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**Indeterminacy and Endogenous Fluctuations
With Arbitrarily Small Liquidity Constraint**

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Indeterminacy and Endogenous Fluctuations with Arbitrarily Small Liquidity Constraint[*]

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Abstract

The empirical relevance of indeterminacy and sunspot fluctuations has been often questioned on the basis of the implausibly high degrees of increasing returns to scale or unconventional calibrations for the fundamentals required. In this paper we study a one-sector economy with partial cash-in-advance constraint on consumption expenditures and show how such phenomena are by contrast quite pervasive: In fact, their scope improves as soon as the liquidity constraint is set smaller and smaller and finally, for amplitudes of the liquidity constraint small enough, they are bound to prevail for whatever fundamentals specification.

Keywords: cash-in-advance, indeterminacy, endogenous fluctuations

JEL Classification: D90, E32, E41

1 Introduction

In this paper we present a one-sector infinite horizon economy with capital accumulation and liquidity constraint on consumption expenditures. Yet, we depart from similar model economies (Wilson, 1979; Stockman, 1981; Abel, 1985; Svensson, 1985; Lucas and Stokey, 1987; Coleman, 1987; Cooley and Hansen, 1989; Woodford, 1994) by assuming, in the spirit of Grandmont and Younès (1972), that only a share between zero and one of current consumption must be paid by cash holding previously accumulated. We then study the conditions generating equilibrium indeterminacy and sunspot fluctuations.

We establish a result which we believe is quite surprising with regard to the literature on endogenous fluctuations: It is sufficient an arbitrarily small departure from the traditional Ramsey-Cass-Koopmans model - namely the requirement that an arbitrarily small amount of consumption purchases must be financed out of cash balances - to obtain multiple equilibria and sunspot fluctuations which as a consequence turn out to be very pervasive phenomena. Such a result is even more attracting since we observe that it does not rely upon any specific calibration for the fundamentals: Once one has fixed technology and preferences, all what is needed to get indeterminacy is to relax the liquidity constraint enough[1].

Our findings challenge the widespread viewpoint according to which indeterminacy would require empirically implausible features. Since the seminal papers by Benhabib and Farmer (1994), Farmer and Guo (1994) and Galí (1994), the conditions under which these successful models may generate an indeterminate equilibrium have been in fact widely criticized for being implausible and largely at odds with data. For example, in Benhabib and Farmer (1994), increasing returns to scale exploited to obtain indeterminacy must be high enough to imply that the labor demand curve is upward sloping, and slopes even steeper than the labor supply curve. The required degree of increasing returns to scale is around 60%, which is considerably higher than what is suggested by most empirical studies, which find - if any - much smaller markups and increasing returns to scale of the order of 3% (see Basu and Fernald, 1997, and Burnside *et al.*, 1995). Wen (1998) shows that adding a variable capital utilization rate to a standard one-sector model may make indeterminacy occurring for lower degrees of increasing returns to scale, yet

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[1] In a closely related paper, Bosi *et al.* (2002), asses the cyclical properties of the model by allowing simultaneously technological and beliefs disturbances and observe that it performs pretty well.

remaining of the order of 10%.

A more recent generation of papers attempt to avoid the large increasing returns problem, but they introduce other non-standard features in the production or utility functions, which are not totally convincing. For example, Bennett and Farmer (2000) show that non-separability between consumption and leisure may allow for an indeterminate equilibrium when externalities are of the order of 3% ; Still, the requirement that the Frisch labor supply curve is downward sloping remains rather strong. Similarly, Benhabib and Farmer (2000) show that indeterminacy may occur under constant returns to scale in a monetary model in which money is included as an argument of the production function. The requirement is now that money is sufficiently complementary with labor as a productive input. Farmer (1997) shows that a model with money in the utility function may also be indeterminate if non-standard specifications of the utility function are considered. In the liquidity constrained economies studied in Grandmont, Pintus and de Vilder (1998) and Barinci and Chéron (2001) - and closer to our - indeterminacy requires, respectively, low elasticities in factors substitution and low elasticities of substitution in consumption.

