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An Overlapping Generations Model with Endogenous Labor Supply: a Dynamic Equilibrium Analysis*

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Abstract:

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1 Introduction

This paper deals with the canonical framework of overlapping generations model with productive capital. The Diamond [8] formulation is considered, but augmented to include endogenous labor supply. We consider the general formulation of preferences over the life cycle of each agent given by non separable utility functions. Our aim is to study the conditions of the possible emergence of endogenous, deterministic or stochastic, fluctuations. We will analyse the determinacy properties of equilibrium paths, as well as the potential existence of local bifurcations. A given configuration will be refered to as indeterminate as soon as there exists a multiplicity of distinct equilibrium paths starting from the same initial value for the capital stock. It is well known that indeterminacy of perfect foresight equilibria is a sufficient condition for the existence of sunspot equilibria and stochastic fluctuations based upon extrinsic uncertainty.[1] The analysis of endogenous fluctuations, not only near an indeterminate steady state but also along local bifurcations, will be conducted.

In the recent period, the Ramsey one-sector growth model augmented to include endogenous labor supply has become a strandard framework for the analysis of local indeterminacy and fluctuations based on the existence of sunspot equilibria.[2] It is proved indeed that locally indeterminate equilibria may occur under increasing returns to scale based on capital and labor externalities in the production technologies. A similar level of interest has not been experienced by the overlapping generations model with endogenous labor supply, despite the fact that Kehoe and Levine [15] exhibit some robust examples of pure exchange overlapping generations economies with a continuum of equilibria.[3]

[1] See Cass and Shell [2] and Woodford [24].

[2] See the initial contribution of Benhabib and Farmer [3] and their recent survey [4] for additional references.

[3] See also Geanakoplos and Polemarchakis [11, 12]. Note that Muller and Woodford [20] extend this result to production multisector economies in which the labor supply is

Most of the earlier papers dealing with an overlapping generations model with production and endogenous labor supply are based on particular assumptions which either restrict preferences to be additively separable or to exclude first period consumption, or restrict technologies to Leontief or Cobb-Douglas formulations.[4] Moreover the demands for consumption and leisure are generally assumed to satisfy the gross substitute axiom. A related literature based on the Woodford [25] model, in which two classes of representative gents, namely "workers" and "capitalists", coexist, considers a similar formulation.[5] Our goal is on the contrary to consider a general overlapping generations model and to provide an analysis of the dynamic properties of equilibrium paths which is a counterpart to the results on the two-sector overlapping generations model with inelastic labor supplied by Galor [9].

In a one dimensional standard Diamond model, local indeterminacy of the steady state is not possible. However, as shown in Galor and Ryder [10], when gross substitutability between consumptions is not assumed, global indeterminacy of perfect foresight equilibria may emerge. In a two-sector formulation with inelastic labor, Galor [9] shows on the contrary that local indeterminacy may appear even under the gross subsitute axiom. It is now well-known since Reichlin [22] that a similar result holds in a one-sector OLG model with endogenous labor supply. However, the specific assumptions on preferences and technology used in this contribution and in related papers prevent to obtain a general picture of the dynamic properties of equilibrium paths.

Cazzavillan and Pintus [6] consider an OLG model with endogenous labor supply under assumptions of additively separable preferences and gross substitutability of consumptions and leisure. These restrictions allow the authors to perform the dynamical analysis using a simple geometrical method

assumed to be inelastic.

[4]See Azariadis [1], Cazzavillan and Pintus [6], Grandmont [13], Lloyd-Braga [17, 18], Medio and Negroni [19], Reichlin [22] and de Vilder [23].

[5]See Grandmont, Pintus and de Vilder [14], Cazzavillan, Lloyd-Braga and Pintus [5].

recently developed by Grandmont, Pintus and de Vilder [14]. Such a methodology is well adapted to models in which the steady state only depends on the properties of technology, as for instance in the Woodford [25] model in which it is given by the modified golden rule. This property however does not hold in an OLG model where the steady state is defined from both preferences and the technology. Restricting the utility function to a formulation in which the Arrow-Pratt index for consumptions and leisure are defined independently, Cazzavillan and Pintus [6] have been able to adapt the geometrical method. On the contrary, with general non separable preferences, this methogology cannot in general be performed. One additional difficulty comes from the fact that we will define the dynamical system from the saving and labor supply functions which are endogenous. We will however consider a CES economy to illustrate our general results which will be analysed with the simple geometrical method.

