau} \equiv-(1+r)^{\tau+1} \lambda_{t+\tau}=1+C^{\prime}\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right)
$$

and in particular

$$
\begin{aligned}
q_{t} & =\sum_{\tau=1}^{\infty}\left(\frac{1}{1+r}\right)^{\tau}\left[\frac{\partial Y_{t+\tau}}{\partial K_{t+\tau}}-\delta+C^{\prime}\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right) \frac{I_{n, t+\tau}}{K_{t+\tau}}-C\left(\frac{I_{n, t+\tau}}{K_{t+\tau}}\right)\right] \\
& =1+C^{\prime}\left(\frac{I_{n t}}{K_{t}}\right) .
\end{aligned}
$$

The marginal value of the firm (the marginal intertemporal return of the investment) at the optimum equals the marginal cost of the investment (the price of the capital good (1) plus the marginal adjustment cost per unit of capital $\left.C^{\prime}\left(I_{n t} / K_{t}\right)\right)$.

Investment Dynamics. Let the investment be defined as a function of $q$.

$$
\begin{align*}
q_{t} & =1+C^{\prime}\left(I_{n t} / K_{t}\right), \\
C^{\prime}\left(I_{n t} / K_{t}\right) & =q_{t}-1, \\
I_{n t} / K_{t} & =C^{\prime-1}\left(q_{t}-1\right), \\
I_{n t} & =\varphi\left(q_{t}-1\right) K_{t}  \tag{3.8}\\
K_{t+1} & =K_{t}+I_{n t}=\left[1+\varphi\left(q_{t}-1\right)\right] K_{t} . \tag{3.9}
\end{align*}
$$

$\varphi$ is well defined because $C$ is strictly convex.
From (3.7) and (3.8) we obtain

$$
\begin{aligned}
& 0=\lambda_{t}-\lambda_{t+1}+\left(\frac{1}{1+r}\right)^{2}\left[\frac{\partial Y_{t+1}}{\partial K_{t+1}}-\delta+C^{\prime}\left(\frac{I_{n, t+1}}{K_{t+1}}\right) \frac{I_{n, t+1}}{K_{t+1}}-C\left(\frac{I_{n, t+1}}{K_{t+1}}\right)\right], \\
& 0=-(1+r)^{2} \lambda_{t}+(1+r)^{2} \lambda_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
& -\left[\frac{\partial Y_{t+1}}{\partial K_{t+1}}-\delta+C^{\prime}\left(\frac{I_{n, t+1}}{K_{t+1}}\right) \frac{I_{n, t+1}}{K_{t+1}}-C\left(\frac{I_{n, t+1}}{K_{t+1}}\right)\right] \\
= & (1+r) q_{t}-q_{t+1} \\
& -\left[\partial Y_{t+1} / \partial K_{t+1}-\delta+\left(q_{t+1}-1\right) \varphi\left(q_{t+1}-1\right)-C\left(\varphi\left(q_{t+1}-1\right)\right)\right] \\
= & (1+r) q_{t}-\partial Y_{t+1} / \partial K_{t+1}-\psi\left(q_{t+1}\right),
\end{aligned}
$$

where

$$
\psi\left(q_{t+1}\right) \equiv q_{t+1}+\left(q_{t+1}-1\right) \varphi\left(q_{t+1}-1\right)-C\left(\varphi\left(q_{t+1}-1\right)\right)-\delta .
$$

From (3.9) we have

$$
(1+r) q_{t}-\frac{\partial Y_{t+1}}{\partial K_{t+1}}\left(\left[1+\varphi\left(q_{t}-1\right)\right] K_{t}\right)-\psi\left(q_{t+1}\right)=0
$$

or more compactly

$$
G\left(K_{t}, q_{t}, q_{t+1}\right)=0 .
$$

The non-linear dynamic system gets the following form:

$$
\begin{align*}
K_{t+1} & =\left[1+\varphi\left(q_{t}-1\right)\right] K_{t}  \tag{3.10}\\
G\left(K_{t}, q_{t}, q_{t+1}\right) & =0 \tag{3.11}
\end{align*}
$$

We adopt now a useful geometrical technique to analyze the local dynamics. The reader is referred to the chapter 1 for more mathematical details.

The steady state is computed as follows

$$
\begin{aligned}
\varphi(q-1) & =0 \\
q & =1+C^{\prime}(0)=1
\end{aligned}
$$

under the assumption $C^{\prime}(0)=0$. Moreover

$$
\begin{aligned}
(1+r) q-\partial F / \partial K-\psi(q) & =0 \\
1+r-\partial F / \partial K-\psi(1) & =0
\end{aligned}
$$

But we observe that

$$
\psi(1)=1+(1-1) \varphi(1-1)-C(0)-\delta=1-\delta
$$

Therefore the steady state is given by

$$
\begin{align*}
q & =1  \tag{3.12}\\
\partial F / \partial K & =r+\delta \tag{3.13}
\end{align*}
$$

(the second condition looks like that appearing in the simple dynamic optimization without adjustment costs: see (3.5))
(3.12) and (3.13) describe long run dynamics.

Local information about the non linear dynamics We want to linearize system (3.10-3.11) around the steady state

$$
\left[\begin{array}{c}
K_{t+1} \\
q_{t+1}
\end{array}\right]-\left[\begin{array}{c}
K^{*} \\
q^{*}
\end{array}\right] \approx\left[\begin{array}{cc}
\partial K_{t+1} / \partial K_{t} & \partial K_{t+1} / \partial q_{t} \\
\partial q_{t+1} / \partial K_{t} & \partial q_{t+1} / \partial q_{t}
\end{array}\right]_{K, q}\left[\left[\begin{array}{c}
K_{t} \\
q_{t}
\end{array}\right]-\left[\begin{array}{c}
K^{*} \\
q^{*}
\end{array}\right]\right] .
$$

All the local information is contained in the Jacobian matrix evaluated at the steady state:

$$
J=\left[\begin{array}{cc}
\partial K_{t+1} / \partial K_{t} & \partial K_{t+1} / \partial q_{t} \\
\partial q_{t+1} / \partial K_{t} & \partial q_{t+1} / \partial q_{t}
\end{array}\right]_{K, q} .
$$

We compute its components:

$$
\begin{aligned}
\left.\frac{\partial K_{t+1}}{\partial K_{t}}\right|_{K, q} & =1+\varphi(q-1)=1, \\
\left.\frac{\partial K_{t+1}}{\partial q_{t}}\right|_{K, q} & =\varphi^{\prime}(q-1) K=\varphi^{\prime}(0) K, \\
\left.\frac{\partial q_{t+1}}{\partial K_{t}}\right|_{K, q} & =-\left.\frac{\partial G / \partial K_{t}}{\partial G / \partial q_{t+1}}\right|_{K, q}=-\frac{-\left(\partial^{2} Y / \partial K^{2}\right)[1+\varphi(q-1)]}{-\psi^{\prime}(q)}=-\frac{\partial^{2} Y}{\partial K^{2}}, \\
\left.\frac{\partial q_{t+1}}{\partial q_{t}}\right|_{K, q} & =-\left.\frac{\partial G / \partial q_{t}}{\partial G / \partial q_{t+1}}\right|_{K, q}=-\frac{1+r-\left(\partial^{2} Y / \partial K^{2}\right) \varphi^{\prime}(q-1) K}{-\psi^{\prime}(q)} \\
& =1+r-\frac{\partial^{2} Y}{\partial K^{2}} \varphi^{\prime}(0) K,
\end{aligned}
$$

because at the steady state $q=1$ and $\varphi(q-1)=0$, and as

$$
\psi\left(q_{t+1}\right) \equiv q_{t+1}+\left(q_{t+1}-1\right) \varphi\left(q_{t+1}-1\right)-C\left(\varphi\left(q_{t+1}-1\right)\right)-\delta,
$$

then

$$
\psi^{\prime}(q)=1+\varphi(q-1)-(q-1) \varphi^{\prime}(q-1)-C^{\prime}(\varphi(q-1)) \varphi^{\prime}(q-1)=1
$$

(we notice that $C^{\prime}(0)=0$ ).
The Jacobian matrix becomes

$$
J=\left[\begin{array}{cc}
1 & \varphi^{\prime}(0) K  \tag{3.14}\\
-\left(\partial^{2} Y / \partial K^{2}\right) & 1+r-\left(\partial^{2} Y / \partial K^{2}\right) \varphi^{\prime}(0) K
\end{array}\right]_{K, q} .
$$

Therefore the trace and the determinant are given by

$$
\begin{aligned}
& T=1+1+r-\left(\partial^{2} Y / \partial K^{2}\right) \varphi^{\prime}(0) K \\
& D=1+r-\left(\partial^{2} Y / \partial K^{2}\right) \varphi^{\prime}(0) K+\left(\partial^{2} Y / \partial K^{2}\right) \varphi^{\prime}(0) K=1+r .
\end{aligned}
$$

More precisely

$$
T=1+D-\left(\partial^{2} Y / \partial K^{2}\right) \varphi^{\prime}(0) K
$$

where

$$
\varphi^{\prime}(0)=\frac{1}{C^{\prime \prime}\left(I_{n} / K\right)}
$$

because

$$
\varphi^{\prime}\left(q_{t}-1\right) \equiv C^{\prime-1}\left(q_{t}-1\right)
$$

Hence

$$
\begin{aligned}
D & =1+r \\
D & =T-1+\frac{\partial^{2} Y / \partial K^{2}}{C^{\prime \prime}} K
\end{aligned}
$$

As by assumption

$$
\begin{aligned}
\partial^{2} Y / \partial K^{2} & <0 \\
C^{\prime \prime} & >0
\end{aligned}
$$

we get

$$
\begin{aligned}
& D>1 \\
& D<T-1
\end{aligned}
$$

According to figure 5 we are in a saddle region.
A saddle configuration in two-dimensional dynamics means that the stable manifold (the union of the converging paths to the stationary state) is one-
dimensional.


Figure 16. Phase diagram.
As the capital is a pre-determined variable, while the marginal value of the firm $q_{t}$ is not, the intersection between the vertical line for a given $K_{t}$ and this stable manifold in the plane $\left\{\left(K_{t}, q_{t}\right)\right\}$ is a unique point defining a unique trajectory.

It is possible to show that the converging path is downward-sloped.
Consider the Jacobian $J$ in (3.14) and set $j_{11}=1$ and $j_{12}=\varphi^{\prime}(0) K=$ $K / C^{\prime \prime}$. Without loss of generality let $\lambda_{1}$ be the stable eigenvalue and $\lambda_{2}$ be the explosive eigenvalue. If $\Lambda$ is the Jordan canonical form and $V \equiv\left[v_{1}, v_{2}\right]$ is the transformation matrix, where $v_{1}$ and $v_{2}$ are the eigenvectors, convergence requirement to lie on the saddle path is

$$
\lim _{t \rightarrow \infty}\left(V \Lambda^{t} V^{-1}\left[\begin{array}{c}
K_{0}-K \\
q_{0}-q
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
0
\end{array}\right] .
$$

In other words the second component of the vector

$$
V^{-1}\left[\begin{array}{c}
K_{0}-K \\
q_{0}-q
\end{array}\right]
$$

must equal zero, or equivalently

$$
\begin{aligned}
q_{0}-q & =\frac{\lambda_{1}-j_{11}}{j_{12}}\left(K_{0}-K\right) \\
& =\frac{\lambda_{1}-1}{K / C^{\prime \prime}}\left(K_{0}-K\right)
\end{aligned}
$$

The slope of the linearized saddle path is given by

$$
\frac{\lambda_{1}-1}{K / C^{\prime \prime}}<0
$$

because $\lambda_{1} \in(0,1)$ and $C^{\prime \prime}>0$.
If $K_{t}<K^{*}$, the steady state, then the marginal productivity of the capital is greater than the interest rate. Therefore the producer wants to increase the investment. He faces the adjustment costs and the investment is implemented over more periods. As the marginal productivity of the capital is higher than its stationary state, the marginal value $q_{t}$ is higher than its steady state $q^{*}$. However the further capital accumulation entailed by this mechanism implies a reduction of the marginal productivity of capital and finally a progressive contraction of the marginal value $q$ towards its stationary level $q^{*}$.

### 3.6 The Neutrality Theorem of ModiglianiMiller

We roughly follow Modigliani and Miller (1961).
We assume that there are three ways of financing investment.

$$
\begin{aligned}
p_{k t} I_{t}= & B_{t+1}-B_{t} \\
& +p_{a t}\left(A_{t+1}-A_{t}\right) \\
& +\left(p_{t} Y_{t}-w_{t} N_{t}-i B_{t}\right)-d_{t} A_{t},
\end{aligned}
$$

where $B_{t+1}-B_{t}$ is the loan the firm takes out, $p_{a t}\left(A_{t+1}-A_{t}\right)$ is the nominal increase of capital on the stock market, and $\left(p_{t} Y_{t}-w_{t} N_{t}-i B_{t}\right)-d_{t} A_{t}$ is the self-financing.

Rearranging the previous expression we get

$$
\begin{equation*}
B_{t+1}+p_{a t} A_{t+1}=(1+i) B_{t}+\left(p_{t}+d_{t}\right) A_{t}-\left(p_{t} Y_{t}-w_{t} N_{t}-p_{k t} I_{t}\right) . \tag{3.15}
\end{equation*}
$$

An equilibrium condition under the assumption of perfect markets is

$$
\begin{equation*}
1+i=\frac{p_{a, t+1}+d_{t+1}}{p_{a t}} . \tag{3.16}
\end{equation*}
$$

That is a no-arbitrage condition: the gross return on bonds must equal the gross return on stocks.

We obtain

$$
p_{a, t+1}+d_{t+1}=(1+i) p_{a t}
$$

and from (3.15)

$$
\begin{align*}
B_{t+1}+p_{a t} A_{t+1}= & (1+i) B_{t}+(1+i) p_{a, t-1} A_{t} \\
& -\left(p_{t} Y_{t}-w_{t} N_{t}-p_{k t} I_{t}\right) \\
= & (1+i)\left(B_{t}+p_{a, t-1} A_{t}\right) \\
& -\left(p_{t} Y_{t}-w_{t} N_{t}-p_{k t} I_{t}\right) . \tag{3.17}
\end{align*}
$$

Let

$$
V_{t} \equiv B_{t}+p_{a, t-1} A_{t}
$$

be the gross firm value. Therefore (3.17) gives the law of motion for this value

$$
V_{t+1}=(1+i) V_{t}-\left(p_{t} Y_{t}-w_{t} N_{t}-p_{k t} I_{t}\right) .
$$

Solving towards the future we get

$$
\begin{aligned}
V_{t}= & \frac{1}{1+i} V_{t+1}+\frac{1}{1+i}\left(p_{t} Y_{t}-w_{t} N_{t}-p_{k t} I_{t}\right) \\
= & \frac{1}{1+i}\left[\frac{1}{1+i} V_{t+2}+\frac{1}{1+i}\left(p_{t+1} Y_{t+1}-w_{t+1} N_{t+1}-p_{k, t+1} I_{t+1}\right)\right] \\
& +\frac{1}{1+i}\left(p_{t} Y_{t}-w_{t} N_{t}-p_{k t} I_{t}\right) \\
= & \left(\frac{1}{1+i}\right)^{2} V_{t+2}+\left(\frac{1}{1+i}\right)^{2}\left(p_{t+1} Y_{t+1}-w_{t+1} N_{t+1}-p_{k, t+1} I_{t+1}\right) \\
& +\frac{1}{1+i}\left(p_{t} Y_{t}-w_{t} N_{t}-p_{k t} I_{t}\right) \\
= & \lim _{T \rightarrow \infty}\left(\frac{1}{1+i}\right)^{T-t} V_{T} \\
& +\sum_{\tau=0}\left(\frac{1}{1+i}\right)^{\tau+1}\left(p_{t+\tau} Y_{t+\tau}-w_{t+\tau} N_{t+\tau}-p_{k, t+\tau} I_{t+\tau}\right) .
\end{aligned}
$$

Under the assumption of no speculative bubbles

$$
\lim _{T \rightarrow \infty}\left(\frac{1}{1+i}\right)^{T-t} V_{T}=0
$$

we have

$$
V_{t}=\sum_{\tau=0}\left(\frac{1}{1+i}\right)^{\tau+1}\left(p_{t+\tau} Y_{t+\tau}-w_{t+\tau} N_{t+\tau}-p_{k, t+\tau} I_{t+\tau}\right)
$$

as seen in the section on the intertemporal profit maximization.
Similar computations provide from (3.16) the price of the assets:

$$
\begin{aligned}
p_{a t} & =\frac{1}{1+i} p_{a, t+1}+\frac{1}{1+i} d_{t+1} \\
& =\frac{1}{1+i}\left(\frac{1}{1+i} p_{a, t+2}+\frac{1}{1+i} d_{t+2}\right)+\frac{1}{1+i} d_{t+1} \\
& =\left(\frac{1}{1+i}\right)^{2} p_{a, t+2}+\left(\frac{1}{1+i}\right)^{2} d_{t+2}+\frac{1}{1+i} d_{t+1} \\
& =\lim _{T \rightarrow \infty}\left(\frac{1}{1+i}\right)^{T-t} p_{a T}+\sum_{\tau=1}\left(\frac{1}{1+i}\right)^{\tau} d_{t+\tau}
\end{aligned}
$$

The no-bubbles condition entails that

$$
\lim _{T \rightarrow \infty}\left(\frac{1}{1+i}\right)^{T-t} p_{a T}=0
$$

and then

$$
p_{a t}=\sum_{\tau=1}\left(\frac{1}{1+i}\right)^{\tau} d_{t+\tau} .
$$

Theorem 3 (Modigliani-Miller) If the financial markets are perfect, the financial structure of the firm does not matter for real choices.

Heuristic hint. Consider the equations

$$
\begin{align*}
V_{t} & \equiv B_{t}+p_{a, t-1} A_{t}  \tag{3.18}\\
p_{a t} & =\sum_{\tau=1}\left(\frac{1}{1+i}\right)^{\tau} d_{t+\tau} \tag{3.19}
\end{align*}
$$

There are two situations.
(i) $A_{t}$ increases. Then the dividend per stock $d_{t+\tau}$ decreases for every $\tau>0$ and the price $p_{a t}$ as well according to (3.19). It is possible to show
that this price reduction will compensate exactly the rise of $A_{t}$ in (3.18) and that $V_{t}$ will remain unchanged.
(ii) $B_{t}$ increases. The payment of more interests will reduce the dividend $d_{t+\tau}$ for every $\tau>0$ and the price $p_{a t}$ as well according to (3.19). It is possible to show that this price reduction will compensate exactly the rise of $B_{t}$ in (3.18) and that $V_{t}$ will remain unchanged.

Therefore the financial structure does not affect the value of the firm $V_{t}$.

## Part II

## General Equilibrium

## Chapter 4

## Exogenous Saving

### 4.1 Growth Accounting

Which is a good measure of welfare? And of country productivity?

We deal with a big problem in measuring growth.

First, the income (GDP) per capita is a rough measure of welfare, even if a positive and significative correlation with other measures of life quality such as life expectancy, is obtained from data.

Second, international comparisons are biased by the exchange rate volatility. An exchange rate based on the purchasing power parity is a good indicator to convert and compare GDPs per capita between countries.

### 4.1.1 Statistics on growth

We follow Jones (1998).

|  | $G D P / N: 1990$ | $G D P / L: 1990$ | $g: 1960 / 90$ |
| :--- | :--- | :--- | :--- |
|  |  |  |  |
| Rich countries |  |  |  |
| USA | 18073 | 36810 | 1.4 |
| West Germany | 14331 | 29488 | 2.5 |
| Japan | 14317 | 22602 | 5.0 |
| France | 13896 | 30340 | 2.7 |
|  |  |  |  |
| Poor countries |  |  |  |
| China | 1324 | 2189 | 2.4 |
| India | 1262 | 3230 | 2.0 |
| Zimbabwe | 1181 | 2435 | 0.2 |
| Uganda | 554 | 1142 | -0.2 |
|  |  |  |  |
| Growth miracles |  |  |  |
| Hong Kong | 14854 | 22835 | 5.7 |
| Singapore | 11698 | 24344 | 5.3 |
| Taiwan | 8067 | 18418 | 5.7 |
| South Corea | 6665 | 16003 | 6.0 |

The $G D P$ data are in 1985 dollars.

$$
g \equiv \ln (G D P / L)_{1}-\ln (G D P / L)_{0} \approx \frac{(G D P / L)_{1}-(G D P / L)_{0}}{(G D P / L)_{1}}
$$

### 4.1.2 Facts

We still follow Jones (1998).
(1) Large variation in per capita income across economies.
(2) Large variation of economic growth rates across countries.
(3) Growth rates are not constant over time.
(4) Countries' relative positions in the world distribution of per capita income varies over time.
(5) The real rate of return to capital show no trend in the US over the last century (Kaldor, 1958).
(6) The capital and labor shares on total income show no trend in the US over the last century (Kaldor, 1958).
(7) The growth rate of output per person has been positive and relatively constant in the US over the last century (Kaldor, 1958).
(8) Growth in output and growth in the volume of international trade are closely related.
(9) Both skilled and unskilled workers tend to migrate from poor to rich countries or regions.

### 4.1.3 Comments

(1) Large variation in per capita income across economies.

To compare countries' GDP the purchasing power parity exchange rate is required. How much does the same representative bundle cost in terms of yens or dollars?

Consider the table.
The representative worker in Uganda must work a month and a half to earn what the typical worker in United States earns in a day.

In 1988 half of the world population lived in countries with less than $10 \%$ of the U.S. GDP per worker.

China and India account for nearly $40 \%$ of the world population and had a GDP per worker less than one-tenth that of the United States.

The newly industrializing countries (NICs) are Hong Kong, Singapore, Taiwan, and South Corea. By 1990 Hong Kong had a per capita GDP $(G D P / N)$ close to that of West Germany. However the GDP per worker is relatively smaller because of the higher labor force participation $(L / N)$.

The success of the NICs depends on a trade policy based on the export instead of on the substitution of import with domestic productions as in India and Latin America.
(2) Large variation of economic growth rates across countries.

The poorest countries of the world exhibited varied growth performance. China and India grew faster than the United States from 1960 to 1990, but their growth rates were less than half those of the NICs.
(3) Growth rates are not constant over time.

In the United States and in many of the poorest countries of the world growth rates have not changed much over the last century. On the other hand, growth rates have increased dramatically in countries such as Japan
and NICs. According to several accounts China's annual growth rate has been nearly $10 \%$ in recent years.
(4) Countries' relative positions in the world distribution of per capita income varies over time.
(5) The real rate of return to capital show no trend in the US over the last century (Kaldor, 1958).
(6) The capital and labor shares on total income show no trend in the US over the last century (Kaldor, 1958).

For the United States, one can calculate labor share of GDP by looking at wage and salary payments and compensation for the self-employed as a share of GDP. These calculations reveal that the labor share has been relatively constant over time, at a value of around 0.7.
(7) The growth rate of output per person has been positive and relatively constant in the US over the last century (Kaldor, 1958).
(8) Growth in output and growth in the volume of international trade are closely related.
(9) Both skilled and unskilled workers tend to migrate from poor to rich countries or regions.

In terms of skilled labor, this raises an interesting puzzle. Presumably skilled labor is scarce in developing countries and simple theories predict that factor returns are highest where factors are scarce. Why, then, doesn't skilled labor migrate from the United States to Sub-Saharan Africa?

### 4.2 The Solow Model

After the break of Second World War several western countries experienced increasing growth rates. A new hope for growth arose for under-developed countries too. Fifties were years of unconstrained optimism. The positive view was reflected also by economic thought. Economic policy was largely inspired by Keynesian ideas about welfare state virtues. The models of Solow (1956) and Swan (1956) are typical intellectual outcomes of the period. These seminal contributions constitute the benchmark for all the subsequent growth theories. They remain the simplest way to describe capital accumulation and growth and claim the catch-up with the rich countries by the poor. In these models preferences are very simply specified: the saving rate is exogenously fixed. Thereby there is a trivial possibility of oversaving and in general dynamic inefficiency, if this rate does not maximize the stationary
consumption. Growth is exogenous too, because it is driven by exogenous laws such as a demographic evolution or an exogenous technical progress.

We enter now the formal details of a discrete time version of Solow (1956).

$$
\begin{aligned}
K_{t+1} & =(1-\delta) K_{t}+\left(Y_{t}-C_{t}\right) \\
& =(1-\delta) K_{t}+S_{t} \\
& =(1-\delta) K_{t}+s Y_{t} \\
& =(1-\delta) K_{t}+s F\left(K_{t}, N_{t}\right)
\end{aligned}
$$

We normalize by $N_{t}$.

$$
\begin{aligned}
\frac{K_{t+1}}{N_{t}} & =(1-\delta) \frac{K_{t}}{N_{t}}+s \frac{F\left(K_{t}, N_{t}\right)}{N_{t}} \\
\frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_{t}} & =(1-\delta) \frac{K_{t}}{N_{t}}+s F\left(\frac{K_{t}}{N_{t}}, 1\right)
\end{aligned}
$$

Let

$$
\begin{aligned}
k_{t} & \equiv \frac{K_{t}}{N_{t}} \\
f\left(k_{t}\right) & \equiv F\left(\frac{K_{t}}{N_{t}}, 1\right)
\end{aligned}
$$

and we obtain

$$
k_{t+1}(1+n)=(1-\delta) k_{t}+s f\left(k_{t}\right)
$$

The law of motion is

$$
k_{t+1}=\frac{1-\delta}{1+n} k_{t}+\frac{s}{1+n} f\left(k_{t}\right)
$$

Usual conditions

$$
\begin{align*}
f^{\prime} & >0  \tag{4.1}\\
f^{\prime \prime} & <0 \tag{4.2}
\end{align*}
$$

Inada conditions (sufficient conditions for the existence of at least one nontrivial steady state).

$$
\begin{aligned}
f(0) & =0 \\
f^{\prime}(0) & =\infty \\
f^{\prime}(\infty) & =0
\end{aligned}
$$

Steady States. Trivial steady state:

$$
k=0
$$

Non-trivial steady state.

$$
\begin{aligned}
k & =\frac{1-\delta}{1+n} k+\frac{s}{1+n} f(k) \\
1 & =\frac{1-\delta}{1+n}+\frac{s}{1+n} \frac{f(k)}{k}
\end{aligned}
$$

Let us define

$$
g(k) \equiv \frac{f(k)}{k}
$$

Observe that

$$
g^{\prime}<0
$$

(prove it by means of a graphic). Therefore

$$
\begin{align*}
\frac{s}{1+n} g(k) & =1-\frac{1-\delta}{1+n}=\frac{\delta+n}{1+n} \\
g(k) & =\frac{\delta+n}{s} \tag{4.3}
\end{align*}
$$

$g$ is monotonic, then invertible:

$$
k=g^{-1}\left(\frac{\delta+n}{s}\right)
$$

Comparative statics:

$$
\begin{align*}
\frac{\partial k}{\partial(\delta+n)} & =\frac{\partial g^{-1}((\delta+n) / s)}{\partial(\delta+n)}=\frac{1}{g^{\prime}(k)} \frac{1}{s}=\frac{1}{s g^{\prime}\left(g^{-1}((\delta+n) / s)\right)}  \tag{4.4}\\
\frac{\partial k}{\partial s} & =\frac{\partial g^{-1}((\delta+n) / s)}{\partial s} \\
& =\frac{1}{g^{\prime}(k)}\left(-\frac{\delta+n}{s^{2}}\right)=-\frac{\delta+n}{s^{2} g^{\prime}\left(g^{-1}((\delta+n) / s)\right)} \tag{4.5}
\end{align*}
$$

Provide a graphic proof.
The stationary production is given by

$$
y=f(k)=f\left(g^{-1}\left(\frac{\delta+n}{s}\right)\right)
$$

while the stationary consumption is provided by

$$
\begin{equation*}
c=(1-s) y=(1-s) f\left(g^{-1}\left(\frac{\delta+n}{s}\right)\right) \tag{4.6}
\end{equation*}
$$

Global Dynamics. Reconsider the law of motion.

$$
k_{t+1}=\frac{1-\delta}{1+n} k_{t}+\frac{s}{1+n} f\left(k_{t}\right) \equiv \varphi\left(k_{t}\right)
$$

Clearly from (4.1) et (4.2)

$$
\begin{aligned}
\varphi^{\prime} & >0 \\
\varphi^{\prime \prime} & <0
\end{aligned}
$$

The Inada conditions entail

$$
\begin{aligned}
\varphi(0) & =0 \\
\varphi^{\prime}(0) & =\infty \\
\varphi^{\prime}(\infty) & =\frac{1-\delta}{1+n}<1
\end{aligned}
$$

Plot the function $\varphi$ and notice that it converges from below to a slanting asymptote with slope $(1-\delta) /(1+n)<1$ and positive intercept. Plot also in the plane $(k, \varphi)$ the bisector of the first orthant. The abscissas of the two intersections are respectively the trivial steady state 0 and the non-trivial $k>0$. Plot the global dynamics and the converging path.

We observe that if the initial condition $k_{0} \in(0, k)$, then

$$
\begin{aligned}
k_{0} & <\varphi\left(k_{0}\right)=k_{1} \\
k_{t} & <\varphi\left(k_{t}\right)=k_{t+1}
\end{aligned}
$$

and the capital-labor ratio increases.
If $k_{0} \in(k,+\infty)$, then

$$
\begin{aligned}
k_{0} & >\varphi\left(k_{0}\right)=k_{1} \\
k_{t} & >\varphi\left(k_{t}\right)=k_{t+1}
\end{aligned}
$$

and the capital-labor ratio decreases.
Thus the stationary state $k$ is globally stable for $k_{0}>0$.
Balanced Growth. At the steady state we get

$$
\frac{K_{t}}{N_{t}}=\frac{K_{t+1}}{N_{t+1}}=k
$$

Then

$$
\frac{K_{t+1}}{K_{t}}=\frac{N_{t+1}}{N_{t}}=1+n
$$

The growth factor equals the population growth factor, which is exogenously given.

Moreover

$$
\begin{aligned}
\frac{Y_{t+1}}{Y_{t}} & =\frac{F\left(K_{t+1}, N_{t+1}\right)}{F\left(K_{t}, N_{t}\right)}=\frac{N_{t+1} f(k)}{N_{t} f(k)}=1+n \\
\frac{C_{t+1}}{C_{t}} & =\frac{(1-s) Y_{t+1}}{(1-s) Y_{t}}=1+n
\end{aligned}
$$

Then the growth is balanced.
Local Dynamics. A similar conclusion is obtained by linearizing the dynamics around the steady state. Dynamics are one-dimensional. The stability condition requires the unique eigenvalue to have modulus less than one:

$$
|\lambda|=\left|\varphi^{\prime}(k)\right|=\left|\varphi^{\prime}\left(g^{-1}\left(\frac{\delta+n}{s}\right)\right)\right|<1
$$

Golden Rule. We are interested in founding the saving rate which maximizes the stationary consumption (the consumption per capita is a rough measure of human welfare).

From (4.6) the stationary consumption is

$$
c=(1-s) f\left(g^{-1}\left(\frac{\delta+n}{s}\right)\right)
$$

We want to find

$$
s^{*} \equiv \arg \max _{s}(1-s) f\left(g^{-1}\left(\frac{\delta+n}{s}\right)\right)
$$

Concavity of $f$ ensures the second order conditions to be satisfied. Therefore we have

$$
\begin{aligned}
\frac{\partial}{\partial s}(1-s) f\left(g^{-1}\left(\frac{\delta+n}{s}\right)\right) & =0 \\
-f(k)+(1-s) f^{\prime}(k)\left[\left(g^{-1}\right)^{\prime}\left(\frac{\delta+n}{s}\right)\right]\left(-\frac{\delta+n}{s^{2}}\right) & =0
\end{aligned}
$$

$$
\begin{aligned}
& -f(k)-(1-s) f^{\prime}(k) \frac{1}{g^{\prime}(k)} \frac{\delta+n}{s^{2}}=0 \\
& -f(k)-(1-s) f^{\prime}(k) \frac{1}{g^{\prime}(k)} \frac{\delta+n}{s^{2}}=0
\end{aligned}
$$

where

$$
g^{\prime}(k)=\left[\frac{f(k)}{k}\right]^{\prime}=\frac{f^{\prime}(k) k-f(k)}{k^{2}}
$$

Therefore

$$
\begin{aligned}
-f(k)-(1-s) f^{\prime}(k) \frac{k^{2}}{f^{\prime}(k) k-f(k)} \frac{\delta+n}{s^{2}} & =0 \\
(1-s) f^{\prime}(k) \frac{k^{2}}{f(k)-f^{\prime}(k) k} \frac{\delta+n}{s^{2}} & =f(k) \\
(1-s) \frac{k}{f(k) /\left[f^{\prime}(k) k\right]-1} \frac{\delta+n}{s^{2}} & =f(k) \\
(1-s) \frac{k}{1 / \varepsilon-1} \frac{\delta+n}{s^{2}} & =f(k)
\end{aligned}
$$

where

$$
\varepsilon \equiv \frac{f^{\prime}(k) k}{f(k)}
$$

is the elasticity of production function with respect to capital, i.e. in economic terms, as the production function is homogeneous of degree 1 , the capital share on total income.

We have

$$
\begin{aligned}
(1-s) \frac{1}{1 / \varepsilon-1} \frac{\delta+n}{s^{2}} & =\frac{f(k)}{k} \\
\frac{1-s}{s} \frac{\varepsilon}{1-\varepsilon} \frac{\delta+n}{s} & =g(k) \\
\frac{1-s}{s} \frac{\varepsilon}{1-\varepsilon} \frac{\delta+n}{s} & =g(k)=\frac{\delta+n}{s}
\end{aligned}
$$

because (4.3). Finally we get

$$
\frac{1-s}{s} \frac{\varepsilon}{1-\varepsilon}=1
$$

which holds if and only if

$$
\begin{equation*}
s^{*}=\varepsilon \tag{4.7}
\end{equation*}
$$

Speed of Convergence. Consider the (one-dimensional) dynamic system:

$$
k_{t+1}=\frac{1-\delta}{1+n} k_{t}+\frac{s}{1+n} f\left(k_{t}\right) \equiv \varphi\left(k_{t}\right)
$$

The linearized dynamics are characterized by the eigenvalue

$$
|\lambda|=\left|\varphi^{\prime}(k)\right|=\left|\varphi^{\prime}\left(g^{-1}\left(\frac{\delta+n}{s}\right)\right)\right|<1
$$

Remember the meaning of linearized dynamics in the general case

$$
x_{t+1}-x \approx J\left(x_{t}-x\right)
$$

where $x_{t}$ is a vector state variable at time $t, x$ is the steady state and $J$ is the Jacobian matrix evaluated at the steady state. Then the non-linear dynamics are approximated in a neighborhood of the steady state by the following ones

$$
x_{t}-x \approx J^{t}\left(x_{0}-x\right)
$$

where the initial condition $k_{0}$ is assumed to belong to a neighborhood of $x$. In our case the Jacobian matrix is one-dimensional. Then

$$
J=\varphi^{\prime}\left(g^{-1}\left(\frac{\delta+n}{s}\right)\right)
$$

and we obtain

$$
k_{t}-k \approx \varphi^{\prime}\left(g^{-1}\left(\frac{\delta+n}{s}\right)\right)^{t}\left(k_{0}-k\right)
$$

If we want to know the time to cover a share $\sigma \in(0,1)$ of the distance between the initial condition and the steady state we must solve the following equation:

$$
\begin{aligned}
\sigma & =\frac{k_{t}-k_{0}}{k-k_{0}} \\
& =\frac{k_{t}-k+k-k_{0}}{k-k_{0}}=\frac{k_{t}-k}{k-k_{0}}+\frac{k-k_{0}}{k-k_{0}} \\
& =1-\frac{k_{t}-k}{k_{0}-k}=1-\varphi^{\prime}\left(g^{-1}\left(\frac{\delta+n}{s}\right)\right)^{t}
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\varphi^{\prime}\left(g^{-1}\left(\frac{\delta+n}{s}\right)\right)^{t} & =1-\sigma \\
t \ln \varphi^{\prime}\left(g^{-1}\left(\frac{\delta+n}{s}\right)\right) & =\ln (1-\sigma)
\end{aligned}
$$

and finally

$$
\begin{equation*}
t=\frac{\ln (1-\sigma)}{\ln \varphi^{\prime}\left(g^{-1}((\delta+n) / s)\right)} \tag{4.8}
\end{equation*}
$$

This time is an inverse measure of the speed of convergence.

### 4.3 Exogenous Technical Progress

The basic model claims that poor countries catch-up developed nations if the fundamentals $(f(k), \delta, n, s)$ are identical. This was the naive hope of the Fifties according to Solow predictions. Broadly speaking we know now that not only the gap between southern and western countries has not been reduced in the last three decades but after the Eighties it rather increased. Two main answers have been provided to solve the paradox. On the one hand side Solow (1957) augmented his basic setup by assuming an exogenous technical progress as engine of differential growth. On the other side the article of Romer (1986) opened the new field of studies on endogenous growth. The latter approach will be treated later on in the fifth chapter.

Different evolution laws for technical progress justify the gap between countries with identical fundamentals.

Let us now consider the discrete time version of the augmented Solow model.

$$
\begin{aligned}
K_{t+1} & =(1-\delta) K_{t}+\left(Y_{t}-C_{t}\right) \\
& =(1-\delta) K_{t}+S_{t} \\
& =(1-\delta) K_{t}+s Y_{t} \\
& =(1-\delta) K_{t}+s F\left(K_{t}, A_{t} N_{t}\right)
\end{aligned}
$$

where we assume

$$
A_{t+1} / A_{t}=1+a
$$

for every $t$. In other terms there is an exponential growth of the technical progress

$$
\begin{aligned}
A_{t} & =(1+a)^{t} A_{0} \\
& =(1+a)^{t}
\end{aligned}
$$

where $A_{0} \equiv 1$.
We normalize by $A_{t} N_{t}$.

$$
\begin{aligned}
\frac{K_{t+1}}{A_{t} N_{t}} & =(1-\delta) \frac{K_{t}}{A_{t} N_{t}}+s \frac{F\left(K_{t}, N_{t}\right)}{A_{t} N_{t}} \\
\frac{K_{t+1}}{A_{t+1} N_{t+1}} \frac{A_{t+1}}{A_{t}} \frac{N_{t+1}}{N_{t}} & =(1-\delta) \frac{K_{t}}{A_{t} N_{t}}+s F\left(\frac{K_{t}}{A_{t} N_{t}}, 1\right)
\end{aligned}
$$

Let

$$
\begin{aligned}
k_{t} & \equiv \frac{K_{t}}{A_{t} N_{t}} \\
f\left(k_{t}\right) & \equiv F\left(\frac{K_{t}}{A_{t} N_{t}}, 1\right)
\end{aligned}
$$

and we obtain

$$
k_{t+1}(1+a)(1+n)=(1-\delta) k_{t}+s f\left(k_{t}\right)
$$

The law of motion is

$$
k_{t+1}=\frac{1-\delta}{(1+a)(1+n)} k_{t}+\frac{s}{(1+a)(1+n)} f\left(k_{t}\right)
$$

Usual conditions

$$
\begin{align*}
f^{\prime} & >0  \tag{4.9}\\
f^{\prime \prime} & <0 \tag{4.10}
\end{align*}
$$

Inada conditions (sufficient conditions for the existence of at least one nontrivial steady state).

$$
\begin{aligned}
f(0) & =0 \\
f^{\prime}(0) & =\infty \\
f^{\prime}(\infty) & =0
\end{aligned}
$$

Steady States. Trivial steady state:

$$
k=0
$$

Non-trivial steady state.

$$
\begin{aligned}
k & =\frac{1-\delta}{(1+a)(1+n)} k+\frac{s}{(1+a)(1+n)} f(k) \\
1 & =\frac{1-\delta}{(1+a)(1+n)}+\frac{s}{(1+a)(1+n)} \frac{f(k)}{k}
\end{aligned}
$$

Let us define

$$
g(k) \equiv \frac{f(k)}{k}
$$

Observe that

$$
g^{\prime}<0
$$

(prove it by means of a graphic). Therefore

$$
\begin{aligned}
\frac{s}{(1+a)(1+n)} g(k) & =1-\frac{1-\delta}{(1+a)(1+n)}=\frac{(1+a)(1+n)-1+\delta}{(1+a)(1+n)} \\
& =\frac{1+a+n+a n-1+\delta}{(1+a)(1+n)}=\frac{a+n+a n+\delta}{(1+a)(1+n)} \\
& \approx \frac{a+n+\delta}{(1+a)(1+n)} \\
s g(k) & =a+n+a n+\delta \\
g(k) & =\frac{a+n+a n+\delta}{s}
\end{aligned}
$$

$g$ is monotonic, then invertible:

$$
k=g^{-1}\left(\frac{a+n+a n+\delta}{s}\right)
$$

Comparative statics: verify that

$$
\begin{aligned}
\frac{\partial k}{\partial a}, \frac{\partial k}{\partial n}, \frac{\partial k}{\partial \delta} & <0 \\
\frac{\partial k}{\partial s} & =\frac{\partial g^{-1}((a+n+a n+\delta) / s)}{\partial s}=\frac{1}{g^{\prime}(k)}\left(-\frac{a+n+a n+\delta}{s^{2}}\right) \\
& =-\frac{a+n+a n+\delta}{s^{2} g^{\prime}\left(g^{-1}((a+n+a n+\delta) / s)\right)}>0
\end{aligned}
$$

because

$$
\left(g^{-1}\right)^{\prime}<0
$$

Provide also a graphic proof.
The stationary production is given by

$$
y=f(k)=f\left(g^{-1}\left(\frac{a+n+a n+\delta}{s}\right)\right)
$$

while the stationary consumption is provided by

$$
c=(1-s) y=(1-s) f\left(g^{-1}\left(\frac{a+n+a n+\delta}{s}\right)\right)
$$

Global Dynamics. The law of motion is

$$
k_{t+1}=\frac{1-\delta}{(1+a)(1+n)} k_{t}+\frac{s}{(1+a)(1+n)} f\left(k_{t}\right) \equiv \varphi\left(k_{t}\right)
$$

We observe that

$$
\begin{aligned}
& \varphi^{\prime}>0 \\
& \varphi^{\prime \prime}<0
\end{aligned}
$$

The Inada conditions entail

$$
\begin{aligned}
\varphi(0) & =0 \\
\varphi^{\prime}(0) & =\infty \\
\varphi^{\prime}(\infty) & =\frac{1-\delta}{(1+a)(1+n)}<1
\end{aligned}
$$

Plot the function $\varphi$ and notice that it converges from below to a slanting asymptote with slope $(1-\delta) /[(1+a)(1+n)]<1$ and positive intercept. Plot also in the plane $(k, \varphi)$ the bisector of the first orthant. The abscissas of the two intersections are respectively the trivial steady state 0 and the non-trivial $k>0$. Plot the global dynamics and the converging path.

We observe that if the initial condition $k_{0} \in(0, k)$, then

$$
\begin{aligned}
& k_{0}<\varphi\left(k_{0}\right)=k_{1} \\
& k_{t}<\varphi\left(k_{t}\right)=k_{t+1}
\end{aligned}
$$

and the capital-labor ratio increases.
If $k_{0} \in(k,+\infty)$, then

$$
\begin{aligned}
k_{0} & >\varphi\left(k_{0}\right)=k_{1} \\
k_{t} & >\varphi\left(k_{t}\right)=k_{t+1}
\end{aligned}
$$

and the capital-labor ratio decreases.
Thus the stationary state $k$ is globally stable for $k_{0}>0$.

Balanced Growth. At the steady state we get

$$
\frac{K_{t}}{A_{t} N_{t}}=\frac{K_{t+1}}{A_{t+1} N_{t+1}}=k
$$

Then

$$
\frac{K_{t+1}}{K_{t}}=\frac{A_{t+1}}{A_{t}} \frac{N_{t+1}}{N_{t}}=(1+a)(1+n)
$$

The growth factor equals the population growth factor, which is exogenously given.

Moreover

$$
\begin{aligned}
\frac{Y_{t+1}}{Y_{t}} & =\frac{F\left(K_{t+1}, A_{t+1} N_{t+1}\right)}{F\left(K_{t}, A_{t} N_{t}\right)}=\frac{A_{t+1} N_{t+1} f(k)}{A_{t} N_{t} f(k)}=(1+a)(1+n) \\
\frac{C_{t+1}}{C_{t}} & =\frac{(1-s) Y_{t+1}}{(1-s) Y_{t}}=(1+a)(1+n)
\end{aligned}
$$

Then the growth is balanced.

Local Dynamics. A similar conclusion is obtained by linearizing the dynamics around the steady state. Dynamics are one-dimensional. The stability condition requires the unique eigenvalue to have modulus less than one:

$$
|\lambda|=\left|\varphi^{\prime}(k)\right|=\left|\varphi^{\prime}\left(g^{-1}\left(\frac{a+n+a n+\delta}{s}\right)\right)\right|<1
$$

At the steady state the aggregate variables grow at rate $a+n$. This is due to the constant returns to scale in aggregate production, a neoclassical assumption. In the sixth chapter this restriction will be removed and we shall see that increasing returns to scale allows for endogenous growth, i.e. the aggregate variables can grow even in absence of exogenous technical
progress and demographic growth. Remember that in the first Solow model the economy converges to a balanced growth, i.e. the growth rate for all the aggregate variables is always $n$ and does not depend on the initial conditions. That's a strong implication of the model and its weakness. Comparing the long term growth rates of world countries one observes no significative convergence and the explanation provided by the Solow residual, i.e. a diverging exogenous growth is not satisfactory because the technical progress is not justified by the theory, i.e. micro-founded. The endogenous growth models we investigate in the sixth chapter, try to shed a light on this point.

The reader, who is interested in more details about the Solow models is referred to the first chapter of Romer (1996). In particular this handbook analyzes the transition due to shocks on the exogenous saving rate, the measure of the speed of convergence and the contribution of the technical progress in the growth accounting (Solow residual).

### 4.4 Endogenous Growth in a Solow Framework

Assume now another law for the technical progress:

$$
A_{t}=K_{t}
$$

The production function becomes:

$$
Y_{t}=F\left(K_{t}, K_{t} N_{t}\right)
$$

Consider the Cobb-Douglas case:

$$
F\left(K_{t}, K_{t} N_{t}\right)=K_{t}^{\alpha}\left(K_{t} N_{t}\right)^{1-\alpha}=N_{t}^{1-\alpha} K_{t}
$$

If $n=0$ we obtain the $A k$ model

$$
\begin{aligned}
Y_{t} & =N^{1-\alpha} K_{t} \\
& =A K_{t}
\end{aligned}
$$

where $A \equiv N^{1-\alpha}$. Moreover

$$
K_{t+1}=(1-\delta) K_{t}+\left(Y_{t}-C_{t}\right)
$$

$$
\begin{aligned}
& =(1-\delta) K_{t}+S_{t} \\
& =(1-\delta) K_{t}+s Y_{t} \\
& =(1-\delta) K_{t}+s F\left(K_{t}, A_{t} N_{t}\right) \\
& =(1-\delta) K_{t}+s A K_{t} \\
& =(1-\delta+s A) K_{t}
\end{aligned}
$$

The aggregate capital growth factor is

$$
\frac{K_{t+1}}{K_{t}}=1-\delta+s A
$$

The production growth factor is

$$
\frac{Y_{t+1}}{Y_{t}}=\frac{A K_{t+1}}{A K_{t}}=1-\delta+s A
$$

Growth turns out to be still balanced. But now it is not generated by an exogenous population growth or an exogenous technical progress. Now the growth rate depends endogenously on $\delta$ and $s$.

### 4.5 Open Economy

### 4.5.1 Human Capital

This model is inspired by the continuous time model of Mankiw, Romer and Weil (1992).