At the same time, another line of research shows that indeterminacy could be easier to obtain only when small increasing returns to scale are coupled with two-sector versions of the original Benhabib and Farmer's 1994 model (among the others, Benhabib and Farmer, 1996; Perli, 1998; Venditti, 1998). Still, the cyclical properties of such models - when compared with US data - are not fully satisfactory, at least when solely driven by beliefs shocks.

Our model provides some strong argument against these skeptical attitudes towards the empirical plausibility of indeterminacy: In fact, if, on the one hand, it is true that for amplitudes of the liquidity constraint close to one such a phenomenon requires strong degrees of complementariness in consumption, on the other one it becomes compatible with whatever fundamentals specification when the amount of consumption to be payed cash is set sufficiently small[2]. It is worth noticing here that we also improve the results obtained in a recent paper by Carlstrom and Fuerst (2000), since, when the share of consumption to be paid cash is low enough, we observe indeterminacy for whatever elasticity of intertemporal substitution in consumption,

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[2]The model is not continuous when the amplitude of the liquidity constraint it set zero, since it loses one dimension collapsing into the standard Ramsey-Cass-Koopmans one in which equilibrium is determinate.

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and not only for some, as in their case.

The economic intuition of our results works, approximately, as follows. First of all, let us observe that the return of reducing foregoing current consumption is, loosely speaking, a convex combination between the real interest rate (weighted by $1 - q$, i.e. the share of tomorrow consumption that can be financed out of physical investment) and the difference, weighted by the share of consumption to be paid cash, q, between today and tomorrow nominal interest rate. This is due to the fact that the nominal interest rate - roughly, the cost of money - measures, in the respective periods, the benefit of reducing today consumption and the cost of increasing tomorrow one. Now, let us suppose that the system is initially at the steady state and let us try to construct an alternative equilibrium in which agents face a fall in next period nominal interest rate. Under such an assumption, they will rationally react by driving up tomorrow consumption and today investment: Accordingly, next period real interest rate falls. Yet, in order to equalize the cost of current foregoing consumption with the benefit of higher investment, in the light of the previous considerations, the current nominal interest rate must increase to compensate, in synergy with next period lower nominal interest rate, the diminished real one. If q is close to one (and thus the weight of the real interest rate in determining investment benefits is relatively small), the adjustment required will be rather slight, in particular smaller than the corresponding decrease in tomorrow nominal interest rate. This, of course, will generate explosive dynamics. On the other hand, when q is small, it will be necessary a sharp increase in today nominal interest rate, in particular larger than the corresponding contraction of tomorrow one, giving thus rise to convergent, although oscillatory, dynamics.

The remainder of the paper is organized as follows. In Section 2 we present the model. Section 3 is devoted to the study of the local dynamics, while Section 4 concludes the paper.

2   The model

2.1   The environment

The economy is populated by a large number of identical long-lived agents and identical firms. Time is discrete and the environment deterministic. Agents are endowed with perfect foresight, supply inelastically one unit of

labor in each period, and maximize $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where $0 < \beta < 1$ denotes the discount factor, c consumption and u the per-period utility of consumption featuring the following standard properties.

Assumption 1. $u : R_+ \rightarrow R$ *is smooth, strictly increasing and strictly concave with* $\lim_{c \to 0^+} u'(c) = +\infty$.

In each period $t \geq 0$ the representative agent is subject to the budget constraint

$$p_t c_t + p_t [k_{t+1} - (1-\delta) k_t] + M_{t+1} = p_t r_t k_t + p_t w_t + M_t + \tau_t \qquad (1)$$

where p denotes the price of the good, k physical equipment, M money balances, r the real rental price of capital, w the real wage, $\delta \in [0,1]$ the depreciation rate of capital and τ nominal lump-sum transfers issued by the government. A share $q \in (0,1]$ of the purchases of the consumption good requires money balances accumulated in the previous period: This implies that agents when maximizing must also take into account the liquidity constraint