Our main result is that we completely characterize the local dynamics of the model and establish, thereby, the necessary and sufficient conditions for the existence of indeterminate equilibria, as well as endogenous fluctuations. We prove that the condition for the existence of a unique equilibrium path converging to a given steady state in the standard Diamond model can be generalized in the model augmented to consider endogenous labor supply. We consider alternatively that the elasticity of the labor supply with respect to the interest factor is positive or negative. In each case, we provide the necessary and sufficient conditions for the steady state to be a saddle-point stable, locally indeterminate or locally unstable. These results allow to conclude that some Hopf and Flip bifurcations could appear when the inputs elasticity of substitution is modified.

The paper is organized as follows. In the next section we shall present the model. In section 3, we focus on the steady state and establish an existence result. In section 4, we analyse the local dynamics and derive our main results. Section 5 provides a detailled analysis of a CES economy. Section 6 gives some concluding comments. All the proofs are gathered in section 7.

2 The model with endogenous labor supply

Consider a perfectly competitive world where economic activity is performed over infinite discrete time in which there are identical non altruistic agents. Each agent lives for two periods: he works during the first, supplying elastically a portion l of one unit of labor: $0 \leq l \leq 1$. He has preferences for his consumption (c, when he is young, and d, when he is old), and for his leisure ($\mathcal{L} = 1 - l$) which are summarized by the utility functions $u(c, \mathcal{L}, d)$.

Assumption 1 . *$u(c, \mathcal{L}, d)$ is strictly increasing with respect to each argument ($u_1(c, \mathcal{L}, d) > 0$, $u_2(c, \mathcal{L}, d) > 0$ and $u_3(c, \mathcal{L}, d) > 0$), $\mathbf{C}^2$, with negative definite Hessian matrix, over the interior of the set $\mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+$. Moreover, for all consumption and leisure levels $c, \mathcal{L}, d > 0$, $u_1(0, \mathcal{L}, d) = u_2(c, 0, d) = u_3(c, \mathcal{L}, 0) = \infty$.*

Each agent is assumed to have $1 + n$ children, with $n \geq 0$, and the number of individuals born in period t is denoted N_t. Considering the wage rate w_t and the expected interest factor R^e_{t+1} as given, he maximizes his utility function over his life-cycle as follows:

$$\begin{array}{cl} \max\limits_{c_t, \mathcal{L}_t, d_{t+1}} & u(c_t, \mathcal{L}_t, d_{t+1}) \\ s.t. & w_t(1 - \mathcal{L}_t) = c_t + s_t \\ & R^e_{t+1} s_t = d_{t+1} \\ & 0 \leq \mathcal{L}_t \leq 1 \end{array} \tag{1}$$

Agents expect perfectly the interest factor $R^e_{t+1} = R_{t+1}$. Assumption 1 implies the existence and uniqueness of interior solutions for optimal saving and labor supply (s_t, l_t).[6] The first order conditions:

$$\left\{ \begin{array}{ll} -u_1(w_t l_t - s_t, 1 - l_t, s_t R_{t+1}) + R_{t+1} u_3(w_t l - s_t, 1 - l_t, s_t R_{t+1}) & = 0 \\ w_t u_1(w_t l - s_t, 1 - l_t, s_t R_{t+1}) - u_2(w_t l - s_t, 1 - l_t, s_t R_{t+1}) & = 0 \end{array} \right. \tag{2}$$

[6]A solution $l_t = 0$, i.e. $\mathcal{L}_t = 1$, cannot hold since it would imply from the assumptions on the technology that $y_t = k_{t+1} = 0$ and that second period consumption is equal to zero. But this is excluded by Assumptions 1.

give the saving and labor supply of each agent as functions:

$$s_t = s(w_t, R_{t+1}) \quad (3)$$

$$l_t = l(w_t, R_{t+1}) \quad (4)$$

In Nourry [21] it is proved that under Assumption 1, the saving function $s(.,.)$ and the labor supply function $l(.,.)$ are differentiable for $(w, R) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$, with values respectively in $\mathbb{R}_+$ and $(0, 1)$. To derive more information on these functions we introduce the following property.