We consider an augmented production function with human capital.

$$
Y_{t}=K_{t}^{\alpha} H_{t}^{\beta}\left(E_{t} L_{t}\right)^{1-\alpha-\beta}
$$

$H$ is the human capital.

$$
\begin{aligned}
\frac{Y_{t}}{E_{t} L_{t}} & =\frac{K_{t}^{\alpha}}{\left(E_{t} L_{t}\right)^{\alpha}} \frac{H_{t}^{\beta}}{\left(E_{t} L_{t}\right)^{\beta}} \\
y_{t} & =k_{t}^{\alpha} h_{t}^{\beta}
\end{aligned}
$$

where

$$
x_{t} \equiv \frac{X_{t}}{E_{t} L_{t}}
$$

Human capital is assumed to depreciate at the same rate $\delta$ as physical capital. Saving rates: $s_{K}$ and $s_{K}$.

$$
\begin{aligned}
H_{t+1}-H_{t} & =s_{H} Y_{t}-\delta H_{t} \\
K_{t+1}-K_{t} & =s_{K} Y_{t}-\delta K_{t}
\end{aligned}
$$

Consider the human capital accumulation:

$$
\begin{aligned}
\frac{E_{t+1} L_{t+1}}{E_{t} L_{t}} \frac{H_{t+1}}{E_{t+1} L_{t+1}}-\frac{H_{t}}{E_{t} L_{t}} & =s_{H} \frac{Y_{t}}{E_{t} L_{t}}-\delta \frac{H_{t}}{E_{t} L_{t}} \\
(1+e)(1+n) h_{t+1}-h_{t} & =s_{H} y_{t}-\delta h_{t} \\
(1+e)(1+n)\left(h_{t+1}-h_{t}\right) & =s_{H} y_{t}-(\delta+e+n+e n) h_{t}
\end{aligned}
$$

Let $g \equiv e+n+e n$.

$$
h_{t+1}-h_{t}=\frac{s_{H}}{1+g} y_{t}-\frac{\delta+g}{1+g} h_{t}
$$

Similarly

$$
\begin{equation*}
k_{t+1}-k_{t}=\frac{s_{K}}{1+g} y_{t}-\frac{\delta+g}{1+g} k_{t} \tag{4.11}
\end{equation*}
$$

Dynamic system

$$
\begin{aligned}
h_{t+1} & =\frac{s_{H}}{1+g} k_{t}^{\alpha} h_{t}^{\beta}+\frac{1-\delta}{1+g} h_{t} \\
k_{t+1} & =\frac{s_{K}}{1+g} k_{t}^{\alpha} h_{t}^{\beta}+\frac{1-\delta}{1+g} k_{t}
\end{aligned}
$$

Stationary state.

$$
\begin{aligned}
& (\delta+g) h=s_{H} k^{\alpha} h^{\beta} \\
& (\delta+g) k=s_{K} k^{\alpha} h^{\beta}
\end{aligned}
$$

Therefore

$$
\frac{h}{k}=\frac{s_{H}}{s_{K}}
$$

By substitution

$$
\begin{aligned}
(\delta+g) h & =s_{H} k^{\alpha} h^{\beta} \\
(\delta+g) h & =s_{H}\left(\frac{s_{K}}{s_{H}} h\right)^{\alpha} h^{\beta} \\
h & =\left(\frac{s_{K}^{\alpha} s_{H}^{1-\alpha}}{\delta+g}\right)^{1 /(1-\alpha-\beta)}
\end{aligned}
$$

and symmetrically

$$
k=\left(\frac{s_{H}^{\beta} s_{K}^{1-\beta}}{\delta+g}\right)^{1 /(1-\alpha-\beta)}
$$

The local analysis is left to the reader.

### 4.5.2 Taxes and Absolute Convergence

According to the empirical evidence (Romer, 1986; De Long, 1988) there is no absolute convergence during the 20th century (more precisely 1870 to 1979) within a representative worldwide sample of developing and rich countries.

Among the possible obstacles the imperfect international capital mobility is pointed out. Taxes are recognized as an obstacle to capital inflow.

If $\rho$ is the pre-tax return rate on capital and $\tau$ is the capital tax rate, then $r=(1-\tau) \rho$ is the after-tax return rate. Assuming a constant return to scale production function in aggregate capital and labor yields $\rho=f^{\prime}(k)$, where $k \equiv K / L$. Consider now two countries $A$ and $B$ with similar fundamentals and different capital tax rates. In the equilibrium $r_{A}=r_{B}$, i.e.

$$
\left(1-\tau_{A}\right) f^{\prime}\left(k_{A}\right)=\left(1-\tau_{B}\right) f^{\prime}\left(k_{B}\right)
$$

With the same Cobb-Douglas technology for both countries we have

$$
\begin{aligned}
\frac{f^{\prime}\left(k_{A}\right)}{f^{\prime}\left(k_{B}\right)} & =\frac{1-\tau_{B}}{1-\tau_{A}} \\
\frac{\alpha k_{A}^{\alpha-1}}{\alpha k_{B}^{\alpha-1}} & =\frac{1-\tau_{B}}{1-\tau_{A}} \\
\frac{k_{A}}{k_{B}} & =\left(\frac{1-\tau_{A}}{1-\tau_{B}}\right)^{1 /(1-\alpha)}
\end{aligned}
$$

Therefore $\tau_{A}>\tau_{B}$ entails $k_{A}<k_{B}$. Effective 1980 tax rate on capital (King and Fullerton, 1984): U.K. $4 \%$, U.S. $37 \%$, Germany $48 \%$.

### 4.5.3 Transition Dynamics in a Closed Economy

In the augmented Solow model without human capital accumulation is provided by a difference equation similar to (4.11).

$$
k_{t+1}=\frac{s}{1+g} y_{t}+\frac{1-\delta}{1+g} k_{t} \equiv \varphi\left(k_{t}\right)
$$

(simply set $s=s_{K}$ ). Linearized dynamics are one-dimensional and the real eigenvalue $\varphi^{\prime}(k)$, where $k$ is the stationary state, is negatively related to the speed of convergence. More precisely

$$
\begin{aligned}
k_{t+1} & =\frac{s f\left(k_{t}\right)}{1+g}+\frac{1-\delta}{1+g} k_{t} \\
k_{t+1}-k & \approx\left[\frac{s}{1+g} f^{\prime}(k)+\frac{1-\delta}{1+g}\right]\left(k_{t}-k\right)
\end{aligned}
$$

A speed of convergence:

$$
\begin{aligned}
\sigma & \equiv \frac{k_{t+1}-k_{t}}{k_{t+1}-k}=1-\frac{k_{t}-k}{k_{t+1}-k}=1-\varphi^{\prime}(k) \\
& =1-\left[\frac{s}{1+g} f^{\prime}(k)+\frac{1-\delta}{1+g}\right] \\
& =\frac{\delta+g-s f^{\prime}(k)}{1+g}
\end{aligned}
$$

The Cobb-Douglas case $\left(y_{t}=k_{t}^{\alpha}\right)$ yields

$$
\begin{aligned}
k & =\left(\frac{s}{\delta+g}\right)^{1 /(1-\alpha)} \\
\sigma & =\frac{\delta+g-s\left(\alpha k^{\alpha-1}\right)}{1+g} \\
& =\frac{\delta+g-s \alpha\left\{[s /(\delta+g)]^{1 /(1-\alpha)}\right\}^{\alpha-1}}{1+g} \\
& =\frac{(1-\alpha)(\delta+g)}{1+g}
\end{aligned}
$$

The measured speed of convergence of U.S. in the past thirty years is about 0.02 . Data yields: $\alpha=1 / 3, \delta=0.03, g \equiv e+n+e n \approx e+n=$ $0.02+0.01=0.03$. Therefore our simple model gives

$$
\sigma=\frac{(1-\alpha)(\delta+g)}{1+g}=\frac{(1-1 / 3)(0.03+0.03)}{1+0.03}=0.038835>0.02
$$

The resulting speed (0.038835) overestimates the observed speed (0.02). Other assumptions are required.

### 4.5.4 Transition Dynamics in an Open Economy

One of the most interesting answer is to assume impediments to financing human capital formation. Human capital is inalienable. Creditors can seize physical capital but not human capital, i.e. the present value of future income stream.

We assume that the economy is small and open and that an agent can borrow in the international credit market up to the stock of physical capital.

The production function with human capital is still

$$
\begin{equation*}
y_{t}=k_{t}^{\alpha} h_{t}^{\beta} \tag{4.12}
\end{equation*}
$$

The world interest rate is given and equal to $r$. Profit maximization equals the interest rate and the physical capital marginal productivity: this is always possible, even if the human capital accumulation is financially constrained, because there are no credit constraints to physical capital accumulation. In other words physical capital adjusts to allow the following equality

$$
\begin{equation*}
\alpha k_{t}^{\alpha-1} h_{t}^{\beta}=r \tag{4.13}
\end{equation*}
$$

The crucial hypothesis is that the human capital accumulation is financially constrained and for simplicity we assume that the aggregate net claims on foreigners $B_{t}$ at the end of date $t$ have the aggregate physical capital as collateral $K_{t}$.

$$
\begin{equation*}
B_{t} \geq-K_{t} \tag{4.14}
\end{equation*}
$$

The agent is financially constrained if and only if the return on human capital is greater than the world interest rate

$$
\beta k_{t}^{\alpha} h_{t}^{\beta-1}>r
$$

In this case constraint (4.14) turns out to be binding:

$$
\begin{equation*}
B_{t}=-K_{t} \tag{4.15}
\end{equation*}
$$

The wealth accumulation identity is

$$
\begin{equation*}
H_{t+1}-H_{t}+K_{t+1}-K_{t}+B_{t+1}-B_{t}=Y_{t}+r B_{t}-C_{t}-\delta H_{t} \tag{4.16}
\end{equation*}
$$

where $r B_{t}$ is negative. For simplicity we assume that the physical capital does not depreciate. Using (4.15) and (4.16) yields

$$
H_{t+1}-H_{t}+\delta H_{t}=Y_{t}-r K_{t}-C_{t}
$$

The saving rate is defined by

$$
s \equiv \frac{Y_{t}-r K_{t}-C_{t}}{Y_{t}-r K_{t}}
$$

Therefore

$$
H_{t+1}-(1-\delta) H_{t}=s\left(Y_{t}-r K_{t}\right)
$$

We normalize as usual by $E_{t} L_{t}$.

$$
\begin{aligned}
\frac{E_{t+1} L_{t+1}}{E_{t} L_{t}} \frac{H_{t+1}}{E_{t+1} L_{t+1}}-(1-\delta) \frac{H_{t}}{E_{t} L_{t}} & =s\left(\frac{Y_{t}}{E_{t} L_{t}}-r \frac{K_{t}}{E_{t} L_{t}}\right) \\
(1+e)(1+n) h_{t+1}-(1-\delta) h_{t} & =s\left(y_{t}-r k_{t}\right) \\
(1+g) h_{t+1}-(1-\delta) h_{t} & =s\left(y_{t}-r k_{t}\right)
\end{aligned}
$$

From (4.12) and (4.13) we have

$$
\begin{aligned}
& k_{t}=\frac{\alpha}{r} y_{t} \\
& y_{t}=k_{t}^{\alpha} h_{t}^{\beta}=\left(\frac{\alpha}{r} y_{t}\right)^{\alpha} h_{t}^{\beta}=y_{t}^{\alpha}\left(\frac{\alpha}{r}\right)^{\alpha} h_{t}^{\beta} \\
& y_{t}=\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)} h_{t}^{\beta /(1-\alpha)} \\
& k_{t}=\frac{\alpha}{r} y_{t}=\frac{\alpha}{r}\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)} h_{t}^{\beta /(1-\alpha)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
(1+g) h_{t+1}-(1-\delta) h_{t} & =s\left(y_{t}-r k_{t}\right) \\
(1+g) h_{t+1}-(1-\delta) h_{t}+(1+g) h_{t}-(1+g) h_{t} & =s\left(y_{t}-r k_{t}\right) \\
(1+g) h_{t+1}-(1+g) h_{t}= & s\left[\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)} h_{t}^{\beta /(1-\alpha)}-r \frac{\alpha}{r}\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)} h_{t}^{\beta /(1-\alpha)}\right] \\
& -(\delta+g) h_{t} \\
(1+g) h_{t+1}= & (1+g) h_{t}+s(1-\alpha)\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)} h_{t}^{\beta /(1-\alpha)} \\
& -(\delta+g) h_{t}
\end{aligned}
$$

Law of motion.

$$
h_{t+1}=h_{t}+\frac{s(1-\alpha)}{1+g}\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)} h_{t}^{\beta /(1-\alpha)}-\frac{\delta+g}{1+g} h_{t}
$$

Steady state.

$$
\begin{aligned}
s(1-\alpha)\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)} h^{\beta /(1-\alpha)} & =(\delta+g) h \\
h^{1-\beta /(1-\alpha)} & =\frac{s(1-\alpha)}{\delta+g}\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)} \\
h & =\left[\frac{s(1-\alpha)}{\delta+g}\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)}\right]^{(1-\alpha) /(1-\alpha-\beta)}
\end{aligned}
$$

We linearize around the steady state.

$$
\begin{aligned}
h_{t+1}-h & \approx\left[1+\frac{s(1-\alpha)}{1+g}\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)} \frac{\beta}{1-\alpha} h^{\beta /(1-\alpha)-1}-\frac{\delta+g}{1+g}\right]\left(h_{t}-h\right) \\
& =\left[1+\frac{s(1-\alpha)}{1+g}\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)} \frac{\beta}{1-\alpha} h^{-(1-\alpha-\beta) /(1-\alpha)}-\frac{\delta+g}{1+g}\right]\left(h_{t}-h\right) \\
& =\left(1-\frac{\delta+g}{1+g}+\frac{s(1-\alpha)}{1+g}\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)} \frac{\beta}{1-\alpha}\right. \\
& \left.=\left\{\left[\frac{s(1-\alpha)}{\delta+g}\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)}\right]^{(1-\alpha) /(1-\alpha-\beta)}\right\}^{-(1-\alpha-\beta) /(1-\alpha)}\right)\left(h_{t}-h\right) \\
& =\left(1-\frac{\delta+g}{1+g}+\frac{s(1-\alpha)}{1+g}\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)} \frac{\beta}{1-\alpha}\right. \\
& \left.=\left[\frac{s(1-\alpha)}{\delta+g}\left(\frac{\alpha}{r}\right)^{\alpha /(1-\alpha)}\right]^{-1}\right)\left(h_{t}-h\right) \\
& =\left(1-\frac{\delta+g}{1+g}+\frac{\delta+g}{1+g} \frac{\beta}{1-\alpha}\right)\left(h_{t}-h\right) \\
& =\left[1-\frac{\delta+g}{1+g}\left(1-\frac{\beta}{1-\alpha}\right)\right]\left(h_{t}-h\right) \\
& =\left[1-\frac{1-\alpha-\beta}{1-\alpha} \frac{\delta+g}{1+g}\right]\left(h_{t}-h\right)
\end{aligned}
$$

So the speed of convergence is given by

$$
\begin{aligned}
1-\sigma^{\prime} & =1-\left[1-\frac{1-\alpha-\beta}{1-\alpha} \frac{\delta+g}{1+g}\right] \\
& =\frac{1-\alpha-\beta}{1-\alpha} \frac{\delta+g}{1+g}
\end{aligned}
$$

We notice the negative effect of $\beta$. This conclusion fits the data better than the model of the previous section.

As economic interpretation one can say that the impediments to borrowing against human capital slow down the accumulation of physical capital, because the two factors are complements in the production process. A similar mechanism is in Barro, Mankiw and Sala-i-Martin (1995).

## Chapter 5

## Endogenous Saving

### 5.1 Two-Period Equilibrium Model

### 5.1.1 Decentralized Equilibrium

Consumer's program

$$
\max \ln c_{0}+\beta \ln c_{1}
$$

$$
\begin{aligned}
c_{0}+p_{0}\left(b_{1}-b_{0}\right)+\left[k_{1}-(1-\delta) k_{0}\right] & \leq r_{k 0} k_{0}+w_{0} l_{0}+p_{0} r_{b 0} b_{0} \\
c_{1} & \leq\left(1-\delta+r_{k 1}\right) k_{1}+w_{1} l_{1}+p_{1}\left(1+r_{b 1}\right) b_{1}
\end{aligned}
$$

Lagrangian function:

$$
\begin{aligned}
\Lambda= & u\left(c_{0}\right)+\beta u\left(c_{1}\right) \\
& +\lambda_{0}\left[r_{k 0} k_{0}+w_{0} l_{0}+p_{0} r_{b 0} b_{0}-c_{0}-p_{0}\left(b_{1}-b_{0}\right)-k_{1}+(1-\delta) k_{0}\right] \\
& +\lambda_{1}\left[\left(1-\delta+r_{k 1}\right) k_{1}+w_{1} l_{1}+p_{1}\left(1+r_{b 1}\right) b_{1}-c_{1}\right]
\end{aligned}
$$

$k_{0}$ is given.
First order conditions:

$$
\begin{aligned}
& \frac{\partial \Lambda}{\partial k_{1}}=-\lambda_{0}+\lambda_{1}\left(1-\delta+r_{k 1}\right)=0 \\
& \frac{\partial \Lambda}{\partial b_{1}}=-\lambda_{0} p_{0}+\lambda_{1} p_{1}\left(1+r_{b 1}\right)=0 \\
& \frac{\partial \Lambda}{\partial c_{0}}=u^{\prime}\left(c_{0}\right)-\lambda_{0}=0 \\
& \frac{\partial \Lambda}{\partial c_{1}}=\beta u^{\prime}\left(c_{1}\right)-\lambda_{1}=0
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\lambda_{0}}{\lambda_{1}} & =1-\delta+r_{k 1} \\
\frac{\lambda_{0}}{\lambda_{1}} & =\frac{p_{1}}{p_{0}}\left(1+r_{b 1}\right) \\
u^{\prime}\left(c_{0}\right) & =\lambda_{0} \\
\beta u^{\prime}\left(c_{1}\right) & =\lambda_{1}
\end{aligned}
$$

We have the Euler equation

$$
\frac{u^{\prime}\left(c_{0}\right)}{\beta u^{\prime}\left(c_{1}\right)}=\frac{\lambda_{0}}{\lambda_{1}}=1-\delta+r_{k 1}
$$

I equilibrium a no-arbitrage condition holds

$$
1-\delta+r_{k 1}=\frac{p_{1}}{p_{0}}\left(1+r_{b 1}\right)
$$

(the real return factor on capital equals the real return factor on bonds).
As there is a representative agent, all the agents in equilibrium will demand the same quantities $b_{0}$ and $b_{1}$ (maybe negative). Therefore in equilibrium

$$
b_{0}=b_{1}=0
$$

The budget constraints become

$$
\begin{aligned}
c_{0}+\left[k_{1}-(1-\delta) k_{0}\right] & =r_{k 0} k_{0}+w_{0} l_{0} \\
c_{1} & =\left(1-\delta+r_{k 1}\right) k_{1}+w_{1} l_{1}
\end{aligned}
$$

Consider now the firm equilibrium

$$
\begin{aligned}
r_{k 0} & =f^{\prime}\left(k_{0}\right) \\
r_{k 1} & =f^{\prime}\left(k_{1}\right) \\
w_{0} & =f\left(k_{0}\right)-f^{\prime}\left(k_{0}\right) k_{0} \\
w_{1} & =f\left(k_{1}\right)-f^{\prime}\left(k_{1}\right) k_{1}
\end{aligned}
$$

(show the steps).
Therefore

$$
\begin{aligned}
r_{k 0} k_{0}+w_{0} l_{0} & =f^{\prime}\left(k_{0}\right) k_{0}+\left[f\left(k_{0}\right)-f^{\prime}\left(k_{0}\right) k_{0}\right] 1=f\left(k_{0}\right) \\
r_{k 1} k_{1}+w_{1} l_{1} & =f^{\prime}\left(k_{1}\right) k_{1}+\left[f\left(k_{1}\right)-f^{\prime}\left(k_{1}\right) k_{1}\right] 1=f\left(k_{1}\right)
\end{aligned}
$$

The constraints become

$$
\begin{aligned}
c_{0}+\left[k_{1}-(1-\delta) k_{0}\right] & =f\left(k_{0}\right) \\
c_{1} & =(1-\delta) k_{1}+f\left(k_{1}\right)
\end{aligned}
$$

The Euler equation becomes

$$
\frac{u^{\prime}\left(c_{0}\right)}{\beta u^{\prime}\left(c_{1}\right)}=1-\delta+f^{\prime}\left(k_{1}\right)
$$

We obtain the following system

$$
\begin{aligned}
\frac{u^{\prime}\left(c_{0}\right)}{u^{\prime}\left(c_{1}\right)} & =\beta\left[1-\delta+f^{\prime}\left(k_{1}\right)\right] \\
c_{0}+\left[k_{1}-(1-\delta) k_{0}\right] & =f\left(k_{0}\right) \\
c_{1} & =(1-\delta) k_{1}+f\left(k_{1}\right)
\end{aligned}
$$

with three equations and three unknowns: $c_{0}, k_{1}, c_{1}$ (notice that $k_{0}$ is given).

### 5.1.2 Planner's Problem

A benevolent planner maximizes the welfare function, which coincides now with the intertemporal utility function of the representative agent:

$$
\max \ln c_{0}+\beta \ln c_{1}
$$

He cares only about the respect of the resource constraints:

$$
\begin{aligned}
c_{0}+\left[k_{1}-(1-\delta) k_{0}\right] & =f\left(k_{0}\right) \\
c_{1} & =(1-\delta) k_{1}+f\left(k_{1}\right)
\end{aligned}
$$

The Lagrangian is

$$
\begin{aligned}
\Lambda= & u\left(c_{0}\right)+\beta u\left(c_{1}\right) \\
& +\lambda_{0}\left[f\left(k_{0}\right)-c_{0}-k_{1}+(1-\delta) k_{0}\right] \\
& +\lambda_{1}\left[(1-\delta) k_{1}+f\left(k_{1}\right)-c_{1}\right]
\end{aligned}
$$

The first order conditions are

$$
\begin{aligned}
& \frac{\partial \Lambda}{\partial k_{1}}=-\lambda_{0}+\lambda_{1}\left(1-\delta+f^{\prime}\left(k_{1}\right)\right)=0 \\
& \frac{\partial \Lambda}{\partial c_{0}}=u^{\prime}\left(c_{0}\right)-\lambda_{0}=0 \\
& \frac{\partial \Lambda}{\partial c_{1}}=\beta u^{\prime}\left(c_{1}\right)-\lambda_{1}=0
\end{aligned}
$$

Then we have the usual Euler equation

$$
\frac{u^{\prime}\left(c_{0}\right)}{\beta u^{\prime}\left(c_{1}\right)}=\frac{\lambda_{0}}{\lambda_{1}}=1-\delta+f^{\prime}\left(k_{1}\right)
$$

and finally with the binding resource constraints

$$
\begin{aligned}
\frac{u^{\prime}\left(c_{0}\right)}{u^{\prime}\left(c_{1}\right)} & =\beta\left[1-\delta+f^{\prime}\left(k_{1}\right)\right] \\
c_{0}+\left[k_{1}-(1-\delta) k_{0}\right] & =f\left(k_{0}\right) \\
c_{1} & =(1-\delta) k_{1}+f\left(k_{1}\right)
\end{aligned}
$$

which is exactly the system we obtained in the decentralized economy case.
Now we can explicitly solve the problem by putting

$$
\begin{aligned}
u\left(c_{0}\right)+\beta u\left(c_{1}\right) & =\ln c_{0}+\beta \ln c_{1} \\
f(k) & =k^{\alpha}
\end{aligned}
$$

### 5.2 Infinite-Lived Agents

### 5.2.1 Decentralized Equilibrium

We explore the class of the so-called infinite-horizon models that are characterized by infinite-lived agents, endogenous saving and exogenous growth. The seminal papers on the topic are Ramsey (1928), Cass (1965) and Koopmans (1965).

In this class of equivalent models the equilibrium conceived as transition path exists, is unique and Pareto-optimal. The first welfare theorem holds and the market decentralizes the planner's solution ${ }^{1}$. In a centralized economy the benevolent planner decides for all and his choice is Pareto-optimal. There are infinitely many deciders in a decentralized economy. Information circulates in form of prices and each price-taker household decides without caring about the others. In the class of models of Ramsey type the market implements the optimal solution as well as the planner.

In a simplest version this model is characterized by a planar system and a saddle equilibrium path converging to the stationary state. Capital plays as a predetermined variable and the equilibrium is typically determinate. It

[^6]is always possible to enrich the model with market imperfections such as externalities, asymmetric information and missing markets and to observe easily indeterminacy. There are no unambiguous definitions of imperfection. We adopt the viewpoint that a market imperfection is what implies the failure of the first welfare theorem. In this sense it becomes central to investigate the relation between optimality and determinacy. Whenever the first welfare theorem fails, the basic role of restoring efficiency may be played by the government.

In what follows we consider a centralized economy. All the results hold as well in a market economy. A monetary example of decentralized economy is provided in section 9.5 (see "The Clower Constraint").

Let me focus now on the planner problem. An infinite-lived representative agent is considered (agents are assumed to be symmetric, i.e. to have same tastes and endowments). The agent maximizes the intertemporal functional.

The program.

$$
\begin{gathered}
\max \sum_{t=0}^{\infty} \beta^{t} u\left(c_{t}\right) \\
p_{t}\left[k_{t+1}-(1-\delta) k_{t}\right]+p_{t} c_{t} \leq r_{t} k_{t}+w_{t} l_{t}
\end{gathered}
$$

Lagrangian functional.

$$
\sum_{t=0}^{\infty} \beta^{t} u\left(c_{t}\right)+\sum_{t=0}^{\infty} \lambda_{t}\left\{r_{t} k_{t}+w_{t} l_{t}-p_{t}\left[k_{t+1}-(1-\delta) k_{t}\right]-p_{t} c_{t}\right\}
$$

First order conditions.

$$
\begin{gathered}
\frac{\partial \Lambda}{\partial k_{t}}=-\lambda_{t-1} p_{t-1}+\lambda_{t} r_{t}+\lambda_{t} p_{t}(1-\delta)=0 \\
\frac{\partial \Lambda}{\partial c_{t}}=\beta^{t} u^{\prime}\left(c_{t}\right)-\lambda_{t} p_{t}=0 \\
\lambda_{t-1} p_{t-1}=\lambda_{t} r_{t}+\lambda_{t} p_{t}(1-\delta) \\
\beta^{t} u^{\prime}\left(c_{t}\right)=\lambda_{t} p_{t}
\end{gathered}
$$

Then

$$
\frac{\lambda_{t-1} p_{t-1}}{\lambda_{t} p_{t}}=\frac{\lambda_{t} r_{t}}{\lambda_{t} p_{t}}+(1-\delta)
$$

$$
\begin{aligned}
\frac{\lambda_{t-1} p_{t-1}}{\lambda_{t} p_{t}} & =1-\delta+\frac{r_{t}}{p_{t}} \\
\frac{\beta^{t-1} u^{\prime}\left(c_{t-1}\right)}{\beta^{t} u^{\prime}\left(c_{t}\right)} & =1-\delta+\frac{r_{t}}{p_{t}} \\
\frac{u^{\prime}\left(c_{t-1}\right)}{\beta u^{\prime}\left(c_{t}\right)} & =1-\delta+\frac{r_{t}}{p_{t}} \\
\frac{u^{\prime}\left(c_{t-1}\right)}{u^{\prime}\left(c_{t}\right)} & =\beta\left(1-\delta+\frac{r_{t}}{p_{t}}\right) \\
\frac{u^{\prime}\left(c_{t}\right)}{u^{\prime}\left(c_{t+1}\right)} & =\beta\left(1-\delta+\frac{r_{t+1}}{p_{t+1}}\right)
\end{aligned}
$$

(Euler equation which is a consumption demand: interpret the impact of the interest rate and time preference on the consumption smoothing).

Firm equilibrium

$$
\begin{aligned}
p_{t} \frac{\partial F}{\partial K_{t}} & =r_{t} \\
p_{t} \frac{\partial F}{\partial N_{t}} & =w_{t} \\
f^{\prime}\left(k_{t}\right) & =\frac{\partial F}{\partial K_{t}}=\frac{r_{t}}{p_{t}} \\
f\left(k_{t}\right)-f^{\prime}\left(k_{t}\right) k_{t} & =\frac{\partial F}{\partial N_{t}}=\frac{w_{t}}{p_{t}}
\end{aligned}
$$

Labor market equilibrium:

$$
l_{t}=1
$$

Good market equilibrium.

$$
\begin{aligned}
p_{t}\left[k_{t+1}-(1-\delta) k_{t}\right]+p_{t} c_{t} & =r_{t} k_{t}+w_{t} l_{t} \\
k_{t+1}-(1-\delta) k_{t}+c_{t} & =\frac{r_{t}}{p_{t}} k_{t}+\frac{w_{t}}{p_{t}} \\
k_{t+1}-(1-\delta) k_{t}+c_{t} & =f^{\prime}\left(k_{t}\right) k_{t}+\left[f\left(k_{t}\right)-f^{\prime}\left(k_{t}\right) k_{t}\right] \\
\frac{u^{\prime}\left(c_{t}\right)}{u^{\prime}\left(c_{t+1}\right)} & =\beta\left(1-\delta+\frac{r_{t+1}}{p_{t+1}}\right)=\beta\left[1-\delta+f^{\prime}\left(k_{t+1}\right)\right]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
k_{t+1}-(1-\delta) k_{t}+c_{t} & =f\left(k_{t}\right) \\
\frac{u^{\prime}\left(c_{t}\right)}{u^{\prime}\left(c_{t+1}\right)} & =\beta\left[1-\delta+f^{\prime}\left(k_{t+1}\right)\right]
\end{aligned}
$$

Steady state.

$$
\beta\left[1-\delta+f^{\prime}(k)\right]=1
$$

(modified golden rule) and

$$
c=f(k)-\delta k
$$

To explicitly solve we require now the fundamentals specification:

$$
\begin{aligned}
f(k) & =k^{\alpha} \\
u(c) & =\ln c
\end{aligned}
$$

Moreover we know that

$$
\delta=1
$$

Then we obtain

$$
\begin{aligned}
\frac{c_{t+1}}{c_{t}} & =\beta \alpha k_{t+1}^{\alpha-1} \\
c_{t+1} & =\beta \alpha k_{t+1}^{\alpha-1} c_{t} \\
k_{t+1}+c_{t} & =k_{t}^{\alpha}
\end{aligned}
$$

Dynamic system

$$
\begin{aligned}
k_{t+1} & =k_{t}^{\alpha}-c_{t} \\
c_{t+1} & =\beta \alpha\left(k_{t}^{\alpha}-c_{t}\right)^{\alpha-1} c_{t}
\end{aligned}
$$

Steady state (explicit)

$$
\begin{aligned}
\beta\left[1-\delta+\alpha k^{\alpha-1}\right] & =1 \\
\alpha k^{\alpha-1} & =\frac{1-\beta(1-\delta)}{\alpha \beta} \\
k & =\left(\frac{1-\beta(1-\delta)}{\alpha \beta}\right)^{\frac{1}{\alpha-1}} \\
& =\left(\frac{1}{\alpha \beta}\right)^{\frac{1}{\alpha-1}}=(\alpha \beta)^{1 /(1-\alpha)}
\end{aligned}
$$

(modified golden rule) and

$$
\begin{aligned}
c & =f(k)-\delta k \\
& =(\alpha \beta)^{\alpha /(1-\alpha)}-(\alpha \beta)^{1 /(1-\alpha)} \\
& =(1-\alpha \beta)(\alpha \beta)^{\alpha /(1-\alpha)} \\
& =(1-\alpha \beta) k^{\alpha}
\end{aligned}
$$

Local dynamics.
Linearize the dynamic system

$$
\begin{aligned}
k_{t+1} & =k_{t}^{\alpha}-c_{t} \\
c_{t+1} & =\beta \alpha\left(k_{t}^{\alpha}-c_{t}\right)^{\alpha-1} c_{t}
\end{aligned}
$$

around the steady state and verify the saddle configuration.
Global dynamics.
Reconsider the dynamic system

$$
\begin{aligned}
k_{t+1} & =k_{t}^{\alpha}-c_{t} \\
c_{t+1} & =\beta \alpha\left(k_{t}^{\alpha}-c_{t}\right)^{\alpha-1} c_{t}
\end{aligned}
$$

We want to show that

$$
c_{t}=(1-\alpha \beta) k_{t}^{\alpha}
$$

is a solution. Substitute as follows

$$
\begin{aligned}
c_{t+1} & =(1-\alpha \beta) k_{t+1}^{\alpha} \\
\beta \alpha\left(k_{t}^{\alpha}-c_{t}\right)^{\alpha-1} c_{t} & =(1-\alpha \beta)\left(k_{t}^{\alpha}-c_{t}\right)^{\alpha} \\
\beta \alpha\left(k_{t}^{\alpha}-c_{t}\right)^{-1} c_{t} & =1-\alpha \beta
\end{aligned}
$$

Substitute again

$$
k_{t}^{\alpha}=\frac{c_{t}}{1-\alpha \beta}
$$

in

$$
\beta \alpha\left(k_{t}^{\alpha}-c_{t}\right)^{-1} c_{t}=1-\alpha \beta
$$

to obtain

$$
\begin{aligned}
\beta \alpha\left(\frac{c_{t}}{1-\alpha \beta}-c_{t}\right)^{-1} c_{t} & =1-\alpha \beta \\
\beta \alpha\left(\frac{1}{1-\alpha \beta}-1\right)^{-1} & =1-\alpha \beta \\
1-\alpha \beta & =1-\alpha \beta
\end{aligned}
$$

We want to show that the path described by

$$
c_{t}=(1-\alpha \beta) k_{t}^{\alpha}
$$

does not violate the transversality condition and converges to the steady state.

$$
\begin{aligned}
k_{t+1} & =k_{t}^{\alpha}-c_{t} \\
& =k_{t}^{\alpha}-(1-\alpha \beta) k_{t}^{\alpha} \\
& =k_{t}^{\alpha}[1-(1-\alpha \beta)] \\
& =\alpha \beta k_{t}^{\alpha}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
k_{1}= & \alpha \beta k_{0}^{\alpha} \\
k_{2}= & \alpha \beta k_{1}^{\alpha}=\alpha \beta\left(\alpha \beta k_{0}^{\alpha}\right)^{\alpha} \\
= & (\alpha \beta)^{1+\alpha} k_{0}^{\alpha^{2}} \\
& \vdots \\
k_{t}= & (\alpha \beta)^{1+\alpha+\ldots+\alpha^{t-1}} k_{0}^{\alpha^{t}}
\end{aligned}
$$

We observe that

$$
1+\alpha+\ldots+\alpha^{t-1}=\frac{1-\alpha^{t}}{1-\alpha}
$$

Therefore

$$
k_{t}=(\alpha \beta)^{\frac{1-\alpha^{t}}{1-\alpha}} k_{0}^{\alpha^{t}}
$$

To obtain the convergence to the steady state we take the limit

$$
\begin{aligned}
\lim _{t \rightarrow \infty} k_{t} & =\lim _{t \rightarrow \infty}(\alpha \beta)^{\frac{1-\alpha^{t}}{1-\alpha}} k_{0}^{\alpha^{t}} \\
& =(\alpha \beta)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

because $\alpha \in(0,1)$, which is exactly the steady state of modified golden rule we have previously found.

Therefore the given path is the saddle path in the phase diagram.

### 5.2.2 Central Planner

Let me focus now on the planner problem. An infinite-lived representative agent is considered (agents are assumed to be symmetric, i.e. to have same tastes and endowments). The agent maximizes the intertemporal functional

$$
\sum_{t=1}^{\infty} \beta^{-t} u\left(c_{t}\right)
$$

under the resource constraint

$$
f\left(k_{t}\right)=\left[k_{t+1}-(1-\delta) k_{t}\right]+c_{t} .
$$

To simplify formulas the capital depreciation rate is set equal to zero. A reduced production function is obtained by normalizing a constant return to scale production function: $f\left(k_{t}\right)=F\left(K_{t}, L_{t}\right) / L_{t}$. The planner cares about the utility level of the representative agent. The Lagrangian for the program is given by

$$
\sum_{t=1}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right)+\sum_{t=1}^{\infty} \lambda_{t}\left[f\left(k_{t}\right)-k_{t+1}+(1-\delta) k_{t}-c_{t}\right]
$$

We derive the first order conditions. The derivative with respect to $k_{t}$ is:

$$
\begin{aligned}
-\lambda_{t-1}+\lambda_{t} f^{\prime}\left(k_{t}\right)+(1-\delta) \lambda_{t} & =0 \\
\lambda_{t-1} / \lambda_{t} & =1-\delta+f^{\prime}\left(k_{t}\right)
\end{aligned}
$$

The derivative with respect to $c_{t}$ gives $\lambda_{t}=\beta^{-t} u^{\prime}\left(c_{t}\right)$, i.e. jointly with the previous condition

$$
\frac{u^{\prime}\left(c_{t-1}\right)}{u^{\prime}\left(c_{t}\right)}=\beta\left[1-\delta+f^{\prime}\left(k_{t}\right)\right]
$$

Moreover we must reconsider the law of motion for capital: $k_{t+1}-(1-\delta) k_{t}=$ $f\left(k_{t}\right)-c_{t}$ and the transversality condition ${ }^{2} \lim _{t \rightarrow \infty} \lambda_{t} k_{t}=0$. This allows us to write the modified golden rule computed at the stationary state: $u^{\prime}(c) / u^{\prime}(c)=$ $\beta\left[1-\delta+f^{\prime}(k)\right]$, i.e.

$$
\begin{aligned}
\beta\left[1-\delta+f^{\prime}(k)\right] & =1 \\
f^{\prime}(k) & =\frac{1}{\beta}-(1-\delta) \\
& \equiv \rho
\end{aligned}
$$

[^7]The stationary capital becomes

$$
k=f^{\prime-1}(\rho)
$$

with $f^{\prime \prime}<0$.
The consumption of stationary state is directly obtained from the motion law:

$$
\begin{aligned}
c & =f(k)-\delta k \\
\frac{c}{k} & =\frac{f(k)}{k}-\delta \\
& =\frac{f}{f^{\prime} k} \rho-\delta \\
& =\frac{\rho}{s}-\delta
\end{aligned}
$$

where $\rho$ is the stationary interest rate, while $s$ is the capital share on total income. Let us stress the fact that this stationary consumption of modified golden rule maximizes the intertemporal utility functional of a representative agent, i.e. the welfare. The alternative stationary state is trivial as in the Solow model: $k=c=0$, with $f(0)=0$.

Dynamic system.

$$
\begin{aligned}
k_{t+1} & =(1-\delta) k_{t}+f\left(k_{t}\right)-c_{t} \\
\beta\left[1-\delta+f^{\prime}\left(k_{t+1}\right)\right] u^{\prime}\left(c_{t+1}\right) & =u^{\prime}\left(c_{t}\right)
\end{aligned}
$$

Local analysis.
Total differential.

$$
\begin{aligned}
& d k_{t+1}=(1-\delta) d k_{t}+f^{\prime}(k) d k_{t}-d c_{t} \\
& \beta f^{\prime \prime}(k) u^{\prime}(c) d k_{t+1}+\beta[1-\delta\left.+f^{\prime}(k)\right] u^{\prime \prime}(c) d c_{t+1}=u^{\prime \prime}(c) d c_{t} \\
& d k_{t+1}=\left(1-\delta+f^{\prime}\right) d k_{t}-d c_{t} \\
& \beta f^{\prime \prime} u^{\prime} d k_{t+1}+u^{\prime \prime} d c_{t+1}=u^{\prime \prime} d c_{t} \\
& \beta f^{\prime \prime} d k_{t+1}+\frac{u^{\prime \prime}}{u^{\prime}} d c_{t+1}=\frac{u^{\prime \prime}}{u^{\prime}} d c_{t} \\
& \frac{d k_{t+1}}{k}=\left(1-\delta+f^{\prime}\right) \frac{d k_{t}}{k}-\frac{c}{k} \frac{d c_{t}}{c} \\
& \beta f^{\prime \prime} u^{\prime} d k_{t+1}+u^{\prime \prime} d c_{t+1}=u^{\prime \prime} d c_{t} \\
& \beta f^{\prime \prime} k \frac{d k_{t+1}}{k}+\frac{u^{\prime \prime} c}{u^{\prime}} \frac{d c_{t+1}}{c}=\frac{u^{\prime \prime} c}{u^{\prime}} \frac{d c_{t}}{c}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d k_{t+1}}{k} & =\frac{1}{\beta} \frac{d k_{t}}{k}-\frac{c}{k} \frac{d c_{t}}{c} \\
\frac{f^{\prime \prime} k}{f^{\prime}} \beta f^{\prime} \frac{d k_{t+1}}{k}-\left(-\frac{u^{\prime \prime} c}{u^{\prime}}\right) \frac{d c_{t+1}}{c} & =-\left(-\frac{u^{\prime \prime} c}{u^{\prime}}\right) \frac{d c_{t}}{c} \\
\frac{f^{\prime \prime} k}{f^{\prime}} \beta \rho \frac{d k_{t+1}}{k}-\left(-\frac{u^{\prime \prime} c}{u^{\prime}}\right) \frac{d c_{t+1}}{c} & =-\left(-\frac{u^{\prime \prime} c}{u^{\prime}}\right) \frac{d c_{t}}{c}
\end{aligned}
$$

Let

$$
\begin{aligned}
\varepsilon_{\rho} & \equiv \frac{f^{\prime \prime} k}{f^{\prime}} \\
\sigma & \equiv-\frac{u^{\prime}}{u^{\prime \prime} c}
\end{aligned}
$$

We have

$$
\begin{aligned}
\frac{d k_{t+1}}{k} & =\frac{1}{\beta} \frac{d k_{t}}{k}-\left(\frac{\rho}{s}-\delta\right) \frac{d c_{t}}{c} \\
\beta \rho \varepsilon_{\rho} \frac{d k_{t+1}}{k}-\frac{1}{\sigma} \frac{d c_{t+1}}{c} & =-\frac{1}{\sigma} \frac{d c_{t}}{c}
\end{aligned}
$$

Jacobian matrix.

$$
\begin{aligned}
J & =\left[\begin{array}{cc}
1 & 0 \\
\beta \rho \varepsilon_{\rho} & -\frac{1}{\sigma}
\end{array}\right]^{-1}\left[\begin{array}{cc}
\frac{1}{\beta} & -\left(\frac{\rho}{s}-\delta\right) \\
0 & -\frac{1}{\sigma}
\end{array}\right] \\
& =\left[\begin{array}{cc}
\frac{1}{\beta} & -\frac{\rho}{s}+\delta \\
\rho \varepsilon_{\rho} \sigma & 1+\beta \rho \sigma \varepsilon_{\rho}\left(\delta-\frac{\rho}{s}\right)
\end{array}\right]
\end{aligned}
$$

Trace.

$$
T=\frac{1}{\beta}+1+\beta \rho \sigma \varepsilon_{\rho}\left(\delta-\frac{\rho}{s}\right)
$$

Determinant.

$$
D=\frac{1}{\beta}
$$

Therefore

$$
T=1+D+\beta \rho \sigma \varepsilon_{\rho}\left(\delta-\frac{\rho}{s}\right)
$$

But

$$
\varepsilon_{\rho}<0
$$

because $f^{\prime \prime}<0$, and

$$
\delta-\frac{\rho}{s}=-\frac{c}{k}<0
$$

Then

$$
\begin{aligned}
& D=T-1-\beta \rho \sigma \varepsilon_{\rho}\left(\delta-\frac{\rho}{s}\right)<T-1 \\
& D=\frac{1}{\beta}>1
\end{aligned}
$$

Locating the point in the $(T, D)$-space, we conclude that the stationary state is a saddle. The point belongs to the shaded cone in figure 17.


Figure17. Saddle equilibrium.
The central planner selects $c_{t}$, given $k_{t}$, such that $\left(k_{t}, c_{t}\right)$ belongs to the saddle path. Diverging trajectories do not satisfy the transversality condition or the positivity constraints for the relevant economic variables.

Notice that in the example with

$$
\begin{aligned}
f\left(k_{t}\right) & =A k_{t}^{\alpha} \\
u\left(c_{t}\right) & =\ln c_{t}
\end{aligned}
$$

we have

$$
\begin{aligned}
s & =\alpha \\
\varepsilon_{\rho} & =-(1-\alpha)<0 \\
\sigma & =1
\end{aligned}
$$

Long-run effects.

$$
\begin{aligned}
f^{\prime}(k) & =\frac{1}{\beta}-(1-\delta) \\
\left(A k^{\alpha}\right)^{\prime} & =\frac{1}{\beta}-(1-\delta) \\
\alpha A k^{\alpha-1} & =\frac{1}{\beta}-(1-\delta) \\
k & =\left\{\frac{1}{\alpha A}\left[\frac{1-\beta(1-\delta)}{\beta}\right]\right\}^{\frac{1}{\alpha-1}} \\
k & =\left\{\frac{\alpha A \beta}{1-\beta(1-\delta)}\right\}^{\frac{1}{1-\alpha}}
\end{aligned}
$$

Comparative statics.

$$
\begin{aligned}
& \frac{\partial k}{\partial A}>0 \\
& \frac{\partial k}{\partial \beta}>0
\end{aligned}
$$

Notice that

$$
\frac{\partial}{\partial A} \frac{c}{k}=\frac{\partial}{\partial A}\left(\frac{\rho}{\alpha}-\delta\right)<0
$$

as

$$
\frac{\partial \rho}{\partial A}<0
$$

because $f^{\prime \prime}$ and thereby

$$
\frac{\partial \rho(k)}{\partial A}<0
$$

To characterize the transition compute the eigenvectors of the Jacobian matrix and the slope of the saddle path. Let $k_{0}$ be the old steady state and $k_{1}$ the new one. Let $s_{1}(k)$ the saddle path converging to the new steady state and set

$$
c_{0} \equiv s_{1}\left(k_{0}\right)
$$

The transition is from the initial condition after the shock

$$
\left(k_{0}, c_{0}\right)=\left(k_{0}, s_{1}\left(k_{0}\right)\right)
$$

to the new steady state

$$
\left(k_{1}, c_{1}\right)=\left(k_{1}, s_{1}\left(k_{1}\right)\right)
$$

In a basic decentralized economy, where agents are price-takers, the market performs exactly the planner solution we characterized above. A decentralized economy with money will be presented later on as well as the differences with the non-monetary decentralized models.

### 5.2.3 Open Economy

Even if the Ramsey-Cass-Koopmans framework constitutes a significant step ahead with respect to the basic Solow setup, it presents some non-negligeable controversial sides. The open economy version for instance implies counterfactual outcomes.

We assume for instance goods to be mobile across national borders of a small economy. Moreover we allow for international borrowing and lending. The interest rate is exogenously fixed by the international market. The representative agent of country $i$ solves the program $\max \sum_{t=1}^{\infty}\left(1+\theta_{i}\right)^{-t} u_{i}\left(c_{t}\right)$ under the budget constraint $a_{i t+1}-a_{i t}=w+r a_{i}-c_{i}$ where $a_{i}=k_{i}-d_{i}$, $d_{i} \equiv D_{i} / L_{i}$ is the net wealth and $D_{i}$ is the debt. By simplicity population growth is set equal to zero. Notice that $r$ is given, thus the productivity $f^{\prime}(k)$ is fixed, as well as the equilibrium capital $k$ and the real wage $w=f(k)-k f^{\prime}(k)$. The modified golden rule is no longer respected: $r=f^{\prime}(k) \neq \theta_{i}$ and the first welfare theorem does not apply because of this price rigidity. Thereby the market economy does not perform the planner solution. Two possible cases matters.
(i) $r<\theta_{i}$. Hence $k_{i, \text { open }}^{*}>k_{i, M G R}^{*}$ and $c_{i}$ converges to zero. To appreciate this result, notice that $c_{t+1} / c_{t}=\left[(1+r) /\left(1+\theta_{i}\right)\right]^{\sigma_{i}}<1$, where $\sigma_{i}$ is the elasticity of intertemporal substitution specific to country $i$.
(ii) If $r>\theta_{i}$ the domestic country cumulates enough assets to violate the small country assumption. At equilibrium if $r>\theta_{i}$ the domestic country grows and country consumption would eventually exceed world output. Before this happened, the world interest rate would adjust downward $(r \downarrow)$. So at steady state, $\theta_{i} \geq r$ for all countries and $r=\min \left\{\theta_{i}\right\}$. Without loss of generality let $\theta_{1}=\min \left\{\theta_{i}\right\}$. Asymptotically country 1 owns all the wealth in the sense of the claims on capital and the present value of wage income in all countries. All other countries own nothing (per person) in the long-run and their consumption converges towards 0 .

Such a simple extension of the Ramsey-Cass-Koopmans paradigm provides counter-factual results: convergence speeds for capital stock and output are infinite and, except for the most patient country, consumption (per unit of effective labor) tends to zero and assets become negative. The most patient country asymptotically will own everything and will consume nearly all of the world's output. To avoid this paradox we must enrich the basic setup. With imperfect international credit markets, the infinite speeds of convergence for capital would not apply to countries that were effectively constrained in their ability to borrow.

### 5.2.4 $A k$ Model

The Solow model (1956) claimed that poor countries catch-up developed nations if the fundamentals $(f(k), \delta, n, s)$ are identical. The evidence of the last decades contradicted this prevision. The augmented Solow model (1957) tried to explain the gap by introducing as residual an exogenous growth rate for technical progress. The answer was ingenious but unsatisfactory. In reality the technical progress is recognized to be endogenous. Even if rare pioneers (among the others Arrow in 1962) wrote about the endogenous progress before the Eighties, the literature on endogenous growth actually took-off after the seminal article of Romer (1986) on the increasing returns to scale in production due to externalities.

The most simple model of the endogenous growth literature is the $A k$ model due to Rebelo (1991). The name refers to a linear intensive production function. We consider here the central planner's solution. It is possible to show that the competitive equilibrium allocation coincides with the optimal planner's solution. Two models of endogenous growth in market economies, derived from the basic $A k$ are treated later. The first is a model with public spending externalities which displays the same reduced form of the $A k$ setup, the second is an $A k$ 's monetary version.