$$q p_t c_t \leq M_t. \qquad (2)$$

The FOC's of the household maximization problem, in addition to constraints (1) and (2), write:

$$
\begin{aligned}
u'(c_t) &= p_t (\lambda_t + q\mu_t), & (3) \\
\lambda_t p_t &= \beta (\lambda_{t+1} p_{t+1} R_{t+1}), & (4) \\
\lambda_t &= \beta (\lambda_{t+1} + \mu_{t+1}) & (5)
\end{aligned}
$$

where λ and μ are real non-negative Lagrange multipliers associated to, respectively, budget constraint and cash-in-advance and $R \equiv 1 - \delta + r$. Equations (3)-(5) are usual no-arbitrage conditions. In particular, according to a commonly shared view in monetary economics, (5) establishes that the price of money at time t, λ_t, is equal to its value in the following period (reflecting the fact that money is a long-lived asset) plus the value of the implicit dividends μ_{t+1} this asset will pay off. Whenever dividends are positive, money is not seen as a speculative bubble. Observe that $p_t \lambda_t$ can be interpreted as the marginal indirect utility of real income in period t but, according to (3), it is not equalized by the individual to the marginal utility of consumption, since part of income cannot be transformed into consumption unless it is first

used to purchase money balances. Condition (4) says that no intertemporal transfers of real income are possible to increase total utility. Finally, optimal plans for the single household must satisfy the transversality condition

$$\lim_{t \to +\infty} \left[\beta^t u'(c_t)(k_{t+1} + \pi_{t+1} m_{t+1}) \right] = 0 \qquad (6)$$

where $m_t \equiv M_t/p_t$ are the real balances held by the representative agent at the outset of period t. Constraint (2) binds as long as money is dominated by capital in terms of returns, which is true when the nominal interest factor $i_t \equiv (1 - \delta + r_t)\pi_t$ is greater than one, $\pi_t \equiv p_t/p_{t-1}$ being the inflation factor between periods $t - 1$ and t. By manipulating conditions (3)-(5), we obtain the following Euler equation for the consumer:

$$u'(c_t) = \beta u'(c_{t+1}) R_{t+1} \frac{q\pi_t R_t + 1 - q}{q\pi_{t+1} R_{t+1} + 1 - q}. \qquad (7)$$

Output is produced according to a constant returns to scale aggregate production function $F(K, L)$, where K and L denote, respectively, aggregate capital and labor. Production can be expressed in the intensive form $f(a)$, with $a \equiv K/L$, exhibiting the usual neoclassical features.

Assumption 2. *The intensive production function $f : R_+ \to R$ is smooth, strictly increasing and strictly concave. Moreover $f(0) = 0$, $\lim_{k \to 0} f'(a) = +\infty$ and $\lim_{k \to +\infty} f'(a) = 0$.*

Assumption 2 ensures on the one hand firms maximization problem to be well defined and on the other one the existence of a unique stationary solution. Profit maximization of the firms implies that the real interest rate and the real wage equalize, respectively, the marginal productivity of capital and the marginal productivity of labor:

$$r = f'(a), \qquad (8)$$
$$w = f(a) - f'(a)a. \qquad (9)$$

Government follows a simple monetary rule: In each period it issues lump-sum transfers of money balances at the constant rate $\mu - 1 > 0$, so that in period t the supply of money, M_t^s, satisfies $M_t^s = \mu^t M_0^s$, where M_0^s is the initial amount of nominal balances. Thus nominal transfers are given by $\tau_t = (\mu - 1)M_t^s$.

2.2 Equilibrium

Equilibrium in factors market implies $K_t = k_t$, $L_t = 1$, and therefore $a_t = k_t$, for every $t \geq 0$. Money market equilibrium requires $\pi_{t+1} = \mu m_t / m_{t+1}$, and good market clears by Walras law. Thereby, when constraint (2) binds, so $m_t = qc_t$ for all $t \geq 0$, intertemporal competitive equilibria can be described by the dynamic evolution of (π_t, k_t, c_t). As a matter of fact, setting $g(k) \equiv f(k) + (1 - \delta)$, we have the following.