Definition 1 . Strong normality
Consider the following maximisation program

$$\begin{array}{ll} \max_{c,\mathcal{L},d} & u(c, \mathcal{L}, d) \\ s.t. & w = c + w\mathcal{L} + d/R \\ & 0 \leq \mathcal{L} \leq 1 \end{array}$$

and the corresponding optimal demand functions $c(w, R)$, $\mathcal{L}(w, R)$ and $d(w, R)$. The consumptions c, d and leisure $\mathcal{L}$ will be called strongly normal goods if $(\partial c/\partial w) \geq 0$, $(\partial d/\partial w) \geq 0$ and $(\partial \mathcal{L}/\partial w) \geq 0$.[7]

We will use throughout the paper the following restriction:

Assumption 2 . *Consumptions c and d are strongly normal goods.*

[7]The standard definition of normality is based on the following program

$$\begin{array}{ll} \max_{c,\mathcal{L},d} & u(c, \mathcal{L}, d) \\ s.t. & \Omega = c + w\mathcal{L} + d/R \\ & 0 \leq \mathcal{L} \leq 1 \end{array}$$

and implies that the demand functions satisfy $(\partial c/\partial \Omega) \geq 0$, $(\partial d/\partial \Omega) \geq 0$ and $(\partial \mathcal{L}/\partial \Omega) \geq 0$. Besides this standard income effect, our definition includes also a price effect which corresponds to the wage rate interpreted as the price of leisure. Strong normality, corresponding to the sum of these both effects which have opposite sign, implies therefore that the income effect is greater than the price effect. Strong normality thus implies normality.

Under Assumption 2, it is easy to prove that the saving function is increasing with respect to the wage rate, i.e. $s_w(w, R) \geq 0$. Similarly, if leisure is a strongly normal good, then $\mathcal{L}_w(w, R) \geq 0$ and the labour demand function is decreasing with respect to the wage rate, i.e. $l_w(w, R) \leq 0$. We will not however impose this restriction since we want to discuss the effects of the labour demand function which will be successively considered as decreasing or increasing with respect to the wage rate.

The production function of a representative firm, denoted $F(K, L)$, depends on the stock of capital K and labor L, and is assumed to be homogeneous of degree one with $F(K, 0) = 0$ for any $K \geq 0$. Assuming also that capital depreciation is complete in each period, and denoting, for any $L \neq 0$, $k = K/L$ the capital stock per labor unit, we may define the production function in intensive form as $f(k) = F(k, 1)$.

Assumption 3 . *$f(k)$ is positively valued, $\mathbf{C}^2$, strictly increasing, strictly concave over $\mathbb{R}_{++}$, and satisfies $f(0) = 0$, $\lim_{k \to 0} f'(k) = \infty$ and $\lim_{k \to \infty} f'(k) = 0$.*

The competitive equilibrium conditions imply that the interest factor R_t and the wage rate w_t satisfy:

$$R_t = f'(k_t) \equiv R(k_t) \tag{5}$$

$$w_t = f(k_t) - k_t f'(k_t) \equiv w(k_t) \tag{6}$$

Thus, the capital accumulation equation simply states as:

$$K_{t+1} = N_t s(w_t, R_{t+1}) \tag{7}$$

Since each young agent supplies l_t unit of labor, the total labor in the economy during period t is:

$$L_t = N_t l_t$$

and equation (7) becomes:

$$(1+n)k_{t+1} l\Big(w(k_{t+1}), R(k_{t+2})\Big) = s_t\Big(w(k_t), R(k_{t+1})\Big) \tag{8}$$

This is a second order difference equation.

3 Steady state

The capital accumulation equation (8), together with the factor market equilibrium conditions (4), (5), (6) and the initial condition for the capital stock, fully characterize the intertemporal equilibria with perfect foresight.