We consider a simple infinite horizon economy. The program of the representative agent consists in maximizing an intertemporal utility functional as in the Koopmans model:

$$
\sum_{t=1}^{\infty} \beta^{t} u\left(c_{t}\right)
$$

subject to the resource constraint

$$
f\left(k_{t}\right)=\left[k_{t+1}-(1-\delta) k_{t}\right]+c_{t} .
$$

The notation is usual (see the third chapter). $\beta$ is the subjective discount factor, an inverse measure of consumer's impatience. Capital depreciates at rate $\delta$. The intensive production function is linear: $f\left(k_{t}\right)=A k_{t}$. This is a crucial assumption: the aggregate production function displays increasing returns to scale indeed. The Lagrangian is set as follows:

$$
\sum_{t=1}^{\infty} \beta^{t} u\left(c_{t}\right)+\sum_{t=1}^{\infty} \lambda_{t}\left[f\left(k_{t}\right)-k_{t+1}+(1-\delta) k_{t}-c_{t}\right] .
$$

There are two sequences of first order conditions (still compare with the Cass-Koopmans model in the fifth chapter). Deriving with respect to $k_{t}$ and respect to $c_{t}$ we obtain respectively $\lambda_{t-1} / \lambda_{t}=1-\delta+f^{\prime}\left(k_{t}\right)$ and

$$
\beta^{t-1} u^{\prime}\left(c_{t-1}\right) /\left[(1+\theta)^{-t} u^{\prime}\left(c_{t}\right)\right]=\lambda_{t-1} / \lambda_{t}=1-\delta+f^{\prime}\left(k_{t}\right) .
$$

Thereby

$$
\frac{u^{\prime}\left(c_{t-1}\right)}{u^{\prime}\left(c_{t}\right)}=\beta\left[1-\delta+f^{\prime}\left(k_{t}\right)\right]
$$

This is the usual Euler condition we take in account jointly with the law of motion: $k_{t+1}-(1-\delta) k_{t}=f\left(k_{t}\right)-c_{t}$, and the necessary transversality condition $\lim _{t \rightarrow \infty} \lambda_{t} k_{t}=0$. The period utility is assumed to have a constant elasticity of intertemporal substitution, i.e.

$$
u\left(c_{t}\right)=\frac{c_{t}^{1-1 / \sigma}-1}{1-1 / \sigma}
$$

We know that $u^{\prime}\left(c_{t-1}\right) / u^{\prime}\left(c_{t}\right)=\left(c_{t} / c_{t-1}\right)^{1 / \sigma}$. In the case of logarithmic utility $u\left(c_{t}\right)=\ln c_{t}$, the elasticity is equal to 1 . From the Euler condition the consumption growth gross rate is obtained as

$$
\frac{c_{t+1}}{c_{t}}=\left\{\beta\left[1-\delta+f^{\prime}\left(k_{t+1}\right)\right]\right\}^{\sigma} .
$$

Notice that the productivity does not depend on the capital level: $f^{\prime}\left(k_{t+1}\right)=$ $A$. This is the source of endogenous growth and constitutes a simple but radical change with respect to the neoclassical assumption in the Solow model as well as in the class of endogenous saving models à la Ramsey-Cass-Koopmans and Diamond. Let $1+\gamma_{t+1}^{c} \equiv c_{t+1} / c_{t}$ be the consumption growth factor. So

$$
1+\gamma^{c}=[\beta(1-\delta+A)]^{\sigma} .
$$

There is no transition at all. The economy directly jumps on its stationary growth rate $\gamma^{c}$. Growth is exponential: $c_{t}=[\beta(1-\delta+A)]^{\sigma t} c_{0}$. From $k_{t+1}=$ $(1-\delta) k_{t}+A k_{t}-c_{t}$ we compute the capital growth gross rate: $1+\gamma_{t+1}^{k} \equiv$ $k_{t+1} / k_{t}=1-\delta+A-c_{t} / k_{t}$. The only possibility for growth are the balanced paths, i.e. a common growth rate for capital, production and consumption: $\gamma^{y}=\gamma^{k}=\gamma^{c} \equiv \gamma$. The balanced dynamics imply that the initial condition $k_{0}$ determines without ambiguity the initial product and consumption: $y_{0}=$ $f\left(k_{0}\right)$ and $[\beta(1-\delta+A)]^{\sigma}=1-\delta+A-c_{0} / k_{0}$, i.e. $c_{0}=(A-\gamma) k_{0}=$ $k_{0}\left\{1-\delta+A-[\beta(1-\delta+A)]^{\sigma}\right\}$. Finally we obtain the explicit trajectories.

$$
\begin{aligned}
c_{t} & =[\beta(1-\delta+A)]^{\sigma t} k_{0}\left\{1-\delta+A-[\beta(1-\delta+A)]^{\sigma}\right\}, \\
k_{t} & =[\beta(1-\delta+A)]^{\sigma t} k_{0}, \\
y_{t} & =[\beta(1-\delta+A)]^{\sigma t} f\left(k_{0}\right) .
\end{aligned}
$$

### 5.2.5 Transaction Costs and Indeterminacy

Business cycles and growth are two major fields of investigation in macroeconomic theory. The setup we provide allows to study the occurrence of endogenous fluctuations around an endogenous growth path in a monetary economy.

In literature the need of money is usually rationalized by putting money into either the objective functions such as the utility function (Sidrauski, 1967) and the production function (Dornbusch and Frenkel, 1973), or the constraints (Clower (1967), Stockman (1981)). In our work the monetary equilibrium is due to a negative impact of costly purchasing transactions.

More precisely we assume that the possession of liquidity reduces only the transaction costs of buying consumption goods without any effect on the costs of the transactions involving capital goods. The rationale is that, in presence of well functioning credit markets, liquidity is not essential for buying capital (see among others Dotsey and Sarte (2000)). The purchased capital plays the role of credit collateral. The consumer without capital provides no financial guarantees to gain access to the credit market and needs real balances to reduce the transaction costs. In this section the consumer endowed with money faces less obstacles during transaction and we assume for the sake of simplicity that he enjoys more consumption ${ }^{3}$.

[^8]Growth is endogenous. A large class of models displays a linear production function as reduced form for technology, and the adoption of the $A k$ shortcut (Rebelo, 1991) in the paper is justified to simplify the pure technological aspects and to focus instead on more complex monetary mechanisms.

Our contribution is above all an investigation about the occurrence of indeterminacy viewed as multiplicity of equilibrium growth paths, and the political solutions to select a unique equilibrium.

The incomplete markets' theory suggests some equivalence between market perfection (or completeness), equilibrium determinacy and Pareto- optimality. Even if a priori there is no indisputable definition of imperfection, the failure of the first welfare theorem requires by definition the existence of imperfections. In this sense incompleteness, externalities and market power, financial and monetary constraints can be viewed as imperfections. However imperfection does not entail automatically indeterminacy. Literature shows examples of dynamically inefficient but determinate equilibria (Cass, 1972). Conversely indeterminacy, as equilibrium multiplicity, implies sub-optimality and thereby requires some imperfection (see for instance Woodford, 1986a, for a financial imperfection).

Restricting attention to one-sector models displaying endogenous growth, the recent literature has identified several mechanisms at the origin of multiple equilibria. Among the others a chief type relies on the presence of increasing returns in production, which can be external (in a perfectly competitive framework) as well as internal (in a monopolistic competitive market) to the single economic units. Externalities in production as well as monopolistic competition on the one hand imply a marginal product of capital large enough for endogenous growth (Benhabib and Rustichini, 1994) and on the other hand they matter for equilibrium indeterminacy.

These models suggest that slight departures from the Real Business Cycle model are consistent with the idea that economic fluctuations may be driven not only by productivity disturbances, but also by the self-fulfilling beliefs of the agents. However such models lack predictive power and cannot therefore be helpful in shedding any light on the behavior of the economic equilibrium (Benhabib and Rustichini, 1994). An open question is whether the economic policy may only rule out the indeterminacy as condition for undesirable fluctuations or a more fine tuning is required which consists in selecting the Pareto-optimal path and coordinating the agents on the right starting point by means of some extra-signal.

In the Nineties the impact of the monetary growth rate on the endogenous
growth rate has been investigated. Conclusions are not unanimous.
Marquis and Reffett (1991) enrich a basic two-sector model with human capital accumulation through a cash-in-advance constraint, and obtain a superneutrality result. The benchmark of Lucas (1988) is revised by Wang and Yip (1991) to incorporate money in the production function still with a superneutrality result, and by Van der Ploeg and Alogoskoufis (1994) to introduce money in the utility function without superneutrality effects. They observe instead a positive impact of money growth on real growth. To the contrary Marquis and Reffett (1994) within a monetary version of Romer (1990) highlight a negative effect essentially because of the negative channel of the inflationary tax. Finally Mino and Shibata (1995) set up an overlapping-generations model with money in the utility function and find a positive effect of money growth (and inflation) on the long run growth rate.

In this section monetary imperfections are specified as flexible transaction costs of purchase (the cash-in-advance is a limit case). The necessary conditions for endogenous growth self-fulfilling fluctuations are characterized for a monetary economy within a simple discrete time setting. The monetary imperfections are properly captured in discrete time. The model provides an example of local indeterminacy and fluctuations of money velocity and inflation due to shocks on the beliefs. In economies endowed with only one consumption good velocity fluctuations are excluded by the cash-in-advance which fix the money velocity to one. A flexible transaction technology generalizes the basic cash-in-advance and allows a variable velocity. In this context the inflation rate displays a counter-cyclical impact on consumption. We shall focus on the transmission mechanism for local real indeterminacy. The role of consumption intertemporal substitution is interpreted as a consumer's ability to make ineffective the monetary constraint as market imperfection and source of local indeterminacy.

The ideal neoclassical worlds of Arrow-Debreu on the microeconomic side, and of Ramsey-Cass-Koopmans on the macroeconomic side, are characterized by existence, optimality, sometime uniqueness and stability of the general equilibrium. When these charming intellectual constructions are enriched in a very broad sense by market imperfections, there is room for Keynesian features such as disequilibrium phenomena, equilibrium multiplicity, suboptimality and instability.

The introduction of money in the general equilibrium theory is not a plain task. The cash-in-advance as well as the transaction technology we assume, are intellectual expedients which capture only a part of money complexity.

The following model will not provide definitive answers, but will shed a light on both these grounds. Under the play of a flexible transaction technology, we will investigate one special interference of money within a real economy and the action of a specific market imperfection for equilibrium multiplicity.

From now on $m_{t} \equiv M_{t-1} / p_{t}$ will denote the real balances. The velocity of circulation of money with respect to consumption is properly defined by

$$
\begin{equation*}
v_{t}=c_{t} / m_{t} \tag{5.1}
\end{equation*}
$$

according to the quantity identity ${ }^{4}$. The transaction costs of consumption purchase are assumed to be homogeneous of degree one in consumption and money:

$$
S\left(c_{t}, m_{t}\right) \equiv s\left(c_{t} / m_{t}\right) c_{t}
$$

where $s_{t}=s\left(v_{t}\right)$ represents the transaction cost to buy one unit of consumption good. The money employed in period $t$ to purchase $c_{t}$ at price $p_{t}$ is set aside at the end of period $t-1$.

Assumption 1. The intensive transaction cost function $s(v)$ satisfies the constraints

$$
\begin{align*}
s(0) & =0,  \tag{5.2}\\
s^{\prime}(v) & >0,  \tag{5.3}\\
2 s^{\prime}(v)+s^{\prime \prime}(v) & >0,  \tag{5.4}\\
2 s^{\prime}(v)+v s^{\prime \prime}(v) & >0 . \tag{5.5}
\end{align*}
$$

The first equality means that if the agent does not consume or he is not financially constrained, he pays no transaction cost. The first inequality claims that more the individual is financially constrained, i.e. the lower is the ratio $c_{t} / m_{t}$, the higher turns out to be the transaction cost per consumption unit. Inequalities (5.4) and (5.5) are mild restrictions and require the transaction costs to be not too concave. For instance they are satisfied by every convex function. The power function

$$
s\left(v_{t}\right)=v_{t}^{\alpha}
$$

satisfies (5.2), (5.3) and (5.5) whatever $\alpha>0$, and satisfies (5.4) for $\alpha>$ $1-2 v$.

[^9]The problem a representative agent faces consists in selecting the functions $m_{t} \equiv M_{t-1} / p_{t}, k_{t}, c_{t}$ (intertemporal trajectories for real balances, productive capital and consumption), in order to maximize the usual intertemporal utility functional

$$
\sum_{t=1}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right)
$$

under a budget constraint which incorporates the consumption transaction costs:

$$
\begin{equation*}
\left(1+\pi_{t+1}\right) m_{t+1}+k_{t+1}+\left[1+s\left(c_{t} / m_{t}\right)\right] c_{t} \leq\left(1+r_{t}\right) k_{t}+m_{t}+\tau_{t} \tag{5.6}
\end{equation*}
$$

Tastes at period $t$ are represented by the utility function $u\left(c_{t}\right)$ and the rate of time preference $\theta$. Utility gets the usual $C E S$ form. The elasticity of intertemporal substitution is set equal to $\sigma$ :

$$
u\left(c_{t}\right) \equiv \frac{c_{t}^{1-1 / \sigma}-1}{1-1 / \sigma}
$$

For the sake of simplicity in (5.6) we assume that the capital does not depreciate ${ }^{5}$.

The inflation factor and the real interest rate are denoted respectively by $1+\pi_{t+1} \equiv p_{t+1} / p_{t}$ and $r_{t} . \tau_{t} \equiv\left(M_{t}-M_{t-1}\right) / p_{t}$ represents the real transfers from the monetary authority to consumers at period $t$. As above a simple monetary rule is adopted: $1+\mu=M_{t} / M_{t-1}$. Initial conditions are specified by the nominal money $M_{0}$ and capital $k_{0}$.

Setting the infinite horizon Lagrangian and rearranging the first order conditions, we obtain the Euler condition, which describes the evolution of consumption across the time.

$$
\begin{equation*}
\frac{c_{t+1}}{c_{t}}=\left[\frac{1+r_{t+1}}{1+\theta} \frac{1+s\left(v_{t}\right)+s^{\prime}\left(v_{t}\right) v_{t}}{1+s\left(v_{t+1}\right)+s^{\prime}\left(v_{t+1}\right) v_{t+1}}\right]^{\sigma} \tag{5.7}
\end{equation*}
$$

The gross inflation factor depends on the velocity of money:

$$
\begin{equation*}
1+\pi_{t+1}=\frac{1+s^{\prime}\left(v_{t+1}\right) v_{t+1}^{2}}{1+r_{t+1}} \tag{5.8}
\end{equation*}
$$

[^10]The qualitative results we shall obtain, will not depend on $\delta$.

In the well known Rebelo's (1991) $A k$ model, a world with no money, the equilibrium interest rate is pegged by the technology: $r_{t}=A$, and the relevant Euler condition fixes a unique endogenous growth factor:

$$
\begin{equation*}
\frac{c_{t+1}}{c_{t}}=1+\gamma \tag{5.9}
\end{equation*}
$$

where

$$
\begin{equation*}
1+\gamma \equiv\left(\frac{1+A}{1+\theta}\right)^{\sigma} \tag{5.10}
\end{equation*}
$$

The economy jumps on this unique growth path. There is no room for transition.

To the converse the monetary version we are considering, allows a transition. Even if at the steady state the velocity of money is constant and equation (5.9) still holds from (5.7), in the short run the growth rate, as shown by (5.7), may deviate from the stationary rate $\gamma$. More precisely if the velocity of money differs from its long run value (this is possible if and only if the equilibrium is indeterminate), then the velocity interferes with the consumption growth rate. Thereby the economy no longer jumps from the beginning on the balanced growth path and a transition actually arises during which the monetary imperfection is no longer neutral.

In our model if the equilibrium is determinate, agents' coordination under rational expectations selects the unique equilibrium, i.e. the stationary one and fixes the non-predetermined velocity to the steady value $v$. The parameter range allowing for determinacy and ruling out transition cycles is provided at the end of the paper.

The intertemporal paths for money and capital is now computed. First notice that $\tau_{t}=\left(1+\pi_{t+1}\right) m_{t+1}-m_{t}$. Thus at equilibrium constraint (5.6) becomes

$$
\begin{equation*}
\left[1+s\left(v_{t}\right)\right] c_{t}+k_{t+1}=\left(1+r_{t}\right) k_{t} \tag{5.11}
\end{equation*}
$$

A linear technology

$$
f\left(k_{t}\right) \equiv A k_{t}
$$

as usual is enough to sustain the endogenous growth. The equilibrium of the firm implies

$$
r_{t}=f^{\prime}\left(k_{t}\right)=A
$$

The relevant dynamics for the velocity of circulation of money with respect to consumption $v_{t}=c_{t} / m_{t}$ is given by the implicit function:

$$
\begin{align*}
\Phi\left(v_{t}, v_{t+1}\right) & \equiv(1+a) \frac{v_{t+1}}{v_{t}}-\left[1+s^{\prime}\left(v_{t+1}\right) v_{t+1}^{2}\right]\left[\frac{1+s\left(v_{t}\right)+s^{\prime}\left(v_{t}\right) v_{t}}{1+s\left(v_{t+1}\right)+s^{\prime}\left(v_{t+1}\right) v_{t+1}}\right]^{\sigma} \\
& =0 \tag{5.12}
\end{align*}
$$

where $a$ is set as follows

$$
\begin{equation*}
1+a \equiv \frac{(1+A)(1+\mu)}{1+\gamma} \tag{5.13}
\end{equation*}
$$

and $\gamma$ is still given by (5.10).
We define

$$
\begin{equation*}
y_{t} \equiv c_{t} / k_{t} . \tag{5.14}
\end{equation*}
$$

From (5.11) and (5.14) we obtain

$$
\begin{equation*}
k_{t+1} / k_{t}=1+A-\left[1+s\left(v_{t}\right)\right] y_{t} . \tag{5.15}
\end{equation*}
$$

As

$$
\frac{y_{t+1}}{y_{t}}=\frac{c_{t+1} / c_{t}}{k_{t+1} / k_{t}},
$$

we get the following discrete time dynamic system from (5.7), (5.12) and (5.15)

$$
\begin{align*}
\Phi\left(v_{t}, v_{t+1}\right) & =0  \tag{5.16}\\
y_{t+1} & =(1+\gamma)\left[\frac{1+s\left(v_{t}\right)+s^{\prime}\left(v_{t}\right) v_{t}}{1+s\left(v_{t+1}\right)+s^{\prime}\left(v_{t+1}\right) v_{t+1}}\right]^{\sigma} \frac{y_{t}}{1+A-\left[1+s\left(v_{t}\right)\right] y_{t}^{(5)}}{ }^{17)}
\end{align*}
$$

The steady state $(v, y)$ of system (5.16-5.17) is implicitly given by

$$
\begin{align*}
s^{\prime}(v) v^{2} & =a,  \tag{5.18}\\
y & =\frac{A-\gamma}{1+s(v)}, \tag{5.19}
\end{align*}
$$

where $\gamma$ is defined by (5.10).
As the consumption-capital ratio $y$ must be positive, we assume

$$
\begin{equation*}
A>\gamma \tag{5.20}
\end{equation*}
$$

Inequality (5.20) always holds if $A>\theta$ and $\sigma<1$.
More explicitly growth is balanced as in the Rebelo's (1991) $A k$ model:

$$
\gamma_{t}^{m}=\gamma_{t}^{k}=\gamma_{t}^{c}=\gamma
$$

where $\gamma_{t}^{m}, \gamma_{t}^{k}, \gamma_{t}^{c}$ denote respectively the growth rates for real balances, capital and consumption (see equations (5.1), (5.15) and (5.7)).

For example the class of power functions $s(v)=v^{\alpha}, \alpha>0$ provides

$$
\begin{aligned}
v & =(a / \alpha)^{1 /(1+\alpha)} \\
y & =\frac{1+A-[(1+A) /(1+\theta)]^{\sigma}}{1+(a / \alpha)^{\alpha /(1+\alpha)}}
\end{aligned}
$$

Under the Assumption 1 the impact of money growth $\mu$ on velocity $v$ is positive, because it raises the inflation rate and the nominal interest rate, i.e. the opportunity cost of holding money. Notice that under Assumption 1 we observed counter-cyclical effects of money velocity. More generally $d v / d a>0$. In particular all the parameters raising $a$, raise $v$ and increase the transaction costs. Furthermore more impatient consumer is, bigger turns out to be the stationary velocity $v$, because $\partial a / \partial \theta>0$, as intuition suggests: the greater the current consumption need, the faster the circulation of money. The impact of the intertemporal substitution $\sigma$ on $a$, and thereby on $v$, is negative, provided that $A>\theta$, i.e. the long term velocity decreases under higher substitution of the present consumption by future purchases. $A$ has a positive impact on $a$ and $v$, if and only if $\sigma<1$, i.e. higher the productivity, higher the current consumption and money velocity, under sufficiently low intertemporal substitution. Otherwise for $\sigma>1$, saving prevails on consumption, and $v$ slows down in our restrictive interpretation of the quantity theory that focuses on consumption transactions.

The Euler condition (5.7) directly provides the stationary consumption growth which is exactly that of the non-monetary version (Rebelo, 1991) in equation (5.9). Growth is balanced: the growth rate is the same for real balances and capital. Even if money turns out to be superneutral at the steady state, it affects the transition. As above, the transversality condition restricts the set of plausible parameters. At the steady state $\lim _{t \rightarrow \infty} \lambda_{t} k_{t}=0$. More explicitly

$$
\lim _{t \rightarrow \infty} \lambda_{t} k_{t}=\lim _{t \rightarrow \infty}(1+\theta)^{-t} u^{\prime}\left(c_{t}\right) k_{t}
$$

$$
\begin{aligned}
& =\lim _{t \rightarrow \infty}(1+\theta)^{-t} c_{t}^{-1 / \sigma} k_{t} \\
& =\lim _{t \rightarrow \infty}(1+\theta)^{-t}\left[(1+\gamma)^{t} c_{0}\right]^{-1 / \sigma}(1+\gamma)^{t} k_{0} \\
& =\lim _{t \rightarrow \infty}\left[(1+\theta)^{-1}(1+\gamma)^{1-1 / \sigma}\right]^{t} c_{0}^{-1 / \sigma} k_{0}=0
\end{aligned}
$$

The term into the brackets must be less than one, i.e.

$$
1+\theta>(1+\gamma)^{1-1 / \sigma}
$$

This inequality constitutes the transversality condition in endogenous growth.
Let us focus now on the occurrence of local real indeterminacy.
The variables $v_{t}=c_{t} / m_{t}$ and $y_{t}=c_{t} / k_{t}$ are independently non- predetermined because $c_{t}$ and $m_{t}$ (i.e. $p_{t}$ ) are independently non predetermined. Local indeterminacy arises if and only if the dimension of the stable manifold is strictly greater than the number of pre-determined variables. In our case this number is zero and hence we require a configuration of either saddle or sink for our stationary state to observe indeterminacy ${ }^{6}$.

The Jacobian matrix of system (5.16-5.17) evaluated at the steady state (5.18-5.19) is given by

$$
J=\left[\begin{array}{cc}
\lambda_{1} & 0 \\
j & \lambda_{2}
\end{array}\right]
$$

where

$$
\begin{align*}
& \lambda_{1}=-\frac{\partial \Phi / \partial v_{t}}{\partial \Phi / \partial v_{t+1}}  \tag{5.21}\\
& \lambda_{2}=\frac{1+A}{1+\gamma}>1 \tag{5.22}
\end{align*}
$$

are the eigenvalues of $J$ and

$$
\begin{equation*}
j \equiv y\left[\frac{y s^{\prime}(v)}{1+\gamma}+\sigma\left(1+\frac{\partial \Phi / \partial v_{t}}{\partial \Phi / \partial v_{t+1}}\right) \frac{2 s^{\prime}(v)+v s^{\prime \prime}(v)}{1+s(v)+v s^{\prime}(v)}\right] \tag{5.23}
\end{equation*}
$$

[^11]The sink configuration is ruled out by $\lambda_{2}>1$, that is the required condition for the consumption to be positive. Therefore local indeterminacy occurs, if and only if the stationary state is a saddle:

$$
\begin{equation*}
\left|-\frac{\partial \Phi / \partial v_{t}}{\partial \Phi / \partial v_{t+1}}\right|_{v}<1 . \tag{5.24}
\end{equation*}
$$

Inequality (5.24) holds if and only if

$$
\begin{equation*}
\left|1-\frac{v^{2} /(1+a)}{1 /\left(2 s^{\prime}+s^{\prime \prime} v\right)+\sigma v /\left(1+s+s^{\prime} v\right)}\right|>1 \tag{5.25}
\end{equation*}
$$

that is a necessary and sufficient condition for local indeterminacy.
We want to prove that in the $\left(v_{t}, y_{t}\right)$-plane the saddle path we obtain under (5.25) and (5.5) is downward-sloped.

Let $v_{1}$ and $v_{2}$ be the eigenvectors respectively associated to the eigenvalues $\lambda_{1}$ and $\lambda_{2}$, and $V$ be the eigenvector matrix. In our case (triangular Jacobian) we get

$$
V \equiv\left[v_{1}, v_{2}\right]=\left[\begin{array}{cc}
\left(\lambda_{1}-\lambda_{2}\right) / j & 0 \\
1 & 1
\end{array}\right]
$$

We have normalized the second component of each eigenvector to one.
In the linearized dynamics the starting point $\left(v_{0}, y_{0}\right)$ belongs to the convergent path if and only if

$$
\lim _{t \rightarrow \infty}\left[\begin{array}{c}
v_{t}-v \\
y_{t}-y
\end{array}\right]=\lim _{t \rightarrow \infty}\left(J^{t}\left[\begin{array}{l}
v_{0}-v \\
y_{0}-y
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

where $(v, y)$ is the stationary state. Let

$$
\Lambda \equiv\left[\begin{array}{cc}
\lambda_{1} & 0 \\
0 & \lambda_{2}
\end{array}\right]
$$

be the Jordan canonical form. We obtain

$$
\lim _{t \rightarrow \infty}\left(V \Lambda^{t} V^{-1}\left[\begin{array}{l}
v_{0}-v \\
y_{0}-y
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

This is possible if and only if the explosive eigenvalue $\left(\lambda_{2}>1\right)$ is ruled out, i.e. if and only if

$$
V^{-1}\left[\begin{array}{l}
v_{0}-v  \tag{5.26}\\
y_{0}-y
\end{array}\right]=\left[\begin{array}{l}
c \\
0
\end{array}\right]
$$

where $c$ is a constant. From (3.15) we compute the second row of the vector equation (5.26):

$$
-j\left(v_{0}-v\right)+\left(\lambda_{1}-\lambda_{2}\right)\left(y_{0}-y\right)=0
$$

i.e. the equation (5.27) of the tangent line to the stable manifold.

Under (5.25) we obtain $\lambda_{1}<\lambda_{2}$, while under (5.25) and (5.5) we get $j>0$ (see (5.23). Hence

$$
\frac{j}{\lambda_{1}-\lambda_{2}}<0
$$

and the saddle path is downward-sloped in a neighborhood of $(v, y)$.
The linearized saddle path is computed:

$$
\begin{equation*}
y_{t}=m v_{t}+n, \tag{5.27}
\end{equation*}
$$

where

$$
\begin{aligned}
m & \equiv \frac{j}{\lambda_{1}-\lambda_{2}}<0 \\
n & \equiv y-\frac{j}{\lambda_{1}-\lambda_{2}} v>0 .
\end{aligned}
$$

$v, y, \lambda_{1}, \lambda_{2}, j$ are respectively provided by (5.18), (5.19), (5.21), (5.22), (5.23).
Assume now that (5.25) is satisfied, i.e. there is local indeterminacy. Rational agents coordinate their initial behavior to stay on the saddle path which is compatible with a long-run equilibrium: $\left(v_{0}, y_{0}\right)$ must belong to the saddle path. As there is local indeterminacy the agents freely implement $v_{0}$, but they are forced to satisfy approximately (5.27) in a neighborhood of the steady state, i.e. to select the convergent equilibrium path:

$$
y_{0} \approx m v_{0}+n
$$

In other words the choice of $y_{0}$ is no longer free. As $y_{0}=c_{0} / k_{0}$ and $k_{0}$ is a predetermined variable, the agents choose the right consumption $c_{0}$ to stay on the saddle path from the beginning on and to converge to the steady state. As the saddle path is locally downward-sloped, a lower initial velocity ( $v_{0}<v$ ) will entail lower transaction costs and then a higher initial consumption $c_{0}$ and a lower initial consumption growth rate $\gamma_{1}^{c}=c_{1} / c_{0}-1<\gamma$ (see also the Euler equation (5.7)).

Moreover we notice that under (5.25) and (5.5)

$$
\begin{equation*}
-1<\lambda_{1}<0 \tag{5.28}
\end{equation*}
$$

Thereby the transition sequence $\left\{v_{t}, y_{t}\right\}_{t=0}^{\infty}$ converges to the steady state $(v, y)$ displaying contracting oscillations of period 2 around $(v, y)$ along the saddle path.

The currency velocity displays counter-cyclical effects for consumption dynamics in a neighborhood of the steady growth. From (5.7) we observe that under the Assumption $1 v_{t+1}>v_{t}$ entails $c_{t+1} / c_{t}<1+\gamma$. The consumption growth rate falls under its balanced long run value. In less formal terms the reduction of real balances, raising the velocity of money and transaction costs, temporarily slows down the consumption growth.

More precisely an increase in money velocity due to a contraction of real balances, is associated to a raise of the opportunity cost of holding money, which is represented by the nominal interest rate $i_{t}=\left(1+\pi_{t}\right)\left(1+r_{t}\right)-1$. The opportunity cost of real balances (and consumption) is interpreted as the relative "price" of the consumption good with respect to capital ${ }^{7}$.

We notice that the real interest rate, $r_{t}=A$, is constant over the time and the nominal interest rate dynamics are due only to inflation movement.

In the very short term there exists a negative relation between the inflation and consumption growth. We observe according to equation (5.8) and Assumption 1 that $\pi_{t+1}>\pi_{t}$ if and only if $v_{t+1}>v_{t}$. Thereby an increasing inflation across the time pulls the consumption growth rate below its long run value and conversely a decreasing inflation pushes this growth rate above the balanced growth rate.

The increase of the opportunity cost of consumption depresses the consumption growth rate below the long run value, but makes the capital relatively cheaper. Thereby capital accumulation is boosted as well as the consumption growth rate of the following period, which will exceed the balanced growth rate.

This is the rationale for the oscillations of period two we have formally obtained in (5.28).

In general the literature is not unanimous about the inflation impact on growth and theoretical models are often powerless to emphasize a strong neg-

[^12]ative relation, basically because of money superneutrality ${ }^{8}$. In the empirical models low inflation are sometime recognized to stimulate growth. Higher inflation rates by confounding relative price signals, make resource allocation inefficient and slow down the growth ${ }^{9}$.

If the equilibria are indeterminate, the agents may individually saturate this degree of freedom by relating their choices to exogenous random signals (sunspots), which do not affect the fundamentals (technology, preferences and endowments). The probability distribution of a sunspot is assumed to be common knowledge and it is inferred from past realizations. In other words the sunspot shocks the believes instead of the fundamentals. If the way of relating the economic future to this distribution is the same for all the agents, the believes are shared. If the choices of the agents and shared believes satisfy the stochastic version of dynamic system (5.12), the shared believes become self-fulfilling prophecies (Azariadis, 1981). Local indeterminacy is the necessary condition to observe stochastic (sunspot) equilibria, i.e. endogenous fluctuations (among the others Grandmont, 1991). According to a Woodford's conjecture (1986b), it turns out to be also sufficient. Under higher transaction costs $\alpha$ and a low elasticity of intertemporal substitution $\sigma$ the economy displays local indeterminacy and possible stochastic (sunspot)

[^13]equilibria of endogenous growth. A higher absolute value of the elasticity of marginal utility is equivalent to either a lower intertemporal substitution or a higher risk aversion across the states of nature. Thus the behavior of a risk averse consumer subject to strong monetary constraints may be a source of endogenous fluctuations.

We stress the possibility of fluctuations of the currency velocity due to shocks on the beliefs. With a standard binding cash-in-advance velocity fluctuations are ruled out $\left(v_{t}=1\right)$. The possibility of exogenous fluctuations with shocks on the fundamentals has already been shown in exogenous growth by Lucas and Stokey (1987). However the authors need two goods (cash and credit good) and fluctuations end up being strictly exogenous. As seen above, in our model the impact of velocity on the transitional consumption growth rate is recognized to be counter-cyclical.

Eventually we notice that the choice of a discrete time setting to study monetary imperfections is not neutral for indeterminacy. Transactions are not continuous in time and usually the consumer does not dispose of liquid amount prior to some instant of cashing. A discrete timing better captures momentary exchanges. Thereby a continuous time approach is a less precise language to describe a sequence of isolated payments. In particular local indeterminacy disappears in the continuous time version of the model and there is no longer room for endogenous fluctuations.

Let us provide now a numerical example. If $s(v)=v^{\alpha}, \alpha>0$, the condition for local indeterminacy (5.25) becomes

$$
\frac{1+a}{a(1+\alpha)}+\frac{\alpha \sigma(1+a)}{\alpha v+a(1+\alpha)}<\frac{1}{2} .
$$

Local indeterminacy arises for instance if $\alpha$ is sufficiently high (higher transaction costs) and $\sigma$ is sufficiently low (difficulty to substitute consumption intertemporally).

We notice that for $\alpha=+\infty$ we obtain the cash-in-advance:

$$
\lim _{\alpha \rightarrow+\infty} v=\lim _{\alpha \rightarrow+\infty}(c / m)=1
$$

and the local indeterminacy condition becomes

$$
\sigma=\lim _{\alpha \rightarrow+\infty} \frac{1+a}{a(1+\alpha)}+\frac{\alpha \sigma(1+a)}{\alpha v+a(1+\alpha)}<\frac{1}{2},
$$

which is exactly the condition we require in the class of endogenous growth models with cash-in-advance (see Bloise, Bosi and Magris (2000) and Bosi (2001)).


Figure 18. Local indeterminacy region.
In the picture the shaded area represents the set of local indeterminacy parameter pairs $(\alpha, \sigma)$. The bifurcation frontier captures the trade-off between the monetary constraint and intertemporal substitution. Productivity, time preference and monetary rule are respectively fixed to $A=10 \%, \theta=1 \%$, $\mu=2 \%$.

If the monetary authority follow an alternative monetary policy which consists in pegging the interest rate ( $i_{t}=i$ ), then the velocity of money turns out to be fixed as well: $1+s^{\prime}\left(v_{t+1}\right) v_{t+1}^{2}=\left(1+\pi_{t+1}\right)(1+A)=1+i$. There is no longer transition. According to equation (5.7) the economy jumps on its long run growth rate defined by (5.10).

### 5.2.6 More on Endogenous Growth

In the Solow model the growth rate results as sum of the demographic growth rate and the rate of technological progress. As these rates are taken to be exogenous, growth is exogenous as well. In the endogenous growth models the growth rate is endogenously explained by the fundamentals. For instance in the continuous time version of the $A k$ model is given by $\gamma^{c}=\sigma(A-\theta)$ with the usual notations (the long run growth rate depends on the parameters that determine the willingness to save and the productivity of capital). The $A k$ model is Pareto-optimal as well as the exogenous growth models of Ramsey-Cass-Koopmans.

Romer (1986) investigates the effects of externalities in production. Profit
maximization under private decreasing returns is compatible with an endogenous growth sustained by aggregate increasing returns. The first welfare theorem fails because of the externality effects and there is a room for the government policy. Romer (1986) and Barro (1990) (with public spending and income taxation) have an identical reduced form, the Rebelo $A k$ and they display no transition as well.

Jones and Manuelli (1990) introduce a flexible technology to explore the transition dynamics: $Y=A K+\Omega(K, L) . \Omega$ is a usual constant returns to scale production function, which affects the short term dynamics and allows a transition, while the $A K$ part can be interpreted as a long term technology.

In the one-sector model with physical $(K)$ and human capital $(H)$ the growth rate of output is inversely related to the ratio $K / H$ as long this ratio is below its steady state value. The relation between the growth rate of output and $H / K$ can be described as an imbalance effect. The greater the imbalance, that is the further $K / H$ is below its steady state value, the higher the growth rate. Rebelo (1991) consider a production world with two sectors. There are two laws of motion for physical and human capital: $Y=C+K^{\prime}+\delta K=$ $A(v K)^{\alpha}(u H)^{1-\alpha}$ and $H^{\prime}+\delta H=B[(1-v) K]^{\beta}[(1-u) H]^{1-\beta}$, where the notation is usual, $(A, \alpha)$ and $(B, \beta)$ are sector specific technological parameters, $(1-v)$ and $(1-u)$ are the residual fractions of resources devoted to education. In the two sectors model of production (one that produced consumables and physical capital and another that created human capital) the growth rate of output (defined broadly to include the production of new human capital) tends to rise with the extent of the imbalance if human capital is relatively abundant, but to decline with the extent of the imbalance if human capital is relatively scarce. This results imply that an economy would recover rapidly in reaction to a war that destroyed primarily physical capital, but would rebound only slowly from an epidemic that eliminated mainly human capital. The key formula is the following: $p^{\prime} / p=A \Phi^{\alpha /(\beta-\alpha)}$ $\left[\alpha \Phi^{1 /(\beta-\alpha)} p^{(1-\alpha) /(\beta-\alpha)}-(1-\alpha) p^{\beta /(\beta-\alpha)}\right] \equiv \varphi(p)$ with $\Phi$ a constant depending on the fundamental parameters and $p=\Phi[(v K) /(u H)]^{\alpha-\beta}$. The consumption growth is described by $\gamma^{c}=\sigma\left\{\alpha A[(u H) /(v K)]^{1-\alpha}-\delta-\theta\right\}$, where $\theta$ captures the time preference. If $\alpha>\beta$ (education sector is relatively intensive in human capital and the good sector is relatively intensive in physical capital) the steady state $p^{*}$ (or equivalently $(K / H)^{*}$ ) is stable. Indeed $\varphi^{\prime}\left(p^{*}\right)<0$. So our previous conclusions follow. If $\alpha=\beta$ the standard $A K$ model arises. In general one can think $K$ and $H$ as two generic produc-
tion factors and no restrictions will be imposed on $\alpha$ and $\beta$. So the steady state could turn out to be unstable. Uzawa (1965) and Lucas (1988) can be reviewed as particular cases of Rebelo (1991) $(\beta=0)$.

Another class of models focuses on the mechanisms of technology diffusion. In a multi-economy setting, the key issue here is how rapidly the discoveries made by leading economies diffuse to followers economies. It is possible to show that if the diffusion of technology occurs gradually, then we get another reason to predict a pattern of convergence across economies. It is possible to study the technological diffusion in the context of the model of variety of intermediate products. Similar results hold in models of quality improvement. The main idea is that follower countries tend to catch up to the leaders because imitation and implementation of discoveries are cheaper than innovation. This mechanism tends to generate convergence even if diminishing returns to capital or to R\&D do not apply.

The "leader - follower" model is inspired to Krugman (1979) and Grossman and Helpman (1991). The leading country is denoted by 1, the follower country by 2 . The level of technology is given by the number of varieties of intermediate products $N_{1}$ that have been discovered by the technological leader. Country 1 innovates, while country 2 imitates. We do not allow for innovation in country 2 or imitation in country 1 . The country 2's cost of imitation is $\nu<\eta$ the country 1 's cost of innovation. In country 1 the inventor of a new type of product is also the monopolistic provider. Country 1's production function is specified as $Y_{1}=A_{1} L_{1}^{1-\alpha} \sum_{j=1}^{N_{1}} X_{1 j}^{\alpha}$, where $X_{1 j}$ is the $j$ th intermediate good. The remaining notations are usual. Country 2's production function is $Y_{2}=A_{2} L_{2}^{1-\alpha} \sum_{j=1}^{N_{2}} X_{2 j}^{\alpha}$. The final goods are tradable across countries and are exchanged at a single world price. In contrast the intermediate products do not flow freely across international borders. It is possible to show that each intermediate good is sold at the monopoly price $1 / \alpha$ (mark-up with the production cost normalized to one). For more details the reader is referred to Barro and Sala-i-Martín (1995, p. 217). The level of output per worker in country 1 is: $y_{1}=A_{1}^{1 /(1-\alpha)} \alpha^{2 /(1-\alpha)} N_{1}$ : the rate of growth is the same of $N_{1}$. The level of output per worker in country 2 is: $y_{2}=A_{2}^{1 /(1-\alpha)} \alpha^{2 /(1-\alpha)} N_{2}$ : the rate of growth is the same of $N_{2}$. The free entry condition for country 1 is $\pi_{1 j}=\eta$. The free entry condition for country 2 is $\pi_{2 j}=\nu$. The two countries do not participate in a common capital market and hence $r_{2}$ can diverge from $r_{1}$. The equilibrium interest rate in country 1 is $r_{1}=L_{1}(1-\alpha) /(\alpha \eta) A_{1}^{1 /(1-\alpha)} \alpha^{2 /(1-\alpha)}$. The equilibrium interest rate in
country 2 whenever $N_{2}<N_{1}$ is $r_{2}=L_{2}(1-\alpha) /(\alpha \eta) A_{2}^{1 /(1-\alpha)} \alpha^{2 /(1-\alpha)}$. The consumers in the two countries are assumed to have the same preferences. The endogenous growth rate in country 1 and country 2 (as in the $A k$ model) turn out to be respectively $\gamma_{1}=\sigma\left(r_{1}-\theta\right)$ and $\gamma_{2}=\sigma\left(r_{2}-\theta\right)$. If $L_{1}=L_{2}$ and $A_{1}=A_{2}$, then $\nu<\eta$ implies $r_{2}>r_{1}$ and hence $\gamma_{2}>\gamma_{1}$. The following conclusions hold. If $\gamma_{2}>\gamma_{1}$, then $N_{2}$ grows faster than $N_{1}$, and eventually rises to equal $N_{1}$. Hence once country 2 learned all of country 1's designs, the two countries would grow at the same rate. Country 1 continues to innovate and country 2 immediately imitates the results. The follower country with $N_{2}<N_{1}$ grows at a faster rate than the leader.

### 5.2.7 More on Indeterminacy

A dynamic general equilibrium can be viewed as a time path of prices and quantities such that all markets clear in each period. A dynamic economy may display a multiplicity of stationary states and a multiplicity of equilibrium paths converging to one particular attractor, which is usually a stationary state. A local real indeterminacy arises as a continuum of equilibrium paths in a neighborhood of the attractor, when the initial conditions (or equivalently the conditions inherited from the previous period) are not sufficient to select a unique sequence for the real quantities.

Nominal indeterminacy (or price indeterminacy) simply means the multiplicity of equilibrium paths for nominal variables. If the real dynamics are determined by the initial conditions, but the initial price level is not, we observe nominal indeterminacy, because the value of the nominal variables is undefined.

In the Real Business Cycle $(R B C)$ literature agents' coordination selects the unique converging path and the equilibrium is determinate. The individuals are not interested in selecting an explosive path. After each technological shock they revise their choice and jump on the saddle path, given the new stock of capital. The RBC economists focus on the propagation of the shocks on fundamentals across the economy and the consequent stabilizing policies, i.e. the policy design reducing the fluctuations.

Whenever the predetermined variables do not select a unique path converging to some attractor, indeterminacy arises. Two relations are investigated in the literature on indeterminacy: $(i)$ the link between the (causes of) suboptimality and the (causes of) indeterminacy, (ii) the relation between indeterminacy and endogenous fluctuations.
(i) A cause of indeterminacy is cause of equilibrium multiplicity and thus generically in the case of a representative agent of inefficiency (to different equilibria are associated different utility levels). However suboptimality could be observed without equilibrium multiplicity.
(ii) There is a conjecture stating that indeterminacy implies the existence stationary sunspot equilibria (invariant transition function) with resulting probability distribution depending on an exogenous random signal (sunspot) (Woodford's conjecture, 1986b).

The Woodford's conjecture is one of the major theoretical contribution to the Endogenous Business Cycle ( $E B C$ ) theory. A sunspot variable is a random variable that conveys no information about fundamentals and thereby does not directly enter the equilibrium conditions for the state variable. However agents are assumed to be able to observe the realizations of the variable and to know its distribution, and to take it into account in making their decision if they choose to. If a rational expectations equilibrium exists in which agents respond to such a variable, it is a sunspot equilibrium. Such equilibria can be considered to represent situations in which speculation may be destabilizing, even when agents optimize and have rational expectations. Woodford focuses on the existence of stationary sunspot equilibria, i.e. equilibria in which the endogenous variables follow a stationary stochastic process with a time invariant transition function. He presents a necessary and sufficient condition for the existence of stationary sunspot equilibria near a deterministic steady state of a stationary economy not subject to exogenous stochastic shocks. However the stationary sunspot equilibria he characterizes depends on the whole infinite past history of the observed sunspots. Spontaneous coordination of agents on one of these equilibria seems to be somewhat problematic since it assumes implicitly economic agents which are able to record and process infinite sets of information and have extremely sophisticated sunspot theories.

Azariadis and Guesnerie (1986) provide explicit rational expectations equilibria, i.e. stochastic processes solving the stochastic difference equation of a monetary model, which are related to exogenous random signals.

Dávila (1997) investigates the existence of Markovian stationary sunspot equilibria with finite support. He shows that stationary sunspot equilibria, which fluctuate between a finite number of states following a Markov chain, exist arbitrarily close to a regular steady state for general one-dimensional one-step forward looking dynamical system depending on predetermined variables under usual assumptions. His result can be generalized, holding for
higher but finite memory processes.
The invariant set arguments have been investigated by Grandmont. Grandmont, Pintus and de Vilder (1998) provide an example of non-explosive randomization in a two-dimensional dynamics. More precisely they substitute the problem of solving the stochastic difference equation with the a priori simpler task of finding sets that are invariant in the forward deterministic perfect foresight dynamics.

We consider now the occurrence of indeterminacy and endogenous fluctuations in monetary economies. Money has two roles. On one side is medium of exchange. On the other side is store of value dominated by many other assets. Related to the first property are the cash-in-advance models. To the second one the portfolio theory.

Tobin (1965) is a monetary version of Solow (1956). The saving remains determined by an exogenous rate. The main result is that money is no longer superneutral. If for instance we assume for instance that the preferences imply a portfolio demand with the reduced form $m_{t} / k_{t}=\Phi\left(\pi_{t}+r_{t}\right)$, with $\Phi^{\prime}<0$ (see chapter two), we obtain a two-dimensional dynamic system. Money is not superneutral at the steady state and the steady state can be stable for some parameter configuration.

Sidrauski (1967) first introduces money in the Ramsey benchmark. Alternatively and equivalently in some sense (Feenstra, 1986) Clower (1967) introduces the idea of cash-in-advance. Sidrauski (1967) puts the real balances in the utility function. At the steady state the modified golden rule holds exactly as in the Ramsey model. This means that the real economy is not affected by the money growth. However money is no longer superneutral during transition whenever the cross derivative $u_{m c}$ is different from zero.

Clower intuition is related to the Baumol (1952) - Tobin (1956) model. In the class of infinite horizon models of this type money is no longer superneutral, indeterminacy arises for low elasticities of intertemporal substitution, and the equilibrium is generically sub-optimal because of constraint. The models with infinite horizon Lagrangian functions or Bellman functional equations provide the same results. In continuous time models indeterminacy may vanish. So the discrete time approach matters allowing for more flexible timing specifications. Similar results hold in endogenous growth.

Samuelson (1958) is the basic monetary OLG model. There are two periods in the life of generation. Each individual is endowed with one unit of a consumption good when young but receives no endowment when old. 1 stored unit gives in the second period of life $1+r$. Population grows at the rate $n$. If
$r=-1$ the good is perishable. In this case the suboptimal barter equilibrium holds: individuals consume all of their endowment when young. Suppose now that the government gives to the old $H$ completely divisible pieces of paper called money. Money restores the Pareto optimum if the economy goes on for ever and at period zero the young agents in $t$ believe that money will be valued at time $(t+1)$. If there is a young which does not believe that money will be valued at time $(t+1)$, then the barter equilibrium may remain an equilibrium even after the introduction of the $H$ pieces of paper. Let assume now $r>-1$. In the case with money $r<n$ implies the existence of a monetary equilibrium that is Pareto-optimal. The barter equilibrium is sub-optimal (dynamic inefficiency). If $r>n$ the monetary equilibrium vanishes and the barter equilibrium becomes Pareto-optimal. Hence if the barter equilibrium is not a Pareto optimum, there exists a monetary equilibrium that leads to a Pareto optimum; if the barter equilibrium is already a Pareto optimum, there cannot be a monetary equilibrium.

Money is said to be neutral if changes in the level of nominal money have no effect on the real equilibrium. It is said to be superneutral if changes in money growth have no effect on the real equilibrium. In this class of $O L G$, money is not superneutral. There is a problematic aspect in Samuelson (1958): money is not dominated as a store of value, an assumption that is both counterfactual and the source of striking results.

How does money enter an endogenous growth setup and which impact have for growth? Bosi (2001) studies the effects of transaction costs on endogenous growth and find a result of indeterminacy for low elasticities of intertemporal substitution.

There are four ways of generating a positive demand for money at equilibrium. As seen above the first one consists in putting money in the objectives (utility function (Sidrauski, 1967) or production function (Dornbusch and Frenkel, 1973)). The second way is to implement monetary constraints (on consumption (cash-in-advance, shopping time) or on production (and similarly capital accumulation) (selling-costs, credits guaranteed by collaterals). In the case of monetary constraints on consumption the elasticity of intertemporal substitution (the degree of consumer's freedom) makes ineffective the constraint as a source of indeterminacy and sub-optimality. In the case of monetary constraints on production (or capital accumulation) the reverse effect is observed: the consumer's freedom provokes indeterminacy (and related sub-optimality). To the converse the freedom of the producer in terms of the factors' elasticity of substitution reduces the possibility of
indeterminacy (however the presence of externalities could be responsible of indeterminacy for high levels of factor substitution).