Definition 1 *An intertemporal equilibrium with perfect foresight is a positive sequence $\{\pi_t, k_t, c_t\}_{t=0}^{\infty}$ satisfying for every $t \geq 0$ equations*

$$k_{t+1} = g(k_t) - c_t, \tag{10}$$

$$\pi_{t+1} c_{t+1} = \mu c_t, \tag{11}$$

$$\frac{\beta u'(c_{t+1}) g'(k_{t+1})}{q \pi_{t+1} g'(k_{t+1}) + 1 - q} = \frac{u'(c_t)}{q \pi_t g'(k_t) + 1 - q} \tag{12}$$

subject to the initial endowment of capital $k_0 > 0$ and the transversality condition (6).

2.3 Steady state

Capital stock, consumption, inflation and real balances are constant at the steady state. Let us denote with a bar over a variable its steady state value. From equation (12), one obtains the stationary level $\bar{k}$ of capital by solving $f'(k) = \beta^{-1} - (1 - \delta) \equiv \theta$ which under Assumption 2 has a unique solution corresponding to the Modified Golden Rule. The stationary consumption $\bar{c}$ is given by output minus investment $f(\bar{k}) - \delta \bar{k}$ and, being constraint (2) binding, one has $\bar{m} = q\bar{c}$. Finally, the stationary inflation factor $\bar{\pi}$ is equal to μ. The immediate implication of these results is that the amplitude q of the liquidity constraint does not affect the steady state value of any variable other than real balances, while the rate of money growth $\mu - 1$ produces the unique consequence of fixing the stationary inflation rate.[3]

One also readily verifies that constraint (2) binds at the steady state if and only if the discount factor β is lower than the factor μ of money growth.

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[3]The effects of inflation in the case $q = 1$ have been studied by Stockman (1981) and Abel (1985). It is shown that money is superneutral at the steady state as well as along the transitional dynamics.

Under this condition, which we will assume to be satisfied throughout the remainder of the paper, system defined by (10)-(12) is consistent with intertemporal equilibria remaining in a small neighborhood of the steady state.

3 Stability analysis

In order to study the occurrence of (local) indeterminacy, we study the stability of the system defined by (10)-(12) around the steady state. Identifying x with (π, k, c), we can define $G_0(x_t)$ and $G_1(x_{t+1})$ as the right-hand sides and left-hand sides, respectively, of equations (10)-(12). Therefore an intertemporal equilibrium of the economy can be now written in the more compact form as a sequence $\{x_t\}_{t=0}^{\infty}$ satisfying

$$G_1(x_{t+1}) = G_0(x_t) \tag{13}$$

for all $t \geq 0$. The steady state of the economy is said to be locally indeterminate if there exists a continuum of sequences $\{x_t\}_{t=0}^{\infty}$ satisfying system (13) for all $t \geq 0$, subject to the initial stock of capital, k_0, all of which converging to the steady state $\bar{x}$. Following the usual procedure, the study of (local) indeterminacy requires an exam of the linear operator

$$A = [DG_1(\bar{x})]^{-1} DG_0(\bar{x}) \tag{14}$$

which regulates the linear tangent motion of (13) near its steady state.[4] Since in system (13) k_0 is the unique pre-determined variable, indeterminacy occurs if and only if the dimension of the stable manifold of A is greater than one, i.e. if and only if A possesses at least two eigenvalues lying inside the unit circle. In this case, there will exist different possible choices for placing the remaining initial conditions, c_0 and π_0, in such a way that the equilibrium dynamics converge to the stationary solution and respect the transversality condition (6). In the opposite case, equilibrium of system (13) will be determinate, i.e. there will be only one pair (c_0, π_0) ensuring the convergence of the system towards its steady state. To carry out the stability analysis, it is useful to define here the elasticity of intertemporal substitution in consumption, $\sigma \equiv -u'(u''\bar{c})^{-1}$, the share of profit in total income, $s \equiv f^{-1}f'\bar{k}$, the elasticity of the real interest rate, $\rho \equiv -(f')^{-1}f''\bar{k}$, all evaluated at the steady state