Definition 2 . *A perfect foresight equilibrium is a sequence* $\{k_t\}_{t=0}^{\infty}$ *such that*

$$(1+n)k_{t+1}l\Big(f(k_{t+1})-k_{t+1}f'(k_{t+1}),f'(k_{t+2})\Big)=s\Big(f(k_t)-k_tf'(k_t),f'(k_{t+1})\Big) \quad (9)$$

where k_0 *is exogenously given.*

Definition 3 . *A steady state is a stationary capital-labor ratio,* $\bar{k}$*, such that*

$$(1+n)\bar{k}l\Big(f(\bar{k})-\bar{k}f'(\bar{k}),f'(\bar{k})\Big)=s\Big(f(\bar{k})-\bar{k}f'(\bar{k}),f'(\bar{k})\Big) \quad (10)$$

The second order difference equation (9) may be characterized by either the non existence, uniqueness or multiplicity of non-trivial steady states.

Proposition 1 . *Under Assumptions 1-3, there exists a trivial steady state* $\hat{k}=0$.

As in the model with exogenous labor supply, the existence of an a non-trivial steady state is not guaranted even with some strengthened Inada condition.[8] Restrictions on the interactions between preferences and technology are required. Let us denote

$$\phi(k)=(1+n)k-\frac{s\left(f(k)-kf'(k),f'(k)\right)}{l\left(f(k)-kf'(k),f'(k)\right)} \quad (11)$$

Proposition 2 . *Under Assumptions 1-3,*

i) if $\lim_{k\to 0}\phi'(k)<0$ *then there exists a non-trivial steady state. Generically, the number of non-trivial steady states is odd;*

ii) if $\lim_{k\to 0}\phi'(k)>0$ *then the number of non-trivial steady states is generically even, and can be zero.*

[8] See Galor and Ryder [10].

We will analyse in the next section the local dynamics of equilibrium paths around a non-trivial steady state.

4 Analysis of the local dynamics

Consider $\epsilon_{l,w}$, $\epsilon_{l,R}$, $\epsilon_{s,w}$, $\epsilon_{s,R}$ the elasticities of the labor supply, $l(w,R)$, and saving, $s(w,R)$, functions with respect to w and R evaluated at a non-trivial steady state $\bar{k}$. We have, for example, $\epsilon_{l,w} = (\bar{w}/\bar{l})(\partial l(\bar{w},\bar{R})/\partial \bar{w})$, where $\bar{w}$ and $\bar{l}$ are the values of the wage rate and the labor supply at the steady state. We also consider the share of capital in total income $\epsilon_f = \bar{k}f'(\bar{k})/f(\bar{k})$ and the index $A_f = -\bar{k}f''(\bar{k})/f'(\bar{k})$. Denoting ς the inputs elasticity of substitution in the production function, it is easy to show that $\varsigma = (1-\epsilon_f)/A_f$.

We will assume that the labor supply is a non constant function of the interest factor, i.e. $\epsilon_{l,R} \neq 0$.[9] To study the local stability properties of the steady state $\bar{k}$, we linearize the dynamical equation (9) in the neighbourhood of this equilibrium. Some straightforward algebra allows to write the corresponding characteristic polynomial in terms of elasticities:

$$\mathcal{P}(\lambda) = \lambda^2 - \lambda \frac{\varsigma + \epsilon_f \epsilon_{l,w} + (1-\epsilon_f)\epsilon_{s,R}}{(1-\epsilon_f)\epsilon_{l,R}} + \frac{\epsilon_f \epsilon_{s,w}}{(1-\epsilon_f)\epsilon_{l,R}} \tag{12}$$

Definition 4 . *Let $\{k_t\}_{t=0}^{\infty}$ denote an equilibrium for an economy with initial condition k_0. We say that it is a locally indeterminate equilibrium if for every $\epsilon > 0$ there exists another sequence $\{k'_t\}_{t=0}^{\infty}$, with $0 < |k'_1 - k_1| < \epsilon$ and $k'_0 = k_0$, which is also an equilibrium.*

If an equilibrium is not indeterminate, then we call it determinate. The dimension of local indeterminacy cannot be greater than one. Actually, the steady state $\bar{k}$ is locally indeterminate if and only if the local stable manifold is two-dimensional. We introduce the following definition:

[9] If $\epsilon_{l,R} = 0$, then it is easy to show that one root of the characteristic polynomial is always equal to zero and the dynamical system becomes one-dimensional. The stability analysis is thus the same as in the Diamond model. This configuration appears for instance when the utility function is Cobb-Douglas.