### 5.3 Overlapping Generations Models and Bubbles

In macrodynamics two classes of models have been the most influential: the infinite horizon models and the overlapping generations models ( $O L G$ ) . All the results of the previous sections are now reinterpreted in the new lights of finite-lived overlapping generations. Some important differences arise in terms of equilibrium inefficiency due to the constrained nature of a finite life horizon.

Even if Allais (1947) and Samuelson (1958) elaborated the seminal overlapping generations frameworks, the most quoted article in this class of models remains that of Diamond (1965). The latter is presented in the current chapter. We will stress the possibility of dynamic inefficiency. Oversaving may arise in a decentralized economy. It provides a good example of failure of first welfare theorem. Such a sub-optimality does not emerge in a decentralized economy à la Ramsey where the modified golden rule holds. In contrast the oversaving could arise in the Solow model in a trivial sense because of the arbitrary saving rate. The surprising result in the $O L G$ model is that oversaving may occur even though households choose saving optimally. This possibility exists because households have a finite horizon, corresponding to the two-period length of life in Diamond (1965), whereas the economy goes on forever. It is known that the introduction of inter-generational altruism such as the bequests, restores the efficiency (Barro, 1974). It links the overlapping generations and transforms a finite horizon consumer's program in an infinite horizon reduced form à la Ramsey (1928). More precisely if the altruistic linkage from parents to children is strong enough to generate intergenerational transfers, that is if the typical person does not end up at a corner solution in which these transfers are zero, then the finite-horizon effect turns out to vanish and we return effectively to Ramsey's conclusions.

### 5.3.1 Rational Bubbles

Diamond (1965) constitutes the seminal model which introduces the capital accumulation in an overlapping generations economy. Tirole (1985) is the
seminal model on the existence of rational bubbles. Since Diamond (1965) is nested in Tirole (1985) we roughly follow the latter.

We deal with a market economy where saving decisions are decentralized. Each household is a price taker decision center. Prices are given by the real wage $w$ and the rental factors $R^{b}$ and $R^{k}$ on bonds and capital. The economy is constituted by individuals and firms. Individuals born at time $t$ live two periods: they consume $c_{t}$ in period $t$ and $d_{t+1}$ in period $t+1$. The problem the representative household born in $t$ faces, is the following.

$$
\begin{align*}
\max & U\left(c_{t}, d_{t+1}\right)  \tag{5.29}\\
\frac{B_{t+1}}{N_{t}}+\frac{K_{t+1}}{N_{t}}+c_{t} & \leq w_{t} \\
d_{t+1} & \leq R_{t+1}^{b} \frac{B_{t+1}}{N_{t}}+R_{t+1}^{k} \frac{K_{t+1}}{N_{t}}
\end{align*}
$$

where $B_{t+1} / N_{t}$ and $K_{t+1} / N_{t}$ represent respectively the individual real demand for public bonds and for capital as financial assets at time $t$. Capital letters denote the aggregate levels, while $N_{t}$ is the size of the generation born at time $t$. The agent works only in the first part of his life and furnishes inelastically one unit of labor. With respect to Solow capital accumulation model, now the saving turns out to be endogenous as in the Ramsey model, i.e. the consumption propensity is the result of a utility maximization program.

Lagrangian.

$$
\begin{aligned}
& U\left(c_{t}, d_{t+1}\right) \\
& +\lambda\left[w_{t}-\frac{B_{t+1}}{N_{t}}-\frac{K_{t+1}}{N_{t}}-c_{t}\right] \\
& +\mu\left[R_{t+1}^{b} \frac{B_{t+1}}{N_{t}}+R_{t+1}^{k} \frac{K_{t+1}}{N_{t}}-d_{t+1}\right]
\end{aligned}
$$

We obtain the following first order conditions.

$$
\begin{aligned}
\frac{B_{t+1}}{N_{t}} & :-\lambda+\mu R_{t+1}^{b}=0 \\
\frac{K_{t+1}}{N_{t}} & :-\lambda+\mu R_{t+1}^{k}=0 \\
c_{t} & : U_{1}-\lambda=0 \\
d_{t+1} & : U_{2}-\mu=0
\end{aligned}
$$

Then

$$
\begin{aligned}
R_{t+1}^{b} & =\lambda / \mu \\
R_{t+1}^{k} & =\lambda / \mu \\
\lambda & =U_{1} \\
\mu & =U_{2}
\end{aligned}
$$

No-Arbitrage-Condition:

$$
R_{t+1}^{b}=R_{t+1}^{k} \equiv R_{t+1}
$$

NAC allows us to rewrite the program in a simpler way.

$$
\begin{aligned}
\max U & \left(c_{t}, d_{t+1}\right) \\
s_{t}+c_{t} & \leq w_{t} \\
d_{t+1} & \leq R_{t+1} s_{t}
\end{aligned}
$$

where

$$
s_{t} \equiv \frac{B_{t+1}}{N_{t}}+\frac{K_{t+1}}{N_{t}}
$$

which is equivalent to a free optimization:

$$
\max _{s_{t}} U\left(w_{t}-s_{t}, R_{t+1} s_{t}\right)
$$

We obtain

$$
\frac{U_{1}}{U_{2}}=R_{t+1}
$$

The representative firm maximizes the profit.

$$
\begin{aligned}
& \max F\left(K_{t}, N_{t}\right)-\left(R_{t}-1\right) K_{t}-\delta K_{t}-w_{t} N_{t} \\
& \max F\left(K_{t}, N_{t}\right)-\left[R_{t}-(1-\delta)\right] K_{t}-w_{t} N_{t}
\end{aligned}
$$

where $K_{t}$ denotes the aggregate factor demand for physical capital at time $t$ and $N_{t}$ is the size of the generation born at time $t$.

As the length of the period is the half life, we assume that the physical capital fully depreciates: $\delta=1$. Then

$$
\max F\left(K_{t}, N_{t}\right)-R_{t} K_{t}-w_{t} N_{t}
$$

We obtain the usual first order conditions.

$$
\begin{aligned}
& \frac{\partial F\left(K_{t}, N_{t}\right)}{\partial K_{t}}=R_{t} \\
& \frac{\partial F\left(K_{t}, N_{t}\right)}{\partial N_{t}}=w_{t}
\end{aligned}
$$

For simplicity let the production function display constant returns to scale (homogeneity of degree one).

$$
\begin{aligned}
R_{t} & =\frac{\partial}{\partial K_{t}}\left[N_{t} F\left(\frac{K_{t}}{N_{t}}, 1\right)\right]=\frac{\partial}{\partial K_{t}}\left[N_{t} f\left(k_{t}\right)\right] \\
w_{t} & =\frac{\partial}{\partial N_{t}}\left[N_{t} F\left(\frac{K_{t}}{N_{t}}, 1\right)\right]=\frac{\partial}{\partial N_{t}}\left[N_{t} f\left(k_{t}\right)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
k & \equiv K / N \\
f & \equiv F / N
\end{aligned}
$$

Therefore

$$
\begin{align*}
R_{t} & =N_{t} f^{\prime}\left(k_{t}\right) \frac{1}{N_{t}}=f^{\prime}\left(k_{t}\right) \equiv R\left(k_{t}\right)  \tag{5.30}\\
w_{t} & =f\left(k_{t}\right)+N_{t} f^{\prime}\left(k_{t}\right)\left(-\frac{K_{t}}{N_{t}^{2}}\right)=f\left(k_{t}\right)-k_{t} f^{\prime}\left(k_{t}\right) \equiv w\left(k_{t}\right) \tag{5.31}
\end{align*}
$$

In equilibrium we obtain

$$
\begin{equation*}
\frac{U_{1}\left(w_{t}-s_{t}, R_{t+1} s_{t}\right)}{U_{2}\left(w_{t}-s_{t}, R_{t+1} s_{t}\right)}=R_{t+1} \tag{5.32}
\end{equation*}
$$

We apply the implicit function theorem to define the saving function:

$$
s_{t}=s\left(R_{t+1}, w_{t}\right)=s\left(R\left(k_{t+1}\right), w\left(k_{t}\right)\right)
$$

Moreover

$$
s_{t} \equiv \frac{B_{t+1}}{N_{t}}+\frac{K_{t+1}}{N_{t}}=n\left(b_{t+1}+k_{t+1}\right)
$$

where

$$
\begin{aligned}
b_{t} & \equiv \frac{B_{t}}{N_{t}} \\
k_{t} & \equiv \frac{K_{t}}{N_{t}}
\end{aligned}
$$

and

$$
n \equiv \frac{N_{t+1}}{N_{t}}
$$

is the demographic growth factor.
We assume no tax and no public spending. Hence the new national debt serves to pay the interest on the current one.

$$
\begin{aligned}
B_{t+1} & =R_{t} B_{t} \\
\frac{N_{t+1}}{N_{t}} \frac{B_{t+1}}{N_{t+1}} & =R_{t} \frac{B_{t}}{N_{t}} \\
n b_{t+1} & =R_{t} b_{t}
\end{aligned}
$$

The dynamic system gets a straightforward form.

$$
\begin{aligned}
s_{t} & =n\left(b_{t+1}+k_{t+1}\right) \\
n b_{t+1} & =R_{t} b_{t}
\end{aligned}
$$

More precisely we have

$$
\begin{align*}
n\left(b_{t+1}+k_{t+1}\right)-s\left(R\left(k_{t+1}\right), w\left(k_{t}\right)\right) & =0  \tag{5.33}\\
n b_{t+2}-R\left(k_{t+1}\right) b_{t+1} & =0 \tag{5.34}
\end{align*}
$$

and, using (5.30) and (5.31):

$$
\begin{align*}
n\left(b_{t+1}+k_{t+1}\right)-s\left(f^{\prime}\left(k_{t+1}\right), f\left(k_{t}\right)-k_{t} f^{\prime}\left(k_{t}\right)\right) & =0  \tag{5.35}\\
n b_{t+2}-f^{\prime}\left(k_{t+1}\right) b_{t+1} & =0 \tag{5.36}
\end{align*}
$$

The system has an implicit structure:

$$
G\left(\left[\begin{array}{c}
b_{t+2} \\
k_{t+1}
\end{array}\right],\left[\begin{array}{c}
b_{t+1} \\
k_{t}
\end{array}\right]\right)=0
$$

where $k_{t}$ is a predetermined variable and $b_{t+1}$ is non-predetermined.

## Diamond Regime and Golden Rule

Stationary equilibria are solutions of the following algebraic system.

$$
\begin{aligned}
n(b+k)-s(R, w) & =0 \\
n b-R b & =0
\end{aligned}
$$

We observe that

$$
\left(b_{0}, k_{0}\right)=(0,0)
$$

is a solution.
The Diamond regime is a non-trivial steady state given by

$$
\begin{aligned}
b & =0 \\
n k-s\left(f^{\prime}(k), f(k)-k f^{\prime}(k)\right) & =0
\end{aligned}
$$

The steady state $k_{1}$ we obtain in this case is just that of Diamond (1965) with no national debt.

The steady states $(0,0)$ and $\left(0, k_{1}\right)$ are said to be inside-money equilibria because households hold no public asset.

Let us focus now on the alternative regime of golden rule (Phelps, 1961): $\left(b_{2}, k_{2}\right)$.

$$
\begin{aligned}
f^{\prime}(k) & =R=n \\
k_{2} & =f^{\prime-1}(n) \\
n b & =s(n, w)-n k \\
b_{2} & =\frac{s\left(n, f\left(k_{2}\right)-n k_{2}\right)}{n}-k_{2}
\end{aligned}
$$

Summarizing, we have three steady states:

$$
\begin{aligned}
& \left(b_{0}, k_{0}\right)=(0,0) \\
& \left(b_{1}, k_{1}\right)=\left(0, k_{1}\right) \\
& \left(b_{2}, k_{2}\right)=\left(\frac{s\left(n, f\left(f^{\prime-1}(n)\right)-n f^{\prime-1}(n)\right)}{n}-f^{\prime-1}(n), f^{\prime-1}(n)\right)
\end{aligned}
$$

where $k_{1}$ solves the implicit equation

$$
n k-s\left(f^{\prime}(k), f(k)-k f^{\prime}(k)\right)=0
$$

## Local Dynamics

We linearize the dynamic system (5.35-5.36).
First equation.

$$
\begin{aligned}
b_{t+2} & : 0 \\
k_{t+1} & : n-s_{R} f^{\prime \prime} \\
b_{t+1} & : n \\
k_{t} & :-s_{w}\left(f^{\prime}-f^{\prime}-k f^{\prime \prime}\right)=s_{w} k f^{\prime \prime}
\end{aligned}
$$

Second equation.

$$
\begin{aligned}
b_{t+2} & : n \\
k_{t+1} & :-f^{\prime \prime} b \\
b_{t+1} & :-f^{\prime} \\
k_{t} & : 0
\end{aligned}
$$

We derive the Jacobian matrix:

$$
\begin{gathered}
{\left[\begin{array}{cc}
0 & n-s_{R} f^{\prime \prime} \\
n & -f^{\prime \prime} b
\end{array}\right]\left[\begin{array}{c}
d b_{t+2} \\
d k_{t+1}
\end{array}\right]+\left[\begin{array}{cc}
n & s_{w} k f^{\prime \prime} \\
-f^{\prime} & 0
\end{array}\right]\left[\begin{array}{c}
d b_{t+1} \\
d k_{t}
\end{array}\right]=0} \\
J=-\left[\begin{array}{cc}
0 & n-s_{R} f^{\prime \prime} \\
n & -f^{\prime \prime} b
\end{array}\right]^{-1}\left[\begin{array}{cc}
n & s_{w} k f^{\prime \prime} \\
-f^{\prime} & 0
\end{array}\right]
\end{gathered}
$$

whose trace and determinant are

$$
\begin{aligned}
T & =\frac{f^{\prime}}{n}-\frac{s_{w} f^{\prime \prime} k}{n-s_{R} f^{\prime \prime}}-\frac{f^{\prime \prime} b}{n-s_{R} f^{\prime \prime}} \\
D & =-\frac{f^{\prime}}{n} \frac{s_{w} f^{\prime \prime} k}{n-s_{R} f^{\prime \prime}}
\end{aligned}
$$

The Jacobian matrix is evaluated in the golden rule:

$$
f^{\prime}=n
$$

Then

$$
\begin{aligned}
T & =1-\frac{s_{w} f^{\prime \prime} k}{n-s_{R} f^{\prime \prime}}-\frac{f^{\prime \prime} b}{n-s_{R} f^{\prime \prime}}=1+D-\frac{f^{\prime \prime} b}{n-s_{R} f^{\prime \prime}} \\
D & =-\frac{s_{w} f^{\prime \prime} k}{n-s_{R} f^{\prime \prime}}
\end{aligned}
$$

The consumption good is plausibly assumed to be normal

$$
s_{w}>0
$$

and the saving to be increasing with respect to the interest factor.

$$
\begin{aligned}
n-s_{R} f^{\prime \prime} & >0 \\
s_{R} & >\frac{n}{f^{\prime \prime}}<0
\end{aligned}
$$

Then

$$
D=T-1+\frac{f^{\prime \prime} b}{n-s_{R} f^{\prime \prime}}<T-1
$$

and

$$
\begin{aligned}
D & >0 \\
2 D & >-2+\frac{f^{\prime \prime} b}{n-s_{R} f^{\prime \prime}} \\
D & >-\left(1+D-\frac{f^{\prime \prime} b}{n-s_{R} f^{\prime \prime}}\right)-1=-T-1
\end{aligned}
$$

In conclusion we obtain

$$
\begin{aligned}
& D<T-1 \\
& D>-T-1
\end{aligned}
$$

The golden rule is a saddle point.

## Global Dynamics

To study the global dynamics, we require more information about fundamentals. Therefore we specify them with Cobb-Douglas production functions.

$$
\begin{align*}
F\left(K_{t}, N_{t}\right) & \equiv K_{t}^{\alpha} N_{t}^{1-\alpha} \\
U\left(c_{t}, d_{t+1}\right) & \equiv c_{t}^{1-\sigma} d_{t+1}^{\sigma} \tag{5.37}
\end{align*}
$$

where $\alpha$ is the capital share on total income and $\sigma$ the propensity to saving. In other terms

$$
\begin{aligned}
f\left(k_{t}\right) & =k_{t}^{\alpha} \\
\ln U\left(c_{t}, d_{t+1}\right) & =(1-\sigma) \ln c_{t}+\sigma \ln d_{t+1}
\end{aligned}
$$

The firm's equilibrium is now

$$
\begin{aligned}
R\left(k_{t}\right) & \equiv f^{\prime}\left(k_{t}\right)=\alpha k_{t}^{\alpha-1} \\
w\left(k_{t}\right) & \equiv f\left(k_{t}\right)-k_{t} f^{\prime}\left(k_{t}\right)=(1-\alpha) k_{t}^{\alpha}
\end{aligned}
$$

We solve explicitly the consumer's program:

$$
\max (1-\sigma) \ln c_{t}+\sigma \ln d_{t+1}
$$

$$
\begin{aligned}
c_{t}+s_{t} & \leq w_{t} \\
d_{t+1} & \leq R_{t+1} s_{t}
\end{aligned}
$$

Equivalently we have the unconstrained program

$$
\max (1-\sigma) \ln \left(w_{t}-s_{t}\right)+\sigma \ln \left(R_{t+1} s_{t}\right)
$$

entailing the tangency condition

$$
-(1-\sigma) \frac{1}{w_{t}-s_{t}}+\sigma \frac{1}{R_{t+1} s_{t}} R_{t+1}=0
$$

that is

$$
\begin{equation*}
s_{t}=\sigma w_{t} \tag{5.38}
\end{equation*}
$$

We observe that

$$
\sigma=\frac{s_{t}}{w_{t}}
$$

can actually be interpreted as a propensity to saving.
Then

$$
s_{t}=\sigma(1-\alpha) k_{t}^{\alpha}
$$

Substituting in the dynamic system (5.35-5.36) we obtain

$$
\begin{aligned}
n\left(b_{t+1}+k_{t+1}\right)-\sigma(1-\alpha) k_{t}^{\alpha} & =0 \\
n b_{t+2}-f^{\prime}\left(k_{t+1}\right) b_{t+1} & =0
\end{aligned}
$$

First, we study the curve

$$
b_{t+2}=b_{t+1}
$$

We have

$$
\begin{aligned}
f^{\prime}\left(k_{t+1}\right) & =\alpha k_{t+1}^{\alpha-1}=n \\
k_{t+1} & =\left(\frac{\alpha}{n}\right)^{1 /(1-\alpha)}=k_{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
n\left(b_{t+1}+k_{2}\right)-\sigma(1-\alpha) k_{t}^{\alpha} & =0 \\
b_{t+1} & =\frac{\sigma(1-\alpha)}{n} k_{t}^{\alpha}-k_{2}
\end{aligned}
$$

We study now the curve

$$
\begin{aligned}
& k_{t+1}=k_{t} \\
& n\left(b_{t+1}+k_{t}\right)-\sigma(1-\alpha) k_{t}^{\alpha}=0 \\
& b_{t+1}=\frac{\sigma(1-\alpha)}{n} k_{t}^{\alpha}-k_{t}
\end{aligned}
$$

Phase diagram.

$$
\begin{aligned}
& b_{t+1}=\frac{\sigma(1-\alpha)}{n} k_{t}^{\alpha}-\left(\frac{\alpha}{n}\right)^{1 /(1-\alpha)} \\
& b_{t+1}=\frac{\sigma(1-\alpha)}{n} k_{t}^{\alpha}-k_{t}
\end{aligned}
$$

We set for instance
$\alpha=1 / 3$
$n=1+1 / 2$
and we study two regimes: underaccumulation, overaccumulation:
$\sigma_{1}=1 / 3$
$\sigma_{2}=2 / 3$



Figure 19. Underaccumulation: $\sigma_{1}=1 / 3$ Figure 20. Overaccumulation: $\sigma_{2}=2 / 3$
The upward sloped one is $b_{t+2}=b_{t+1}$, while the downward sloped curve is $k_{t+1}=k_{t}$. The intersection between $k_{t+1}=k_{t}$ and $b_{t+1}=0$ is the Diamond
regime $\left(b_{1}, k_{1}\right)$. The intersection between $b_{t+2}=b_{t+1}$ and $k_{t+1}=k_{t}$ is the golden rule $\left(b_{2}, k_{2}\right)$. The saddle path passes through $\left(b_{2}, k_{2}\right)$, is positively but less sloped than $b_{t+2}=b_{t+1}$. The golden rule regime is possible if and only if $k_{1}>k_{2}$, i.e. if and only if there is overaccumulation. Otherwise $b_{2}<0$, a meaningless solution.

The steady state $k_{1}$ with no debt is given by

$$
\begin{aligned}
\frac{\sigma(1-\alpha)}{n} k^{\alpha}-k & =0 \\
k_{1} & =\left[\frac{\sigma(1-\alpha)}{n}\right]^{1 /(1-\alpha)}
\end{aligned}
$$

We compare $k_{1}$ with the golden rule

$$
k_{2}=\left(\frac{\alpha}{n}\right)^{1 /(1-\alpha)}
$$

A long-run strictly positive bubble $b_{2}$ of golden rule is possible, if and only if

$$
k_{1}>k_{2}
$$

In this case

$$
\begin{aligned}
{\left[\frac{\sigma(1-\alpha)}{n}\right]^{1 /(1-\alpha)} } & >\left(\frac{\alpha}{n}\right)^{1 /(1-\alpha)} \\
\sigma & >\frac{\alpha}{1-\alpha}
\end{aligned}
$$

Summarizing, we have that an economy with a strictly positive bubble is possible if and only if

$$
\begin{equation*}
\sigma>\frac{\alpha}{1-\alpha} \tag{5.39}
\end{equation*}
$$

(see figure 20 with $\sigma=2 / 3>1 / 2$ ). In this case a feasible saddle path converges to a positive steady state of golden rule

$$
\left(k_{2}, b_{2}\right)
$$

In this case the golden rule dominates the Diamond equilibrium $\left(0, k_{1}\right)$ in terms of welfare, because the bubble absorbs the oversaving and reduces the capital overaccumulation generating the Pareto sub-optimality of Diamond equilibrium.

If

$$
\sigma<\frac{\alpha}{1-\alpha}
$$

we obtain an impossible

$$
b_{2}<0
$$

There is a unique non-trivial equilibrium

$$
\left(k_{1}, 0\right)
$$

which is now efficient because it is not dominated by $\left(k_{2}, b_{2}\right)$. Moreover as $k_{1}<k_{2}$ the zero debt regime is a regime of Pareto-efficient underaccumulation because no redistribution is possible to attain $k_{2}$.

### 5.3.2 Rational Bubbles and Growth

## The Engine of Growth

We roughly follow Grossman and Yanagawa (1993) and consider a representative firm with constant private returns to scale and aggregate externalities.

$$
Y_{t}=A_{t} F\left(K_{t}, N_{t}\right)
$$

where $F$ is homogeneous of degree one, the external effects depend on the capital intensity

$$
\begin{aligned}
A_{t} & =A k_{t}^{\varepsilon} \\
k_{t} & \equiv \frac{K_{t}}{N_{t}}
\end{aligned}
$$

$\varepsilon$ is interpreted as an externality measure.
Firms do not take into account the impact of factors demand on $A_{t}$. Let

$$
f\left(k_{t}\right)=\frac{F\left(K_{t}, N_{t}\right)}{N_{t}} \equiv F\left(\frac{K_{t}}{N_{t}}, 1\right)
$$

Private profit is the firm's objective.

$$
\begin{aligned}
\Pi_{p t} & \equiv A_{t} F\left(K_{t}, N_{t}\right)-\left(R_{t}-1\right) K_{t}-\delta K_{t}-w_{t} N_{t} \\
& =A_{t} F\left(K_{t}, N_{t}\right)-\left[R_{t}-(1-\delta)\right] K_{t}-w_{t} N_{t}
\end{aligned}
$$

where the wage is per unit of labor services.

As the length of the period is equal to the half-life of a generation, we plausibly assume the capital depreciation $\delta=1$. Then the program becomes:

$$
\begin{aligned}
& \max _{K_{t}, N_{t}} A_{t} F\left(K_{t}, N_{t}\right)-R_{t} K_{t}-w_{t} N_{t} \\
& A_{t} \frac{\partial F\left(K_{t}, N_{t}\right)}{\partial K_{t}}=R_{t} \\
& A_{t} \frac{\partial F\left(K_{t}, N_{t}\right)}{\partial N_{t}}=w_{t}
\end{aligned}
$$

Private returns to scale are constant. Hence

$$
\begin{aligned}
R_{t} & =A_{t} \frac{\partial}{\partial K_{t}}\left[N_{t} F\left(\frac{K_{t}}{N_{t}}, 1\right)\right]=A_{t} \frac{\partial}{\partial K_{t}}\left[N_{t} f\left(k_{t}\right)\right]=A_{t} N_{t} f^{\prime}\left(k_{t}\right) \frac{1}{N_{t}}=A_{t} f^{\prime}\left(k_{t}\right) \\
w_{t} & =A_{t} \frac{\partial}{\partial N_{t}}\left[N_{t} F\left(\frac{K_{t}}{N_{t}}, 1\right)\right]=A_{t} \frac{\partial}{\partial N_{t}}\left[N_{t} f\left(k_{t}\right)\right]=A_{t} f\left(k_{t}\right)+A_{t} N_{t} f^{\prime}\left(k_{t}\right)\left(-\frac{K_{t}}{N_{t}}\right) \\
& =A_{t}\left[f\left(k_{t}\right)-k_{t} f^{\prime}\left(k_{t}\right)\right]
\end{aligned}
$$

For simplicity we focus on the Cobb-Douglas case:

$$
\begin{aligned}
A_{t} F\left(K_{t}, N_{t}\right) & =A_{t} K_{t}^{\alpha} N_{t}^{1-\alpha} \\
A_{t} \frac{F\left(K_{t}, N_{t}\right)}{N_{t}} & =A_{t}\left(\frac{K_{t}}{N_{t}}\right)^{\alpha} \\
& =A k_{t}^{\varepsilon} k_{t}^{\alpha}
\end{aligned}
$$

For simplicity we set also

$$
\varepsilon=1-\alpha
$$

to obtain a reduced form close to Rebelo's $A k$ model.

$$
y_{t} \equiv \frac{Y_{t}}{N_{t}}=A k_{t}
$$

Moreover

$$
\begin{aligned}
R_{t} & =A_{t} f^{\prime}\left(k_{t}\right)=A k_{t}^{1-\alpha} \alpha k_{t}^{\alpha-1}=\alpha A \equiv R \\
w_{t} & =A_{t}\left[f\left(k_{t}\right)-k_{t} f^{\prime}\left(k_{t}\right)\right]=A k_{t}^{1-\alpha}(1-\alpha) k_{t}^{\alpha}=(A-\alpha A) k_{t} \\
& \equiv(A-R) k_{t}
\end{aligned}
$$

We observe that

$$
R_{t} k_{t}+w_{t}=R k_{t}+(A-R) k_{t}=A k_{t}=y_{t}
$$

(product exhaustion).
Social profit.

$$
\begin{aligned}
A_{t} F\left(K_{t}, N_{t}\right)-R_{t} K_{t}-w_{t} N_{t} & =N_{t}\left[A_{t} f\left(k_{t}\right)-R_{t} k_{t}-w_{t}\right] \\
& =N_{t}\left[A k_{t}-R_{t} k_{t}-w_{t}\right] \\
& \equiv N_{t} \pi_{s t}
\end{aligned}
$$

The maximization of the social profit with respect to $k_{t}$ gives in equilibrium:

$$
R_{t}=A
$$

The externalities $(\varepsilon)$ constitute a wedge between social $(A)$ and private returns $(R)$ to capital:

$$
\frac{A-R}{A}=1-\alpha=\varepsilon
$$

## Consumers

We assume the same household's preferences of the previous section. Program (5.29) still holds.

The No-Arbitrage-Condition allows us to define a common return on public bonds and private capital.

$$
R_{t+1}^{b}=R_{t+1}^{k} \equiv R_{t+1}
$$

Then consumer's program simplifies and we obtain the Euler equation (5.32):

$$
\frac{U_{1}\left(w_{t}-s_{t}, R_{t+1} s_{t}\right)}{U_{2}\left(w_{t}-s_{t}, R_{t+1} s_{t}\right)}=R_{t+1}
$$

Therefore the saving depends on the return on financial wealth and on the labor income.

$$
s_{t}=s\left(R_{t+1}, w_{t}\right)
$$

## General Equilibrium

As above we have

$$
s_{t}=n\left(b_{t+1}+k_{t+1}\right)
$$

and since we assume no tax and no public spending, the new national debt pays the interest on the current one.

$$
B_{t+1}=R_{t} B_{t}
$$

Dynamic system (5.33-5.34) holds, but now $R_{t+1}$ no longer depends on $k_{t}$ :

$$
\begin{aligned}
n\left(b_{t+1}+k_{t+1}\right)-s\left(R,(A-R) k_{t}\right) & =0 \\
n b_{t+2}-R b_{t+1} & =0
\end{aligned}
$$

Then

$$
\begin{aligned}
n\left(b_{t+1}+k_{t+1}\right)-\sigma\left(k_{t}\right) & =0 \\
n b_{t+2}-R b_{t+1} & =0
\end{aligned}
$$

where

$$
\sigma\left(k_{t}\right) \equiv s\left(R,(A-R) k_{t}\right)
$$

Consumption normality (and in particular utility separability) entails that

$$
s_{w} \in(0,1)
$$

We observe that

$$
\sigma^{\prime}\left(k_{t}\right) \equiv(A-R) s_{w}>0
$$

## Steady States

Assume now the propensity to saving $\sigma$ to be constant as follows

$$
\sigma\left(k_{t}\right)=s\left(R, w_{t}\right) \equiv \sigma w_{t}
$$

This is actually the case, when a Cobb-Douglas specification is retained for the utility function (see equations (5.37) and (5.38)).

The system becomes

$$
\begin{align*}
n\left(b_{t+1}+k_{t+1}\right)-\sigma(A-R) k_{t} & =0  \tag{5.40}\\
n b_{t+2}-R b_{t+1} & =0 \tag{5.41}
\end{align*}
$$

We study three possible stationary equilibria: the trivial one, the Diamond regime, the Grossman-Yanagawa regime. In the following we set

$$
\begin{aligned}
g_{t+1} & \equiv \frac{k_{t+1}}{k_{t}} \\
h_{t+1} & \equiv \frac{b_{t+1}}{b_{t}}
\end{aligned}
$$

(0) Trivial steady state. $k_{t}=0$ implies $w_{t}=(A-R) k_{t}=0$. Then $b_{t+1}+k_{t+1}=0$ and $b_{t+1}=k_{t+1}=0$. Equation (5.41) entails $b_{t+2}=0$. The trivial steady state $\left(b_{0}, k_{0}\right) \equiv(0,0)$ is then feasible.
(1) Diamond regime (bubbleless steady state). If $b_{t+1}=0$, then $b_{t+2}=$ 0 , since (5.41) holds. Equation (5.40) implies $n k_{t+1}=\sigma(A-R) k_{t}$ and eventually

$$
g=\sigma \frac{A-R}{n}
$$

We notice that the steady state is in terms of growth factors (growth is endogenous).
(2) Grossman-Yanagawa regime.

$$
\begin{aligned}
n b_{t+2}-R b_{t+1} & =0 \\
h & =\frac{R}{n}
\end{aligned}
$$

Growth must be balanced:

$$
g=\frac{R}{n}
$$

Summarizing, we have the following three stationary equilibria.

$$
\begin{aligned}
\left(b_{0}, k_{0}\right) & \equiv(0,0) \\
\left(b_{1}, g_{1}\right) & \equiv\left(0, \sigma \frac{A-R}{n}\right) \\
\left(h_{2}, g_{2}\right) & \equiv\left(\frac{R}{n}, \frac{R}{n}\right)
\end{aligned}
$$

Characterization of the bubbly equilibrium.

$$
\begin{aligned}
g & =h=\frac{R}{n} \\
n\left(b_{t+1}+k_{t+1}\right) & =\sigma(A-R) k_{t}
\end{aligned}
$$

$$
\begin{aligned}
n\left(\frac{b_{t+1}}{k_{t}}+g_{t+1}\right) & =n\left(\frac{R}{n} \frac{b_{t}}{k_{t}}+\frac{R}{n}\right)=\sigma(A-R) \\
\frac{b_{t}}{k_{t}} & =\sigma \frac{A-R}{R}-1
\end{aligned}
$$

A strictly positive stationary bubble exists, if and only if

$$
\begin{aligned}
\sigma \frac{A-R}{R} & >1 \\
\sigma & >\frac{\alpha}{1-\alpha}
\end{aligned}
$$

This is precisely condition (5.39).

## No Transition

( $i$ Let us consider the bubbleless regime. The equilibrium without bubble is the following

$$
\begin{aligned}
n\left(b_{t+1}+k_{t+1}\right) & =\sigma(A-R) k_{t} \\
b_{t+1} & =0 \\
\frac{k_{t+1}}{k_{t}} & =\sigma \frac{A-R}{n}
\end{aligned}
$$

We observe no transition. The economy jumps from the beginning on the steady state.

$$
k_{t}=\left(\sigma \frac{A-R}{n}\right)^{t} k_{0}
$$

(ii) We study now the bubbly regime. We have

$$
\begin{aligned}
n\left(\frac{b_{t+1}}{k_{t}}+g_{t+1}\right) & =\sigma(A-R) \\
\frac{b_{t}}{k_{t}} R+n g_{t+1} & =\sigma(A-R) \\
\frac{b_{t}}{k_{t}} & =\frac{1}{R}\left[\sigma(A-R)-n g_{t+1}\right]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{R}{n} & =h_{t+1}=\frac{b_{t+1} / k_{t+1}}{b_{t} / k_{t}} \frac{k_{t+1}}{k_{t}}=\frac{\sigma(A-R)-n g_{t+2}}{\sigma(A-R)-n g_{t+1}} g_{t+1} \\
g_{t+2} & =\frac{R}{n}+\frac{\sigma(A-R)}{n}-\frac{R}{n} \frac{\sigma(A-R)}{n} \frac{1}{g_{t+1}}
\end{aligned}
$$

We observe that

$$
g=\frac{R}{n}
$$

Local stability around the steady state.

$$
\begin{aligned}
\frac{\partial g_{t+2}}{\partial g_{t+1}} & =\frac{R}{n} \frac{\sigma(A-R)}{n} \frac{1}{g^{2}} \\
& =\sigma \frac{A-R}{R}=\sigma \frac{1-\alpha}{\alpha}
\end{aligned}
$$

A strictly positive bubble exists, if and only if local dynamics diverge.

$$
\sigma>\frac{\alpha}{1-\alpha}
$$

The only possible bubbly equilibrium is

$$
\begin{aligned}
\frac{b_{t}}{k_{t}} & =\frac{1}{R}[\sigma(A-R)-n g]=\sigma \frac{1-\alpha}{\alpha}-1 \\
\frac{b_{t+1}}{k_{t}} & =\frac{R}{n}\left(\sigma \frac{1-\alpha}{\alpha}-1\right)
\end{aligned}
$$

In the first period the non-predetermined variable $b_{1}$ must adjust.

$$
b_{1}=\frac{R}{n}\left(\sigma \frac{1-\alpha}{\alpha}-1\right) k_{0}
$$

The explicit dynamics become.

$$
\begin{aligned}
k_{t} & =\left(\frac{R}{n}\right)^{t} k_{0} \\
b_{t+1} & =\left(\frac{R}{n}\right)^{t+1}\left(\sigma \frac{1-\alpha}{\alpha}-1\right) k_{0} \\
t & =0,1, \ldots
\end{aligned}
$$

## Welfare

In our context there are two sources of sub-optimality: the finite life of agents and the external effects in production. A perfect endogenous growth
economy with infinite-lived agents and no externalities is characterized by a Pareto-optimal growth factor:

$$
g^{*}=\frac{A}{n}
$$

We have denoted the Diamond bubbleless growth factor with $g_{1}$ and the bubbly equilibrium with $g_{2}$.

We observe that

$$
\begin{aligned}
& g^{*}>\sigma \frac{A}{n}>\sigma\left(\frac{A}{n}-\frac{R}{n}\right)=g_{1} \\
& g^{*}>\frac{\alpha A}{n}=g_{2}
\end{aligned}
$$

Moreover

$$
g_{1}=\sigma \frac{A-R}{n}=\sigma \frac{1-\alpha}{\alpha} \frac{R}{n}=\sigma \frac{1-\alpha}{\alpha} g_{2}
$$

and

$$
g_{1}>g_{2}
$$

if and only if

$$
\sigma>\frac{\alpha}{1-\alpha}
$$

Two cases eventually matter according to the sign of the previous inequality.
(1) High saving rates:

$$
\sigma>\frac{\alpha}{1-\alpha}
$$

In this case the bubbly equilibrium exists and

$$
g^{*}>g_{1}>g_{2}
$$

The bubbleless equilibrium is better than the bubbly one.
The overaccumulation partially internalizes the externalities in production.
(2) Low saving rates:

$$
\sigma<\frac{\alpha}{1-\alpha}
$$

There is no longer the bubbly equilibrium and

$$
g^{*}>g_{1}
$$

## Part III

## Exercises

## Chapter 6

## Exercises

### 6.1 Elements of Dynamics

### 6.1.1 Autonomous Difference Equations

We consider the difference equation $x_{t+1}=x_{t}\left(2-x_{t}\right)$ with $x_{0} \geq 0$ as initial condition. To characterize the sequence $\left\{x_{t}\right\}_{t=0}^{\infty}$, investigate $(i)$ the existence of steady states and their multiplicity, (ii) their stability.

## Solution

(i) Existence and uniqueness-multiplicity of stationary states. The general form for autonomous difference equations is: $x_{t+1}=f\left(x_{t}\right)$. By definition of steady state: $x=f(x) . x=x(2-x)$ implies $x_{0}^{*}=0$ and $x_{1}^{*}=1$ : there are two distinct steady states.
(ii) Stability. In the one-dimensional case the steady state is locally stable, if and only if $\left|f^{\prime}\left(x^{*}\right)\right|<1$. In our case $f^{\prime}(x)=-2 x+2$, i.e. $\left|f^{\prime}\left(x_{0}^{*}\right)\right|$ $=\left|f^{\prime}(0)\right|=2>1$ and $\left|f^{\prime}\left(x_{1}^{*}\right)\right|=\left|f^{\prime}(1)\right|=0<1$. Hence $x_{0}^{*}=0$ is unstable,
while $x_{1}^{*}=1$ is stable.


Figure 21. Convergent path.

### 6.1.2 Autonomous Difference Equations

We consider the implicit difference equation $2 x_{t+1}+e^{x_{t+1} x_{t}}+x_{t}=1$ with $x_{0} \geq 0$ as initial condition. To characterize the sequence $\left\{x_{t}\right\}_{t=0}^{\infty}$, investigate ( $i$ ) the existence of steady states and their multiplicity, (ii) the stability of the highest steady state.

## Solution

The implicit dynamics get the following form:

$$
F\left(x_{t}, x_{t+1}\right) \equiv 2 x_{t+1}+e^{x_{t} x_{t+1}}+x_{t}-1=0 .
$$

Steady states.

$$
\begin{aligned}
F(x, x) & =0 \\
3 x+e^{x^{2}} & =1
\end{aligned}
$$

We observe that

$$
x=0
$$

is a solution. As $f(x) \equiv 3 x+e^{x^{2}}$ is continuous and strictly convex (check the second derivative),

$$
\lim _{x \rightarrow-\infty} 3 x+e^{x^{2}}=+\infty
$$

and

$$
f^{\prime}(0)=3>0
$$

then there are two steady states and $x=0$ is the higher one.
Local dynamics. We need the total differential of dynamics.

$$
\begin{aligned}
F\left(x_{t}, x_{t+1}\right) \equiv & 2 x_{t+1}+e^{x_{t} x_{t+1}}+x_{t}-1=0 \\
\frac{\partial F}{\partial x_{t}} d x_{t}+\frac{\partial F}{\partial x_{t+1}} d x_{t+1} & =0 \\
\frac{d x_{t+1}}{d x_{t}} & =-\frac{\partial F / \partial x_{t}}{\partial F / \partial x_{t+1}}=-\frac{1+x_{t+1} e^{x_{t} x_{t+1}}}{2+x_{t} e^{x_{t} x_{t+1}}}
\end{aligned}
$$

We evaluate this eigenvalue at the steady state:

$$
\left.\frac{d x_{t+1}}{d x_{t}}\right|_{x=0}=-\left.\frac{1+x e^{x^{2}}}{2+x e^{x^{2}}}\right|_{x=0}=-\frac{1}{2}
$$

The steady state $x=0$ is stable.

### 6.1.3 Planar Systems

The following planar system is non-linear.

$$
\left[\begin{array}{l}
x_{1, t+1} \\
x_{2, t+1}
\end{array}\right]=\left[\begin{array}{c}
x_{1 t}+e^{x_{2 t}-2}-1 \\
\left(3-x_{1 t}\right)\left[1+\ln \left(x_{2 t}-1\right)\right]
\end{array}\right] .
$$

Characterize the stationary states and their stability with the geometrical method of triangle.

## Solution

The stationary state is $x=(1,2)$. The Jacobian matrix evaluated at this steady state is

$$
J=\left[\begin{array}{cc}
1 & 1 \\
-1 & 2
\end{array}\right]
$$

The image of the eigenvalues in the $(T, D)$-plane is $(T, D)=(3,3)$. Then $(T, D)$ belongs to the class of sources, i.e. unstable points in all directions. Moreover $D>T^{2} / 4$. The two eigenvalues are complex and conjugated: the explosive path is a spiral.

### 6.1.4 Planar Systems

Another non-linear planar system is considered:

$$
\left[\begin{array}{c}
x_{1, t+1} \\
x_{2, t+1}
\end{array}\right]=\left[\begin{array}{c}
e^{x_{1 t}+x_{2 t}} \\
x_{1 t} x_{2 t}
\end{array}\right] .
$$

As above characterize the stationary states and their stability.

## Solution

The unique stationary state is now $x=(1,-1)$. The Jacobian matrix is

$$
J=\left[\begin{array}{cc}
1 & 1 \\
-1 & 1
\end{array}\right]
$$

The eigenvalue image in the $(T, D)$-plane is $(T, D)=(2,2)$, which belongs to the source region. Moreover $D>T^{2} / 4$. The two eigenvalues are still complex and conjugated and the path is an explosive spiral.

### 6.1.5 Planar Systems

Without computing the stationary state, show that the stationary state is not stable for the following non-linear dynamics.

$$
\left[\begin{array}{c}
x_{1, t+1} \\
x_{2, t+1}
\end{array}\right]=\left[\begin{array}{c}
e^{x_{1 t}}-4 x_{2 t} \\
-x_{1 t}+2 x_{2 t}
\end{array}\right]
$$

## Solution

It is not difficult to verify that a steady state exists. The Jacobian matrix evaluated at the steady state $\left(x_{1}, x_{2}\right)$ is

$$
J=\left[\begin{array}{cc}
e^{x_{1}^{*}} & -4 \\
-1 & 2
\end{array}\right]
$$

We obtain $D=2 e^{x_{1}^{*}}-4, T=2+e^{x_{1}^{*}}$ and then $D=2 T-8$. This straight line lies outside the triangle of stability. Hence the stationary state can not be a sink.

### 6.1.6 Planar Systems

Specify the second order implicit dynamics:

$$
F\left(\left[\begin{array}{l}
x_{1 t+1} \\
x_{2 t+1}
\end{array}\right],\left[\begin{array}{l}
x_{1 t} \\
x_{2 t}
\end{array}\right]\right)=0
$$

as follows:

$$
\begin{aligned}
& F_{1}\left(x_{1 t+1}, x_{2 t+1}, x_{1 t}, x_{2 t}\right) \equiv x_{1 t+1}^{3} x_{2 t}+x_{2 t+1}^{2}-1=0 \\
& F_{2}\left(x_{1 t+1}, x_{2 t+1}, x_{1 t}, x_{2 t}\right) \equiv x_{2 t} e^{x_{1 t+1}}-\ln \left(x_{1 t}+x_{2 t+1}\right)-1=0
\end{aligned}
$$

Find the steady states and characterize their local stability by mean of the triangle method.

## Solution

We apply the implicit function theorem.

$$
\begin{aligned}
& \frac{\partial F_{1}}{\partial x_{1 t+1}} d x_{1 t+1}+\frac{\partial F_{1}}{\partial x_{2 t+1}} d x_{2 t+1}+\frac{\partial F_{1}}{\partial x_{1 t}} d x_{1 t}+\frac{\partial F_{1}}{\partial x_{2 t}} d x_{2 t}=0 \\
& \frac{\partial F_{2}}{\partial x_{1 t+1}} d x_{1 t+1}+\frac{\partial F_{2}}{\partial x_{2 t+1}} d x_{2 t+1}+\frac{\partial F_{2}}{\partial x_{1 t}} d x_{1 t}+\frac{\partial F_{2}}{\partial x_{2 t}} d x_{2 t}=0
\end{aligned}
$$

In matrix form:

$$
\begin{aligned}
{\left[\begin{array}{ll}
\frac{\partial F_{1}}{\partial x_{1+1}} & \frac{\partial F_{1}}{\partial x_{1}+1} \\
\frac{\partial F_{2}}{\partial x_{1 t+1}} & \frac{\partial F_{2}}{\partial x_{2 t+1}}
\end{array}\right]\left[\begin{array}{l}
d x_{1 t+1} \\
d x_{2 t+1}
\end{array}\right] } & =-\left[\begin{array}{ll}
\frac{\partial F_{1}}{\partial x_{t}} & \frac{\partial F_{1}}{\partial x_{2 t}} \\
\frac{\partial F_{2}}{\partial x_{1 t}} & \frac{\partial F_{2}}{\partial x_{2 t}}
\end{array}\right]\left[\begin{array}{l}
d x_{1 t} \\
d x_{2 t}
\end{array}\right] \\
{\left[\begin{array}{l}
d x_{1 t+1} \\
d x_{2 t+1}
\end{array}\right] } & =-\left[\begin{array}{lll}
\frac{\partial F_{1}}{\partial x_{1+1}} & \frac{\partial F_{1}}{\partial x_{2 t+1}} \\
\frac{\partial F_{2}}{\partial x_{1 t+1}} & \frac{\partial F_{2}}{\partial x_{2 t+1}}
\end{array}\right]^{-1}\left[\begin{array}{ll}
\frac{\partial F_{1}}{\partial x_{t} t} & \frac{\partial F_{1}}{\partial x_{2 t}} \\
\frac{\partial F_{2}}{\partial x_{1 t}} & \frac{\partial F_{2}}{\partial x_{2 t}}
\end{array}\right]\left[\begin{array}{l}
d x_{1 t} \\
d x_{2 t}
\end{array}\right]
\end{aligned}
$$

The Jacobian matrix is

$$
J=-\left[\begin{array}{cc}
\frac{\partial F_{1}}{\partial x_{1 t+1}} & \frac{\partial F_{1}}{\partial x_{1} t+1} \\
\frac{\partial F_{2}}{\partial x_{1 t+1}} & \frac{\partial F_{2}}{\partial x_{2 t+1}}
\end{array}\right]^{-1}\left[\begin{array}{ll}
\frac{\partial F_{1}}{\partial x_{1 t}} & \frac{\partial F_{1}}{\partial x_{2} t} \\
\frac{\partial F_{2}}{\partial x_{1 t}} & \frac{\partial F_{2}}{\partial x_{2 t}}
\end{array}\right]
$$

which is evaluated at the steady state.
In the given example the steady state is given by:

$$
\left(x_{1}, x_{2}\right)=(0,1)
$$

We compute then the Jacobian matrix:

$$
\begin{aligned}
& F_{1}\left(x_{1 t+1}, x_{2 t+1}, x_{1 t}, x_{2 t}\right) \equiv x_{1 t+1}^{3} x_{2 t}+x_{2 t+1}^{2}-1=0 \\
& F_{2}\left(x_{1 t+1}, x_{2 t+1}, x_{1 t}, x_{2 t}\right) \equiv x_{2 t} e^{x_{1 t+1}}-\ln \left(x_{1 t}+x_{2 t+1}\right)-1=0 \\
J= & -\left[\begin{array}{cc}
3 x_{1 t+1}^{2} x_{2 t} & 2 x_{2 t+1} \\
x_{2 t} e^{x_{1 t+1}} & -\left(x_{1 t}+x_{2 t+1}\right)^{-1}
\end{array}\right]^{-1}\left[\begin{array}{cc}
0 & x_{1 t+1}^{3} \\
-\left(x_{1 t}+x_{2 t+1}\right)^{-1} & e^{x_{1 t+1}}
\end{array}\right]
\end{aligned}
$$

We evaluate this Jacobian at the steady state:

$$
\begin{aligned}
J & =-\left[\begin{array}{cc}
3 x_{1}^{2} x_{2} & 2 x_{2} \\
x_{2} e^{x_{1}} & -\left(x_{1}+x_{2}\right)^{-1}
\end{array}\right]^{-1}\left[\begin{array}{cc}
0 & x_{1}^{3} \\
-\left(x_{1}+x_{2}\right)^{-1} & e^{x_{1}}
\end{array}\right] \\
& =-\left[\begin{array}{cc}
0 & 2 \\
1 & -1
\end{array}\right]^{-1}\left[\begin{array}{cc}
0 & 0 \\
-1 & 1
\end{array}\right] \\
& =\left[\begin{array}{cc}
1 & -1 \\
0 & 0
\end{array}\right]
\end{aligned}
$$

The sum of eigenvalues is equal to one (trace) while their product is equal to zero (determinant). Therefore

$$
\begin{aligned}
& \lambda_{1}=0 \\
& \lambda_{2}=1
\end{aligned}
$$

### 6.2 The Consumption Function

### 6.2.1 The Life-Cycle Hypothesis

Let the credit market interest rate be equal to zero and the entry be free. No collateral guarantees are required to finance the consumption. Individuals are supposed to live 80 years. During the initial twenty years of life they earn nothing. From 21 to 60 their income increases linearly as follows:

$$
y_{t}=1000+10 *(t-20)
$$

where $t$ denotes the worker's age. During the last twenty years of life (retirement phase) the income per year is equal to 1300 . Time is discrete.