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[4]$DG_i(x)$, with $i = 0, 1$, denotes the matrix of the derivatives of G_i with respect to x.

equilibrium. Taking into account that $\overline{c}/\overline{k} = \theta s^{-1} - \delta$, we obtain the following expression[5] for A:

$$A = \begin{bmatrix} -\zeta & -\frac{1}{k}\beta^2\rho\theta\frac{1-q}{q}\zeta & \frac{1}{k}\beta\rho\theta\left(\mu+\frac{1-q}{q}\right)\zeta \\ \frac{\bar{c}}{\mu}\zeta & 1+\frac{\bar{c}}{k}\frac{\beta^2\rho\theta}{\mu}\frac{1-q}{q}\zeta & -\frac{\bar{c}}{k}\frac{\beta\rho\theta}{\mu}\left(\mu+\frac{1-q}{q}\right)\zeta \\ 0 & -1 & 1/\beta \end{bmatrix}$$

where

$$\zeta \equiv \sigma\left[1 - \sigma + \frac{1-q}{q}\frac{\beta}{\mu}\right]^{-1}. \tag{15}$$

The characteristic polynomial of A is

$$P(\xi) \equiv \xi^3 - T\xi^2 + \Sigma\xi - D \tag{16}$$

where T, Σ, D are, respectively, the trace, the sum of the principal minors of order two and the determinant of A. Straightforward although tedious computations show that T, Σ and D write, respectively:

$$\begin{aligned} T &= \frac{1+\beta}{\beta} - \zeta\left[1 - \beta\rho\theta\left(\frac{\theta}{s} - \delta\right)\frac{1-q}{q}\frac{\beta}{\mu}\right], \\ \Sigma &= \frac{1}{\beta} - \zeta\left[\frac{1+\beta}{\beta} + \beta\rho\theta\left(\frac{\theta}{s} - \delta\right)\right], \\ D &= -\zeta/\beta. \end{aligned}$$

The next proposition characterizes the modulus of the eigenvalues of A. It is shown that when the amplitude of liquidity constraint q is close to one indeterminacy comes about only for low elasticities of intertemporal substitution in consumption σ. This result is analogous to those found in the cash-in-advance economies ($q = 1$) studied in Bloise *et al.* (2000) and Barinci and Chéron (2001). On the other hand, and more surprisingly, when q is continuously relaxed the range for σ generating indeterminacy becomes larger and larger and includes eventually, for q low enough, the whole domain of definition of σ.

In order to prove this, it is useful to introduce here the following critical values:

$$\sigma^* \equiv \frac{1 + \frac{1-q}{q}\frac{\beta}{\mu}}{2 + b\left(1 - \frac{1-q}{q}\frac{\beta}{\mu}\right)}, \tag{17}$$

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[5]Provided that $\sigma \neq 1+\frac{\beta}{\mu}\frac{1-q}{q}$, DG_1 is a linear isomorphism and, so, the formula defining A above is meaningful. In addition, we assume that A has no eigenvalues on the unit circle.

$$q^* \equiv \frac{1}{1 + \frac{\mu}{\beta} \frac{(2+b)\sigma - 1}{b\sigma + 1}}, \tag{18}$$

$$q^{**} \equiv \frac{1}{1 + \frac{\mu}{\beta} \frac{2+b}{b}} \tag{19}$$

where $b \equiv \beta^2 \rho \theta \left(\theta/s - \delta\right) / \left[2\left(1 + \beta\right)\right]$.

Proposition 2 *The linear operator A possesses a real eigenvalue belonging to $(0, 1)$ and an eigenvalue with modulus greater than one. In addition:*

(i) When $0 < q < q^{**}$, the third eigenvalue belongs to $(-1, 0)$ and the steady state of system (13) is locally indeterminate.