Definition 5 . *A steady state $\bar{k}$ of the second order difference equation (9) is saddle-point stable if and only if the dimension of the local stable manifold is equal to 1.*

We will discuss the dynamic properties of the equilibrium depending on the sign of the elasticity of labor supply with respect to the interest factor, $\epsilon_{l,R}$. Standard assumptions of gross substitutability for consumptions in both periods and labor imply that the saving and labor supply functions are increasing with respect to the interest factor R. However, we do not want to impose such a strong restriction and we will study both configurations in which $\epsilon_{l,R}$ is successively assumed to be positive and negative.

4.1 The case $\epsilon_{l,R} > 0$.

In the standard Diamond model, under the assumption $\epsilon_{s,R} \geq 0$, the following condition ensures the local stability of equilibrium paths:

$$\Lambda_D \equiv \varsigma + (1-\epsilon_f)\epsilon_{s,R} - \epsilon_f \epsilon_{s,w} > 0 \tag{13}$$

We may derive some conditions in the model with endogenous labor which can be interpreted as extensions of the above inequality. We obtain indeed the following local characterization of the steady state $\bar{k}$:

Proposition 3 . *Let $\Lambda_L = \Lambda_D + \epsilon_f \epsilon_{l,w} - (1-\epsilon_f)\epsilon_{l,R}$. Under Assumptions 1-3 and $\epsilon_{l,R} > 0$, the following cases hold:*

(1) If $\Lambda_L + \epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R} > 0$, the steady state $\bar{k}$ is

i) saddle-point stable if and only if $\Lambda_L > 0$;

ii) locally indeterminate if and only if $\Lambda_L < 0$ and $\epsilon_f \epsilon_{s,w} < (1-\epsilon_f)\epsilon_{l,R}$;

iii) locally unstable if and only if $\Lambda_L < 0$ and $\epsilon_f \epsilon_{s,w} > (1-\epsilon_f)\epsilon_{l,R}$.

(2) If $\Lambda_L + \epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R} < 0$, the steady state $\bar{k}$ is

i) saddle-point stable if and only if $\Lambda_L + 2[\epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R}] < 0$;

ii) locally indeterminate if and only if $\Lambda_L + 2[\epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R}] > 0$ and $\epsilon_f \epsilon_{s,w} < (1-\epsilon_f)\epsilon_{l,R}$;

iii) locally unstable if and only if $\Lambda_L + 2[\epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R}] > 0$ *and* $\epsilon_f \epsilon_{s,w} > (1-\epsilon_f)\epsilon_{l,R}$.

Note that in case (1) we provide some conditions which appear to be simple extensions of the standard Diamond condition. Λ_D is indeed augmented by additive terms related to the elasticities of the labor supply. In particular the condition for saddle-point stability is very similar to the stability condition of the Diamond model.

Assume that there exist n non-trivial steady states, $n \in \mathbb{N}^*_+$ and that the steady states are ordered as $k_1 > k_2 > ... > k_n$. We provide the following characterization:[10]

Proposition 4 . *Under Assumptions 1-3, let* $\epsilon_{l,R} > 0$. *Then all the steady states with an odd index are saddle-point stable, whereas the steady states with an even index are locally indeterminate or locally unstable. Moreover, the trivial steady state* $\hat{k} = 0$ *is saddle-point stable if* $\lim_{k \to 0} \phi'(k) > 0$, *and locally indeterminate or locally unstable if* $\lim_{k \to 0} \phi'(k) < 0$.