According to the life-cycle hypothesis, which is the stationary consumption the agents desire?

## Solution

The cumulated revenues of the three life phases are the following.

1. Youth.

$$
\sum_{t=1}^{20} y_{t}=0 * 20=0
$$

2. Active life.

$$
\begin{aligned}
\sum_{t=21}^{60} y_{t} & =\sum_{t=21}^{60}[1000+10 *(t-20)] \\
& =\sum_{t=1}^{40}[1000+10 * t] \\
& =\sum_{t=1}^{40} 1000+10 * \sum_{t=1}^{40} t \\
& =40 * 1000+10 * \frac{40 * 41}{2} \\
& =48200
\end{aligned}
$$

because

$$
\sum_{t=1}^{T} t=\frac{T(T+1)}{2}
$$

3. Retirement phase.

$$
\sum_{t=61}^{80} y_{t}=1300 * 20=26000
$$

Therefore we obtain the following smoothed consumption:

$$
\begin{aligned}
c_{t} & =c=\frac{\sum_{t=1}^{80} y_{t}}{80}=\frac{0+48200+26000}{80} \\
& =927.5 .
\end{aligned}
$$

### 6.2.2 The Life-Cycle Hypothesis

Let us consider a continuous time version of the consumer's behavior.

Compute the permanent income of a one-period life with an income flow in continuous time specified as follows:

$$
y_{t}=t-t^{2} .
$$



Figure 22.

## Solution

The permanent consumption is an average:

$$
\begin{aligned}
c & =\frac{1}{1-0} \int_{0}^{1} t-t^{2} d t \\
& =\left[\frac{1}{2} t^{2}-\frac{1}{3} t^{3}\right]_{0}^{1} \\
& =\frac{1}{2}-\frac{1}{3} \\
& =1 / 6 .
\end{aligned}
$$

### 6.2.3 The Life-Cycle Hypothesis

Compute the permanent income of a two-period life with an income flow in continuous time specified as follows:

$$
y_{t}=\sqrt{1-(t-1)^{2}}
$$



Figure 23.

## Solution

The income average is

$$
\begin{aligned}
c & =\frac{1}{2-0} \int_{0}^{2} \sqrt{1-(t-1)^{2}} d t \\
& =\frac{1}{2}\left[\frac{1}{2}[(t-1) \sqrt{t(2-t)}+\arcsin (t-1)]\right]_{0}^{2} \\
& =\frac{1}{4}[(t-1) \sqrt{t(2-t)}+\arcsin (t-1)]_{0}^{2} \\
& =\frac{1}{4}[0+\arcsin 1-0-\arcsin (-1)] \\
& =\frac{1}{4}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) \\
& =\frac{\pi}{4} .
\end{aligned}
$$

If each period is 40 years we have an annual consumption of

$$
\frac{\pi}{160}
$$

### 6.2.4 The Permanent Income Hypothesis

A consumer lives 3 periods: youth ( 20 years), working age ( 40 years), retirement phase (20 years). The respective revenues per year are $0, y,(3 / 4) y$. During his initial half life the annual credit market interest rate is equal to $r$. In the second part of his life he faces a double interest rate $2 r$.

Compute the permanent income $c$ and saving $s_{t}$ for each period. Which is the value of the permanent consumption $c$, if we set $y=20000$ euros and $r=2 \%$ ?

## Solution

Life-cycle budget constraint.

$$
\begin{aligned}
& \sum_{t=1}^{40} \frac{c}{(1+r)^{t}}+\sum_{t=41}^{80} \frac{c}{(1+r)^{40}(1+2 r)^{t-40}} \\
= & \sum_{t=1}^{20} \frac{0}{(1+r)^{t}}+\sum_{t=21}^{40} \frac{y}{(1+r)^{t}}+\sum_{t=41}^{60} \frac{y}{(1+r)^{40}(1+2 r)^{t-40}} \\
& +\sum_{t=61}^{80} \frac{(3 / 4) y}{(1+r)^{40}(1+2 r)^{t-40}}
\end{aligned}
$$

Consider the LHS.

$$
\begin{aligned}
& \sum_{t=1}^{40} \frac{c}{(1+r)^{t}}+\sum_{t=41}^{80} \frac{c}{(1+r)^{40}(1+2 r)^{t-40}} \\
= & c \sum_{t=1}^{40} \frac{1}{(1+r)^{t}}+\frac{c}{(1+r)^{40}} \sum_{t=41}^{80} \frac{1}{(1+2 r)^{t-40}} \\
= & c \sum_{t=1}^{40} \frac{1}{(1+r)^{t}}+\frac{c}{(1+r)^{40}} \sum_{t=1}^{40} \frac{1}{(1+2 r)^{t}} \\
= & \frac{c}{1+r} \sum_{t=0}^{39} \frac{1}{(1+r)^{t}}+\frac{c}{(1+r)^{40}(1+2 r)} \sum_{t=0}^{39} \frac{1}{(1+2 r)^{t}}
\end{aligned}
$$

We know that

$$
\begin{aligned}
(1-x) \sum_{t=0}^{T} x^{t} & =\sum_{t=0}^{T} x^{t}-\sum_{t=1}^{T+1} x^{t} \\
& =x^{0}+\sum_{t=1}^{T} x^{t}-\left(\sum_{t=1}^{T} x^{t}+x^{T+1}\right) \\
& =1+\left(\sum_{t=1}^{T} x^{t}-\sum_{t=1}^{T} x^{t}\right)-x^{T+1}
\end{aligned}
$$

$$
=1-x^{T+1}
$$

Hence

$$
\begin{equation*}
\sum_{t=0}^{T} x^{t}=\frac{1-x^{T+1}}{1-x} \tag{6.1}
\end{equation*}
$$

Let us apply formula (6.1).

$$
\begin{aligned}
x & =\frac{1}{1+r} \\
\sum_{t=0}^{39}\left(\frac{1}{1+r}\right)^{t} & =\frac{1-\left(\frac{1}{1+r}\right)^{39+1}}{1-\left(\frac{1}{1+r}\right)}=\frac{1+r}{r}\left[1-\left(\frac{1}{1+r}\right)^{40}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
x & =\frac{1}{1+2 r} \\
\sum_{t=0}^{39} \frac{1}{(1+2 r)^{t}} & =\frac{1-\left(\frac{1}{1+2 r}\right)^{39+1}}{1-\left(\frac{1}{1+2 r}\right)}=\frac{1+2 r}{2 r}\left[1-\left(\frac{1}{1+2 r}\right)^{40}\right]
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
L H S= & \frac{c}{1+r} \sum_{t=0}^{39} \frac{1}{(1+r)^{t}}+\frac{c}{(1+r)^{40}(1+2 r)} \sum_{t=0}^{39} \frac{1}{(1+2 r)^{t}} \\
= & \frac{c}{1+r} \frac{1+r}{r}\left[1-\left(\frac{1}{1+r}\right)^{40}\right] \\
& +\frac{c}{(1+r)^{40}(1+2 r)} \frac{1+2 r}{2 r}\left[1-\left(\frac{1}{1+2 r}\right)^{40}\right] \\
= & \frac{c}{r}\left[1-\left(\frac{1}{1+r}\right)^{40}\right]+\frac{c}{2 r(1+r)^{40}}\left[1-\left(\frac{1}{1+2 r}\right)^{40}\right]
\end{aligned}
$$

We compute now the RHS.

$$
\begin{aligned}
& \sum_{t=1}^{20} \frac{0}{(1+r)^{t}}+\sum_{t=21}^{40} \frac{y}{(1+r)^{t}} \\
& +\sum_{t=41}^{60} \frac{y}{(1+r)^{40}(1+2 r)^{t-40}}+\sum_{t=61}^{80} \frac{(3 / 4) y}{(1+r)^{40}(1+2 r)^{t-40}}
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{y}{(1+r)^{20}} \sum_{t=21}^{40} \frac{1}{(1+r)^{t-20}}+\frac{y}{(1+r)^{40}} \sum_{t=41}^{60} \frac{1}{(1+2 r)^{t-40}} \\
& +\frac{3 y}{4(1+r)^{40}(1+2 r)^{20}} \sum_{t=61}^{80} \frac{1}{(1+2 r)^{t-60}} \\
= & \frac{y}{(1+r)^{20}} \sum_{t=1}^{20} \frac{1}{(1+r)^{t}}+\frac{y}{(1+r)^{40}} \sum_{t=1}^{20} \frac{1}{(1+2 r)^{t}} \\
& +\frac{3 y}{4(1+r)^{40}(1+2 r)^{20}} \sum_{t=1}^{20} \frac{1}{(1+2 r)^{t}} \\
= & \frac{y}{(1+r)^{20}} \sum_{t=1}^{20} \frac{1}{(1+r)^{t}} \\
& +\left[1+\frac{3}{4(1+2 r)^{20}}\right] \frac{y}{(1+r)^{40}} \sum_{t=1}^{20} \frac{1}{(1+2 r)^{t}} \\
= & \frac{y}{(1+r)^{21}} \sum_{t=0}^{19} \frac{1}{(1+r)^{t}} \\
& +\left[1+\frac{3}{4(1+2 r)^{20}}\right] \frac{y}{(1+r)^{40}(1+2 r)} \sum_{t=0}^{19} \frac{1}{(1+2 r)^{t}}
\end{aligned}
$$

As

$$
\begin{aligned}
& \sum_{t=0}^{19}\left(\frac{1}{1+r}\right)^{t}=\frac{1+r}{r}\left[1-\left(\frac{1}{1+r}\right)^{20}\right] \\
& \sum_{t=0}^{19} \frac{1}{(1+2 r)^{t}}=\frac{1+2 r}{2 r}\left[1-\left(\frac{1}{1+2 r}\right)^{20}\right]
\end{aligned}
$$

we have

$$
\begin{aligned}
R H S= & \frac{y}{(1+r)^{21}} \sum_{t=0}^{19} \frac{1}{(1+r)^{t}} \\
& +\left[1+\frac{3}{4(1+2 r)^{20}}\right] \frac{y}{(1+r)^{40}(1+2 r)} \sum_{t=0}^{19} \frac{1}{(1+2 r)^{t}} \\
= & \frac{y}{(1+r)^{21}} \frac{1+r}{r}\left[1-\left(\frac{1}{1+r}\right)^{20}\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\left[1+\frac{3}{4(1+2 r)^{20}}\right] \frac{y}{(1+r)^{40}(1+2 r)} \frac{1+2 r}{2 r}\left[1-\left(\frac{1}{1+2 r}\right)^{20}\right] \\
= & \frac{y}{r(1+r)^{20}}\left[1-\left(\frac{1}{1+r}\right)^{20}\right] \\
& +\left[1+\frac{3}{4(1+2 r)^{20}}\right] \frac{y}{2 r(1+r)^{40}}\left[1-\left(\frac{1}{1+2 r}\right)^{20}\right]
\end{aligned}
$$

We are now able to compute the permanent income:

$$
\begin{aligned}
& \text { LHS }=\text { RHS } \\
& \frac{c}{r}\left[1-\left(\frac{1}{1+r}\right)^{40}\right]+\frac{c}{2 r(1+r)^{40}}\left[1-\left(\frac{1}{1+2 r}\right)^{40}\right] \\
& =\frac{y}{r(1+r)^{20}}\left[1-\left(\frac{1}{1+r}\right)^{20}\right] \\
& +\left[1+\frac{3}{4(1+2 r)^{20}}\right] \frac{y}{2 r(1+r)^{40}}\left[1-\left(\frac{1}{1+2 r}\right)^{20}\right] \\
& c=y \frac{\frac{1}{r(1+r)^{20}}\left[1-\left(\frac{1}{1+r}\right)^{20}\right]+\left[1+\frac{3}{4(1+2 r)^{20}}\right] \frac{1}{2 r(1+r)^{40}}\left[1-\left(\frac{1}{1+2 r}\right)^{20}\right]}{\frac{1}{r}\left[1-\left(\frac{1}{1+r}\right)^{40}\right]+\frac{1}{2 r(1+r)^{40}}\left[1-\left(\frac{1}{1+2 r}\right)^{40}\right]} \\
& =y \frac{2(1+r)^{20}\left[1-\left(\frac{1}{1+r}\right)^{20}\right]+\left[1+\frac{3}{4(1+2 r)^{20}}\right]\left[1-\left(\frac{1}{1+2 r}\right)^{20}\right]}{2(1+r)^{40}\left[1-\left(\frac{1}{1+r}\right)^{40}\right]+\left[1-\left(\frac{1}{1+2 r}\right)^{40}\right]} \\
& =y \frac{2\left[(1+r)^{20}-1\right]+\left[1+\frac{3}{4(1+2 r)^{20}}\right]\left[1-\left(\frac{1}{1+2 r}\right)^{20}\right]}{2\left[(1+r)^{40}-1\right]+\left[1-\left(\frac{1}{1+2 r}\right)^{40}\right]} \\
& =y \frac{2(1+2 r)^{40}\left[(1+r)^{20}-1\right]+(1+2 r)^{40}\left[1+\frac{3}{4(1+2 r)^{20}}\right]\left[1-\left(\frac{1}{1+2 r}\right)^{20}\right]}{2(1+2 r)^{40}\left[(1+r)^{40}-1\right]+(1+2 r)^{40}\left[1-\left(\frac{1}{1+2 r}\right)^{40}\right]} \\
& =y \frac{2(1+2 r)^{40}\left[(1+r)^{20}-1\right]+\left[(1+2 r)^{20}+3 / 4\right]\left[(1+2 r)^{20}-1\right]}{2(1+2 r)^{40}\left[(1+r)^{40}-1\right]+\left[(1+2 r)^{40}-1\right]}
\end{aligned}
$$

$$
\begin{aligned}
& =y \frac{2(1+2 r)^{40}\left[(1+r)^{20}-1\right]+\left[(1+2 r)^{20}+3 / 4\right]\left[(1+2 r)^{20}-1\right]}{2(1+2 r)^{40}\left[(1+r)^{40}-1\right]+\left[(1+2 r)^{40}-1\right]} \\
& =y \frac{2(1+2 r)^{40}(1+r)^{20}-2(1+2 r)^{40}+(1+2 r)^{40}-(1 / 4)(1+2 r)^{20}-3 / 4}{2(1+r)^{40}(1+2 r)^{40}-2(1+2 r)^{40}+(1+2 r)^{40}-1} \\
& =y \frac{2(1+2 r)^{40}(1+r)^{20}-(1+2 r)^{40}-(1 / 4)(1+2 r)^{20}-3 / 4}{2(1+r)^{40}(1+2 r)^{40}-2(1+2 r)^{40}+(1+2 r)^{40}-1} \\
& =y \frac{2(1+2 r)^{40}(1+r)^{20}-(1+2 r)^{40}-(1 / 4)(1+2 r)^{20}-3 / 4}{2(1+r)^{40}(1+2 r)^{40}-(1+2 r)^{40}-1} \\
& =y \frac{(1+2 r)^{40}\left[2(1+r)^{20}-1\right]-\left[3+(1+2 r)^{20}\right] / 4}{(1+2 r)^{40}\left[2(1+r)^{40}-1\right]-1} \\
& r=0.02 \\
& y=20000 \\
& \text { we obtain }
\end{aligned}
$$

$$
c=10609
$$

euros per year
Annual saving.

$$
\begin{aligned}
0-20 & : \\
0-c & =-10609 \\
21-60 & : \\
y-c & =20000-10609=9391 \\
61-80 & : \\
(3 / 4) y-c & =(3 / 4) 20000-10609=4391
\end{aligned}
$$

Therefore the consumer during the youth borrows 10609 euros per year. During his active life and the retirement phase he repays the initial debt and the relevant interests with the saving effort.

A more realistic model would take in account the parental effort to finance the young's consumption.

### 6.2.5 The Permanent Income Hypothesis

Insert a continuous time version with an exponential discounting.

### 6.2.6 The Permanent Income Hypothesis

We assume that the capital return and labor income of a infinite-lived household grow according respectively to the constant rates $\gamma^{k}$ and $\gamma^{l}$. Which is the permanent income, if the individual wants to implement a consumption growing at a constant rate $\gamma^{c}$ ?

## Solution

As we have seen in the course

$$
c_{t}=c_{1}\left(1+\gamma^{c}\right)^{t-1}
$$

and substituting in

$$
\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t} c_{t}=W
$$

we obtain

$$
\begin{align*}
\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t}\left[c_{1}\left(1+\gamma^{c}\right)^{t-1}\right] & =\frac{c_{1}}{1+\gamma^{c}} \sum_{t=1}^{T}\left(\frac{1+\gamma^{c}}{1+r}\right)^{t} \\
& =\frac{c_{1}}{1+\gamma^{c}} \frac{1+\gamma^{c}}{1+r} \sum_{t=0}^{T-1}\left(\frac{1+\gamma^{c}}{1+r}\right)^{t} \\
& =\frac{c_{1}}{1+r} \frac{1-\left[\left(1+\gamma^{c}\right) /(1+r)\right]^{T}}{1-\left(1+\gamma^{c}\right) /(1+r)} \\
& =\frac{c_{1}}{r-\gamma^{c}}\left[1-\left(\frac{1+\gamma^{c}}{1+r}\right)^{T}\right] \\
& =W . \tag{6.2}
\end{align*}
$$

Let $r>\gamma^{c}$. In this case

$$
\lim _{T \rightarrow \infty}\left[1-\left(\frac{1+\gamma^{c}}{1+r}\right)^{T}\right]=1
$$

For an infinite-lived agent $(T=\infty)$ equation (6.2) becomes

$$
c_{1}=\left(r-\gamma^{c}\right) W
$$

Thereby the higher the preferred consumption growth rate, the lower the initial consumption.

Let us now compute the wealth.
The capital income is:

$$
y_{t}^{k}=y_{1}^{k}\left(1+\gamma^{k}\right)^{t-1} .
$$

The labor income is

$$
y_{t}^{l}=y_{1}^{l}\left(1+\gamma^{l}\right)^{t-1}
$$

$$
\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t}\left(y_{t}^{k}+y_{t}^{l}\right)=\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t}\left[y_{1}^{k}\left(1+\gamma^{k}\right)^{t-1}+y_{1}^{l}\left(1+\gamma^{l}\right)^{t-1}\right]
$$

$$
=\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t} y_{1}^{k}\left(1+\gamma^{k}\right)^{t-1}
$$

$$
+\sum_{t=1}^{T}\left(\frac{1}{1+r}\right)^{t} y_{1}^{l}\left(1+\gamma^{l}\right)^{t-1}
$$

$$
=\frac{y_{1}^{k}}{1+\gamma^{k}} \sum_{t=1}^{T}\left(\frac{1+\gamma^{k}}{1+r}\right)^{t}
$$

$$
=\frac{y_{1}^{k}}{1+\gamma^{k}} \frac{1+\gamma^{k}}{1+r} \sum_{t=0}^{T-1}\left(\frac{1+\gamma^{k}}{1+r}\right)^{t}
$$

$$
+\frac{y_{1}^{l}}{1+\gamma^{l}} \frac{1+\gamma^{l}}{1+r} \sum_{t=0}^{T-1}\left(\frac{1+\gamma^{l}}{1+r}\right)^{t}
$$

$$
=\frac{y_{1}^{k}}{1+r} \frac{1-\left[\left(1+\gamma^{k}\right) /(1+r)\right]^{T}}{1-\left(1+\gamma^{k}\right) /(1+r)}
$$

$$
+\frac{y_{1}^{l}}{1+r} \frac{1-\left[\left(1+\gamma^{l}\right) /(1+r)\right]^{T}}{1-\left(1+\gamma^{l}\right) /(1+r)}
$$

$$
=\frac{y_{1}^{k}}{r-\gamma^{k}}\left[1-\left(\frac{1+\gamma^{k}}{1+r}\right)^{T}\right]
$$

$$
+\frac{y_{1}^{l}}{r-\gamma^{l}}\left[1-\left(\frac{1+\gamma^{l}}{1+r}\right)^{T}\right]
$$

$$
=N+H=W
$$

We assume that $r>\max \left\{\gamma^{k}, \gamma^{l}, \gamma^{c}\right\}$.
Let the consumer be infinite-lived $(T=\infty)$. Hence

$$
W=\frac{y_{1}^{k}}{r-\gamma^{k}}+\frac{y_{1}^{l}}{r-\gamma^{l}}
$$

and eventually

$$
\frac{c_{1}}{r-\gamma^{c}}=W=\frac{y_{1}^{k}}{r-\gamma^{k}}+\frac{y_{1}^{l}}{r-\gamma^{l}} .
$$

We obtain

$$
c_{1}=\frac{r-\gamma^{c}}{r-\gamma^{k}} y_{1}^{k}+\frac{r-\gamma^{c}}{r-\gamma^{l}} y_{1}^{l} .
$$

Notice that

$$
\begin{aligned}
& \frac{\partial c_{1}}{\partial \gamma^{c}}<0 \\
& \frac{\partial c_{1}}{\partial \gamma^{k}}>0 \\
& \frac{\partial c_{1}}{\partial \gamma^{l}}>0
\end{aligned}
$$

The sign of

$$
\frac{\partial c_{1}}{\partial r}
$$

is ambiguous. For instance it is always positive if $\gamma^{c}>\gamma^{k}, \gamma^{l}$.
You can provide an economic explanation..
The entire consumption path becomes:

$$
\begin{aligned}
c_{t} & =c_{1}\left(1+\gamma^{c}\right)^{t-1} \\
& =\left(\frac{r-\gamma^{c}}{r-\gamma^{k}} y_{1}^{k}+\frac{r-\gamma^{c}}{r-\gamma^{l}} y_{1}^{l}\right)\left(1+\gamma^{c}\right)^{t-1} .
\end{aligned}
$$

### 6.2.7 The Permanent Income Hypothesis

Which are the impacts on the consumption of a temporary change in the revenue and of a permanent variation?

## Solution

By simplicity we consider an infinite-lived consumer and a temporary shock at period $t$.

$$
\begin{gathered}
y_{t} \rightarrow y_{t}+\Delta . \\
W=\sum_{t=1}^{\infty}\left(\frac{1}{1+r}\right)^{t} y_{t}
\end{gathered}
$$

Then

$$
\frac{\partial W}{\partial y_{t}}=\left(\frac{1}{1+r}\right)^{t}
$$

and as

$$
y^{p}=r W \text {, }
$$

we get

$$
\frac{\partial y^{p}}{\partial y_{t}}=r \frac{\partial W}{\partial y_{t}}=r\left(\frac{1}{1+r}\right)^{t} .
$$

Therefore

$$
\Delta c=\Delta y^{p} \approx r\left(\frac{1}{1+r}\right)^{t} \Delta
$$

Let now the shock in $t$ be permanent.

$$
y_{\tau} \rightarrow y_{\tau}+\Delta
$$

with $\tau=t, t+1, \ldots$ Let by simplicity $\Delta$ be constant and independent on $t$. So if $r>0$

$$
\begin{aligned}
\Delta c & =\Delta y^{p} \approx \sum_{\tau=t}^{\infty} \frac{\partial y^{p}}{\partial y_{t}} \Delta \\
& =\sum_{\tau=t}^{\infty} r\left(\frac{1}{1+r}\right)^{\tau} \Delta \\
& =r\left(\frac{1}{1+r}\right)^{t} \Delta \sum_{\tau=0}^{\infty}\left(\frac{1}{1+r}\right)^{\tau} \\
& =r\left(\frac{1}{1+r}\right)^{t} \Delta \frac{1}{1-(1+r)^{-1}} \\
& =r\left(\frac{1}{1+r}\right)^{t} \Delta \frac{1+r}{r}
\end{aligned}
$$

$$
=\left(\frac{1}{1+r}\right)^{t-1} \Delta
$$

You can now compare the temporary and the permanent shock

$$
\begin{aligned}
T S & =r\left(\frac{1}{1+r}\right)^{t} \Delta \\
P S & =\left(\frac{1}{1+r}\right)^{t-1} \Delta
\end{aligned}
$$

In other terms

$$
\frac{T S}{P S}=\frac{r(1+r)^{-t}}{(1+r)^{-t+1}}=\frac{r}{1+r}
$$

This ratio does not depend on $t$.

### 6.2.8 The Permanent Income Hypothesis

Let us consider a continuous time version of the consumer's behavior according to the permanent income hypothesis. Friedman (1957) enriches the Modigliani and Brumberg (1954) model by taking in account the role of the credit market. In other words we can though the life-cycle theory as a particular case of the permanent income hypothesis with $r=0$.

For simplicity we reconsider a previous exercise augmented now by the assumption of a positive and constant interest rate across the life cycle.

Compute the permanent income of a one-period life with an income flow in continuous time specified as follows:

$$
y_{t}=\left(t-t^{2}\right) 10^{7}
$$

euros and an annual credit market interest rate $r=4 \%$. Life horizon is assumed to be equal to 80 years.

## Solution

Let $\rho$ be the one-period ( 80 years) interest rate. The life-cycle budget constraint under a free credit market entry is

$$
\int_{0}^{1} e^{-\rho t} c d t=\int_{0}^{1} e^{-\rho t}\left(t-t^{2}\right) 10^{7} d t
$$

$$
\begin{aligned}
c\left[-\frac{e^{-\rho t}}{\rho}\right]_{0}^{1} & =\left[e^{-\rho t}\left(\rho^{2} t^{2}+(2-\rho)(1+\rho t)\right) / \rho^{3}\right]_{0}^{1} 10^{7} \\
c\left(-\frac{e^{-\rho}}{\rho}+\frac{1}{\rho}\right) & =\left\{e^{-\rho}\left[\rho^{2}+(2-\rho)(1+\rho)\right] / \rho^{3}-(2-\rho) / \rho^{3}\right\} 10^{7} \\
c \rho^{2}\left(1-e^{-\rho}\right) & =\left\{e^{-\rho}\left[\rho^{2}+(2-\rho)(1+\rho)\right]-(2-\rho)\right\} 10^{7} \\
c & =\frac{e^{-\rho}\left[\rho^{2}+(2-\rho)(1+\rho)\right]-(2-\rho)}{\rho^{2}\left(1-e^{-\rho}\right)} 10^{7}
\end{aligned}
$$

Annual discount factor

$$
e^{-0.04}
$$

One-period discount factor:

$$
\left(e^{-0.04}\right)^{80}=e^{-3.2}
$$

Therefore the one-period consumption is

$$
\begin{aligned}
c & =\frac{e^{-\rho}\left[\rho^{2}+(2-\rho)(1+\rho)\right]-(2-\rho)}{\rho^{2}\left(1-e^{-\rho}\right)} 10^{7} \\
& =1437500
\end{aligned}
$$

euros and the annual consumption will be equal to

$$
\frac{c}{80}=17969
$$

euros.


Figure 24.

### 6.2.9 Three-Period Utility Maximization

The consumer maximizes an isoelastic utility function

$$
u\left(c_{0}, c_{1}, c_{2}\right)=\frac{c_{0}^{1-1 / \sigma}-1}{1-1 / \sigma}+\frac{1}{1+\theta} \frac{c_{1}^{1-1 / \sigma}-1}{1-1 / \sigma}+\frac{1}{(1+\theta)^{2}} \frac{c_{2}^{1-1 / \sigma}-1}{1-1 / \sigma}
$$

over three periods, and he must satisfy a unique intertemporal budget constraint, i.e. his discounted consumption must not be greater than his wealth (the consumer can freely borrow and lend in the credit market):

$$
c_{0}+\frac{c_{1}}{1+r}+\frac{c_{2}}{(1+r)^{2}} \leq y_{0}+\frac{y_{1}}{1+r}+\frac{y_{2}}{(1+r)^{2}}
$$

Find the consumption demand function as well as the saving for the three periods.

### 6.2.10 Infinite Horizon Utility Maximization

An infinite-lived consumer maximizes the following utility functional:

$$
\sum_{t=1}^{\infty}(1+\theta)^{-t} \ln c_{t} .
$$

His intertemporal budget constraint has a usual form:

$$
\sum_{t=1}^{\infty} \frac{1}{(1+r)^{t}} c_{t} \leq \sum_{t=1}^{\infty} \frac{1}{(1+r)^{t}} y_{t}
$$

where we specify

$$
y_{t}=\begin{aligned}
& y_{0}(1+\gamma)^{t}\left(1-b^{t}\right), \text { if } t \text { is even, } \\
& y_{0}(1+\gamma)^{t}\left(1+b^{t}\right), \text { if } t \text { is odd }
\end{aligned}
$$

with

$$
\begin{aligned}
& \gamma<r \\
& b<1
\end{aligned}
$$

Notice that the income path oscillates around and converges to the long run path $\left\{y_{0}(1+\gamma)^{t}\right\}_{t=1}^{\infty}$.
(i) Compute the consumption path.
(ii) Which is the impact of the parameters $b, \gamma, \theta, r, y_{0}$ on the shape of the consumption path?
(iii) Compute numerically the shape of the consumption path, if $\theta=0.01$, $r=0.03, y_{0}=100, \gamma=0.03, b=0.5$.

## Hint

Maximization gives

$$
c_{t}=c_{1}\left(\frac{1+r}{1+\theta}\right)^{t-1}
$$

and

$$
\begin{aligned}
\sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} & =\sum_{t=1}^{\infty} \frac{c_{1}[(1+r) /(1+\theta)]^{t-1}}{(1+r)^{t}} \\
& =\frac{c_{1}}{1+r} \sum_{t=1}^{\infty}\left(\frac{1}{1+\theta}\right)^{t-1} \\
& =\frac{c_{1}}{1+r} \sum_{t=0}^{\infty}\left(\frac{1}{1+\theta}\right)^{t} \\
& =\frac{c_{1}}{1+r} \frac{1+\theta}{\theta} .
\end{aligned}
$$

We obtain

$$
\begin{aligned}
c_{1} & =(1+r) \frac{\theta}{1+\theta} \sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} \\
c_{t} & =(1+r) \frac{\theta}{1+\theta}\left(\frac{1+r}{1+\theta}\right)^{t-1} \sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} \\
& =\theta\left(\frac{1+r}{1+\theta}\right)^{t} \sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} .
\end{aligned}
$$

We want now to compute the sum of discounted revenues..

$$
\sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}}=\sum_{t=1}^{\infty} \frac{y_{2 t}}{(1+r)^{2 t}}+\sum_{t=1}^{\infty} \frac{y_{2 t-1}}{(1+r)^{2 t-1}}
$$

$$
\begin{aligned}
= & \sum_{t=1}^{\infty} \frac{y_{0}(1+\gamma)^{2 t}\left(1-b^{2 t}\right)}{(1+r)^{2 t}}+\sum_{t=1}^{\infty} \frac{y_{0}(1+\gamma)^{2 t-1}\left(1+b^{2 t-1}\right)}{(1+r)^{2 t-1}} \\
= & y_{0}\left[\sum_{t=1}^{\infty}\left(\frac{1+\gamma}{1+r}\right)^{2 t}+\sum_{t=1}^{\infty}\left(\frac{1+\gamma}{1+r}\right)^{2 t-1}\right. \\
& \left.-\sum_{t=1}^{\infty}\left(b \frac{1+\gamma}{1+r}\right)^{2 t}+\sum_{t=1}^{\infty}\left(b \frac{1+\gamma}{1+r}\right)^{2 t-1}\right] \\
= & y_{0}\left[\left(\frac{1+\gamma}{1+r}\right)^{2} \sum_{t=0}^{\infty}\left(\frac{1+\gamma}{1+r}\right)^{2 t}\right. \\
& +\frac{1+r}{1+\gamma}\left(\frac{1+\gamma}{1+r}\right)^{2} \sum_{t=0}^{\infty}\left(\frac{1+\gamma}{1+r}\right)^{2 t} \\
& -\left(b \frac{1+\gamma}{1+r}\right)^{2} \sum_{t=0}^{\infty}\left(b \frac{1+\gamma}{1+r}\right)^{2 t} \\
= & \left.\frac{1+r}{b(1+\gamma)}\left(b \frac{1+\gamma}{1+r}\right)^{2} \sum_{t=0}^{\infty}\left(b \frac{1+\gamma}{1+r}\right)^{2 t}\right] \\
& +\frac{1+r}{1+\gamma}\left(\frac{1+\gamma}{1+r}\right)^{2} \frac{1+\gamma}{1-[(1+\gamma) /(1+r)]^{2}} \\
& -\left(b \frac{1+\gamma}{1+r}\right)^{2} \frac{1}{1-[b(1+\gamma) /(1+r)]^{2}} \\
& \left.+\frac{1+r}{b(1+\gamma)}\left(b \frac{1+\gamma}{1+r}\right)^{2} \frac{1}{1-[b(1+\gamma) /(1+r)]^{2}}\right\} \\
= & y_{0}\left(\frac{1}{[(1+r) /(1+\gamma)]^{2}-1}\left(\frac{1+r}{1+\gamma}+1\right)\right. \\
& \left.+\frac{1}{\{(1+r) /[b(1+\gamma)]\}^{2}-1}\left[\frac{1+r}{b(1+\gamma)}-1\right]\right)
\end{aligned}
$$

Therefore

$$
\sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}}=y_{0}\left\{\frac{1}{(1+r) /(1+\gamma)-1}+\frac{1}{(1+r) /[b(1+\gamma)]+1}\right\}
$$

$$
\begin{aligned}
& =y_{0}\left\{\frac{1+\gamma}{1+r-(1+\gamma)}+\frac{b(1+\gamma)}{1+r+b(1+\gamma)}\right\} \\
& =y_{0}(1+\gamma)\left\{\frac{1}{1+r-(1+\gamma)}+\frac{b}{1+r+b(1+\gamma)}\right\} \\
& =y_{0}(1+\gamma)\left\{\frac{1+r+b(1+\gamma)+b[1+r-(1+\gamma)]}{[1+r-(1+\gamma)][1+r+b(1+\gamma)]}\right\} \\
& =y_{0} \frac{(1+b)(1+\gamma)(1+r)}{(r-\gamma)[1+r+b(1+\gamma)]} .
\end{aligned}
$$

In conclusion

$$
\begin{aligned}
c_{t} & =\theta\left(\frac{1+r}{1+\theta}\right)^{t} \sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} \\
& =y_{0} \frac{\theta(1+b)(1+\gamma)(1+r)}{(r-\gamma)[1+r+b(1+\gamma)]}\left(\frac{1+r}{1+\theta}\right)^{t} .
\end{aligned}
$$

### 6.2.11 Constant Elasticity of Intertemporal Substitution

Show that an utility function characterized by a constant elasticity of intertemporal substitution $\sigma(C E S)$ gets the following shape:

$$
\frac{c_{t}^{1-1 / \sigma}-1}{1-1 / \sigma}
$$

Represent this curve for $\sigma=0.5$ and $\sigma=2$.

## Solution

As usual in macrodynamics we consider an utility function displaying a constant elasticity of intertemporal substitution $(C E S)$. The elasticity of substitution between the consumption at time $s$ and consumption at time $t$ is given by

$$
\sigma \equiv-\frac{u^{\prime}\left(c_{s}\right) / u^{\prime}\left(c_{t}\right)}{c_{s} / c_{t}} \frac{d\left(c_{s} / c_{t}\right)}{d\left[u^{\prime}\left(c_{s}\right) / u^{\prime}\left(c_{t}\right)\right]}
$$

Taking the limit for $s$ converging to $t$, one obtains in continuous time $\sigma\left(c_{t}\right)$ $=-u^{\prime}\left(c_{t}\right) /\left[u^{\prime \prime}\left(c_{t}\right) c_{t}\right]=-\left\{u^{\prime \prime}\left(c_{t}\right) /\left[u^{\prime}\left(c_{t}\right) / c_{t}\right]\right\}^{-1}$, that is the negative inverse of the elasticity of marginal utility (for more details see Blanchard and

Fischer (1989), chapter 2). In discrete time we adopt the latter formula as a definition. An isoelastic function with elasticity $\sigma$ has the form $u\left(c_{t}\right)=$ $C_{1} c_{t}^{1-1 / \sigma} /(1-1 / \sigma)+C_{2}$, where $C_{1}$ and $C_{2}$ are integration constants. To see that, reconsider the definition of elasticity: $-u^{\prime}\left(c_{t}\right) /\left[u^{\prime \prime}\left(c_{t}\right) c_{t}\right]=\sigma$, hence $-u^{\prime \prime}\left(c_{t}\right) / u^{\prime}\left(c_{t}\right)=1 /\left(\sigma c_{t}\right)$. We can write $-d \ln u^{\prime}\left(c_{t}\right) / d c_{t}=(1 / \sigma) d \ln c_{t} / d c_{t}$. The indefinite integral is

$$
-\int \frac{d}{d c_{t}} \ln u^{\prime}\left(c_{t}\right) d c_{t}=\frac{1}{\sigma} \int \frac{d}{d c_{t}} \ln c_{t} d c_{t}
$$

Thereby $-\ln u^{\prime}\left(c_{t}\right)=\left(\ln c_{t}\right) / \sigma+c$, where $c$ is an indefinite integration constant. Taking the power with base $e$ we obtain $e^{-\ln u^{\prime}\left(c_{t}\right)}=e^{\left(\ln c_{t}\right) / \sigma+c}$ and $e^{\ln \left[u^{\prime}\left(c_{t}\right)\right]^{-1}}=e^{c} e^{\ln c_{t}^{1 / \sigma}}$, i.e. $\left[u^{\prime}\left(c_{t}\right)\right]^{-1}=e^{c} c_{t}^{1 / \sigma}$ and $u^{\prime}\left(c_{t}\right)=e^{-c} c_{t}^{-1 / \sigma}$. The integral is now defined between 0 and $c_{t}: \int_{c_{0}}^{c_{t}} u^{\prime}\left(x_{t}\right) d x_{t}=e^{-c} \int_{c_{0}}^{c_{t}} x_{t}^{-1 / \sigma} d x_{t}$. Finally $\left[u\left(x_{t}\right)\right]_{c_{0}}^{c_{t}}=e^{-c}\left[x_{t}^{1-1 / \sigma} /(1-1 / \sigma)\right]_{c_{0}}^{c_{t}}$ and

$$
u\left(c_{t}\right)-u\left(c_{0}\right)=e^{-c}\left[c_{t}^{1-1 / \sigma} /(1-1 / \sigma)-c_{0}^{1-1 / \sigma} /(1-1 / \sigma)\right]
$$

Hence $u\left(c_{t}\right)=e^{-c} c_{t}^{1-1 / \sigma} /(1-1 / \sigma)-e^{-c} c_{0}^{1-1 / \sigma} /(1-1 / \sigma)+u\left(c_{0}\right)$. We can specify the two integration constants as $c=0$ and $u\left(c_{0}\right)=\left(c_{0}^{1-1 / \sigma}-1\right)$ $/(1-1 / \sigma)$, to obtain the standard $C E S$ function

$$
u\left(c_{t}\right)=\frac{c_{t}^{1-1 / \sigma}-1}{1-1 / \sigma}
$$

By applying the definition, it is possible to check that the elasticity of intertemporal substitution is actually $\sigma$. For $\sigma=1$, this isoelastic function is replaced by the logarithm in the function space:

$$
\frac{c_{t}^{1-1 / \sigma}-1}{1-1 / \sigma} \rightarrow \ln c_{t}
$$

Check that the logarithm function has a constant elasticity of intertemporal substitution just equal to one.

### 6.2.12 Infinite Horizon Utility Maximization

An infinite-lived household computes the intertemporal consumption demand as a function of the prices which are given by the future interest rates, and
of the future revenues $\left\{y_{t}\right\}_{1}^{\infty}$ he will receive during his life. We assume that the household has a perfect foresight of the future revenues as well as of the interest rates, and he can freely enter the credit market.

More precisely the intertemporal utility functional gets the following form:

$$
U\left(c_{1}, c_{2}, \ldots\right) \equiv \sum_{t=1}^{\infty}(1+\theta)^{-t} \ln c_{t} .
$$

For the sake of simplicity we assume that the market interest rate is constant at each period and equal to $r$.
(i) Compute and interpret the Euler equation.
(ii) Compute the consumption and saving as time functions.

## Solution

The household maximizes an intertemporal utility function. By simplicity we assume that his life goes on forever and that the utility functional is additively separable.

$$
\begin{equation*}
U\left(c_{1}, c_{2}, \ldots\right) \equiv \sum_{t=1}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right) \tag{6.3}
\end{equation*}
$$

The utility function $u$ is assumed to be increasing and strictly concave. The consumer has a free access to credit market as lender or borrower, so he faces an intertemporal budget constraint.

$$
\sum_{t=1}^{\infty} \frac{1}{(1+r)^{t}} c_{t} \leq \sum_{t=1}^{\infty} \frac{1}{(1+r)^{t}} y_{t}
$$

The revenue at period $t$ is given by the capital and labor income.

$$
y_{t} \equiv y_{t}^{k}+y_{t}^{l} .
$$

The Lagrangian for the program is given by

$$
\Lambda=\sum_{t=1}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right)+\lambda\left[\sum_{t=1}^{\infty} \frac{1}{(1+r)^{t}} y_{t}-\sum_{t=1}^{\infty} \frac{1}{(1+r)^{t}} c_{t}\right] .
$$

Notice that $\lambda$ is independent on time. Deriving with respect to the generic choice variable $c_{t}$, we get the corresponding first order condition

$$
\frac{\partial \Lambda}{\partial c_{t}}=0
$$

i.e.

$$
(1+\theta)^{-t} u^{\prime}\left(c_{t}\right)=\lambda(1+r)^{-t}
$$

To eliminate the multiplier we compute the intertemporal marginal rate of substitution:

$$
I M R S_{t+1}=\frac{(1+\theta)^{-t} u^{\prime}\left(c_{t}\right)}{(1+\theta)^{-t-1} u^{\prime}\left(c_{t+1}\right)}=\frac{(1+r)^{t+1}}{(1+r)^{t}}
$$

Notice that the right-hand side is just the price ratio. We obtain

$$
\begin{equation*}
u^{\prime}\left(c_{t}\right)=\frac{1+r}{1+\theta} u^{\prime}\left(c_{t+1}\right) \tag{6.4}
\end{equation*}
$$

This is the non-stochastic Euler equation.

$$
\sum_{t=1}^{\infty} \frac{c_{t}}{(1+r)^{t}}=\sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}}
$$

The constraint is now binding because the utility function is monotonic.
To provide an explicit solution we consider a particular class of utility functions.

As usual in macrodynamics an utility function with a constant elasticity of intertemporal substitution $\sigma(C E S)$ is employed:

$$
u\left(c_{t}\right)=\frac{c_{t}^{1-1 / \sigma}}{1-1 / \sigma} .
$$

From (6.4), we write

$$
\frac{c_{t}^{-1 / \sigma}}{c_{t+1}^{-1 / \sigma}}=\frac{1+r}{1+\theta}
$$

The consumption growth rate is given by

$$
\frac{c_{t+1}}{c_{t}}=\left(\frac{1+r}{1+\theta}\right)^{\sigma}
$$

Therefore

$$
c_{t}=c_{1}\left(\frac{1+r}{1+\theta}\right)^{\sigma(t-1)}
$$

and

$$
\begin{aligned}
\sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} & =\sum_{t=1}^{\infty} \frac{c_{1}[(1+r) /(1+\theta)]^{\sigma(t-1)}}{(1+r)^{t}} \\
& =\frac{c_{1}}{1+r} \sum_{t=1}^{\infty} \frac{[(1+r) /(1+\theta)]^{\sigma(t-1)}}{(1+r)^{t-1}} \\
& =\frac{c_{1}}{1+r} \sum_{t=1}^{\infty}\left[\frac{(1+r)^{\sigma-1}}{(1+\theta)^{\sigma}}\right]^{t-1} \\
& =\frac{c_{1}}{1+r} \sum_{t=0}^{\infty}\left[\frac{(1+r)^{\sigma-1}}{(1+\theta)^{\sigma}}\right]^{t}
\end{aligned}
$$

The series converges if and only if

$$
\begin{align*}
\frac{(1+r)^{\sigma-1}}{(1+\theta)^{\sigma}} & <1 \\
(1+r)^{\sigma-1} & <(1+\theta)^{\sigma} \\
(\sigma-1) \ln (1+r) & <\sigma \ln (1+\theta) \\
\frac{\sigma-1}{\sigma} & <\frac{\ln (1+\theta)}{\ln (1+r)} \tag{6.5}
\end{align*}
$$

We assume that $r>0$. Then

$$
\frac{\ln (1+\theta)}{\ln (1+r)}>0
$$

The inequality (6.5) is for instance respected if $\sigma<1$ (weak elasticity of intertemporal substitution).

Under inequality (6.5) we obtain

$$
\begin{aligned}
\sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} & =\frac{c_{1}}{1+r} \sum_{t=0}^{\infty}\left[\frac{(1+r)^{\sigma-1}}{(1+\theta)^{\sigma}}\right]^{t} \\
& =\frac{c_{1}}{1+r} \frac{1}{1-(1+r)^{\sigma-1} /(1+\theta)^{\sigma}} \\
& =\frac{c_{1}}{1+r-[(1+r) /(1+\theta)]^{\sigma}}
\end{aligned}
$$

We are able now to determine the initial consumption and then the entire path.

$$
\begin{aligned}
c_{1} & =\left[1+r-\left(\frac{1+r}{1+\theta}\right)^{\sigma}\right] \sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} \\
c_{t} & =\left(\frac{1+r}{1+\theta}\right)^{\sigma(t-1)} c_{1} \\
& =\left(\frac{1+r}{1+\theta}\right)^{\sigma(t-1)}\left[1+r-\left(\frac{1+r}{1+\theta}\right)^{\sigma}\right] \sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} \\
& =\left[(1+r)\left(\frac{1+r}{1+\theta}\right)^{\sigma(t-1)}-\left(\frac{1+r}{1+\theta}\right)^{\sigma t}\right] \sum_{t=1}^{\infty} \frac{y_{t}}{(1+r)^{t}} .
\end{aligned}
$$

It is possible to perform the comparative statics by evaluating the impact of $r$ and $\theta$ on the path $\left\{c_{t}\right\}_{t=1}^{\infty}$.