(ii) When $q^{**} < q < 1$, for $0 < \sigma < \sigma^*$, the third eigenvalue belongs to $(-1, 0)$ and the steady state of system (13) is locally indeterminate. Otherwise, it has modulus greater than one and the steady state is locally determinate. Moreover, when σ goes through σ^*, the steady state undergoes a flip bifurcation.

Proof. The eigenvalues of A correspond to the roots of the characteristic polynomial (16). Performing simple computations we obtain

$$
\begin{aligned}
P\left(-1\right) &= \zeta \left[4\frac{1+\beta}{\beta} + \beta\rho\theta \left(\frac{\theta}{s} - \delta\right) \left(1 - \frac{1-q}{q}\frac{\beta}{\mu}\right) - \frac{2}{\sigma}\frac{1+\beta}{\beta} \left(1 + \frac{1-q}{q}\frac{\beta}{\mu}\right) \right], \\
P\left(0\right) &= \zeta/\beta, \\
P\left(1\right) &= -\zeta\beta\rho\theta \left(\frac{\theta}{s} - \delta\right) \left(1 + \frac{1-q}{q}\frac{\beta}{\mu}\right).
\end{aligned}
$$

Observe that $\lim_{\xi \to +\infty} P\left(\xi\right) = +\infty$, $\lim_{\xi \to -\infty} P\left(\xi\right) = -\infty$ and that the polynomial is a continuous function and its domain is connected. One can easily verify that $P\left(1\right)P\left(0\right) < 0$. This implies that there is always a real root, say ξ_1, in $(0, 1)$. At the same time straightforward computations show that $P\left(-1\right)P\left(1\right) > 0$ either when $0 < q < q^{**}$ or, for $q^{**} < q \le 1$, when $\sigma < \sigma^*$. Therefore two main regimes are possible.

(i) $0 < q < q^{**}$. Then $P\left(-1\right)P\left(1\right) > 0$ for all σ. It follows that there is a root belonging to $(-1, 0)$ and by continuity of the polynomial, a third real root with modulus greater than one.

(ii) $q^{**} < q \le 1$. Then $P\left(-1\right)P\left(1\right) > 0$ if and only if $\sigma < \sigma^*$. In such a case there is a root belonging to $(-1, 0)$ and a third one with modulus

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greater than one. The steady state is thus locally indeterminate. When $\sigma > \sigma^*$ observe in addition that $D = -P(0) > 1$ and that the determinant corresponds to the product of the eigenvalues. It follows that there exists at least one root with modulus greater than one. If such a root is real and positive, by the continuity of the polynomial, one has $\xi_i > 1$ for $i = 2, 3$. If it is real and negative, one has $\xi_i < -1$ for $i = 2, 3$. If such a root is complex, there are two conjugate eigenvalues ξ_2, ξ_3 such that $|\xi_2| = |\xi_3| > 1$. In either cases, the steady state is locally determinate. Finally, it is immediate to verify that when $\sigma = \sigma^*$ one eigenvalue goes through -1 and the system undergoes a flip bifurcation. ∎

Remark 1 *The expression for q^* is simply obtained from that for σ^* and defines a meaningful bifurcation value (in the sense that it is between zero and one) in terms of q provided that $\sigma > 1/(2+b)$. Clearly $q < q^*$ entails indeterminacy. Finally, observe that*

$$q^{**} = \lim_{\sigma \to +\infty} q^*$$

and so for q less than q^{**} we observe indeterminacy for all σ's.

Proposition 2 states that when $0 < q < q^{**}$ indeterminacy comes about for all σ's and when $q^{**} < q \leq 1$ for all $\sigma < \sigma^*$. Thus a small departure from the traditional Ramsey-Cass-Koopmans model - the requirement that an arbitrarily small amount of consumption purchases must be paid by cash holdings accumulated from the previous periods - is sufficient to make the equilibrium indeterminate and to allow for self-fulfilling revisions in expectations to be consistent with rational expectations. What makes such a result even more attracting is that it does not rely upon any specific calibration for the fundamentals: All what is needed here is to relax the liquidity constraint enough.[6]

In order to complete the characterization of local indeterminacy, let us observe that σ^* is decreasing in $q \in [q^{**}, 1]$: It follows that the range for σ giving raise to indeterminacy improves continuously as soon as q is relaxed,

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[6] In a precedent version of the paper, Bosi and Magris (2001) show that all these results in terms of indeterminacy do still hold in a context of endogenous growth promoted by the existence of spillover effects in aggregate capital, with the unique exception that the elasticity of intertemporal substitution in consumption, σ, must be bounded from above in order the transversality condition to be respected.

before including eventually all possible σ's $(0 < q < q^{**})$. (To have an idea of this picture see Fig. A).