The saving at each period is given by

$$
\begin{aligned}
s_{t} & =y_{t}-c_{t} \\
& =y_{t}-\left(\frac{1+r}{1+\theta}\right)^{\sigma(t-1)}\left[1+r-\left(\frac{1+r}{1+\theta}\right)^{\sigma}\right] \sum_{\tau=1}^{\infty} \frac{y_{\tau}}{(1+r)^{\tau}} .
\end{aligned}
$$

### 6.2.13 Two-Period Stochastic Maximization

Consider a problem of consumption choice over two periods under uncertainty. The individual consumes $c_{0}$ during the first period and $c_{1}$ during the second. He receives at the end of the first period a revenue $y_{1}$ with probability $\pi_{1}$, or $y_{2}$ with probability $\pi_{2}=1-\pi_{1}$. The interest rate in the credit market is constant and equal to $r$. The intertemporal utility function gets the following form:

$$
U\left(c_{0}, c_{1}\right) \equiv \ln c_{0}+\frac{1}{1+\theta} \ln c_{1}
$$

where $\theta$ measures the consumer's impatience.
(i) Determine the optimal stochastic consumption. Show that the interest rate does not affect the consumption $c_{0}$ (under the logarithm specification the revenue and substitution effects exactly compensate).
(ii) Compute the numeric solution, if

$$
\begin{aligned}
\pi_{1} & =\pi_{2} \\
y_{1} & =1 \\
y_{2} & =2 \\
\theta & =r=1 \% .
\end{aligned}
$$

## Solution

(i) The expected intertemporal utility is

$$
\ln c_{0}+\frac{1}{1+\theta}\left(\pi_{1} \ln c_{11}+\pi_{2} \ln c_{12}\right)
$$

There are two states of nature, two histories and then two intertemporal budget constraints:

$$
\begin{aligned}
& c_{0}+\frac{1}{1+r} c_{11} \leq y_{1} \\
& c_{0}+\frac{1}{1+r} c_{12} \leq y_{2}
\end{aligned}
$$

We observe that the second period consumption is a random variable.
The Lagrangian function has two multipliers (as many as the histories):

$$
\begin{aligned}
& \ln c_{0}+\frac{1}{1+\theta}\left(\pi_{1} \ln c_{11}+\pi_{2} \ln c_{12}\right) \\
& +\lambda_{1}\left(y_{1}-c_{0}-\frac{1}{1+r} c_{11}\right)+\lambda_{2}\left(y_{2}-c_{0}-\frac{1}{1+r} c_{12}\right)
\end{aligned}
$$

We obtain the following first order conditions:

$$
\begin{aligned}
\frac{1}{c_{0}} & =\lambda_{1}+\lambda_{2} \\
\frac{\pi_{1}}{1+\theta} \frac{1}{c_{11}} & =\frac{\lambda_{1}}{1+r} \\
\frac{\pi_{2}}{1+\theta} \frac{1}{c_{12}} & =\frac{\lambda_{2}}{1+r}
\end{aligned}
$$

Then

$$
\frac{\pi_{1}}{1+\theta} \frac{1}{c_{11}}+\frac{\pi_{2}}{1+\theta} \frac{1}{c_{12}}=\frac{1}{1+r}\left(\lambda_{1}+\lambda_{2}\right)=\frac{1}{1+r} \frac{1}{c_{0}} .
$$

Rearranging we get the stochastic Euler equation:

$$
\begin{aligned}
\frac{1}{c_{0}} & =\frac{1+r}{1+\theta}\left(\pi_{1} \frac{1}{c_{11}}+\pi_{2} \frac{1}{c_{12}}\right), \\
u^{\prime}\left(c_{0}\right) & =\frac{1+r}{1+\theta} E_{0} u^{\prime}\left(c_{1}\right) .
\end{aligned}
$$

The constraints are binding because the utility is monotonic.
Therefore we have now three equations

$$
\begin{aligned}
\frac{1}{c_{0}} & =\frac{1+r}{1+\theta}\left(\pi_{1} \frac{1}{c_{11}}+\pi_{2} \frac{1}{c_{12}}\right), \\
c_{0}+\frac{1}{1+r} c_{11} & =y_{1}, \\
c_{0}+\frac{1}{1+r} c_{12} & =y_{2}
\end{aligned}
$$

and three variables: $c_{0}, c_{11}, c_{12}$.
Using the fact that $\pi_{1}+\pi_{2}=1$, we obtain the explicit solution:

$$
\begin{aligned}
c_{0}= & \frac{\left(1+\theta+\pi_{2}\right) y_{1}+\left(1+\theta+\pi_{1}\right) y_{2}}{2(2+\theta)} \\
& \pm \frac{\sqrt{\left[\left(1+\theta+\pi_{2}\right) y_{1}+\left(1+\theta+\pi_{1}\right) y_{2}\right]^{2}-4(1+\theta)(2+\theta) y_{1} y_{2}}}{2(2+\theta)}, \\
c_{11}= & (1+r)\left(y_{1}-c_{0}\right), \\
c_{12}= & (1+r)\left(y_{2}-c_{0}\right) .
\end{aligned}
$$

### 6.2.14 Three-Period Stochastic Maximization

Consider a problem of consumption choice over three periods under uncertainty. The individual consumes $c_{0}$ during the first period, $c_{1}$ during the second, and $c_{3}$ in the third period. He receives at the end of the first period a revenue $y_{0}^{1}$ with probability $\pi_{0}^{1}$, or $y_{0}^{2}$ with probability $\pi_{0}^{2}=1-\pi_{0}^{2}$. He receives at the end of the second period a revenue $y_{1}^{1}$ with probability $\pi_{1}^{1}$, or $y_{1}^{2}$ with probability $\pi_{1}^{2}=1-\pi_{1}^{2}$. We assume that the probability distributions of the two periods are independent. The interest rate in the credit market is constant and equal to $r$. The intertemporal utility function gets the following form:

$$
U\left(c_{0}, c_{1}, c_{3}\right) \equiv u\left(c_{0}\right)+\frac{1}{1+\theta} u\left(c_{1}\right)+\left(\frac{1}{1+\theta}\right)^{2} u\left(c_{2}\right),
$$

where $\theta$ measures the consumer's impatience. For the sake of simplicity we consider a quadratic utility:

$$
u(c)=a c-\frac{b}{2} c^{2}
$$

(i) Compute the stochastic Euler equations.
(ii) Determine the optimal stochastic consumption at each period under the Hall's assumption (1978) $r=\theta$.
(iii) Compute the numerical solution, if

$$
\begin{aligned}
\pi_{t}^{1} & =\pi_{t}^{2} \\
y_{t}^{i} & =i \\
t & =1,2 \\
\theta & =r=1 \% .
\end{aligned}
$$

## Solution

(i) The expected intertemporal utility is:

$$
\begin{aligned}
& u\left(c_{0}\right)+\frac{1}{1+\theta}\left[\pi_{0}^{1} u\left(c_{1}^{1}\right)+\pi_{0}^{2} u\left(c_{1}^{2}\right)\right] \\
& +\left(\frac{1}{1+\theta}\right)^{2}\left[\pi_{0}^{1} \pi_{1}^{1} u\left(c_{2}^{11}\right)+\pi_{0}^{1} \pi_{1}^{2} u\left(c_{2}^{12}\right)+\pi_{0}^{2} \pi_{1}^{1} u\left(c_{2}^{21}\right)+\pi_{0}^{2} \pi_{1}^{2} u\left(c_{2}^{22}\right)\right]
\end{aligned}
$$

There are two states of nature in the first period and two in the second period. Therefore there are four possible histories:

11,
12,
21,
22.

Thereby four are the constraints

$$
\begin{aligned}
& c_{0}+\frac{1}{1+r} c_{1}^{1}+\left(\frac{1}{1+r}\right)^{2} c_{2}^{11} \leq y_{0}^{1}+\frac{1}{1+r} y_{1}^{1} \\
& c_{0}+\frac{1}{1+r} c_{1}^{1}+\left(\frac{1}{1+r}\right)^{2} c_{2}^{12} \leq y_{0}^{1}+\frac{1}{1+r} y_{1}^{2}
\end{aligned}
$$

$$
\begin{aligned}
& c_{0}+\frac{1}{1+r} c_{1}^{2}+\left(\frac{1}{1+r}\right)^{2} c_{2}^{21} \leq y_{0}^{2}+\frac{1}{1+r} y_{1}^{1} \\
& c_{0}+\frac{1}{1+r} c_{1}^{2}+\left(\frac{1}{1+r}\right)^{2} c_{2}^{22} \leq y_{0}^{2}+\frac{1}{1+r} y_{1}^{2}
\end{aligned}
$$

and four are the Lagrangian multipliers.
Hence the Lagrangian function becomes:

$$
\begin{aligned}
& u\left(c_{0}\right)+\frac{1}{1+\theta}\left[\pi_{0}^{1} u\left(c_{1}^{1}\right)+\pi_{0}^{2} u\left(c_{1}^{2}\right)\right] \\
& +\left(\frac{1}{1+\theta}\right)^{2}\left[\pi_{0}^{1} \pi_{1}^{1} u\left(c_{2}^{11}\right)+\pi_{0}^{1} \pi_{1}^{2} u\left(c_{2}^{12}\right)+\pi_{0}^{2} \pi_{1}^{1} u\left(c_{2}^{21}\right)+\pi_{0}^{2} \pi_{1}^{2} u\left(c_{2}^{22}\right)\right] \\
& +\lambda^{11}\left[y_{0}^{1}+\frac{1}{1+r} y_{1}^{1}-c_{0}-\frac{1}{1+r} c_{1}^{1}-\left(\frac{1}{1+r}\right)^{2} c_{2}^{11}\right] \\
& +\lambda^{12}\left[y_{0}^{1}+\frac{1}{1+r} y_{1}^{2}-c_{0}-\frac{1}{1+r} c_{1}^{1}-\left(\frac{1}{1+r}\right)^{2} c_{2}^{12}\right] \\
& +\lambda^{21}\left[y_{0}^{2}+\frac{1}{1+r} y_{1}^{1}-c_{0}-\frac{1}{1+r} c_{1}^{2}-\left(\frac{1}{1+r}\right)^{2} c_{2}^{21}\right] \\
& +\lambda^{22}\left[y_{0}^{2}+\frac{1}{1+r} y_{1}^{2}-c_{0}-\frac{1}{1+r} c_{1}^{2}-\left(\frac{1}{1+r}\right)^{2} c_{2}^{22}\right]
\end{aligned}
$$

We obtain the following first order conditions.

$$
\begin{aligned}
u^{\prime}\left(c_{0}\right) & =\lambda^{11}+\lambda^{12}+\lambda^{21}+\lambda^{22}, \\
\frac{1}{1+\theta} \pi_{0}^{1} u^{\prime}\left(c_{1}^{1}\right) & =\frac{1}{1+r}\left(\lambda^{11}+\lambda^{12}\right), \\
\frac{1}{1+\theta} \pi_{0}^{2} u^{\prime}\left(c_{1}^{2}\right) & =\frac{1}{1+r}\left(\lambda^{21}+\lambda^{22}\right), \\
\left(\frac{1}{1+\theta}\right)^{2} \pi_{0}^{1} \pi_{1}^{1} u^{\prime}\left(c_{2}^{11}\right) & =\left(\frac{1}{1+r}\right)^{2} \lambda^{11}, \\
\left(\frac{1}{1+\theta}\right)^{2} \pi_{0}^{1} \pi_{1}^{2} u^{\prime}\left(c_{2}^{12}\right) & =\left(\frac{1}{1+r}\right)^{2} \lambda^{12}, \\
\left(\frac{1}{1+\theta}\right)^{2} \pi_{0}^{2} \pi_{1}^{1} u^{\prime}\left(c_{2}^{21}\right) & =\left(\frac{1}{1+r}\right)^{2} \lambda^{21},
\end{aligned}
$$

$$
\left(\frac{1}{1+\theta}\right)^{2} \pi_{0}^{2} \pi_{1}^{2} u^{\prime}\left(c_{2}^{22}\right)=\left(\frac{1}{1+r}\right)^{2} \lambda^{22}
$$

From the first three equations we get

$$
\begin{equation*}
u^{\prime}\left(c_{0}\right)=\lambda^{11}+\lambda^{12}+\lambda^{21}+\lambda^{22}=\frac{1+r}{1+\theta}\left[\pi_{0}^{1} u^{\prime}\left(c_{1}^{1}\right)+\pi_{0}^{2} u^{\prime}\left(c_{1}^{2}\right)\right] \tag{6.6}
\end{equation*}
$$

that is the first Euler equation:

$$
u^{\prime}\left(c_{0}\right)=\frac{1+r}{1+\theta} E_{0} u^{\prime}\left(c_{1}\right) .
$$

From the remaining equations we obtain:

$$
\begin{aligned}
& \lambda^{11}+\lambda^{12}+\lambda^{21}+\lambda^{22} \\
= & \left(\frac{1+r}{1+\theta}\right)^{2}\left[\pi_{0}^{1} \pi_{1}^{1} u^{\prime}\left(c_{2}^{11}\right)+\pi_{0}^{1} \pi_{1}^{2} u^{\prime}\left(c_{2}^{12}\right)+\pi_{0}^{2} \pi_{1}^{1} u^{\prime}\left(c_{2}^{21}\right)+\pi_{0}^{2} \pi_{1}^{2} u^{\prime}\left(c_{2}^{22}\right)\right]
\end{aligned}
$$

which gives with (6.6) the second stochastic Euler equation:

$$
\begin{aligned}
& {\left[\pi_{0}^{1} u^{\prime}\left(c_{1}^{1}\right)+\pi_{0}^{2} u^{\prime}\left(c_{1}^{2}\right)\right] } \\
= & {\left[\pi_{0}^{1} \pi_{1}^{1} u^{\prime}\left(c_{2}^{11}\right)+\pi_{0}^{1} \pi_{1}^{2} u^{\prime}\left(c_{2}^{12}\right)+\pi_{0}^{2} \pi_{1}^{1} u^{\prime}\left(c_{2}^{21}\right)+\pi_{0}^{2} \pi_{1}^{2} u^{\prime}\left(c_{2}^{22}\right)\right] } \\
E_{0} u^{\prime}\left(c_{1}\right)= & \frac{1+r}{1+\theta} E_{0} u^{\prime}\left(c_{2}\right)
\end{aligned}
$$

(ii) To explicitly solve the program we arrange the first order conditions as follows:

$$
\begin{aligned}
& \frac{1+r}{1+\theta} \pi_{0}^{1} u^{\prime}\left(c_{1}^{1}\right)=\left(\frac{1+r}{1+\theta}\right)^{2} \pi_{0}^{1} \pi_{1}^{1} u^{\prime}\left(c_{2}^{11}\right)+\left(\frac{1+r}{1+\theta}\right)^{2} \pi_{0}^{1} \pi_{1}^{2} u^{\prime}\left(c_{2}^{12}\right) \\
& \frac{1+r}{1+\theta} \pi_{0}^{2} u^{\prime}\left(c_{1}^{2}\right)=\left(\frac{1+r}{1+\theta}\right)^{2} \pi_{0}^{2} \pi_{1}^{1} u^{\prime}\left(c_{2}^{21}\right)+\left(\frac{1+r}{1+\theta}\right)^{2} \pi_{0}^{2} \pi_{1}^{2} u^{\prime}\left(c_{2}^{22}\right)
\end{aligned}
$$

Simplifying we get seven equation with seven unknowns: $c_{0}, c_{1}^{1}, c_{1}^{2}, c_{2}^{11}$, $c_{2}^{12}, c_{2}^{21}, c_{2}^{22}$ :

$$
\begin{aligned}
u^{\prime}\left(c_{0}\right) & =\frac{1+r}{1+\theta}\left[\pi_{0}^{1} u^{\prime}\left(c_{1}^{1}\right)+\pi_{0}^{2} u^{\prime}\left(c_{1}^{2}\right)\right] \\
u^{\prime}\left(c_{1}^{1}\right) & =\frac{1+r}{1+\theta}\left[\pi_{1}^{1} u^{\prime}\left(c_{2}^{11}\right)+\pi_{1}^{2} u^{\prime}\left(c_{2}^{12}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
u^{\prime}\left(c_{1}^{2}\right) & =\frac{1+r}{1+\theta}\left[\pi_{1}^{1} u^{\prime}\left(c_{2}^{21}\right)+\pi_{1}^{2} u^{\prime}\left(c_{2}^{22}\right)\right] \\
c_{0}+\frac{1}{1+r} c_{1}^{1}+\left(\frac{1}{1+r}\right)^{2} c_{2}^{11} & =y_{0}^{1}+\frac{1}{1+r} y_{1}^{1} \\
c_{0}+\frac{1}{1+r} c_{1}^{1}+\left(\frac{1}{1+r}\right)^{2} c_{2}^{12} & =y_{0}^{1}+\frac{1}{1+r} y_{1}^{2} \\
c_{0}+\frac{1}{1+r} c_{1}^{2}+\left(\frac{1}{1+r}\right)^{2} c_{2}^{21} & =y_{0}^{2}+\frac{1}{1+r} y_{1}^{1} \\
c_{0}+\frac{1}{1+r} c_{1}^{2}+\left(\frac{1}{1+r}\right)^{2} c_{2}^{22} & =y_{0}^{2}+\frac{1}{1+r} y_{1}^{2} .
\end{aligned}
$$

By simplicity we assume

$$
\begin{aligned}
r & =\theta \\
u(c) & =a c-\frac{b}{2} c^{2}
\end{aligned}
$$

The previous system gets a simple form:

$$
\begin{aligned}
c_{0} & =\pi_{0}^{1} c_{1}^{1}+\pi_{0}^{2} c_{1}^{2} \\
c_{1}^{1} & =\pi_{1}^{1} c_{2}^{11}+\pi_{1}^{2} c_{2}^{12} \\
c_{1}^{2} & =\pi_{1}^{1} c_{2}^{21}+\pi_{1}^{2} c_{2}^{2} \\
c_{0}+\frac{1}{1+r} c_{1}^{1}+\left(\frac{1}{1+r}\right)^{2} c_{2}^{11} & =y_{0}^{1}+\frac{1}{1+r} y_{1}^{1} \\
c_{0}+\frac{1}{1+r} c_{1}^{1}+\left(\frac{1}{1+r}\right)^{2} c_{2}^{12} & =y_{0}^{1}+\frac{1}{1+r} y_{1}^{2} \\
c_{0}+\frac{1}{1+r} c_{1}^{2}+\left(\frac{1}{1+r}\right)^{2} c_{2}^{21} & =y_{0}^{2}+\frac{1}{1+r} y_{1}^{1} \\
c_{0}+\frac{1}{1+r} c_{1}^{2}+\left(\frac{1}{1+r}\right)^{2} c_{2}^{22} & =y_{0}^{2}+\frac{1}{1+r} y_{1}^{2}
\end{aligned}
$$

In matrix terms it becomes

$$
\left[\begin{array}{ccccccc}
1 & -\pi_{0}^{1} & -\pi_{0}^{2} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -\pi_{1}^{1} & -\pi_{1}^{2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -\pi_{1}^{1} & -\pi_{1}^{2} \\
1 & \frac{1}{1+r} & 0 & \left(\frac{1}{1+r}\right)^{2} & 0 & 0 & 0 \\
1 & \frac{1}{1+r} & 0 & 0 & \left(\frac{1}{1+r}\right)^{2} & 0 & 0 \\
1 & 0 & \frac{1}{1+r} & 0 & 0 & \left(\frac{1}{1+r}\right)^{2} & 0 \\
1 & 0 & \frac{1}{1+r} & 0 & 0 & 0 & \left(\frac{1}{1+r}\right)^{2}
\end{array}\right]\left[\begin{array}{c}
c_{0} \\
c_{1}^{1} \\
c_{1}^{2} \\
c_{2}^{11} \\
c_{2}^{12} \\
c_{2}^{21} \\
c_{2}^{22}
\end{array}\right]=\left[\begin{array}{c}
0 \\
0 \\
0 \\
y_{0}^{1}+\frac{1}{1+r} y_{1}^{1} \\
y_{0}^{1}+\frac{1}{1+r} y_{1}^{2} \\
y_{0}^{2}+\frac{1}{1+r} y_{1}^{1} \\
y_{0}^{2}+\frac{1}{1+r} y_{1}^{2}
\end{array}\right] .
$$

The analytical solution is

$$
\left[\begin{array}{c}
c_{0} \\
c_{1}^{1} \\
c_{1}^{2} \\
c_{2}^{11} \\
c_{2}^{12} \\
c_{2}^{21} \\
c_{2}^{22}
\end{array}\right]=\left[\begin{array}{ccccccc}
1 & -\pi_{0}^{1} & -\pi_{0}^{2} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -\pi_{1}^{1} & -\pi_{1}^{2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -\pi_{1}^{1} & -\pi_{1}^{2} \\
1 & \frac{1}{1+r} & 0 & \left(\frac{1}{1+r}\right)^{2} & 0 & 0 & 0 \\
1 & \frac{1}{1+r} & 0 & 0 & \left(\frac{1}{1+r}\right)^{2} & 0 & 0 \\
1 & 0 & \frac{1}{1+r} & 0 & 0 & \left(\frac{1}{1+r}\right)^{2} & 0 \\
1 & 0 & \frac{1}{1+r} & 0 & 0 & 0 & \left(\frac{1}{1+r}\right)^{2}
\end{array}\right]^{-1}\left[\begin{array}{c}
0 \\
0 \\
0 \\
y_{0}^{1}+\frac{1}{1+r} y_{1}^{1} \\
y_{0}^{1}+\frac{1}{1+r} y_{1}^{2} \\
y_{0}^{2}+\frac{1}{1+r} y_{1}^{1} \\
y_{0}^{2}+\frac{1}{1+r} y_{1}^{2}
\end{array}\right] .
$$

Numerical computations provide a straightforward solution.

### 6.2.15 Infinite Horizon Stochastic Optimization

Solve the problem of intertemporal consumption choice after replacing the quadratic utility of Hall (1978) by a logarithmic utility. We assume that the revenues are provided by a stochastic return (by simplicity i.i.d) on the residual wealth after the consumption action.

## Solution

The infinite-lived consumer is at period $t$ and from this period on solves a stochastic version of the intertemporal utility optimization program:

$$
\begin{aligned}
& \max _{c_{t}, \ldots, c_{T}} E_{t} \sum_{\tau=t}^{\infty}(1+\theta)^{-(\tau-t)} u\left(c_{\tau}\right), \\
& A_{\tau+1} \leq\left(1+r_{\tau}\right)\left(A_{\tau}-c_{\tau}\right), \\
& A_{t} \text { given }
\end{aligned}
$$

where $\theta$ measures his impatience, $u$ is the utility of a period, $c_{\tau}$ the consumption at period $\tau, A_{\tau}$ is the random wealth at period $\tau$ providing during the period a random rate of return $r_{\tau}$.
$r_{\tau}$ is governed by a first order Markov process.
More precisely the cumulative function of $r_{\tau}$ depends only on $r_{\tau-1}$ :

$$
\operatorname{prob}\left\{r_{\tau} \leq r^{\prime} \mid r_{\tau-1}=r\right\}=F\left(r_{\tau}^{\prime} ; r\right) .
$$

Solving the problem as in Hall (1978) we obtain the stochastic Euler equation

$$
\begin{equation*}
(1+\theta) E_{t} u^{\prime}\left(c_{\tau}\right)=E_{t}\left[u^{\prime}\left(c_{\tau+1}\right)\left(1+r_{\tau}\right)\right] . \tag{6.7}
\end{equation*}
$$

We notice that

$$
E_{t} c_{t}=c_{t}
$$

because $c_{t} \in I_{t}$. Setting $\tau=t$ in (6.7), we obtain

$$
\begin{equation*}
E_{t} u^{\prime}\left(c_{t}\right)=u^{\prime}\left(c_{t}\right)=\frac{1}{1+\theta} E_{t}\left[u^{\prime}\left(c_{t+1}\right)\left(1+r_{t}\right)\right] \tag{6.8}
\end{equation*}
$$

There is a simple economic interpretation of (6.8). If the consumer renounces to one unit of consumption in $t$, he reduces the utility of $u^{\prime}\left(c_{t}\right)$ and increases in $t+1$ the utility of the expected gain $E_{t}\left[u^{\prime}\left(c_{t+1}\right)\left(1+r_{t}\right)\right]$. However the latter expression must be discounted according to the time preference

$$
\frac{1}{1+\theta} E_{t}\left[u^{\prime}\left(c_{t+1}\right)\left(1+r_{t}\right)\right] .
$$

Subjective costs and benefits are equal at optimum.
We develop now the required specific example:

$$
u\left(c_{t}\right)=\ln c_{t} .
$$

$r_{t}$ is assumed to be an independently and identically distributed random variable (with the restriction

$$
1 \leq 1+E r_{t} \leq 1+\theta
$$

to allow for the series convergence).
We guess that the optimal solution gets the form

$$
c_{t}=\gamma A_{t}
$$

(life-cycle-permanent income hypothesis). The problem consists in determining $\gamma$, a constant.

By substitution in the Euler equation we obtain

$$
\begin{aligned}
u^{\prime}\left(c_{t}\right) & =\frac{1}{1+\theta} E_{t}\left[u^{\prime}\left(c_{t+1}\right)\left(1+r_{t}\right)\right] \\
\left(\ln c_{t}\right)^{\prime} & =\frac{1}{1+\theta} E_{t}\left[\left(\ln c_{t+1}\right)^{\prime}\left(1+r_{t}\right)\right] \\
\frac{1}{c_{t}} & =\frac{1}{1+\theta} E_{t} \frac{1+r_{t}}{c_{t+1}} \\
\frac{1}{\gamma A_{t}} & =\frac{1}{1+\theta} E_{t} \frac{1+r_{t}}{\gamma A_{t+1}}
\end{aligned}
$$

We know that

$$
\begin{aligned}
A_{t+1} & =\left(1+r_{t}\right)\left(A_{t}-c_{t}\right) \\
& =\left(1+r_{t}\right)\left(A_{t}-\gamma A_{t}\right) \\
& =(1-\gamma)\left(1+r_{t}\right) A_{t} .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{\gamma A_{t}} & =\frac{1}{1+\theta} E_{t} \frac{1+r_{t}}{\gamma(1-\gamma)\left(1+r_{t}\right) A_{t}} \\
& =\frac{1}{1+\theta} E_{t} \frac{1}{\gamma(1-\gamma) A_{t}}
\end{aligned}
$$

As $A_{t} \in I_{t}, E_{t} A_{t}=A_{t}$. Then

$$
\begin{aligned}
\frac{1}{\gamma A_{t}} & =\frac{1}{1+\theta} \frac{1}{\gamma(1-\gamma) A_{t}} \\
1 & =\frac{1}{1+\theta} \frac{1}{1-\gamma} \\
\gamma & =\frac{\theta}{1+\theta}
\end{aligned}
$$

The evolution of the assets is the following

$$
\begin{aligned}
A_{t+1} & =\left(1+r_{t}\right)\left(A_{t}-c_{t}\right) \\
& =\left(1+r_{t}\right)\left(A_{t}-\gamma A_{t}\right) \\
& =(1-\gamma)\left(1+r_{t}\right) A_{t}
\end{aligned}
$$

$$
\begin{aligned}
& A_{t}=(1-\gamma)^{t} \prod_{\tau=0}^{t-1}\left(1+r_{\tau}\right) A_{0} \\
& c_{t}=\gamma(1-\gamma)^{t} \prod_{\tau=0}^{t-1}\left(1+r_{\tau}\right) A_{0}
\end{aligned}
$$

We can compute the stochastic intertemporal utility

$$
\begin{aligned}
& E_{0} \sum_{t=0}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right) \\
= & \ln \left(\gamma A_{0}\right)+E_{0} \sum_{t=1}^{\infty}(1+\theta)^{-t} \ln \left[\gamma(1-\gamma)^{t} \prod_{\tau=0}^{t-1}\left(1+r_{\tau}\right) A_{0}\right] \\
= & \ln \gamma+\ln A_{0}+\sum_{t=1}^{\infty}(1+\theta)^{-t} \ln \gamma+\sum_{t=1}^{\infty} t(1+\theta)^{-t} \ln (1-\gamma) \\
& +\sum_{t=1}^{\infty} E_{0}(1+\theta)^{-t} \sum_{\tau=0}^{t-1} \ln \left(1+r_{\tau}\right)+\sum_{t=1}^{\infty}(1+\theta)^{-t} \ln A_{0} \\
= & \sum_{t=0}^{\infty}(1+\theta)^{-t} \ln \gamma+\sum_{t=0}^{\infty}(1+\theta)^{-t} \ln A_{0}+\ln (1-\gamma) \sum_{t=1}^{\infty} t(1+\theta)^{-t} \\
& +\sum_{t=1}^{\infty}(1+\theta)^{-t} \sum_{\tau=0}^{t-1} E_{0} \ln \left(1+r_{\tau}\right) \\
= & \ln \gamma \sum_{t=0}^{\infty}(1+\theta)^{-t}+\ln A_{0} \sum_{t=0}^{\infty}(1+\theta)^{-t}+\ln (1-\gamma) \sum_{t=1}^{\infty} t(1+\theta)^{-t} \\
& +\sum_{t=1}^{\infty}(1+\theta)^{-t} t E_{0} \ln (1+r),
\end{aligned}
$$

because $r_{t}$ is identically and independently distributed.
As $\theta>0$, we obtain

$$
\begin{aligned}
E_{0} \sum_{t=0}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right)= & \frac{1+\theta}{\theta} \ln \gamma+\frac{1+\theta}{\theta} \ln A_{0}+\frac{1+\theta}{\theta^{2}} \ln (1-\gamma) \\
& +\left[E_{0} \ln (1+r)\right] \sum_{t=1}^{\infty} t(1+\theta)^{-t}
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1+\theta}{\theta}\left(\ln \gamma+\ln A_{0}\right) \\
& +\frac{1+\theta}{\theta^{2}}\left[\ln (1-\gamma)+E_{0} \ln (1+r)\right],
\end{aligned}
$$

because if $x \in(0,1)$

$$
\begin{aligned}
\sum_{t=1}^{\infty} t x^{t} & =\sum_{t=0}^{\infty} t x^{t}=\sum_{t=1}^{\infty} \frac{x^{t}}{1-x} \\
& =\frac{x}{1-x} \sum_{t=0}^{\infty} x^{t}=\frac{x}{1-x} \frac{1}{1-x} \\
& =\frac{x}{(1-x)^{2}}
\end{aligned}
$$

### 6.3 The Investment Function

### 6.3.1 Static Behavior

Show the duality between the production maximization and the cost minimization in the Cobb-Douglas case.

## Solution

We want to compare the two programs

$$
\begin{aligned}
& \max _{x} f(x), \\
& w x \leq c .
\end{aligned}
$$

and

$$
\begin{gather*}
\min _{x} w x \\
f(x) \geq y . \tag{6.9}
\end{gather*}
$$

in the Cobb-Douglas case.
We observe that in the Cobb-Douglas case the isoquants are convex. This entails that the two programs are dual and give the same factors demands.

Let us verify this point.
First we consider the production maximization:

$$
\max _{x} x_{1}^{\alpha} x_{2}^{1-\alpha}
$$

$$
w_{1} x_{1}+w_{2} x_{2} \leq c
$$

Lagrangian:

$$
\alpha \ln x_{1}+(1-\alpha) \ln x_{2}+\lambda\left[c-w_{1} x_{1}-w_{2} x_{2}\right]
$$

FOC's:

$$
\begin{aligned}
\frac{\alpha}{x_{1}} & =\lambda w_{1} \\
\frac{1-\alpha}{x_{2}} & =\lambda w_{2} \\
w_{1} x_{1}+w_{2} x_{2} & =c
\end{aligned}
$$

We have

$$
\begin{aligned}
\lambda w_{1} x_{1}+\lambda w_{2} x_{2} & =\lambda c \\
\frac{\alpha}{x_{1}} x_{1}+\frac{1-\alpha}{x_{2}} x_{2} & =\lambda c \\
\lambda & =\frac{1}{c}
\end{aligned}
$$

and finally

$$
\begin{aligned}
x_{1} & =\frac{\alpha}{\lambda w_{1}}=\frac{\alpha}{w_{1}} c \\
x_{2} & =\frac{1-\alpha}{\lambda w_{2}}=\frac{1-\alpha}{w_{2}} c
\end{aligned}
$$

and the optimal production is given by

$$
\begin{aligned}
y & =x_{1}^{\alpha} x_{2}^{1-\alpha} \\
& =\left(\frac{\alpha}{w_{1}} c\right)^{\alpha}\left(\frac{1-\alpha}{w_{2}} c\right)^{1-\alpha} \\
& =\left(\frac{\alpha}{w_{1}}\right)^{\alpha}\left(\frac{1-\alpha}{w_{2}}\right)^{1-\alpha} c
\end{aligned}
$$

The cost in term of production level is given by

$$
\begin{equation*}
c=\left(\frac{w_{1}}{\alpha}\right)^{\alpha}\left(\frac{w_{2}}{1-\alpha}\right)^{1-\alpha} y=c(w, y) \tag{6.10}
\end{equation*}
$$

while the demand functions in terms of production level are provided by

$$
\begin{align*}
x_{1} & =\frac{\alpha}{w_{1}} c=\frac{\alpha}{w_{1}}\left(\frac{w_{1}}{\alpha}\right)^{\alpha}\left(\frac{w_{2}}{1-\alpha}\right)^{1-\alpha} y \\
& =\left(\frac{w_{1}}{\alpha}\right)^{\alpha-1}\left(\frac{w_{2}}{1-\alpha}\right)^{1-\alpha} y \\
& =\left(\frac{\alpha}{1-\alpha} \frac{w_{2}}{w_{1}}\right)^{1-\alpha} y=x_{1}(w, y)  \tag{6.11}\\
x_{2} & =\frac{1-\alpha}{w_{2}} c=\frac{1-\alpha}{w_{2}}\left(\frac{w_{1}}{\alpha}\right)^{\alpha}\left(\frac{w_{2}}{1-\alpha}\right)^{1-\alpha} y \\
& =\left(\frac{w_{1}}{\alpha}\right)^{\alpha}\left(\frac{w_{2}}{1-\alpha}\right)^{-\alpha} y \\
& =\left(\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}}\right)^{\alpha} y=x_{2}(w, y) \tag{6.12}
\end{align*}
$$

We want now to show the duality. In other terms the solutions (6.10), (6.11) and (6.12) are the same we obtain in the cost minimization program (6.9):

$$
\begin{gathered}
\min _{x} w_{1} x_{1}+w_{2} x_{2} \\
x_{1}^{\alpha} x_{2}^{1-\alpha} \geq y
\end{gathered}
$$

Lagrangian:

$$
-w_{1} x_{1}-w_{2} x_{2}+\lambda\left[x_{1}^{\alpha} x_{2}^{1-\alpha}-y\right]
$$

FOC's:

$$
\begin{gathered}
-w_{1}+\lambda \alpha x_{1}^{\alpha-1} x_{2}^{1-\alpha}=0 \\
-w_{2}+\lambda(1-\alpha) x_{1}^{\alpha} x_{2}^{-\alpha}=0 \\
w_{1}=\lambda \alpha x_{1}^{\alpha-1} x_{2}^{1-\alpha} \\
w_{2}=\lambda(1-\alpha) x_{1}^{\alpha} x_{2}^{-\alpha} \\
\frac{w_{1}}{w_{2}}=\frac{\alpha}{1-\alpha} \frac{x_{2}}{x_{1}} \\
x_{2}=\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}} x_{1}
\end{gathered}
$$

Substituting that in the constraint we obtain

$$
\begin{aligned}
x_{1}^{\alpha} x_{2}^{1-\alpha} & =y \\
x_{1}^{\alpha}\left(\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}} x_{1}\right)^{1-\alpha} & =y \\
x_{1}\left(\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}}\right)^{1-\alpha} & =y \\
x_{1}^{*} & =\left(\frac{\alpha}{1-\alpha} \frac{w_{2}}{w_{1}}\right)^{1-\alpha} y \\
x_{2}^{*} & =\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}} x_{1} \\
& =\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}}\left(\frac{\alpha}{1-\alpha} \frac{w_{2}}{w_{1}}\right)^{1-\alpha} y \\
& =\left(\frac{\alpha}{1-\alpha} \frac{w_{2}}{w_{1}}\right)^{-\alpha} y \\
& =\left(\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}}\right)^{\alpha} y
\end{aligned}
$$

Eventually we find

$$
\begin{aligned}
c^{*} & =w_{1} x_{1}^{*}+w_{2} x_{2}^{*} \\
& =w_{1}\left(\frac{\alpha}{1-\alpha} \frac{w_{2}}{w_{1}}\right)^{1-\alpha} y+w_{2}\left(\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}}\right)^{\alpha} y \\
& =w_{1}\left(\frac{\alpha}{1-\alpha} \frac{w_{2}}{w_{1}}\right)\left(\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}}\right)^{\alpha} y+w_{2}\left(\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}}\right)^{\alpha} y \\
& =\left[w_{1}\left(\frac{\alpha}{1-\alpha} \frac{w_{2}}{w_{1}}\right)+w_{2}\right]\left(\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}}\right)^{\alpha} y \\
& =\left[\frac{\alpha}{1-\alpha}+1\right] w_{2}\left(\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}}\right)^{\alpha} y \\
& =\frac{w_{2}}{1-\alpha}\left(\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}}\right)^{\alpha} y \\
& =\left(\frac{w_{1}}{\alpha}\right)^{\alpha}\left(\frac{w_{2}}{1-\alpha}\right)^{1-\alpha} y
\end{aligned}
$$

Summing up, we have:

$$
\begin{aligned}
x_{1}^{*} & =\left(\frac{\alpha}{1-\alpha} \frac{w_{2}}{w_{1}}\right)^{1-\alpha} y \\
x_{2}^{*} & =\left(\frac{1-\alpha}{\alpha} \frac{w_{1}}{w_{2}}\right)^{\alpha} y \\
c^{*} & =\left(\frac{w_{1}}{\alpha}\right)^{\alpha}\left(\frac{w_{2}}{1-\alpha}\right)^{1-\alpha} y
\end{aligned}
$$

These are exactly formulas $(6.10),(6.11)$ and (6.12) and the cost minimization program (6.9) is the dual of the production maximization program. This is due, as highlighted above, to the isoquants convexity.

### 6.3.2 Static Behavior

Solve the profit maximization program with a Cobb-Douglas production function, find the profit function (optimal profit) and verify the envelope theorem by computing the derivative of the profit function with respect to the product price and the factor prices.

Compare this program to production maximization and/or cost minimization.

## Solution

Insert.

### 6.3.3 Dynamic Behavior

Solve the firm value maximization without adjustment costs and show the program equivalence with the static profit maximization.

Which kind of dynamic change is concerned by the adjustment costs?

## Solution

Intertemporal profit maximization.

$$
\begin{aligned}
\max _{\left\{K_{j+1}, N_{j}\right\}_{j=t}^{\infty}} \Pi_{t} \equiv & F\left(K_{t}, N_{t}\right)-\left[K_{t+1}-(1-\delta) K_{t}\right]-w_{t} N_{t} \\
& +\sum_{i=t+1}^{\infty} \frac{1}{\prod_{h=t+1}^{i} R_{h}}\left[F\left(K_{i}, N_{i}\right)-\left[K_{i+1}-(1-\delta) K_{i}\right]-w_{i} N_{i}\right]
\end{aligned}
$$

where $R_{h}$ is the interest factor of period $h$.
FOC's.

$$
\begin{aligned}
\frac{\partial \Pi_{t}}{\partial K_{i}} & =-\frac{1}{\prod_{h=t+1}^{i-1} R_{h}}+\frac{1}{\prod_{h=t+1}^{i} R_{h}}\left[\frac{\partial F\left(K_{i}, N_{i}\right)}{\partial K_{i}}+(1-\delta)\right]=0 \\
i & =t+1, \ldots, \infty \\
\frac{\partial \Pi_{t}}{\partial N_{i}} & =\frac{1}{\prod_{h=t+1}^{i} R_{h}} \frac{\partial F\left(K_{i}, N_{i}\right)}{\partial N_{i}}-\frac{1}{\prod_{h=t+1}^{i} R_{h}} w_{i}=0 \\
i & =t, \ldots, \infty
\end{aligned}
$$

We obtain

$$
\begin{aligned}
& \frac{1}{\prod_{h=t+1}^{i} R_{h}}\left[1-\delta+\frac{\partial F\left(K_{i}, N_{i}\right)}{\partial K_{i}}\right]=\frac{1}{\prod_{h=t+1}^{i-1} R_{h}} \\
& i=t+1, \ldots, \infty \\
& \frac{1}{\prod_{h=t+1}^{i} R_{h}} \frac{\partial F\left(K_{i}, N_{i}\right)}{\partial N_{i}}=\frac{1}{\prod_{h=t+1}^{i} R_{h}} w_{i} \\
& i=t, \ldots, \infty \\
& \frac{1}{R_{i}}\left[1-\delta+\frac{\partial F\left(K_{i}, N_{i}\right)}{\partial K_{i}}\right]=1 \\
& i=t+1, \ldots, \infty \\
& \frac{\partial F\left(K_{i}, N_{i}\right)}{\partial N_{i}}=w_{i} \\
& i=t, \ldots, \infty \\
& 1-\delta+\frac{\partial F\left(K_{i}, N_{i}\right)}{\partial K_{i}}= R_{i} \\
& i=t+1, \ldots, \infty \\
& \frac{\partial F\left(K_{i}, N_{i}\right)}{\partial N_{i}}=w_{i} \\
& i=t, \ldots, \infty \\
& F \in H^{1}
\end{aligned}
$$

Let $t$ be a generic period.

$$
\begin{aligned}
1-\delta+\frac{\partial}{\partial K_{t}}\left[N_{t} F\left(\frac{K_{t}}{N_{t}}, 1\right)\right] & =R_{t} \\
\frac{\partial}{\partial N_{t}}\left[N_{t} F\left(\frac{K_{t}}{N_{t}}, 1\right)\right] & =w_{t}
\end{aligned}
$$

$$
\begin{aligned}
& 1-\delta+\frac{\partial}{\partial K_{t}}\left[N_{t} f\left(k_{t}\right)\right]=R_{t} \\
& \frac{\partial}{\partial N_{t}}\left[N_{t} f\left(k_{t}\right)\right]=w_{t} \\
& k \equiv K / N \\
& f \equiv F / N \\
& R_{t}=1-\delta+\frac{\partial}{\partial K_{t}}\left[N_{t} f\left(k_{t}\right)\right]=1-\delta+N_{t} f^{\prime}\left(k_{t}\right) \frac{1}{N_{t}}=1-\delta+f^{\prime}\left(k_{t}\right) \\
& w_{t}=\frac{\partial}{\partial N_{t}}\left[N_{t} f\left(k_{t}\right)\right]=f\left(k_{t}\right)+N_{t} f^{\prime}\left(k_{t}\right)\left(-\frac{K_{t}}{N_{t}^{2}}\right)=f\left(k_{t}\right)-k_{t} f^{\prime}\left(k_{t}\right)
\end{aligned}
$$

that are the usual first order conditions in the static profit maximization.
Firm value.

$$
\begin{aligned}
Z \delta_{t} & \equiv F\left(K_{t}, N_{t}\right)-\left[K_{t+1}-(1-\delta) K_{t}\right]-w_{t} N_{t} \\
R_{t+1}^{e} & =\frac{q_{t+1}^{e}+\delta_{t+1}^{e}}{q_{t}} \\
q_{t} & =\frac{q_{t+1}+\delta_{t+1}}{R_{t+1}}=\frac{\delta_{t+1}}{R_{t+1}}+\frac{q_{t+1}}{R_{t+1}} \\
& =\frac{\delta_{t+1}}{R_{t+1}}+\frac{1}{R_{t+1}} \frac{q_{t+2}+\delta_{t+2}}{R_{t+2}}=\frac{\delta_{t+1}}{R_{t+1}}+\frac{\delta_{t+2}}{R_{t+1} R_{t+2}}+\frac{q_{t+2}}{R_{t+1} R_{t+2}} \\
q_{t}+\delta_{t} & =\delta_{t}+\sum_{i=t+1}^{\infty} \frac{\delta_{i}}{\prod_{h=t+1}^{i} R_{h}}+\lim _{T \rightarrow \infty} \frac{q_{T}}{\prod_{h=t+1}^{T} R_{h}}
\end{aligned}
$$

where $Z$ is the number of shares and $\delta_{t}$ is the dividend per-share at time $t$.
NBC (No-Bubble-Condition).

$$
\lim _{T \rightarrow \infty} \frac{q_{T}}{\prod_{h=t+1}^{T} R_{h}}=0
$$

Under the NBC we get the value:

$$
q_{t}+\delta_{t}=\delta_{t}+\sum_{i=t+1}^{\infty} \frac{\delta_{i}}{\prod_{h=t+1}^{i} R_{h}}
$$

More explicitly

$$
Z q_{t}+Z \delta_{t}=Z \delta_{t}+\sum_{i=t+1}^{\infty} \frac{Z \delta_{i}}{\prod_{h=t+1}^{i} R_{h}}
$$

$$
\begin{aligned}
= & F\left(K_{t}, N_{t}\right)-\left[K_{t+1}-(1-\delta) K_{t}\right]-w_{t} N_{t} \\
& +\sum_{i=t+1}^{\infty} \frac{F\left(K_{i}, N_{i}\right)-\left[K_{i+1}-(1-\delta) K_{i}\right]-w_{i} N_{i}}{\prod_{h=t+1}^{i} R_{h}} \\
\equiv & \Pi_{t}
\end{aligned}
$$

Therefore the value of the firm is just given by the intertemporal profit:

$$
Z\left(q_{t}+\delta_{t}\right)=\Pi_{t}
$$

### 6.4 Exogenous Saving

### 6.4.1 A Static Linear $I S-L M$ Model

Good market ( $I S$ ) and money market ( $L M$ ) :

$$
I S\left\{\begin{array}{l}
Y=C+I+G+X N \\
C=C_{0}+c Y D \\
Y D \equiv(1-t) Y \\
I=I_{0}-i r \\
G=t Y \\
X N \equiv X-M \\
M=M_{0}+m Y D
\end{array} \quad, L M\left\{\begin{array}{l}
L_{t}=\tau_{Y} Y-\tau_{r} r \\
L_{s}=L_{0}-\sigma r \\
L=L_{s}+L_{t} \\
L=M_{s}
\end{array},\right.\right.
$$

where $Y$ is the product, $C$ the consumption, $I$ the investment, $G$ the public spending, $X N$ the net export, $C_{0}$ the autonomous consumption, $c$ denotes the propensity to consumption, $Y D$ the disposable income, $t$ the rate of the income tax, $I_{0}$ the autonomous investment, $i$ the investment sensitivity with respect to the interest rate, $r$ the interest rate, $X$ the export, $M$ the import, $M_{0}$ the autonomous import, $m$ the propensity to import, $L_{t}$ the demand of transaction money and precautionary money, $\tau_{Y}$ the sensitivity of $L_{t}$ with respect to the income, $\tau_{r}$ the sensitivity of $L_{t}$ with respect to the interest rate, $L_{s}$ the demand of money to speculate, $L_{0}$ the autonomous demand of money to speculate, $\sigma$ the sensitivity of money to speculate with respect to the interest rate, $L$ the money demand and finally $M_{s}$ the money supply.

Notice that there are three equilibrium condition: $Y=C+I+G+$ $X N$ (equality between supply and demand in the good and service market), $L=M_{s}$ (equality between demand and supply in the money market), $G=$
$t Y$ (government budget equilibrium). The equilibrium in the implicit bond market is guaranteed by a corollary of the Walras' law.
(i) Write down the reduced system with two equations and two unknowns: the product $Y$ and the interest rate $r$.
(ii) Find the equilibrium solution $\left(Y^{*}, r^{*}\right)$ in the two markets simultaneously with the technique of matrix inversion.
(iii) Measure the impact of a rise of the marginal propensity to consume on the equilibrium income and interest rate $\left(\partial Y^{*} / \partial c, \partial r^{*} / \partial c\right)$, the impact of the increase of the tax rate $\left(\partial Y^{*} / \partial t, \partial r^{*} / \partial t\right)$, and finally the impact a rise of the marginal propensity to liquidity $\left(\partial Y^{*} / \partial \tau_{Y}, \partial r^{*} / \partial \tau_{Y}\right)$.

## Hint

The system becomes

$$
\left\{\begin{array}{lll}
(1-t)(1+m-c) Y & +i r & =C_{0}+I_{0}+X-M_{0} \\
\tau_{Y} Y & -\left(\sigma+\tau_{r}\right) r & =M_{s}-L_{0}
\end{array} .\right.
$$

### 6.4.2 Time to Double

How do you compute the time it takes the per capita income to double?

## Solution

$$
\begin{aligned}
2 y & =(1+g)^{t} y \\
2 & =(1+g)^{t} \\
\ln 2 & =t \ln (1+g) \\
t & =\frac{\ln 2}{\ln (1+g)} \approx \frac{0.7}{g}
\end{aligned}
$$

### 6.4.3 The Solow Model with a Cobb-Douglas Production Function

The neoclassical production function is now specified. We consider a simple intertemporal economy, which is characterized by an exogenous saving rate $s$ and a Cobb-Douglas production function

$$
\begin{aligned}
F\left(K_{t}, N_{t}\right) & \equiv K_{t}^{\alpha} N_{t}^{1-\alpha} \\
\alpha & \in(0,1)
\end{aligned}
$$

$K$ is the aggregate stock of capital and $L$ is the population growing according to a growth rate equal to $n$. The depreciation rate of capital is denoted by $\delta$. i.e.

$$
\begin{aligned}
f\left(k_{t}\right) & =k_{t}^{\alpha} \\
h & \equiv g(k)=k^{\alpha-1} \\
k & =g^{-1}(h)=h^{-1 /(1-\alpha)}
\end{aligned}
$$

Check that the Cobb-Douglas production function satisfies the Inada conditions.

The law of motion becomes

$$
k_{t+1}=\frac{1-\delta}{1+n} k_{t}+\frac{s}{1+n} k_{t}^{\alpha} \equiv \varphi\left(k_{t}\right)
$$

and the steady state

$$
k=\left(\frac{s}{\delta+n}\right)^{1 /(1-\alpha)}
$$

Insert the comparative statics from (4.4) and (4.5).
Local dynamics.
Since

$$
\begin{aligned}
& \varphi^{\prime}(k)=\frac{1-\delta}{1+n}+\frac{s}{1+n} \alpha k^{\alpha-1} \\
&\left|\varphi^{\prime}(k)\right|=\left|\varphi^{\prime}\left(\left(\frac{s}{\delta+n}\right)^{1 /(1-\alpha)}\right)\right|=\left|\frac{1-\delta}{1+n}+\frac{s}{1+n} \alpha\left[\left(\frac{s}{\delta+n}\right)^{1 /(1-\alpha)}\right]^{\alpha-1}\right| \\
&=\left|\frac{1-\delta}{1+n}+\frac{s}{1+n} \alpha \frac{\delta+n}{s}\right|=\left|\frac{1-\delta}{1+n}+\alpha \frac{\delta+n}{1+n}\right| \\
&=\left|\frac{1-\delta+\alpha \delta+\alpha n}{1+n}\right|=\left|\frac{[1-\delta(1-\alpha)]+\alpha n}{1+n}\right|<1
\end{aligned}
$$

Insert the golden rule analysis $\left(s^{*} \equiv \arg \max _{s} c\right)$.
We know from (4.7) that the golden rule is that the saving rate must equal the capital share on total income:

$$
s^{*}=\varepsilon
$$

In the Cobb-Douglas case we obtain

$$
\varepsilon=\frac{f^{\prime}(k) k}{f(k)}=\frac{\left(\alpha k^{\alpha-1}\right) k}{k^{\alpha}}=\alpha
$$

Thereby the golden rule becomes

$$
s^{*}=\alpha
$$

Speed of convergence.
We apply formula (4.8)

$$
t=\frac{\ln (1-\sigma)}{\ln \varphi^{\prime}\left(g^{-1}((\delta+n) / s)\right)}
$$

where we have fixed

$$
\begin{aligned}
\sigma & =90 \% \\
\delta & =5 \% \\
n & =1 \% \\
\alpha & =1 / 3
\end{aligned}
$$

We know that

$$
\varphi^{\prime}(k)=\frac{[1-\delta(1-\alpha)]+\alpha n}{1+n}
$$

Then

$$
\begin{aligned}
t & =\frac{\ln (1-\sigma)}{\ln (\{[1-\delta(1-\alpha)]+\alpha n\} /(1+n))} \\
& \approx 56.98
\end{aligned}
$$

years.

### 6.4.4 $C E S$ Case

We consider the $C E S$ case.

$$
\begin{aligned}
F\left(K_{t}, N_{t}\right) & \equiv\left(a K_{t}^{\frac{\sigma-1}{\sigma}}+b N_{t}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \\
a, b & >0 \\
\sigma & >0 \\
\sigma & \neq 1
\end{aligned}
$$

( $\sigma=0$ is the Leontief case, $\sigma=1$ is the Cobb-Douglas case, $\sigma=\infty$ is the linear case).

The elasticity of substitution is defined as follows

$$
\frac{d\left(N_{t} / K_{t}\right)}{N_{t} / K_{t}} / \frac{d(-T R S)}{-T R S}=\frac{d \ln \left(N_{t} / K_{t}\right)}{d \ln (-T R S)}
$$

Provide a graphic interpretation. We observe that

$$
-T R S=\frac{\partial F / \partial K_{t}}{\partial F / \partial N_{t}}
$$

and that

$$
\begin{aligned}
\frac{\partial F}{\partial K_{t}} & =\frac{\partial\left(a K_{t}^{\frac{\sigma-1}{\sigma}}+b N_{t}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}}{\partial K_{t}} \\
& =\frac{\sigma}{\sigma-1}\left(a K_{t}^{\frac{\sigma-1}{\sigma}}+b N_{t}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} a \frac{\sigma-1}{\sigma} K_{t}^{\frac{\sigma-1}{\sigma}-1} \\
& =\left(a K_{t}^{\frac{\sigma-1}{\sigma}}+b N_{t}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} a K_{t}^{-\frac{1}{\sigma}} \\
\frac{\partial F}{\partial N_{t}} & =\frac{\partial\left(a K_{t}^{\frac{\sigma-1}{\sigma}}+b N_{t}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}}{\partial N_{t}} \\
& =\frac{\sigma}{\sigma-1}\left(a K_{t}^{\frac{\sigma-1}{\sigma}}+b N_{t}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} b \frac{\sigma-1}{\sigma} N_{t}^{\frac{\sigma-1}{\sigma}-1} \\
& =\left(a K_{t}^{\frac{\sigma-1}{\sigma}}+b N_{t}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} b N_{t}^{-\frac{1}{\sigma}}
\end{aligned}
$$

Hence

$$
-T R S=\frac{\left(a K_{t}^{\frac{\sigma-1}{\sigma}}+b N_{t}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} a K_{t}^{-\frac{1}{\sigma}}}{\left(a K_{t}^{\frac{\sigma-1}{\sigma}}+b N_{t}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} b N_{t}^{-\frac{1}{\sigma}}}=\frac{a}{b}\left(\frac{N_{t}}{K_{t}}\right)^{\frac{1}{\sigma}}
$$

Let $x_{t} \equiv \ln \left(N_{t} / K_{t}\right)$. Therefore

$$
\begin{aligned}
\frac{d \ln \left(N_{t} / K_{t}\right)}{d \ln (-T R S)} & =\frac{d \ln \left(N_{t} / K_{t}\right)}{d \ln \left[(a / b)\left(N_{t} / K_{t}\right)_{t}^{1 / \sigma}\right]} \\
& =\frac{d \ln \left(N_{t} / K_{t}\right)}{d\left[\ln (a / b)+(1 / \sigma) \ln \left(N_{t} / K_{t}\right)\right]}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{d x_{t}}{d\left[\ln (a / b)+(1 / \sigma) x_{t}\right]} \\
& =\left\{\frac{d\left[\ln (a / b)+(1 / \sigma) x_{t}\right]}{d x_{t}}\right\}^{-1} \\
& =(1 / \sigma)^{-1}=\sigma
\end{aligned}
$$

Notice that $F$ is homogeneous of degree 1, as required in the Solow model. The per-capita production function is:

$$
\begin{aligned}
\frac{F\left(K_{t}, N_{t}\right)}{N_{t}} & =\frac{1}{N_{t}}\left(a K_{t}^{\frac{\sigma-1}{\sigma}}+b N_{t}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \\
& =\left[a\left(\frac{K_{t}}{N_{t}}\right)^{\frac{\sigma-1}{\sigma}}+b\right]^{\frac{\sigma}{\sigma-1}} \\
& =\left(a k_{t}^{\frac{\sigma-1}{\sigma}}+b\right)^{\frac{\sigma}{\sigma-1}} \equiv f\left(k_{t}\right)
\end{aligned}
$$

Notice that

$$
f(0)=b^{\frac{\sigma}{\sigma-1}}>0
$$

The Inada conditions are not respected. There is no longer the trivial steady state: a positive production can be ensured by a positive labor.

$$
\begin{aligned}
f^{\prime}\left(k_{t}\right) & =\frac{\sigma}{\sigma-1}\left(a k_{t}^{\frac{\sigma-1}{\sigma}}+b\right)^{\frac{\sigma}{\sigma-1}-1} a \frac{\sigma-1}{\sigma} k_{t}^{\frac{\sigma-1}{\sigma}-1} \\
& =\left(a k_{t}^{\frac{\sigma-1}{\sigma}}+b\right)^{\frac{1}{\sigma-1}} a k_{t}^{-\frac{1}{\sigma}}>0 \\
f^{\prime}(0+) & =a^{\sigma /(\sigma-1)}>0 \text { if } \sigma<1 \\
& =\infty \text { if } \sigma>1 \\
f^{\prime}(\infty) & =0 \text { if } \sigma<1 \\
& =a^{\sigma /(\sigma-1)} \text { if } \sigma>1 \\
f^{\prime \prime}\left(k_{t}\right) & =\left[\left(a k_{t}^{\frac{\sigma-1}{\sigma}}+b\right)^{\frac{1}{\sigma-1}} a k_{t}^{-\frac{1}{\sigma}}\right]^{\prime} \\
& =\frac{a}{\sigma} k^{-1 / \sigma}\left(a k_{t}^{\frac{\sigma-1}{\sigma}}+b\right)^{\frac{1}{\sigma-1}-1}\left[a k^{1 / \sigma}-\left(a k_{t}^{\frac{\sigma-1}{\sigma}}+b\right) / k\right]
\end{aligned}
$$

Then $f^{\prime \prime}<0$ if and only if

$$
a k^{(1+\sigma) / \sigma}<a k^{(\sigma-1) / \sigma}+b
$$

Example: let $a=b=1$ and $\sigma=1 / 2$ and $\sigma=2$. We obtain the following figures with respectively $f\left(k_{t}\right)=\left(k_{t}^{-1}+1\right)^{-1}$ and $f\left(k_{t}\right)=\left(k_{t}^{1 / 2}+1\right)^{2}$.


Figure 25.


Figure 26.

The law of motion of the general Solow model holds

$$
k_{t+1}=\frac{1-\delta}{1+n} k_{t}+\frac{s}{1+n} f\left(k_{t}\right)
$$

and becomes in our case

$$
k_{t+1}=\frac{1-\delta}{1+n} k_{t}+\frac{s}{1+n}\left(a k_{t}^{\frac{\sigma-1}{\sigma}}+b\right)^{\frac{\sigma}{\sigma-1}}
$$

Steady state.
There is no longer the trivial steady state $\left(k_{0}=0\right)$ for $b>0$. The existence of a non-trivial one depends on the parameter configuration. From (4.3) we have

$$
\begin{aligned}
\frac{f(k)}{k} & =\frac{\delta+n}{s} \\
\frac{\left(a k^{\frac{\sigma-1}{\sigma}}+b\right)^{\frac{\sigma}{\sigma-1}}}{k} & =\frac{\delta+n}{s}
\end{aligned}
$$

$$
\begin{aligned}
\left(a k^{\frac{\sigma-1}{\sigma}}+b\right)^{\frac{\sigma}{\sigma-1}} & =\frac{\delta+n}{s} k \\
a k^{\frac{\sigma-1}{\sigma}}+b & =\left(\frac{\delta+n}{s} k\right)^{\frac{\sigma-1}{\sigma}} \\
a k^{\frac{\sigma-1}{\sigma}}+b & =\left(\frac{\delta+n}{s}\right)^{\frac{\sigma-1}{\sigma}} k^{\frac{\sigma-1}{\sigma}} \\
{\left[\left(\frac{\delta+n}{s}\right)^{\frac{\sigma-1}{\sigma}}-a\right] k^{\frac{\sigma-1}{\sigma}} } & = \\
k^{\frac{\sigma-1}{\sigma}} & =b /\left[\left(\frac{\delta+n}{s}\right)^{\frac{\sigma-1}{\sigma}}-a\right] \\
k & =\left\{\frac{b}{[(\delta+n) / s]^{(\sigma-1) / \sigma}-a}\right\}^{\sigma /(\sigma-1)}
\end{aligned}
$$

The existence condition

$$
[(\delta+n) / s]^{(\sigma-1) / \sigma}>a
$$

Analyze the comparative statics.
Stationary production.

$$
\begin{aligned}
y & =\left[a\left(\left\{\frac{b}{[(\delta+n) / s]^{(\sigma-1) / \sigma}-a}\right\}^{\sigma /(\sigma-1)}\right)^{\frac{\sigma-1}{\sigma}}+b\right]^{\frac{\sigma}{\sigma-1}} \\
& =\left\{a \frac{b}{[(\delta+n) / s]^{(\sigma-1) / \sigma}-a}+b\right\}^{\frac{\sigma}{\sigma-1}} \\
& =\left(\left\{\frac{a}{[(\delta+n) / s]^{(\sigma-1) / \sigma}-a}+1\right\} b\right)^{\frac{\sigma}{\sigma-1}} \\
& =\left(\left\{\frac{a+[(\delta+n) / s]^{(\sigma-1) / \sigma}-a}{[(\delta+n) / s]^{(\sigma-1) / \sigma}-a}\right\} b\right)^{\frac{\sigma}{\sigma-1}} \\
& =\left\{\frac{[(\delta+n) / s]^{(\sigma-1) / \sigma} b}{[(\delta+n) / s]^{(\sigma-1) / \sigma}-a}\right\}
\end{aligned}
$$

$$
=\left\{\frac{b}{[(\delta+n) / s]^{(\sigma-1) / \sigma}-a}\right\}^{\frac{\sigma}{\sigma-1}} \frac{\delta+n}{s}
$$

Stationary consumption.

$$
c=(1-s)\left\{\frac{b}{[(\delta+n) / s]^{(\sigma-1) / \sigma}-a}\right\}^{\frac{\sigma}{\sigma-1}} \frac{\delta+n}{s}
$$

Local dynamics.
Dynamics are one-dimensional. The local stability condition requires the unique eigenvalue to have modulus less than one:

$$
|\lambda|=\left|\varphi^{\prime}(k)\right|<1
$$

where

$$
\varphi^{\prime}\left(k_{t}\right)=\left[\frac{1-\delta}{1+n} k_{t}+\frac{s}{1+n} f\left(k_{t}\right)\right]^{\prime}=\frac{1-\delta}{1+n}+\frac{s}{1+n} f^{\prime}\left(k_{t}\right)
$$

i.e.

$$
\begin{aligned}
& \left|\frac{1-\delta}{1+n}+\frac{s}{1+n} f^{\prime}(k)\right|=\left|\frac{1-\delta}{1+n}+\frac{s}{1+n}\left(a k^{\frac{\sigma-1}{\sigma}}+b\right)^{\frac{1}{\sigma-1}} a k^{-\frac{1}{\sigma}}\right| \\
= & \left\lvert\, \frac{1-\delta}{1+n}+\frac{s}{1+n}\left[a\left(\left\{\frac{b}{[(\delta+n) / s]^{(\sigma-1) / \sigma}-a}\right\}^{\sigma /(\sigma-1)}\right)^{\frac{\sigma-1}{\sigma}}+b\right]^{\frac{1}{\sigma-1}}\right. \\
& \left.a\left(\left\{\frac{b}{[(\delta+n) / s]^{(\sigma-1) / \sigma}-a}\right\}^{\sigma /(\sigma-1)}\right)^{-\frac{1}{\sigma}} \right\rvert\,<1
\end{aligned}
$$

Find the growth rate of the intensive variables when the steady state does not exists.

### 6.5 Endogenous Saving

### 6.5.1 The Clower Constraint

"Money buys goods, goods buy money, but goods do not buy goods". This was the seminal intuition of Clower (1967). First Stockman (1981) formalizes Clower in a context of dynamic general equilibrium, by adding a cash-inadvance constraint to budget constraint. The representative agent maximizes an intertemporal utility functional

$$
\sum_{t=1}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right)
$$

where $\theta$ is a measure of impatience (time preference) and $c_{t}$ denotes the consumption at period $t$. Individual must satisfy a budget constraint at each period: $M_{t+1}+p_{t} k_{t+1}+p_{t} c_{t} \leq M_{t}+p_{t} r_{t} k_{t}+p_{t} w_{t} l_{t}$, where $M_{t+1}$ is the money demand at period $t, p_{t}$ is the price of a unique produced good which is employed as either consumption or investment good. $k_{t+1}$ is the investment at period $t$, which will generate the production of period $t+1$. Capital entirely depreciates at each period $(\delta=1) . r_{t}$ is the real return on one unit of capital, and $w_{t}$ is the real wage. $l_{t}$ are the labor forces. Individual faces a cash-inadvance constraint on consumption: $p_{t} c_{t} \leq M_{t}$. In other words he must pay cash his consumption. The production function $F\left(k_{t}, l_{t}\right)$ displays constant returns to scale in capital and labor. Labor supply is inelastic: $l_{t}=1$. Money supply is constant across the time: $M_{t}^{s}=M$. Observe that the agent must store at each period $(t-1)$ an amount $M_{t}$ of money to finance his consumption $c_{t}$ in the following period. The existence of a monetary equilibrium is exactly guaranteed by this monetary constraint.

In what follows we shall compute $(i)$ the dynamic system and (ii) the stationary state. We will characterize (iii) the local stability of stationary state and eventually verify ( iv ) equilibrium determinacy.

## Solution

(i) The program of representative household is the following:

$$
\begin{aligned}
& \max _{\left\{\left(M_{t+1}, k_{t+1}, c_{t}\right)\right\}_{t=1}^{\infty}} \sum_{t=1}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right), \\
& M_{t+1}+p_{t} k_{t+1}+p_{t} c_{t} \leq M_{t}+p_{t} r_{t} k_{t}+p_{t} w_{t} l_{t}, \\
& p_{t} c_{t} \leq M_{t} \\
& M_{1}, k_{1} \text { given, } \\
& t=1,2, \ldots
\end{aligned}
$$

The infinite horizon Lagrangian is characterized by two sequences of multipliers:

$$
\begin{aligned}
& \sum_{t=1}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right) \\
& +\sum_{t=1}^{\infty} \lambda_{t}\left[M_{t}+p_{t} r_{t} k_{t}+p_{t} w_{t} l_{t}-M_{t+1}-p_{t} k_{t+1}-p_{t} c_{t}\right] \\
& +\sum_{t=1}^{\infty} \mu_{t}\left[M_{t}-p_{t} c_{t}\right]
\end{aligned}
$$

Deriving with respect to $\left\{\left(M_{t+1}, k_{t+1}, c_{t}\right)\right\}_{t=1}^{\infty}$ one obtains the following first order conditions: $(1+\theta)^{-t} u^{\prime}\left(c_{t}\right)=p_{t}\left(\lambda_{t}+\mu_{t}\right), \lambda_{t-1}=\lambda_{t}+\mu_{t}, \lambda_{t-1} p_{t-1}=$ $r_{t} \lambda_{t} p_{t}, t=1,2, \ldots$ By eliminating the multipliers one obtains: $u^{\prime}\left(c_{t+1}\right) / u^{\prime}\left(c_{t}\right)$ $=(1+\theta)\left(1+\pi_{t+1}\right) /\left[r_{t}\left(1+\pi_{t}\right)\right]$, where $1+\pi_{t} \equiv p_{t} / p_{t-1}$ is the gross inflation rate. The period utility $u$ (felicity) is specified in a $C E S$ (constant elasticity of substitution) form. Thereby $u(c)=\left(c^{1-1 / \sigma}-1\right) /(1-1 / \sigma)$, where $\sigma$ is the elasticity. So the previous equation becomes: $\left(c_{t} / c_{t+1}\right)^{1 / \sigma}$ $=(1+\theta)\left(1+\pi_{t+1}\right) /\left[r_{t}\left(1+\pi_{t}\right)\right]$.

We compute the market clearing solution. Under constant returns to scale in production, profit maximization requires $r_{t}=f^{\prime}\left(k_{t}\right)$ and $w_{t}=$ $f\left(k_{t}\right)-k_{t} f^{\prime}\left(k_{t}\right)$, where $f\left(k_{t}\right)=f\left(k_{t} / l_{t}\right) \equiv F\left(k_{t} / l_{t}, 1\right)$, with $f^{\prime \prime}<0$. The inelastic labor supply is normalized to $l_{t}^{s}=1$. The equilibrium conditions for the firm are satisfied. The opportunity cost of holding money is given by the nominal interest rate and, as we will see later on, it is positive at steady state because of a zero inflation rate. This implies that the cash-in-advance is binding (individuals do not want to hold more real balances): $p_{t} c_{t}=M_{t}$, i.e. $p_{t} c_{t}=M$ at equilibrium because for the sake
of simplicity we assume a constant money supply. Price ratio becomes $1+\pi_{t+1}=c_{t} / c_{t+1}$. Good market equilibrium requires $k_{t+1}+c_{t}=f\left(k_{t}\right)$. As $f\left(k_{t}\right)=r_{t} k_{t}+w_{t} l_{t}$ and $M_{t}=M$, the budget constraint is binding too. Hence $1+\pi_{t}=\left[f\left(k_{t-1}\right)-k_{t}\right] /\left[f\left(k_{t}\right)-k_{t+1}\right]$. Collecting these results and considering the logarithmic felicity case $(\sigma=1)$, one has $(1+\theta) / f^{\prime}\left(k_{t}\right)$ $=1+\pi_{t}=\left[f\left(k_{t-1}\right)-k_{t}\right] /\left[f\left(k_{t}\right)-k_{t+1}\right]$. The reduced dynamics for the state variable $k_{t}$ is then provided by a second order difference equation: $k_{t+1}=f\left(k_{t}\right)-\left[f\left(k_{t-1}\right)-k_{t}\right] f^{\prime}\left(k_{t}\right) /(1+\theta)$. To construct an equivalent twodimensional system of first order, we simply set $h_{t} \equiv k_{t-1}$. Thereby

$$
\begin{aligned}
h_{t+1} & =k_{t} \\
k_{t+1} & =f\left(k_{t}\right)-\left[f\left(h_{t}\right)-k_{t}\right] f^{\prime}\left(k_{t}\right) /(1+\theta) .
\end{aligned}
$$

(ii) The stationary state is implicitly determined by $f^{\prime}(k)=1+\theta, c=$ $f(k)-k, p=M / c, \pi=0$.
(iii) The Jacobian matrix of dynamic system evaluated at the steady state is

$$
\begin{aligned}
J^{*} & =\left[\begin{array}{cc}
0 & 1 \\
-f^{\prime 2} /(1+\theta) & f^{\prime}-f^{\prime \prime}[f-k] /(1+\theta)+f^{\prime} /(1+\theta)
\end{array}\right] \\
& =\left[\begin{array}{cc}
0 & 1 \\
-(1+\theta) & (1+\theta)+1-f^{\prime \prime}(k)[f(k)-k] /(1+\theta)
\end{array}\right] .
\end{aligned}
$$

Straightforward computations give $D=1+\theta$ and $T=1+D+a$, where $a \equiv-f^{\prime \prime}(k)[f(k)-k] /(1+\theta)>0$. Hence $1<D<T-1$ and the steady state is always a saddle point (see figure 18). As the initial condition $k_{1}$ is given, then $h_{t} \equiv k_{t-1}$ is a predetermined variable, while $k_{t}$ is a choice variable, i.e. the control (for $t=2,3, \ldots$ ). The equilibrium dynamics is determinate because the dimension of the stable saddle manifold (1) equals the number of predetermined variables (1)).

Now we can sum up about determinacy and neutrality. Equilibrium determinacy implies that there is a unique equilibrium under rational expectations. Capital $k_{t-1}$ is a predetermined variable and capital $k_{t}$ is not predetermined. At each period there is only one possible choice for consumer to stay on the converging saddle path. Rational expectation arguments allow for this solution, because agents reply to each price path announce only with the consumption paths respecting the transversality condition. The auctioneer will find the equilibrium price paths which clear the market, only among the paths satisfying the transversality condition.

In our case the saddle path is the unique trajectory compatible with the transversality condition and agents are forced by their rational expectation behavior to select $k_{t}$ such that ( $k_{t-1}, k_{t}$ ) belongs to the saddle path, given the capital $k_{t-1}$ inherited from the previous period.

As well as in the standard Real Business Cycle models we can simulate the propagation of shocks on fundamentals (outside money and technology). As in Sidrauski (1967), a continuous time model of money in the utility function, the standard result of the Ramsey-Cass-Koopmans benchmark, i.e. the modified golden rule, holds: the market is efficient and performs the Paretooptimal planner's solution. Observe that the depreciation rate of capital is one and the modified golden rule becomes $f^{\prime}(k)=1+\theta$. Furthermore money is neutral at steady state: $c^{*}=M / p$.

In slightly different monetary models indeterminacy and sub-optimality may arise under low elasticity of intertemporal substitution. Eventually note that in the model there is no monetary growth. Money is no longer superneutral under monetary growth and resource allocation is affected by the monetary growth rate.

### 6.5.2 Barro Model

We provide the discrete time version of an endogenous growth model with taxes and public expenditure due to Barro (1990). The public spending plays as a production externality.

An infinite-lived and representative agent maximizes an intertemporal utility functional

$$
\sum_{t=0}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right)
$$

where $\theta$ measures the time preference, $c_{t}$ denotes the consumption which gives him an utility $u\left(c_{t}\right)$ at period $t$. The utility function is assumed to be increasing and strictly concave. The consumer faces a budget constraint at each period

$$
k_{t+1}-k_{t}+c_{t} \leq(1-\tau)\left(r_{t} k_{t}+w_{t} l_{t}^{s}\right),
$$

where $k_{t+1}-k_{t}$ denotes the investment in capital. The capital by simplicity does not depreciate. On the right-hand side the disposable income is constituted by the capital income $r_{t} k_{t}$ and the labor income $w_{t} l_{t}^{s}$ after the income $\operatorname{tax} \tau . l_{t}^{s}$ is the amount of labor services provided by the representative agent during a period of production. We assume an inelastic labor supply $l_{t}^{s}=1$.

Let us consider the firm equilibrium and the government budget equilibrium. A constant private returns to scale production function is specified as in Barro (1990)

$$
F\left(k_{t}, l_{t}^{d}\right)=A k_{t}^{\alpha}\left(l_{t}^{d}\right)^{1-\alpha} g_{t}^{\varepsilon},
$$

where $l_{t}^{d}$ is the firm's labor demand and $\alpha$ is the capital share on total income. $g_{t}$ is the public spending which plays as a positive externality in production, and $\varepsilon>0$ is the relative elasticity ${ }^{1}$.

Write down the global dynamics and find the tax which maximizes the welfare.

## Solution

The intensive production is obtained, by normalizing the production function by the labor services $l_{t}^{d}$.

$$
f\left(h_{t}\right) \equiv F\left(k_{t}, l_{t}^{d}\right) / l_{t}^{d}=A\left(k_{t} / l_{t}^{d}\right)^{\alpha} g_{t}^{\varepsilon}
$$

where $h_{t} \equiv k_{t} / l_{t}^{d}$.
As in Barro (1990) we set $\varepsilon=1-\alpha$ to allow for a balanced growth. Therefore

$$
f\left(h_{t}\right)=A h_{t}^{\alpha} g_{t}^{1-\alpha} .
$$

Firm equilibrium requires

$$
\begin{aligned}
r_{t} & =f^{\prime}\left(h_{t}\right), \\
w_{t} & =f\left(h_{t}\right)-f^{\prime}\left(h_{t}\right) h_{t} .
\end{aligned}
$$

We obtain

$$
r_{t}=\alpha A h_{t}^{\alpha-1} g_{t}^{1-\alpha} .
$$

Because of the inelastic labor supply at equilibrium we get $l_{t}^{d}=l_{t}^{s}=1$. Therefore

$$
h_{t}=k_{t} .
$$

In this model the income tax is the only way to finance public spending. Budget equilibrium requires

$$
g_{t}=\tau\left(r_{t} k_{t}+w_{t}\right)=\tau f\left(k_{t}\right)=\tau A k_{t}^{\alpha} g_{t}^{1-\alpha} .
$$

[^14]It follows that

$$
\begin{aligned}
g_{t} & =(\tau A)^{1 / \alpha} k_{t}, \\
f\left(k_{t}\right) & =A^{1 / \alpha} \tau^{1 / \alpha-1} k_{t}, \\
r_{t} & =\alpha A^{1 / \alpha} \tau^{1 / \alpha-1} \equiv r .
\end{aligned}
$$

The production per unit of labor services is linear in the intensive capital, while the real interest rate is a constant $(r)$ and depends on the technological parameters ( $\alpha$ and $A$ ) and on the income tax rate $(\tau)$.

The representative agent maximizes the intertemporal utility functional under the budget constraint. $k_{0}$ is given as initial condition. The choice sequences are $\left\{k_{t}\right\}_{t=1}^{\infty},\left\{c_{t}\right\}_{t=0}^{\infty}$. We set the Lagrangian

$$
\begin{aligned}
L= & \sum_{t=0}^{\infty}(1+\theta)^{-t} u\left(c_{t}\right) \\
& +\sum_{t=0}^{\infty} \lambda_{t}\left[(1-\tau)\left(r_{t} k_{t}+w_{t}\right)-k_{t+1}+k_{t}-c_{t}\right]
\end{aligned}
$$

where $\lambda_{t}$ is a non-negative Lagrangian multiplier.
We obtain the following necessary first order conditions which are also sufficient because of the strict concavity of the utility function.

$$
\begin{aligned}
\partial L / \partial k_{t} & =0 \\
\partial L / \partial c_{t} & =0 \\
\lim _{t \rightarrow \infty} \lambda_{t} k_{t} & =0
\end{aligned}
$$

Notice that the first equation must hold for $t=1,2, \ldots$ while the second must hold for $t=0,1, \ldots$ The last equation is the usual transversality condition.

Rearranging we get the relevant Euler equation:

$$
\frac{u^{\prime}\left(c_{t}\right)}{u^{\prime}\left(c_{t+1}\right)}=\frac{1+(1-\tau) r_{t}}{1+\theta}=\frac{1+(1-\tau) \alpha A^{1 / \alpha} \tau^{1 / \alpha-1}}{1+\theta}
$$

The utility function is now assumed to display a constant elasticity of intertemporal substitution $\sigma$ :

$$
u\left(c_{t}\right) \equiv \frac{c_{t}^{1-1 / \sigma}-1}{1-1 / \sigma}
$$

We obtain the consumption dynamics:

$$
\frac{c_{t+1}}{c_{t}}=\left[\frac{1+(1-\tau) \alpha A^{1 / \alpha} \tau^{1 / \alpha-1}}{1+\theta}\right]^{\sigma} .
$$

The economy jumps from its very beginning on the long run growth factor:

$$
\begin{equation*}
1+\gamma \equiv\left[\frac{1+(1-\tau) \alpha A^{1 / \alpha} \tau^{1 / \alpha-1}}{1+\theta}\right]^{\sigma} \tag{6.13}
\end{equation*}
$$

In other words there is no transition.
More explicitly

$$
c_{t}=\left[\frac{1+(1-\tau) \alpha A^{1 / \alpha} \tau^{1 / \alpha-1}}{1+\theta}\right]^{\sigma t} c_{0}
$$

The law of motion for capital is

$$
\begin{aligned}
k_{t+1}-k_{t} & =(1-\tau)\left(r_{t} k_{t}+w_{t}\right)-c_{t} \\
& =(1-\tau) f\left(k_{t}\right)-c_{t}=(1-\tau) A^{1 / \alpha} \tau^{1 / \alpha-1} k_{t}-c_{t} .
\end{aligned}
$$

The only non trivial equilibrium is the balanced growth: $\gamma_{y}=\gamma_{k}=\gamma_{c} \equiv$ $\gamma$, i.e. $k_{t+1} / k_{t}-1=(1-\tau) A^{1 / \alpha} \tau^{1 / \alpha-1}-c_{t} / k_{t}$. Hence $1+\gamma=k_{t+1} / k_{t}$ $=1+(1-\tau) A^{1 / \alpha} \tau^{1 / \alpha-1}-c_{0} / k_{0}$ i.e. $c_{0}=\left[(1-\tau) A^{1 / \alpha} \tau^{1 / \alpha-1}-\gamma\right] k_{0}$. Notice that $k_{0}$ is known as initial condition. The complete solution becomes

$$
\begin{aligned}
k_{t}= & {\left[\frac{1+(1-\tau) \alpha A^{1 / \alpha} \tau^{1 / \alpha-1}}{1+\theta}\right]^{\sigma t} k_{0}, } \\
c_{0}= & {\left[1+(1-\tau) A^{1 / \alpha} \tau^{1 / \alpha-1}-\left[\frac{1+(1-\tau) \alpha A^{1 / \alpha} \tau^{1 / \alpha-1}}{1+\theta}\right]^{\sigma}\right] k_{0}(6.14) } \\
c_{t}= & {\left[\frac{1+(1-\tau) \alpha A^{1 / \alpha} \tau^{1 / \alpha-1}}{1+\theta}\right]^{\sigma t} } \\
& \left\{1+(1-\tau) A^{1 / \alpha} \tau^{1 / \alpha-1}-\left[\frac{1+(1-\tau) \alpha A^{1 / \alpha} \tau^{1 / \alpha-1}}{1+\theta}\right]^{\sigma}\right\} k_{0} .
\end{aligned}
$$

It is possible to measure the welfare at the steady state. We evaluate the utility functional along the balanced growth path.

If $\sigma \neq 1$, we obtain

$$
\begin{aligned}
W & =\sum_{t=0}^{\infty}(1+\theta)^{-t} \frac{c_{t}^{1-1 / \sigma}-1}{1-1 / \sigma} \\
& =\frac{\sigma}{1-\sigma} \frac{1+\theta}{\theta}-\frac{\sigma}{1-\sigma} c_{0}^{-(1-\sigma) / \sigma} \sum_{t=0}^{\infty}\left[\frac{(1+\gamma)^{-(1-\sigma) / \sigma}}{1+\theta}\right]^{t} .
\end{aligned}
$$

The convergence of the series requires

$$
(1+\theta)^{-1}(1+\gamma)^{-(1-\sigma) / \sigma}>1
$$

i.e. exactly the transversality condition. We get

$$
W=\frac{\sigma}{1-\sigma}\left[\frac{1+\theta}{\theta}+\frac{c_{0}^{-(1-\sigma) / \sigma}}{(1+\theta)^{-1}(1+\gamma)^{-(1-\sigma) / \sigma}-1}\right]
$$

where $\gamma$ and $c_{0}$ are respectively defined by (6.13) and (6.14).
If $\sigma=1$, i.e. $u\left(c_{t}\right)=\ln c_{t}$, we obtain

$$
W=\frac{1+\theta}{\theta} \ln k_{0}+\frac{1+\theta}{\theta}\left\{\frac{1}{\theta} \ln (1+\gamma)+\ln \left[1+(1-\tau) \frac{r}{\alpha}-(1+\gamma)\right]\right\} .
$$

Given the other parameters the welfare function can be viewed as a function of the income tax rate.

$$
W=W(\tau) .
$$

Straightforward computations show that the welfare is maximized by

$$
\tau^{*}=1-\alpha
$$

in accordance with Barro's (1990) result.

### 6.5.3 The Diamond Model with Central Planner

Diamond (1965) as Samuelson (1958) is an overlapping generations model where agents have a two-period life. Production activity and capital accumulation are considered. First we study a centralized economy where a benevolent planner takes in account the welfare of future generations and implements a Pareto optimum. The main result of Ramsey model, i.e. the
modified golden rule, holds. However a decentralized economy may be inefficient. In the $O L G$ case the competitive equilibrium decentralized by the market, may be Pareto-inferior and there is room for dynamic inefficiency, i.e. the failure of first welfare theorem (see the next section).

The economy is composed by individuals and firms. Individuals born at time $t$ live two periods: they consume $c_{1, t}$ in period $t$ and $c_{2, t+1}$ in period $t+1$.


Figure 27. Overlapping generations.
The planner's program is the following.

$$
\begin{aligned}
& \left.\max _{\left\{k_{t}, c_{1, t}, c_{2}, t\right.}\right\}_{t=1}^{\infty}(1+\theta)^{-1} u\left(c_{2,0}\right) \\
& +\sum_{t=0}^{\infty}(1+R)^{-(t+1)}\left[u\left(c_{1, t}\right)+(1+\theta)^{-1} u\left(c_{2, t+1}\right)\right] \\
& f\left(k_{t}\right)=(1+n) k_{t+1}-k_{t}+c_{1, t}+c_{2, t} /(1+n), t=1,2, \ldots
\end{aligned}
$$

The objective is a welfare measure as weighted average of generational utilities $u$. The weights are constituted by powers of the social discount factor $(1+R)^{-(t+1)}$, where $R$ is the planner's time preference. Individual time preference is captured by $\theta$. All remaining notations are usual. Notice that a representative agent is considered for each generation. Aggregate demands for investment and consumption equal the aggregate production: $K_{t}+F\left(K_{t}, N_{t}\right)$ $=K_{t+1}+N_{t} c_{1, t}+N_{t-1} c_{2, t}$. Dividing by the generation size $N_{t}$ one obtains the constraint of the program $f\left(k_{t}\right)=(1+n) k_{t+1}-k_{t}+c_{1, t}+c_{2, t} /(1+n)$, i.e. $c_{1, t}=k_{t}+f\left(k_{t}\right)-(1+n) k_{t+1}-c_{2, t} /(1+n)$. The program becomes

$$
\begin{aligned}
& \max _{c_{2, t}}(1+\theta)^{-1} u\left(c_{2,0}\right)+\sum_{t=0}^{\infty}(1+R)^{-(t+1)} \\
& *\left[u\left(k_{t}+f\left(k_{t}\right)-(1+n) k_{t+1}-c_{2, t} /(1+n)\right)+(1+\theta)^{-1} u\left(c_{2, t+1}\right)\right]
\end{aligned}
$$

and is equivalent to two sequences of sub-programs:
(i) $\max _{c_{2, t}}(1+R)^{-t+1-1}$
$+(1+R)^{-t-1} u\left(k_{t}+f\left(k_{t}\right)-(1+n) k_{t+1}-c_{2, t} /(1+n)\right)$,
(ii) $\max _{k_{t}}(1+R)^{-t+1-1} u\left(k_{t-1}+f\left(k_{t-1}\right)-(1+n) k_{t-1+1}-c_{2, t-1} /(1+n)\right)$
$+(1+R)^{-t-1} u\left(k_{t}+f\left(k_{t}\right)-(1+n) k_{t+1}-c_{2, t} /(1+n)\right)$
with $t=1,2, \ldots$ The first order conditions of sequence $(i)$ are: $(1+R)^{-t}$ $(1+\theta)^{-1} u^{\prime}\left(c_{2, t}\right)+(1+R)^{-(t+1)} u^{\prime}\left(c_{1, t}\right)\left[-(1+n)^{-1}\right]=0$, i.e. $(1+\theta)^{-1}$ $u^{\prime}\left(c_{2, t}\right)=(1+R)^{-1}(1+n)^{-1} u^{\prime}\left(c_{1, t}\right)$ describing the allocation between the old and the young at time $t$. The sub-program (ii) provides the second sequence of first order conditions $(1+R)^{-t} u^{\prime}\left(c_{1, t-1}\right)[-(1+n)]+(1+R)^{-1}$ $u^{\prime}\left(c_{1, t}\right)\left[1+f^{\prime}\left(k_{t}\right)\right]=0$. Thus

$$
u^{\prime}\left(c_{1, t-1}\right)(1+n)=(1+R)^{-1} u^{\prime}\left(c_{1, t}\right)\left[1+f^{\prime}\left(k_{t}\right)\right]=0
$$

and $u^{\prime}\left(c_{1, t-1}\right)=(1+R)^{-1}(1+n)^{-1} u^{\prime}\left(c_{1, t}\right)\left[1+f^{\prime}\left(k_{t}\right)\right]$. The first order conditions are summed up by $(i)(1+\theta)^{-1} u^{\prime}\left(c_{2, t}\right)=(1+R)^{-1}(1+n)^{-1}$ $u^{\prime}\left(c_{1, t}\right)$ and (ii) $u^{\prime}\left(c_{1, t-1}\right)=(1+R)^{-1}(1+n)^{-1}\left[1+f^{\prime}\left(k_{t}\right)\right] u^{\prime}\left(c_{1, t}\right)$. At the steady state the last condition becomes $1+f^{\prime}(k)=(1+n)(1+R) \approx 1+$ $n+R$, which constitutes a discrete time restatement of Ramsey's modified golden rule: $f^{\prime}(k)=n+R . R$ is interpreted as an infinite horizon time preference, while the finite horizon time preference $\theta$ does not matter.

### 6.5.4 The Diamond Model with Market Economy

In the market model saving decisions are decentralized. Each household becomes a decision center. Individual are price taker and the price system carries the information. In this real economy prices are given by the real wage $w$ and real interest rate $r$. As above economy is composed by individuals and firms. Individuals born at time $t$ live two periods: they still consume $c_{1, t}$ in period $t$ and $c_{2, t+1}$ in period $t+1$ (see figure 19). The problem the individual born in $t$ faces, is the following.

$$
\begin{aligned}
& \max _{c_{1, t}, c_{2, t+1}, s_{t}} u\left(c_{1, t}\right)+(1+\theta)^{-1} u\left(c_{2, t+1}\right), \\
& c_{1, t}+s_{t}=w_{t} \\
& c_{2, t+1}=\left(1+r_{t+1}\right) s_{t} .
\end{aligned}
$$

He works only in the first period of his life. He furnishes only one unit of labor and receives the wage $w_{t} . r_{t}$ is the return on capital, i.e. the real interest rate. To solve the program we substitute the constraints in the utility function: $\max _{s_{1 t}} u\left(w_{t}-s_{t}\right)+(1+\theta)^{-1} u\left(\left(1+r_{t+1}\right) s_{t}\right)$. Endogenous saving simply means that saving is a choice variable. The first order condition is $u^{\prime}\left(w_{t}-s_{t}\right)-\left(1+r_{t+1}\right) u^{\prime}\left(\left(1+r_{t+1}\right) s_{t}\right) /(1+\theta)=0$. It is an implicit function. We apply the implicit function theorem to describe the local dynamics. Differentiating the first order condition, we obtain the partial derivatives for the implicit function: $s_{t}=s\left(r_{t+1}, w_{t}\right)$. We denote them by $s_{w}$ and $s_{r}: u^{\prime \prime}\left(w_{t}-s_{t}\right) d w_{t}-u^{\prime \prime}\left(w_{t}-s_{t}\right) d s_{t}-\left[\left(1+r_{t+1}\right) /(1+\theta)\right] u^{\prime \prime}\left(\left(1+r_{t+1}\right) s_{t}\right)$ $\left(1+r_{t+1}\right) d s_{t}=0$, i.e.

$$
\frac{d s_{t}}{d w_{t}}=\frac{u^{\prime \prime}\left(w_{t}-s_{t}\right)}{u^{\prime \prime}\left(w_{t}-s_{t}\right)+\left(1+r_{t+1}\right)^{2} u^{\prime \prime}\left(\left(1+r_{t+1}\right) s_{t}\right) /(1+\theta)} \in(0,1)
$$

The impact of real wage on saving is unambiguous. In contrast to know the sign of $d s_{t} / d r_{t}$, more information is required. It depends on the relative importance of substitution and revenue effects.

On the firm side profit is maximized. We compute the equilibrium conditions. A neoclassical production function is adopted: $F\left(K_{t}, N_{t}\right)$. We normalize the aggregate production to obtain $f\left(k_{t}\right) \equiv F\left(k_{t}, 1\right)=F\left(K_{t}, L_{t}\right) / N_{t}$. Profit maximization requires $f^{\prime}\left(k_{t}\right)=r_{t}$ and $w_{t}=f\left(k_{t}\right)-f^{\prime}\left(k_{t}\right) k_{t}$, because the production function if homogeneous of degree 1. The Euler formula implies an optimal zero profit.

The equilibrium in the good market requires that aggregate demand equals aggregate supply and investment equals saving: $I=S$. By definition the investment is the variation of capital. Thereby $\Delta K=S$ and $K_{t+1}-K_{t}=N_{t} s_{t}-K_{t} . N_{t} s_{t}$ is the saving of the young. The old consume $\left(1+r_{t}\right) s_{t-1} N_{t-1}$, i.e. the capital $K_{t}=s_{t-1} N_{t-1}$ and the fruit of capital $K_{t}: r_{t} s_{t-1} N_{t-1}$. Thus the capital increases of $K_{t+1}=N_{t} s_{t}$, the contribution of the young, and decreases of $K_{t}=N_{t-1} s_{t-1}$, the dissaving of the old. Notice that here the capital is assumed to not depreciate $(\delta=0)$. It is consumed by the old. The growth rate for population is assumed to be constant $N_{t+1} / N_{t}=1+n$. The aggregate capital is normalized by the generation size: $k_{t+1}=K_{t+1} / N_{t+1}=s_{t} /\left(N_{t+1} / N_{t}\right)=s_{t} /(1+n)$. Thereby $(1+n) k_{t+1}=s_{t}=s\left(r_{t+1}, w_{t}\right)$.

Local dynamics. We know from the Euler formula that production is shared by capitalists and workers: $f^{\prime}\left(k_{t}\right)=r_{t}=r\left(k_{t}\right)$ and $w_{t}=f\left(k_{t}\right)-$
$f^{\prime}\left(k_{t}\right) k_{t}=w\left(k_{t}\right)$. Substituting these prices in the implicit equation $(1+n) k_{t+1}=$ $s_{t}=s\left(r_{t+1}, w_{t}\right)$ we obtain $k_{t+1}=s\left(r_{t+1}, w_{t}\right) /(1+n)=s_{t}\left(r\left(k_{t+1}\right), w\left(k_{t}\right)\right)$ $/(1+n)$. Hence $k_{t+1}-s\left(f^{\prime}\left(k_{t+1}\right), f\left(k_{t}\right)-f^{\prime}\left(k_{t}\right) k_{t}\right) /(1+n) \equiv \Phi\left(k_{t}, k_{t+1}\right)$ $=0$. This is an implicit difference equation:

$$
\Phi\left(k_{t}, k_{t+1}\right)=0
$$

The stationary state is given by $\Phi(k, k)=0$. We locally know the explicit function $k_{t+1}=\varphi\left(k_{t}\right)$. The steady state is locally stable if and only if the eigenvalue lies inside the unit circle: $\left|\varphi^{\prime}(k)\right|<1$. From the implicit function theorem we obtain: $\varphi^{\prime}(k)=d k_{t+1} /\left.d k_{t}\right|_{k^{*}}=-\left(\partial \Phi / \partial k_{t}\right) /\left.\left(\partial \Phi / \partial k_{t+1}\right)\right|_{k^{*}}$. The total differential is $d \Phi=\left(\partial \Phi / \partial k_{t}\right) d k_{t}+\left(\partial \Phi / \partial k_{t+1}\right) d k_{t+1}=0$. Then $\left|d k_{t+1} / d k_{t}\right|_{k^{*}}\left|=\left|-s_{w} k f^{\prime \prime}(k) /\left[1+n-s_{r} f^{\prime \prime}(k)\right]\right|<1\right.$. Monotonic convergence requires $0<-s_{w} k f^{\prime \prime}(k) /\left[1+n-s_{r} f^{\prime \prime}(k)\right]<1$, where

$$
\begin{aligned}
s_{r} & =\partial s\left(r_{t+1}, w_{t}\right) / \partial r_{t+1} \\
s_{w} & =\partial s\left(r_{t+1}, w_{t}\right) / \partial w_{t}
\end{aligned}
$$

Dynamics are represented in figure 28.


Figure 28. Monotonic convergence.
Dynamic inefficiency. We investigate the condition under which dynamic inefficiency arise. The competitive equilibrium may be different from the central planner's solution and the first welfare theorem may fail. The consumption is $C=Y-I$, i.e. $C_{t}=F\left(K_{t}, N_{t}\right)-\left(K_{t+1}-K_{t}\right)$. Normalizing by
the population size one obtain: $C_{t} / N_{t}=F\left(K_{t}, N_{t}\right) / N_{t}-\left(K_{t+1} / N_{t}-K_{t} / N_{t}\right)$. Thus $c_{t}=f\left(k_{t}\right)-\left[\left(K_{t+1} / N_{t+1}\right)\left(N_{t+1} / N_{t}\right)-k_{t}\right]=f\left(k_{t}\right)-k_{t+1}(1+n)-k_{t}$. At the stationary state we have $c=f(k)-n k$. The capital which maximizes the stationary consumption, is $f^{\prime}(k)=n$, or in terms of gross rates $1+f^{\prime}(k)=1+n$. This is just the golden rule of Solow model ${ }^{2}$. In the CassKoopmans model the modified golden rule is Pareto-optimal: $1+f^{\prime}(k)=$ $(1+n)(1+\theta)$ which is approximated in continuous time by $f^{\prime}(k)=\theta+n$ as in the Ramsey model. Notice that $k_{M G R}<k_{G R}$ because the production function is concave. The saving of modified golden rule is optimal because the time preference is taken into account. The oversaving of golden rule is clearly inefficient. In the Solow model preferences were naively specified: the saving rate was constant, which in terms of time preference means $\theta=0$.

In the $O L G$ planner's case the modified golden rule becomes $1+f^{\prime}(k)=$ $(1+n)(1+R)$, i.e. approximately $f^{\prime}(k) \approx n+R$, where $R$ replaces $\theta$. The social discount factor of the planner's who takes into account an infinite number of generations replaces the discount factor of the representative agent living an infinite number of periods. The planner's case is optimal in the Pareto sense. First notice that $R$ depends on planner's tastes and is discretional. Hence the outcome of a decentralized economy is likely to not coincide with the planner's optimum. In the decentralized case the equilibrium may be sub-optimal and the economy may be over-capitalized at the steady state.

To show that in general we consider the aggregate consumption $C_{t} \equiv$ $N_{t} c_{1 t}+N_{t-1} c_{2 t}$. By dividing by $N_{t}$ one obtains $c_{t}=c_{1 t}+c_{2 t} /(1+n)$ and from $f\left(k_{t}\right)=(1+n) k_{t+1}-k_{t}+c_{1 t}+c_{2 t} /(1+n)$, i.e. $f\left(k_{t}\right)=(1+n) k_{t+1}-k_{t}+c_{t}$. The steady state is given by $c=f(k)-n k$ and the golden rule by $f^{\prime}(k)=n$. Let the economy be at the steady state: $k_{t}=k$. Assume a decrease of $k_{t+1}$. From $f\left(k_{t}\right)=(1+n) k_{t+1}-k_{t}+c_{t}$ one obtain $d c_{t}=-(1+n) d k_{t+1}>$ 0 . Such decrease may be permanent, so for every successive period from $c=f(k)-n k$ one obtains $d c=\left(f^{\prime}-n\right) d k$. As $d k<0$, then $d c>0$ if and only if $f^{\prime}<n$. In this case all generations will be better off and a downward deviation from the inefficient golden rule, i.e. a decrease of stationary capital stock, will constitute a Pareto-improvement. It is clear that in the Cass-Koopmans-Ramsey case $f^{\prime}=n+\theta>n$ and in the planner's

[^15]case $f^{\prime}=n+R>n$. So the equilibrium turns out to be Pareto-efficient. In contrast in a decentralized $O L G$ economy we may observe $f^{\prime}<n$. The general equilibrium is no longer optimal and the first welfare theorem fails. To see more in detail the occurrence of dynamic inefficiency, an example with a Cobb-Douglas production function is provided in the next section. Otherwise the reader is referred to Blanchard and Fischer (1989, p. 103).

### 6.5.5 The Decentralized Equilibrium in an Overlapping Generation Model

By providing an explicit form for fundamentals, we are able to solve the decentralized equilibrium in an overlapping generations model à la Diamond (1965). As above the economy is composed by individuals and firms. The agents born at time $t$ live two periods and consume $c_{1, t}$ in period $t$ and $c_{2, t+1}$ in period $t+1$. The utility of the individual born at time $t$ is specified as follows: $\ln c_{1, t}+(1+\theta)^{-1} \ln c_{2, t+1}$. The agent works only in the first period of his life, supplies inelastically a unit of labor and earns $w_{t}$. He saves $s_{t}$. The real return on his saving is $r_{t+1}$. This saving will finance the consumption of second period. The production function is a Cobb-Douglas: $F\left(K_{t}, N_{t}\right) \equiv K_{t}^{\alpha} N_{t}^{1-\alpha}, 0<\alpha<1$, where $N_{t}$ is the size of generation $t$. Let $n$ be the constant population growth rate. Growth is then exponential. We want $(i)$ to write the individual program, (ii) to evaluate the impact of the interest rate on saving, (iii) to determine the equilibrium of the firm, (iv) to compute the stationary state for capital, production, consumption and prices. Moreover we investigate the stability of the steady state. We want $(v)$ to determine the explicit dynamics for capital, production, consumption and prices, (vi) to explore the sense of a decentralized equilibrium and (vii) dynamic inefficiency.