Proposition 2 also shows that the change in stability which leads to indeterminacy occurs always through a flip bifurcation: For a given $q > q^{**}$, when $\sigma = \sigma^*$ there is one characteristic root going through -1. Given the non-linearity of the system, this implies (see, e.g., Grandmont, 1988) that when σ is arbitrarily close to σ^*, there will generically emerge, according to the direction of the bifurcation, a stable or unstable two-period cycle.

To give just an idea of the plausibility of indeterminacy in the model under study, we calibrate the structural parameters on quarterly data. More precisely we chose $\beta = 0.99$, $\delta = 0.025$, $\mu = 1.015$, $s = 0.33$. We assume in addition a Cobb-Douglas production function, so that $\rho = 1 - s = 0.67$. What we obtain is the following picture:
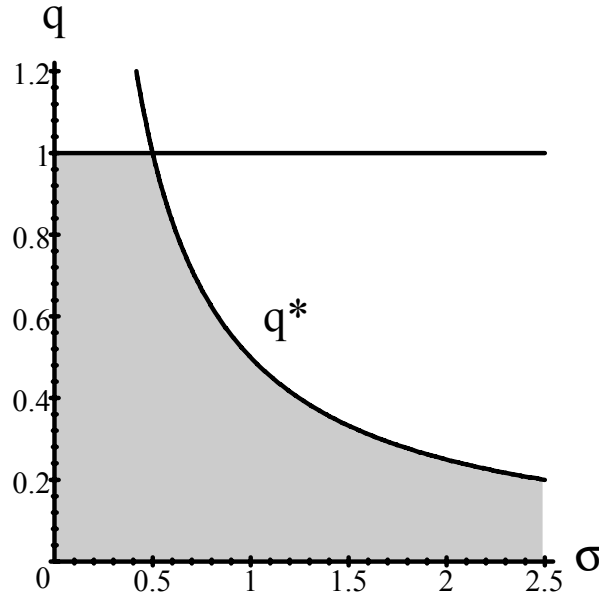
Figure A: Indeterminacy region.

In Fig. A the shaded area corresponds to the indeterminacy region, its boundary coinciding with the bifurcation values σ^*. As it emerges in Fig. A, for $q = 1$ indeterminacy requires σ to be lower than, approximately, $1/2$. As soon as q is relaxed, the range of σ generating indeterminacy improves: It corresponds, for example, to the interval $(0, 1)$ for $q = 0.48$ and to the interval $(0, 2)$ for $q = 0.22$.

3.1 Interpreting conditions for indeterminacy

Results in Proposition 2 are apparently counter-intuitive: According to them, in fact, the scope for indeterminacy improves as the degree of market imperfection - namely the share of consumption purchases q to be paid cash in the hand of the representative agent - is decreased and not increased, as intuition should suggest. Yet, an analysis based on a direct inspection of the Euler equation (7) can provide an heuristic explanation of the main driving force leading to indeterminacy, and make it clear why the latter is easier to obtain when the liquidity constraint is small.