## Solution

(i) The consumer's program gets the following form:

$$
\begin{aligned}
& \max _{c_{1, t,}, c_{2, t}, s_{t}} \ln c_{1, t}+(1+\theta)^{-1} \ln c_{2, t+1}, \\
& c_{1 t}+s_{t}=w_{t} \\
& c_{2, t+1}=\left(1+r_{t+1}\right) s_{t} .
\end{aligned}
$$

An equivalent program is $\max _{s_{t}} \ln \left(w_{t}-s_{t}\right)+(1+\theta)^{-1} \ln \left(\left(1+r_{t+1}\right) s_{t}\right)$. Saving is now endogenously determined. The choice of saving gives the first order condition: $-1 /\left(w_{t}-s_{t}\right)+\left(1+r_{t+1}\right) /\left[(1+\theta)\left(1+r_{t+1}\right) s_{t}\right]=0$, i.e. $s_{t}$ $=w_{t} /(2+\theta)$.
(ii) Notice that $\partial s_{t} / \partial r_{t+1}=0$ : this means that the revenue effect compensates exactly the substitution effect.
(iii) The intensive production is represented by the following function: $f\left(k_{t}\right) \equiv K_{t}^{\alpha} N_{t}^{1-\alpha} / N_{t}=\left(K_{t} / N_{t}\right)^{\alpha}=k_{t}^{\alpha}$. At the equilibrium the profit is maximized: $f^{\prime}\left(k_{t}\right)=r_{t}$ and $w_{t}=f\left(k_{t}\right)-k_{t} f^{\prime}\left(k_{t}\right)$ (Euler formula). Thereby $\alpha k_{t}^{\alpha-1}=r_{t}$ and $w_{t}=k_{t}^{\alpha}-\alpha k_{t}^{\alpha-1} k_{t}=(1-\alpha) k_{t}^{\alpha}$.
(iv) $N_{t} s_{t}$ is the saving of the young. $K_{t}$ is the dissaving of the old. The old people consume $\left(1+r_{t}\right) s_{t-1} N_{t-1}$, i.e. the capital $K_{t}=s_{t-1} N_{t-1}$ and its fruits $r_{t} s_{t-1} N_{t-1}$. Then $K_{t+1}=s_{t} N_{t}$ and $k_{t+1}=K_{t+1} / N_{t+1}=s_{t} /\left(N_{t+1} / N_{t}\right)$ $=s_{t} /(1+n)$. We focus now on the implicit dynamics. $s_{t}=w_{t} /(2+\theta)$, $w_{t}=(1-\alpha) k_{t}^{\alpha}$ and $k_{t+1}=s_{t} /(1+n)$ implies $k_{t+1}=w_{t} /[(1+n)(2+\theta)]$ $=(1-\alpha) k_{t}^{\alpha} /[(1+n)(2+\theta)]$. The capital of steady state is computed as follows: $k=w_{t} /[(1+n)(2+\theta)]=(1-\alpha) k^{\alpha} /[(1+n)(2+\theta)]$, i.e. $k=$ $\{(1-\alpha) /[(1+n)(2+\theta)]\}^{1 /(1-\alpha)}$. The stationary production is given by $y$ $=k^{\alpha}=\{(1-\alpha) /[(1+n)(2+\theta)]\}^{\alpha /(1-\alpha)}$. The stationary consumption is now obtained. Note that $C_{t} \equiv N_{t} c_{1 t}+N_{t-1} c_{2 t}$, i.e. $c_{t}=c_{1 t}+c_{2 t} /(1+n)$. The equilibrium requires $f\left(k_{t}\right)=(1+n) k_{t+1}-k_{t}+c_{1 t}+c_{2 t} /(1+n)$, then $f\left(k_{t}\right)=(1+n) k_{t+1}-k_{t}+c_{t}$. It follows that $c_{t}=f\left(k_{t}\right)+k_{t}-(1+n) k_{t+1}=$ $k_{t}^{\alpha}+k_{t}-(1+n)(1-\alpha) k^{\alpha} /[(1+n)(2+\theta)]=k_{t}+[1-(1-\alpha) /(2+\theta)] k_{t}^{\alpha}$. The consumption of steady state is given by $c=f(k)-n k=k^{\alpha}-n k$ $=\{(1-\alpha) /[(1+n)(2+\theta)]\}^{\alpha /(1-\alpha)}-n\{(1-\alpha) /[(1+n)(2+\theta)]\}^{1 /(1-\alpha)}$. The interest rate is $r=\alpha k^{\alpha-1}=\alpha(1+n)(2+\theta) /(1-\alpha)$, while the stationary wage is $w=(1-\alpha) k^{\alpha}=(1-\alpha)\{(1-\alpha) /[(1+n)(2+\theta)]\}^{\alpha /(1-\alpha)}$. The stability aspect are now investigated.

$$
\left|\partial k_{t+1} / \partial k_{t}\right|^{*}=\left|\alpha(1-\alpha) k^{\alpha-1} /[(1+n)(2+\theta)]\right|=\alpha<1:
$$

the steady state is stable.
$(v)$ The explicit dynamics for all the variables are computed.

$$
\begin{aligned}
k_{1} & =\{(1-\alpha) /[(1+n)(2+\theta)]\} k_{0}^{\alpha}, \\
k_{2} & =\{(1-\alpha) /[(1+n)(2+\theta)]\}\left\{\{(1-\alpha) /[(1+n)(2+\theta)]\} k_{0}^{\alpha}\right\}^{\alpha}, \\
& =\{(1-\alpha) /[(1+n)(2+\theta)]\}^{1+\alpha} k_{0}^{\alpha^{2}}, \\
k_{3} & =\{(1-\alpha) /[(1+n)(2+\theta)]\}^{1+\alpha+\alpha^{2}} k_{0}^{\alpha^{3}} .
\end{aligned}
$$

In general

$$
\begin{aligned}
k_{t} & =\{(1-\alpha) /[(1+n)(2+\theta)]\}_{\tau=0}^{\sum_{\tau=0}^{t-1} \alpha^{\tau}} k_{0}^{\alpha^{t}} \\
& =\{(1-\alpha) /[(1+n)(2+\theta)]\}^{\left(1-\alpha^{t}\right) /(1-\alpha)} k_{0}^{\alpha^{t}} .
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\lim _{t \rightarrow \infty} k_{t} & =\{(1-\alpha) /[(1+n)(2+\theta)]\}^{\lim _{t \rightarrow \infty}\left[\left(1-\alpha^{t}\right) /(1-\alpha)\right]} k_{0}^{\lim _{t \rightarrow \infty} \alpha^{t}} \\
& =\{(1-\alpha) /[(1+n)(2+\theta)]\}^{1 /(1-\alpha)}=k
\end{aligned}
$$

because $\alpha \in(0,1)$. It is possible to know the dynamics of all variables. The production path is: $y_{t}=k_{t}^{\alpha}=\{(1-\alpha) /[(1+n)(2+\theta)]\}^{\alpha\left(1-\alpha^{t}\right) /(1-\alpha)} k_{0}^{\alpha^{t+1}}$. The consumption trajectory:

$$
\begin{aligned}
c_{t}= & k_{t}+[1-(1-\alpha) /(2+\theta)] k_{t}^{\alpha} \\
= & \{(1-\alpha) /[(1+n)(2+\theta)]\}^{\left(1-\alpha^{t}\right) /(1-\alpha)} k_{0}^{\alpha^{t}} \\
& +[1-(1-\alpha) /(2+\theta)]\{(1-\alpha) /[(1+n)(2+\theta)]\}^{\alpha\left(1-\alpha^{t}\right) /(1-\alpha)} k_{0}^{\alpha^{t+1}} .
\end{aligned}
$$

The equilibrium interest rate changes according to

$$
r_{t}=\alpha k_{t}^{\alpha-1}=\alpha\{(1-\alpha) /[(1+n)(2+\theta)]\}^{\alpha^{t}-1} k_{0}^{(\alpha-1) \alpha^{t}} .
$$

Eventually the real wage path is

$$
w_{t}=(1-\alpha) k_{t}^{\alpha}=(1-\alpha)\{(1-\alpha) /[(1+n)(2+\theta)]\}^{\alpha\left(1-\alpha^{t}\right) /(1-\alpha)} k_{0}^{\alpha^{t+1}}
$$

(vi) Agents are price takers. The prices carry all information of the economic system. Price movement clears the market: the equilibrium is general (in every market demand equals supply) and dynamic (in every period demand equals supply). Here the equilibrium is not decided by a central planner. Every agent decides independently from the others, by simply observing the prices.
(vii) The dynamic system is inefficient if the dynamic general equilibrium is not optimal in the Pareto sense, i.e. the first welfare theorem fails. One knows that a sufficient condition to have dynamic inefficiency is $f^{\prime}(k)<n$ (see the previous section). Here $f^{\prime}(k)=r=\alpha k^{\alpha-1}=\alpha(1+n)(2+\theta)$ $/(1-\alpha)<n$. If for instance the technological parameter $\alpha$ is low enough, dynamic inefficiency is observed.

### 6.5.6 Central Planner's Problems

We compare the two planner's solutions for an economy with infinite-lived representative agent and an economy with overlapping generations.

## Solution

First the planner maximizes an infinite horizon intertemporal utility of a representative consumer: $\sum_{t=0}^{\infty}(1+\Theta)^{-t} \ln c_{t}$. The initial endowment for capital is denoted by $k_{0}$. The product is partially consumed $\left(c_{t}\right)$, partially devoted to investment $\left(k_{t+1}-(1-\delta) k_{t}\right.$, where $\delta$ denotes the depreciation rate of capital).

We write the law of motion for capital and determine the first order conditions of maximization.. A simple infinite horizon economy is considered. The program of a representative agent is the following: $\max \sum_{t=0}^{\infty}(1+\Theta)^{-t} \ln c_{t}$ subject to $f\left(k_{t}\right)=\left[k_{t+1}-(1-\delta) k_{t}\right]+c_{t}$ where the notation is usual and $\Theta$ captures the time preference. Capital depreciates at rate $\delta$. The intensive production function is linear: $f\left(k_{t}\right)=A k_{t}$.

The Lagrangian is

$$
\sum_{t=0}^{\infty}(1+\Theta)^{-t} \ln c_{t}+\sum_{t=1}^{\infty} \lambda_{t}\left[f\left(k_{t}\right)-k_{t+1}+(1-\delta) k_{t}-c_{t}\right]
$$

The first order conditions are the following. Deriving with respect to $k_{t}$ : $\lambda_{t-1} / \lambda_{t}=1-\delta+f^{\prime}\left(k_{t}\right)$. Deriving with respect to $c_{t}:(1+\Theta)^{-t+1} u^{\prime}\left(c_{t-1}\right)$ $/\left[(1+\Theta)^{-t} u^{\prime}\left(c_{t}\right)\right]=\lambda_{t-1} / \lambda_{t}=1-\delta+f^{\prime}\left(k_{t}\right)$, i.e. $u^{\prime}\left(c_{t-1}\right) / u^{\prime}\left(c_{t}\right)=$ $\left[1-\delta+f^{\prime}\left(k_{t}\right)\right] /(1+\Theta)$. We must integrate this Euler condition with the law of motion: $k_{t+1}-(1-\delta) k_{t}=f\left(k_{t}\right)-c_{t}$, and the transversality condition $\lim _{t \rightarrow \infty} k_{t} \lambda_{t}(1+\Theta)^{-t}=0$.

We determine the growth rate for consumption, capital and product. The Euler condition becomes $c_{t+1} / c_{t}=\left\{\left[1-\delta+f^{\prime}\left(k_{t+1}\right)\right] /[1+\Theta]\right\}$. In this model $f^{\prime}\left(k_{t+1}\right)=A$ because of linearity assumption. This endogenous growth assumption is crucial and radically changes the conclusion of the exogenous growth benchmark of Solow, Cass-Koopmans-Ramsey and Diamond. Let $1+\gamma_{t+1}^{c} \equiv c_{t+1} / c_{t}$ be the consumption growth factor. So $1+\gamma^{c}$ $=(1-\delta+A) /(1+\Theta)$. There is no transition, economy directly jumps on the stationary growth rate $\gamma^{c}$. Growth is exponential:

$$
c_{t}=[(1-\delta+A) /(1+\Theta)]^{t} c_{0}
$$

From $k_{t+1}=(1-\delta) k_{t}+A k_{t}-c_{t}$ one computes the capital growth factor $1+\gamma_{t+1}^{k} \equiv k_{t+1} / k_{t}=1-\delta+A-c_{t} / k_{t}$.

We compute the initial consumption as a function of initial capital, and the capital, the consumption and the product at each period as a function of the initial capital. The only possibility is a balanced growth, i.e. a common growth rate for capital, consumption and product: $\gamma^{y}=\gamma^{k}=\gamma^{c} \equiv \gamma$. This implies that the initial condition $k_{0}$ determines the initial product and consumption: $y_{0}=f\left(k_{0}\right)=A k_{0}$ and $(1-\delta+A) /(1+\Theta)=1-\delta+A-c_{0} / k_{0}$, i.e. $c_{0}=(A-\gamma) k_{0}=k_{0} \Theta(1-\delta+A) /(1+\Theta)$. Finally

$$
\begin{align*}
c_{t} & =\Theta[(1-\delta+A) /(1+\Theta)]^{t+1} k_{0},  \tag{6.15}\\
k_{t} & =[(1-\delta+A) /(1+\Theta)]^{t} k_{0},  \tag{6.16}\\
y_{t} & =[(1-\delta+A) /(1+\Theta)]^{t} A k_{0} . \tag{6.17}
\end{align*}
$$

An overlapping generation model is now considered. The central planner maximizes the welfare function

$$
\ln c_{2,0}+\sum_{t=0}^{\infty}(1+\Theta)^{-t}\left[\ln c_{1, t}+(1+\theta)^{-1} \ln c_{2, t+1}\right]
$$

where $\ln c_{2,0}$ is the old's utility in the initial period and $\ln c_{1, t}+(1+\theta)^{-1} \ln c_{2, t+1}$ is the utility of the generation born at time $t . c_{1, t}$ is the consumption of the young born at time $t, c_{2, t}$ is the consumption of period $t$ of the old born at time $t-1 . \theta$ denotes the time preference which is common to every generation. $\Theta$ is the social planner's time preference. The welfare function can be interpreted as a weighted average of generational utilities. We set the population growth equal to zero for simplicity. The capital depreciation rate is $\delta$. The intensive production is linear $f\left(k_{t}\right)=\left(A K_{t}\right) / N_{t}=A k_{t}$. Let $c_{t} \equiv c_{1, t}+c_{2, t}$ be the aggregate consumption of period $t$.

We write the law of motion for capital and determine the first order conditions for maximization. The planner's program is: $\max _{\left\{k_{t+1}, c_{1, t}, c_{2, t}\right\}_{t=0}^{\infty}}$ $\ln c_{2,0}+\sum_{t=0}^{\infty}(1+\Theta)^{-t}\left[\ln c_{1, t}+(1+\theta)^{-1} \ln c_{2, t+1}\right]$ subject to the law of motion for capital $k_{t+1}-(1-\delta) k_{t}=f\left(k_{t}\right)-c_{1, t}-c_{2, t}$. We substitute $c_{1, t}$ $=(1-\delta) k_{t}+f\left(k_{t}\right)-k_{t+1}-c_{2, t}$ in the objective. So the program becomes:

$$
\begin{aligned}
& \max _{\left\{k_{t+1}, c_{2, t}\right\}_{t=0}^{\infty}} \ln c_{2,0} \\
& +\sum_{t=0}^{\infty}(1+\Theta)^{-t}\left\{\ln \left[(1-\delta) k_{t}+f\left(k_{t}\right)-k_{t+1}-c_{2, t}\right]+(1+\theta)^{-1} \ln c_{2, t+1}\right\} .
\end{aligned}
$$

Deriving with respect to $c_{2, t}$ we obtain:

$$
(1+\Theta)^{-t+1}(1+\theta)^{-1} u^{\prime}\left(c_{2, t}\right)-(1+\Theta)^{-t} u^{\prime}\left(c_{1, t}\right)=0
$$

i.e. $u^{\prime}\left(c_{1, t}\right) / u^{\prime}\left(c_{2, t}\right)=c_{2, t} / c_{1, t}=(1+\Theta) /(1+\theta)$ describing the optimal allocation between the old and the young at time $t$. Deriving the objective with respect to $k_{t}$, one obtains the following first order condition

$$
-(1+\Theta)^{-t+1} u^{\prime}\left(c_{1, t-1}\right)+(1+\Theta)^{-t} u^{\prime}\left(c_{1, t}\right)\left[1-\delta+f^{\prime}\left(k_{t}\right)\right]=0,
$$

i.e. $u^{\prime}\left(c_{1, t-1}\right) / u^{\prime}\left(c_{1, t}\right)=c_{1, t} / c_{1, t-1}=(1-\delta+A) /(1+\Theta)$ describing the optimal intertemporal allocation.

We determine the growth rate for consumption, capital and product. We compute the capital, the aggregate consumption and the product at each period as functions of the initial capital. We shall compare these result with the corresponding ones in an infinite horizon setup. Note that $c_{2, t}=c_{1, t}(1+\Theta) /(1+\theta)$, so the aggregate consumption is $c_{t} \equiv c_{1, t}+c_{2, t}$ $=[1+(1+\Theta) /(1+\theta)] c_{1, t}=c_{1, t}(2+\theta+\Theta) /(1+\theta)$, i.e. $c_{1, t}=c_{t}(1+\theta)$ $/(2+\theta+\Theta)$ and $c_{2, t}=c_{t}(1+\Theta) /(2+\theta+\Theta)$. The growth factor of $c_{1}$ is given by $c_{1, t} / c_{1, t-1}=(1-\delta+A) /(1+\Theta)$, so $c_{1, t}=[(1-\delta+A) /(1+\Theta)]^{t}$ $c_{1,0}$ and

$$
\begin{aligned}
c_{t} & =[(2+\theta+\Theta) /(1+\theta)][(1-\delta+A) /(1+\Theta)]^{t} c_{1,0} \\
& =[(1-\delta+A) /(1+\Theta)]^{t} c_{0} .
\end{aligned}
$$

One knows that $k_{t+1}-(1-\delta) k_{t}=f\left(k_{t}\right)-c_{1, t}-c_{2, t}=f\left(k_{t}\right)-c_{t}$. Exactly as above from $k_{t+1}=(1-\delta) k_{t}+A k_{t}-c_{t}$ one computes the capital growth factor $1+\gamma_{t+1}^{k} \equiv k_{t+1} / k_{t}=1-\delta+A-c_{t} / k_{t}$. As above the only possibility is a balanced growth, i.e. a common growth rate for capital, consumption and product: $\gamma^{y}=\gamma^{k}=\gamma^{c} \equiv \gamma$. This implies that the initial condition $k_{0}$ determines the initial product and consumption: $y_{0}=A k_{0}$ and $(1-\delta+A) /(1+\Theta)$ $=1-\delta+A-c_{0} / k_{0}$, i.e. $c_{0}=(A-\gamma) k_{0}=k_{0} \Theta(1-\delta+A) /(1+\Theta)$. Finally

$$
\begin{aligned}
c_{t} & =\Theta[(1-\delta+A) /(1+\Theta)]^{t+1} k_{0}, \\
c_{1, t} & =[(1+\theta) /(2+\theta+\Theta)] \Theta[(1-\delta+A) /(1+\Theta)]^{t+1} k_{0}, \\
c_{2, t} & =[(1+\Theta) /(2+\theta+\Theta)] \Theta[(1-\delta+A) /(1+\Theta)]^{t+1} k_{0}, \\
k_{t} & =[(1-\delta+A) /(1+\Theta)]^{t} k_{0}, \\
y_{t} & =[(1-\delta+A) /(1+\Theta)]^{t} A k_{0} .
\end{aligned}
$$

which are exactly formulas (6.15) - (6.17).

## Bibliography

[1] Allais M. 1947. Economie et intérêt. Imprimerie Nationale, Paris.
[2] Arrow K. 1962. The Economic Implications of Learning by Doing. Review of Economic Studies 28, 155-73.
[3] Azariadis C. 1981. Self-Fulfilling Prophecies. Journal of Economic Theory, 25, 380-96.
[4] Azariadis C. and R. Guesnerie. 1986. Sunspots and Cycles. Review of Economic Studies 53, 725-38.
[5] Barro R.J. 1974. Are Government Bonds Net Wealth? Journal of Political Economy 82, 1095-117.
[6] Barro R.J. 1990. Government Spending in a Simple Model of Endogenous Growth. Journal of Political Economy 98, 103-25.
[7] Barro R.J., N.G. Mankiw and X. Sala-i-Martín. 1995. Capital Mobility in Neoclassical Models of Growth. American Economic Review, 85, 10315.
[8] Barro R.J. and X. Sala-i-Martín. 1995. Economic Growth. McGraw-Hill, New York.
[9] Baumol W. 1952. The Transactions Demand for Cash. Quarterly Journal of Economics 67, 545-56.
[10] Bellman R. 1957. Dynamic Programming. Princeton University Press, N.J.
[11] Benhabib J. and A. Rustichini. 1994. Introduction to the Symposium on Growth, Fluctuations, and Sunspot: Confronting the Data. Journal of Economic Theory 63, 1-18.
[12] Blanchard O.J. and S. Fischer. 1989. Lectures on Macroeconomics. MIT Press, Cambridge, Massachusetts.
[13] Bloise G., S. Bosi and F. Magris. 2000. Indeterminacy and Cycles in a Cash-in-Advance Economy with Production. Rivista Internazionale di Scienze Sociali,.CVIII, 3, 263-75.
[14] Bosi S. 2000. Growth Cycles. Document de Recherche EPEE, 00-11, University of Evry.
[15] Bosi S. 2001. Money, Growth and Indeterminacy. Rivista Internazionale di Scienze Sociali,.CIX, 2, 115-36.
[16] Brock W.A. 1975. A Simple perfect Foresight Monetary Model. Journal of Monetary Economics, 1, 133-50.
[17] Cass D. 1965. Optimal Growth in an Aggregate Model of Capital Accumulation. Review of Economic Studies 32, 233-40.
[18] Cass D. 1972. On Capital Overaccumulation in the Aggregative, Neoclassical Model of Economic Growth. A Complete Characterization. Journal of Economic Theory, 4, 200-23.
[19] Chari V., J. Jones and R. Manuelli. 1995. The Growth Effects of Monetary Policy. Federal Reserve Bank of Minneapolis Quarterly Review, 19, 18-32.
[20] Clower R. 1967. A Reconsideration of the Microeconomic Foundations of Monetary Theory. Western Economic Journal 6, 1-8.
[21] Cooley T.F, and G.D. Hansen. 1989. The Inflation Tax in a Real Business Cycle Model. American Economic Review, 79, 492-511.
[22] Correia I. and P. Teles. 1996. Is the Fridmanian Rule Optimal When Money Is an Intermediate Good? Journal of Monetary Economics, 38, 223-44.
[23] Dávila J. 1997. Sunspot Fluctuations in Dynamics with Predetermined Variables. Economic Theory 10, 483-95.
[24] De Long J.B. 1988. Productivity Growth, Convergence, and Welfare: Comments. American Economic Review, 78, 1139-54.
[25] Diamond P. 1965. National Debt in a Neoclassical Growth Model. American Economic Review 55, 1126-50.
[26] Dornbusch R. and J. Frenkel. 1973. Inflation and Growth: Alternative Approaches. Journal of Money, Credit and Banking 50, 141-56.
[27] Dotsey M. and P.D. Sarte. 2000. Inflation Uncertainty in a Cash-inAdvance Economy. Journal of Monetary Economics, 45, 631-55.
[28] Duesemberry J. 1949. Income, Saving and the Theory of Consumer Behavior. Cambridge, Harvard University Press.
[29] Feenstra R.C. 1986. Functional Equivalence between Liquidity Costs and the Utility of Money. Journal of Monetary Economics 17, 271-91.
[30] Fisher I. 1930. The Theory of Interest. New York, Macmillan.
[31] Fischer S. 1993. The Role of Macroeconomics Factors in Growth. Journal of Monetary Economics 32, 485-512.
[32] Friedman M. 1957. A Theory of the Consumption Function. Princeton University Press.
[33] Ghosh A. and S. Phillips. 1998. Warning: Inflation May Be Harmful to Your Growth. IMF Staff Papers, 45, 4.
[34] Grandmont J.M. 1991. Expectations Driven Nonlinear Business Cycles. Rheinisch-Westfalische Akademie der Wissenschaften Papers, Dusseldorf.
[35] Grandmont, J.M., P. Pintus and R. de Vilder. 1998. Capital-Labor Substitution and Competitive Nonlinear Endogenous Business Cycles. Journal of Economic Theory 80, 14-59.
[36] Grossman G.M. and E. Helpman. 1991. Quality Ladders in the Theory of Growth. Review of Economic Studies 58, 43-61.
[37] Grossman G.M. and N. Yanagawa. 1993. Asset Bubbles and Endogenous Growth. Journal of Monetary Economics 31, 3-19.
[38] Hairault J.O. ed. 2000. Analyse Macroéconomique 1. Editions La Découverte.
[39] Hale J. and H. Koçak. 1991. Dynamics and Bifurcations. SpringerVerlag, New York.
[40] Hall R.E. 1978. Stochastic Implications of the Life-Cycle-Permanent Income Hypothesis: Theory and Evidence. Journal of Political Economy, 86, 6.
[41] Jones C.I. 1998. Introduction to Economic Growth. Norton, New York.
[42] Jones L.E. and R.E. Manuelli. 1990. A Convex Model of Equilibrium Growth: Theory and Policy Implications. Journal of Political Economy 98, 1008-38.
[43] Jones L.E. and R.E. Manuelli. 1993. Growth and the Effect of Inflation. NBER Working Paper, 4523, 38.
[44] Kaldor N. 1961. Capital Accumulation and Economic Growth. In "The Theory of Capital", eds. F.A. Lutz and D.C. Hague. St. Martin, New York.
[45] Keynes J.M. 1936. The General Theory of Employment, Interest and Money. London, Macmillan.
[46] Kydland F. and E.C. Prescott. 1982. Time to Build and Aggregate Fluctuations. Econometrica, 50, 1345-70.
[47] Koopmans T. 1965. On the Concept of Optimal Economic Growth, In "The Econometric Approach to Development Planning", North-Holland, Amsterdam.
[48] Krugman P. 1979. A Model of Innovation, Technology Transfer, and the World Distribution of Income. Journal of Political Economy 87,253-66.
[49] Kuznets S. (with L. Epstein and E. Jenks). 1946. National Product Since 1869. New York, NBER.
[50] Long J.B. and C.I. Plosser. 1983. Real Business Cycles. Journal of Political Economy, 39-69.
[51] Lucas R.E. 1988. On the Mechanism of Economic Development. Journal of Monetary Economics 22, 3-42.
[52] Lucas R.E. and N.L. Stockey. 1987. Money and Interest in a Cash-inAdvance Economy. Econometrica 55, 491-513.
[53] Mankiw N.G., D. Romer and D.N. Weil. 1992. A Contribution to the Empirics of Economic Growth. Quarterly Journal of Economics, 107, 407-37.
[54] Marquis M.H. and K.L. Reffett. 1991. Real Interest Rates and Endogenous Growth in a Monetary Economy. Economic Letters, 37, 105-9.
[55] Marquis M.H. and K.L. Reffett. 1994. New Technology Spillovers into the Payment System. Economic Journal, 104, 1123-38.
[56] Matsuyama K. 1991. Endogenous Price Fluctuations in an Optimizing Model of a Monetary Economy. Econometrica, 59, 1617-31.
[57] Mino K. and A. Shibata. 1995. Monetary Policy, Overlapping Generations, and Patterns of Growth. Economica, 62, 179-94.
[58] Modigliani F. and R. Brumberg. 1954. Utility Analysis and the Consumption Function: an Interpretation of Cross-Section Data. In K. Kurihara (ed.), Post-Keynesian Economics, Rutgers University Press.
[59] Modigliani F. and Miller M.H. 1961. The Cost of Capital, Corporation Finance and the Theory of Investment. American Economic Review, 48 (3), 261-97.
[60] Ramsey F. 1928. A Mathematical Theory of Saving. Economic Journal 38, 543-59.
[61] Rebelo S. 1991. Long-Run Policy Analysis and Long-Run Growth. Journal of Political Economy 99, 500-21.
[62] Romer P.M. 1986. Increasing Returns and Long Run Growth. Journal of Political Economy 94, 1002-37.
[63] Romer P.M. 1990. Endogenous Technological Change. Journal of Political Economy, 98, S71-S102.
[64] Romer D. 1996. Advanced Macroeconomics. McGraw-Hill, New York.
[65] Samuelson P.A. 1958. An Exact Consumption Loan Model of Interest with or without the Social Contrivance of Money. Journal of Political Economy 66, 1002-11.
[66] Sidrauski M. 1967. Rational Choice and Patterns of Growth in a Monetary Economy. American Economic Review 57, 534-44.
[67] Solow R. 1956. A Contribution to the Theory of Economic Growth. Quarterly Journal of Economics 70, 65-94.
[68] Solow R.M. 1957. Technical Change and the Aggregate Production Function. Review of Economics and Statistics 39, 312-320.
[69] Stockman A.C. 1981. Anticipated Inflation and the Capital Stock in a Cash-in-Advance Economy. Journal of Monetary Economics 8, 387-93.
[70] Swan T.W. 1956. Economic Growth and Capital Accumulation. Economic Record 32.
[71] Tirole J. 1985. Asset Bubbles and Overlapping Generations. Econometrica 53, 1499-528.
[72] Tobin J. 1956. The Interest Elasticity of the Transactions Demand for Cash. Review of Economics and Statistics 38, 241-7.
[73] Tobin J. 1965. Money and Economic Growth. Econometrica 33, 671-84.
[74] Uzawa H. 1965. Optimum Technical Change in an Aggregative Model of Economic Growth. International Economic Review 6, 18-31.
[75] Van der Ploeg F. and G.S. Alogoskoufis. 1994. Money and Endogenous Growth. Journal of Money Credit and Banking, 26, 4, 771-91.
[76] Varian H. 1992. Microeconomic Analysis. Norton.
[77] Wang P., and C.K. Yip. 1991. Real Effects of Money and Welfare Costs of Inflation in an Endogenously Growing Economy with Transaction Costs. Mimeograph, Pennsylvania State University.
[78] Woodford M. 1986a. Stationary Sunspots Equilibria in a Finance Constrained Economy. Journal of Economic Theory 63, 97-112.
[79] Woodford M. 1986b. Stationary Sunspot Equilibria. The Case of Small Fluctuations around a Deterministic Steady State. Mimeograph, University of Chicago and New York University.

## Documents de recherche EPEE

## 2003

| 03-01 | Basic Income/Minimum Wage Schedule and the Occurrence of Inactivity Traps: |
| :--- | :--- |
| Some Evidence on the French Labor Market |  |
|  | Thierry LAURENT \& Yannick L'HORTY |

## 2002

| 02-01 | Inflation, salaires et SMIC: quelles relations? Yannick L'HORTY \& Christophe RAULT |
| :---: | :---: |
| 02-02 | Le paradoxe de la productivité |
|  | Nathalie GREENAN \& Yannick L'HORTY |
| 02-03 | 35 heures et inégalités |
|  | Fabrice GILLES \& Yannick L'HORTY |
| 02-04 | Droits connexes, transferts sociaux locaux et retour à l'emploi Denis ANNE \& Yannick L'HORTY |
| 02-05 | Animal Spirits with Arbitrarily Small Market Imperfection Stefano BOSI, Frédéric DUFOURT \& Francesco MAGRIS |
| 02-06 | Actualité du protectionnisme : <br> l'exemple des importations américaines d'acier Anne HANAUT |
| 02-07 | The Fragility of the Fiscal Theory of Price Determination Gaetano BLOISE |
| 02-08 | Pervasiveness of Sunspot Equilibria Stefano BOSI \& Francesco MAGRIS |
| 02-09 | Du côté de l'offre, du côté de la demande : quelques interrogations sur la politique française en direction des moins qualifiés <br> Denis FOUGERE, Yannick L'HORTY \& Pierre MORIN |
| 02-10 | A « Hybrid» Monetary Policy Model: Evidence from the Euro Area Jean-Guillaume SAHUC |

02-11 An Overlapping Generations Model with Endogenous Labor Supply: A Dynamic Analysis
Carine NOURRY \& Alain VENDITTI
02-12 Rhythm versus Nature of Technological Change Martine CARRE \& David DROUOT

02-13 Revisiting the « Making Work Pay » Issue:
Static vs Dynamic Inactivity Trap on the Labor Market Thierry LAURENT \& Yannick L'HORTY

02-14 Déqualification, employabilité et transitions sur le marché du travail : une analyse dynamique des incitations à la reprise d'emploi Thierry LAURENT \& Yannick L'HORTY

02-15 Privatization and Investment: Crowding-Out Effect vs Financial Diversification

| 02-16 | Taxation of Savings Products: An International Comparison Thierry LAURENT \& Yannick L'HORTY |
| :---: | :---: |
| 02-17 | Liquidity Constraints, Heterogeneous Households and Sunspots Fluctuations Jean-Paul BARINCI, Arnaud CHERON \& François LANGOT |
| 02-18 | Influence of Parameter Estimation Uncertainty on the European Central Banker Behavior: An Extension Jean-Guillaume SAHUC |
|  | 2001 |
| 01-01 | Optimal Privatisation Design and Financial Markets Stefano BOSI, Guillaume GIRMENS \& Michel GUILLARD |
| 01-02 | Valeurs extrêmes et series temporelles : <br> application à la finance <br> Sanvi AVOUYI-DOVI \& Dominique GUEGAN |
| 01-03 | La convergence structurelle européenne : rattrapage technologique et commerce intra-branche Anne HANAUT \& EI Mouhoub MOUHOUD |
| 01-04 | Incitations et transitions sur le marché du travail : une analyse des stratégies d'acceptation et des refus d'emploi Thierry LAURENT, Yannick L'HORTY, Patrick MAILLE \& Jean-François OUVRARD |
| 01-05 | La nouvelle economie et le paradoxe de la productivité : une comparaison France - Etats-Unis Fabrice GILLES \& Yannick L'HORTY |
| 01-06 | Time Consistency and Dynamic Democracy Toke AIDT \& Francesco MAGRIS |
| 01-07 | Macroeconomic Dynamics Stefano BOSI |
| 01-08 | Règles de politique monétaire en présence d'incertitude : une synthèse <br> Hervé LE BIHAN \& Jean-Guillaume SAHUC |
| 01-09 | Indeterminacy and Endogenous Fluctuations with Arbitrarily Small Liquidity Constraint Stefano BOSI \& Francesco MAGRIS |
| 01-10 | Financial Effects of Privatizing the Production of Investment Goods Stefano BOSI \& Carine NOURRY |
| 01-11 | On the Woodford Reinterpretation of the Reichlin OLG Model : a Reconsideration <br> Guido CAZZAVILLAN \& Francesco MAGRIS |
| 01-12 | Mathematics for Economics Stefano BOSI |
| 01-13 | Real Business Cycles and the Animal Spirits Hypothesis in a Cash-in-Advance Economy Jean-Paul BARINCI \& Arnaud CHERON |
| 01-14 | Privatization, International Asset Trade and Financial Markets Guillaume GIRMENS |
| 01-15 | Externalités liées dans leur réduction et recyclage |

## Carole CHEVALLIER \& Jean DE BEIR

| 01-16 | Attitude towards Information and Non-Expected Utility Preferences : <br> a Characterization by Choice Functions <br> Marc-Arthur DIAYE \& Jean-Max KOSKIEVIC |
| :---: | :--- |
| 01-17 | Fiscalité de l'épargne en Europe : <br> une comparaison multi-produits <br> Thierry LAURENT \& Yannick L'HORTY |
| 01-18 | Why is French Equilibrium Unemployment so High : <br> an Estimation of the WS-PS Model <br> Yannick L'HORTY \& Christophe RAULT |
| 01-19 | La critique du « système agricole » par Smith <br> Daniel DIATKINE |
| 01-20 | Modèle à Anticipations Rationnelles <br> de la COnjoncture Simulée : MARCOS <br> Pascal JACQUINOT \& Ferhat MIHOUBI |
| 01 -21 | Qu'a-t-on appris sur le lien salaire-emploi ? <br> De l'équilibre de sous emploi au chômage d'équilibre : <br> la recherche des fondements microéconomiques <br> de la rigidité des salaires <br> Thierry LAURENT \& Hélène ZAJDELA |
| 01-22 | Formation des salaires, ajustements de l'emploi <br> et politique économique <br> Thierry LAURENT |

2000
00-01 \(\left.\left.$$
\begin{array}{cl}\text { Wealth Distribution and the Big Push } \\
\text { Zoubir BENHAMOUCHE }\end{array}
$$\right] \begin{array}{l}Conspicuous Consumption <br>

Stefano BOSI\end{array}\right]\)| Cible d'inflation ou de niveau de prix : |
| :--- |
| quelle option retenir pour la banque centrale |
| dans un environnement « nouveau keynésien » ? |
| Ludovic AUBERT |

00-04 Soutien aux bas revenus, réforme du RMI et incitations à l'emploi : une mise en perspective Thierry LAURENT \& Yannick L'HORTY

00-05 Growth and Inflation in a Monetary « Selling-Cost » Model Stefano BOSI \& Michel GUILLARD

00-06 Monetary Union : a Welfare Based Approach Martine CARRE \& Fabrice COLLARD

00-07 Nouvelle synthèse et politique monétaire Michel GUILLARD

00-08 Neoclassical Convergence versus Technological Catch-Up : a Contribution for Reaching a Consensus Alain DESDOIGTS

00-09 L'impact des signaux de politique monétaire sur la volatilité intrajournalière du taux de change deutschemark - dollar Aurélie BOUBEL, Sébastien LAURENT \& Christelle LECOURT

| 00-10 | A Note on Growth Cycles <br> Stefano BOSI, Matthieu CAILLAT \& Matthieu LEPELLEY <br> $00-11$ <br> $00-12$ |
| :--- | :--- |
| Growth Cycles <br> Stefano BOSI |  |
| 00-13 | Règles monétaires et prévisions d'inflation en économie ouverte <br> Michel BOUTILLIER, Michel GUILLARD \& Auguste MPACKO PRISO |
|  | Long-Run Volatility Dependencies in Intraday Data <br> and Mixture of Normal Distributions <br> Aurélie BOUBEL \& Sébastien LAURENT |

1999

| 99-01 | Liquidity Constraint, Increasing Returns and Endogenous Fluctuations Stefano BOSI \& Francesco MAGRIS |
| :---: | :---: |
| 99-02 | Le temps partiel dans la perspective des 35 heures Yannick L'HORTY \& Bénédicte GALTIER |
| 99-03 | Les causes du chômage en France : Une ré-estimation du modèle WS - PS Yannick L'HORTY \& Christophe RAULT |
| 99-04 | Transaction Costs and Fluctuations in Endogenous Growth Stefano BOSI |
| 99-05 | La monnaie dans les modèles de choix intertemporels: quelques résultats d'équivalences fonctionnelles Michel GUILLARD |
| 99-06 | Cash-in-Advance, Capital, and Indeterminacy Gaetano BLOISE, Stefano BOSI \& Francesco MAGRIS |
| 99-07 | Sunspots, Money and Capital <br> Gaetano BLOISE, Stefano BOSI \& Francesco MAGRIS |
| 99-08 | Inter-Juridictional Tax Competition in a Federal System of Overlapping Revenue Maximizing Governments Laurent FLOCHEL \& Thierry MADIES |
| 99-09 | Economic Integration and Long-Run Persistence of the GNP Distribution <br> Jérôme GLACHANT \& Charles VELLUTINI |
| 99-10 | Macroéconomie approfondie : croissance endogène Jérôme GLACHANT |
| 99-11 | Growth, Inflation and Indeterminacy in a Monetary «Selling-Cost » Model Stefano BOSI \& Michel GUILLARD |
| 99-12 | Règles monétaires, « ciblage » des prévisions et (in)stabilité de l'équilibre macroéconomique Michel GUILLARD |
| 99-13 | Educating Children : <br> a Look at Household Behaviour in Côte d'Ivoire Philippe DE VREYER, Sylvie LAMBERT \& Thierry MAGNAC |
| 99-14 | The Permanent Effects of Labour Market Entry in Times of High Aggregate Unemployment |


| 99-15 | Allocating and Funding Universal Service Obligations <br> in a Competitive Network Market <br> Philippe CHONE, Laurent FLOCHEL \& Anne PERROT |
| :---: | :--- |
| 99-16 | Intégration économique et convergence <br> des revenus dans le modèle néo-classique <br> Jérôme GLACHANT \& Charles VELLUTINI |
| 99-17 | Convergence des productivités européennes : <br> réconcilier deux approches de la convergence <br> Stéphane ADJEMIAN |
| 99-18 | Endogenous Business Cycles : <br> Capital-Labor Substitution and Liquidity Constraint <br> Stefano BOSI \& Francesco MAGRIS |
| $99-19$ | Structure productive et procyclicité de la productivité <br> Zoubir BENHAMOUCHE |
| $99-20$ | Intraday Exchange Rate Dynamics and Monetary Policy <br> Aurélie BOUBEL \& Richard TOPOL |

1998

| 98-01 | Croissance, inflation et bulles <br> Michel GUILLARD |
| :---: | :--- |
| 98-02 | Patterns of Economic Development and the Formation of Clubs <br> Alain DESDOIGTS |
| 98-03 | Is There Enough RD Spending ? <br> A Reexamination of Romer's (1990) Model <br> Jérôme GLACHANT |
| 98-04 | Spécialisation internationale et intégration régionale. <br> L'Argentine et le Mercosur <br> Carlos WINOGRAD |
| 98-05 | Emploi, salaire et coordination des activités <br> Thierry LAURENT \& Hélène ZAJDELA |
| 98-06 | Interconnexion de réseaux et charge d'accès : <br> une analyse stratégique |
| Laurent FLOCHEL |  |

98-11 Substituabilité des hommes aux heures et ralentissement de la productivité ? Yannick L'HORTY \& Chistophe RAULT

98-12 De l'équilibre de sous emploi au chômage d'équilibre :
la recherche des fondements microéconomiques de la rigidité des salaires Thierry LAURENT \& Hélène ZAJDELA


[^0]:    6.5.6 Central Planner's Problems 244

[^1]:    ${ }^{1}$ Two systems of difference equations $x_{t+1}=f\left(x_{t}\right)$ and $x_{t+1}=g\left(x_{t}\right)$ defined on open subsets $S_{f}$ and $S_{g}$ of $R^{n}$, respectively, are said to be topologically equivalent if there exists a homeomorphism $h: S_{f} \rightarrow S_{h}$ such that $h$ maps the orbits of $f$ onto the orbits of $g$ and preserves the sense of direction of time. If $x$ is an hyperbolic steady state of $f$, then there is a neighborhood of $x$ in which $f$ is topologically equivalent to the linear dynamics $x_{t+1}=x+D_{x_{t}} f(x)\left(x_{t}-x\right)$.

[^2]:    ${ }^{2}$ The steady state is stable (sink) if and only if $\left|\lambda_{1}\right|,\left|\lambda_{2}\right|<1$, i.e. $\left|\operatorname{tr} J / 2 \pm \sqrt{(\operatorname{tr} J / 2)^{2}-\operatorname{det} J}\right|<1$. It is a saddle if and only if either $\left|\lambda_{1}\right|<1,\left|\lambda_{2}\right|>1$, or $\left|\lambda_{1}\right|>1,\left|\lambda_{2}\right|<1$, i.e. either

    $$
    \left|\operatorname{tr} J / 2-\sqrt{(\operatorname{tr} J / 2)^{2}-\operatorname{det} J}\right|<1 \text { and }\left|\operatorname{tr} J / 2+\sqrt{(\operatorname{tr} J / 2)^{2}-\operatorname{det} J}\right|>1
    $$

    or

    $$
    \left|\operatorname{tr} J / 2-\sqrt{(\operatorname{tr} J / 2)^{2}-\operatorname{det} J}\right|>1 \text { and }\left|\operatorname{tr} J / 2+\sqrt{(\operatorname{tr} J / 2)^{2}-\operatorname{det} J}\right|<1 .
    $$

[^3]:    ${ }^{1}$ If, for instance, we assume by simplicity that the probability distributions $\pi_{t}, \pi_{t+1}, \ldots$ are independent, then the probability independence implies

    $$
    \pi\left(s_{t}, \ldots, s_{\tau}\right)=\prod_{i=t}^{\tau} \pi_{i}\left(s_{i}\right)
    $$

[^4]:    ${ }^{2}$ Proof. $x^{*}=\max p y-w x=M(a)$ where $a=(p, w)$. Then $D_{a} x^{*}=D_{b}(p y-w x)$ where $b \equiv(y, x)$.

[^5]:    ${ }^{3}$ This section is inspired by the chapter "L'investissement" by F. Collard in Hairault et al. (2000).

[^6]:    ${ }^{1}$ The first welfare theorem claims the Pareto-optimality of a competitive equilibrium.

[^7]:    ${ }^{2}$ The transversality condition is necessary to ensure the convergence of utility series.

[^8]:    ${ }^{3}$ The usual cash-in-advance is obtained as a limit case with infinitely costly transaction costs. For more details see among the others Correia and Teles (1996).

[^9]:    ${ }^{4}$ The money velocity $v_{t}$ is defined with respect to consumption: $M_{t-1} v_{t} \equiv p_{t} c_{t}$, i.e. $v_{t} \equiv c_{t} /\left(M_{t-1} / p_{t}\right)$.

[^10]:    ${ }^{5}$ Capital depreciation is usually parametrized by a depreciation rate $\delta$ and the budget constraint is reset as follows

    $$
    \left(1+\pi_{t+1}\right) m_{t+1}+k_{t+1}+\left[1+s\left(c_{t} / m_{t}\right)\right] c_{t} \leq\left(1-\delta+r_{t}\right) k_{t}+m_{t}+\tau_{t}
    $$

[^11]:    ${ }^{6}$ A stable manifold is the union of all the convergent trajectories. A variable, which has been determined prior to time $t$, is said to be predetermined at time $t$. For instance in standard macroeconomic dynamics the stock of capital $k_{t}$ plays as a predetermined variable, because it depends on the investment decisions, which has been taken in the previous period $t-1$. In our model as consumption and real balances are not predetermined, neither is the velocity of circulation of money with respect to consumption. Indeterminacy occurs when the dimension of the stable manifold is greater than the number of predetermined variables.

[^12]:    ${ }^{7}$ Capital and consumption good have the same nominal price $p_{t}$. However the real balances and consumption have the same real opportunity cost $i_{t}$ with respect to the productive capital, because of the cash-in-advance.

[^13]:    ${ }^{8}$ By definition the exogenous growth models are not adapted to capture the interplay between monetary growth, inflation and real growth. One prediction from Tobin's model (1965) is that an inflationary money growth positively affects the capital stock. Sidrauski (1967), using a model with money in the utility function, develops long run neutrality results. A negative relationship between money growth and capital is shown in Brock (1975) when the supply of labor is endogenous. In Stockman (1981) a cash-in-advance constraint is applied to consumption and investment. In Cooley and Hansen (1989) money is introduced through a cash-in-advance constraint on consumption. In both of these articles higher inflation rates affect steady-capital/output ratios but not growth rates. In the Real Business Cycle theory as advanced by Kydland and Prescott (1982) and Long and Plosser (1983), money typically plays no role. In Matsuyama (1991) endogenous price fluctuations are associated to a higher money supply growth.

    By construction the endogenous growth models are better to explain the inflation impact on growth. In Jones and Manuelli (1993) the effects of inflation are still evaluated in a model of endogenous growth with increasing returns. Inflation is recognized to induce small growth rate effects and moderate welfare costs. In Van der Ploeg and Alogoskoufis (1994) monetary growth is no longer neutral. It boots real growth and inflation therefore rises by less than the monetary growth.
    ${ }^{9}$ A large evidence points out that $10 \%$ increase in the inflation rate is associated with a decrease in the growth rate of between about 0.2 and $0.7 \%$ (among the others Fischer (1993), Chari et al. (1995)). However authors disagree about the effects of moderate inflation (Ghosh and Phillips, 1998).

[^14]:    ${ }^{1}$ By simplicity we consider a Cobb-Douglas specification instead of a more general production function with constant returns to scale.

[^15]:    ${ }^{2}$ In Solow (1956) the law of motion is $k^{\prime}=s f(k)-(\delta+n) k=f(k)-c-(\delta+n) k$. The stationary consumption is given by $c^{*}=f\left(k^{*}\right)-(\delta+n) k^{*}$ and the golden rule by $f^{\prime}\left(k^{*}\right)=\delta+n$. In our case $\delta=0$.