For sake of simplicity, and without any relevant loss in generality, let us begin our analysis by supposing that utility is logarithmic in consumption. Under this specification, and taking into account that equilibrium in money market implies $c_{t+1}/c_t = \pi_{t+1}^{-1}$, the arbitrage equation (7) boils down to

$$\frac{1}{\pi_{t+1}} = \beta R_{t+1} \frac{q\pi_t R_t + 1 - q}{q\pi_{t+1}R_{t+1} + 1 - q}. \tag{20}$$

Suppose now that the system is initially at the steady state and let us try to construct an alternative equilibrium in which agents in period t anticipate a fall in next period inflation factor π_{t+1}. Under this conjecture, if the liquidity constraint is still binding, today investment and tomorrow consumption will be driven up. Now, since a higher investment implies a lower expected interest factor R_{t+1}, and since in correspondence to a given fall of π_{t+1}, the left-hand side of (20) increases more than its right-hand one, in order to re-establish equation (20), today inflation factor π_t must go up. In particular, the lower q, the sharper the required increase of π_t: In fact, the amplitude of q does not affect the variation of the left-hand side of (20) induced by a fall of π_{t+1}, meanwhile it measures the sensibility of its right-hand side with respect to both π_t and π_{t+1}. Moreover the higher q, the lower the impact of the real interest factor R_{t+1} on the right-hand side of (20).

Assume first that q is close to one. In such a case, for a given fall of π_{t+1}, in order to re-establish (20), it will be then sufficient only a slight increase in today inflation factor π_t, and such an increase will therefore turn out to be lower than tomorrow decrease. This, of course, will generate explosive dynamics. Consider conversely the case in which q is rather small. Under such a configuration, in order to compensate the fall of π_{t+1}, π_t must know a sharp increase, and such an increase will be actually greater than the decrease of π_{t+1}, giving thus rise to convergent, although oscillatory, dynamics.

These considerations also allow us to understand why a more flexible technology, thus an interest rate more reactive to investment, requires a lower q to generate indeterminacy. Indeed, in correspondence to a cheaper future consumption and an increase in foregoing investment, R_{t+1} will be strongly driven down, making as a consequence the decline of the right-hand side of (20) even sharper. Therefore, for a given q, the re-establishment of (20) will require an even sharper increase in today inflation factor π_t and the mechanism above described will lead to convergent dynamics even in correspondence to greater $q's$. Actually, this is in line with what suggested in the expression of q^* in which the latter appears to be increasing in the elasticity of the interest rate ρ.

Carrying out the same kind of reasoning, it is also possible to explain why one needs a lower q to obtain indeterminacy when the elasticity of intertemporal substitution in consumption σ increases, as Proposition 2 claims. Suppose, for sake of simplicity, that utility is isoelastic, i.e. of the form $u(c) = \left(c^{1-1/\sigma} - 1\right) / (1 - 1/\sigma)$. Under this specification, the Euler equation can be opportunely rewritten as

$$\left(\frac{1}{\pi_{t+1}}\right)^{1/\sigma} = \beta R_{t+1} \frac{q\pi_t R_t + 1 - q}{q\pi_{t+1}R_{t+1} + 1 - q}. \tag{21}$$

One may immediately verify that the higher σ, the lower the increase of the left-hand side of (21) induced by a fall of π_{t+1}. It follows that when σ is relatively high, the re-establishment of condition (21), for each q, will require only a smaller increase in π_t. Thus, the mechanism above described will lead to convergent dynamics only in correspondence to very low $q's$. Conversely, when σ is close to zero, indeterminacy can take place also in correspondence to a q very close to one, and even equal to one.

4   Conclusion

In this paper we have presented a one-sector productive economy with partial and variable cash-in-advance constraint on consumption purchases and studied the occurrence of indeterminacy and endogenous fluctuations. We have shown that the relaxation of the liquidity constraint makes indeterminacy more and more likely to occur, in the sense that the range for the parameters values giving raise to it becomes larger and larger. Moreover, when the

amplitude of the liquidity constraint is set sufficiently low, the unique steady state is bound to be indeterminate for whatever fundamentals specification.

These findings seem to suggest that indeterminacy and sunspot fluctuations - very far from being some kind of "exotic" or "pathological" feature in macroeconomic models - are by contrast very pervasive. In fact, it is sufficient to slightly perturb the benchmark Ramsey-Cass-Koopmans model in order to rule out the saddle path stability and obtain multiple deterministic as well as stochastic equilibria.

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