Corollary 1 . *Under Assumptions 1-3, let* $\epsilon_{l,R} > 0$. *If there exists a unique steady state* $\bar{k} > 0$, *then it is saddle-point stable while the trivial steady state* $\hat{k} = 0$ *is locally indeterminate or locally unstable.*

Let us now consider that the inputs elasticity of substitution, ς, is constant. This property is in particular satisfied when we consider a C.E.S. production function. The variations from the elasticity of the production function ϵ_f could allow the appearance of bifurcations. Denoting the sum of eigenvalues as $T(\epsilon_f)$, and the product as $D(\epsilon_f)$, we have:

$$T(\epsilon_f) = \frac{\varsigma + \epsilon_f \epsilon_{l,w} + (1-\epsilon_f)\epsilon_{s,R}}{(1-\epsilon_f)\epsilon_{l,R}}, \quad D(\epsilon_f) = \frac{\epsilon_f \epsilon_{s,w}}{(1-\epsilon_f)\epsilon_{l,R}}$$

We also introduce the following notations which will be convenient to state the results:

[10] Assuming that $\Lambda_L + \epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R} > 0$, Nourry [21] obtains a similar result.

$$\mathcal{N}(\epsilon_f) = \varsigma + \epsilon_f \epsilon_{l,w} + (1-\epsilon_f)\epsilon_{s,R}$$

denotes the numerator of the trace,

$$\mathcal{P}_1(\epsilon_f) = (1-\epsilon_f)\epsilon_{l,R} + \epsilon_f \epsilon_{s,w} - \mathcal{N}(\epsilon_f)$$

denotes the numerator of the characteristic polynomial when $\lambda = 1$, and

$$\mathcal{P}_{-1}(\epsilon_f) = (1-\epsilon_f)\epsilon_{l,R} + \epsilon_f \epsilon_{s,w} + \mathcal{N}(\epsilon_f)$$

denotes the numerator of the characteristic polynomial when $\lambda = -1$. The local stability properties of a non-trivial steady state $\bar{k} > 0$ are summarized as follows:

Proposition 5 . *Under Assumptions 1-3, let $\epsilon_{l,R} > 0$ and the technology have a constant given elasticity of substitution $\varsigma \geq 0$. Then the following cases hold*

i) If $\mathcal{N}(\epsilon_f) > 0$ for any $\epsilon_f \in (0,1)$, $\lim_{\epsilon_f \to 0} \mathcal{P}_1(\epsilon_f) = \epsilon_{l,R} - (\varsigma + \epsilon_{s,R}) < 0$ and $\lim_{\epsilon_f \to 1} \mathcal{P}_1(\epsilon_f) = \epsilon_{s,w} - (\varsigma + \epsilon_{l,w}) > 0$, there exists $\epsilon_f^T \in (0,1)$ such that the steady state is saddle-point stable for any $\epsilon_f \in (0, \epsilon_f^T)$ and locally indeterminate when ϵ_f belongs to a right neighbourhood of ϵ_f^T. An eigenvalue goes through one as ϵ_f crosses ϵ_f^T.[11]

ii) If $\mathcal{N}(\epsilon_f) > 0$ and $\mathcal{P}_1(\epsilon_f) > 0$ for any $\epsilon_f \in (0,1)$, $\lim_{\epsilon_f \to 0} D(\epsilon_f) = 0$ and $\lim_{\epsilon_f \to 1} D(\epsilon_f) = +\infty$, there exists $\epsilon_f^H \in (0,1)$ such that the steady state is locally indeterminate for any $\epsilon_f \in (0, \epsilon_f^H)$ and locally unstable when ϵ_f belongs to a right neighbourhood of ϵ_f^H. A Hopf bifurcation occurs at ϵ_f^H.

iii) If $\mathcal{N}(\epsilon_f) < 0$ for any $\epsilon_f \in (0,1)$, $\lim_{\epsilon_f \to 0} \mathcal{P}_{-1}(\epsilon_f) = \varsigma + \epsilon_{l,R} + \epsilon_{s,R} < 0$ and $\lim_{\epsilon_f \to 1} \mathcal{P}_{-1}(\epsilon_f) = \varsigma + \epsilon_{s,w} + \epsilon_{l,w} > 0$, there exists $\epsilon_f^F \in (0,1)$ such that the steady state is saddle-point stable for any $\epsilon_f \in (0, \epsilon_f^F)$ and locally indeterminate when ϵ_f belongs to a right neighbourhood of ϵ_f^F. A Flip bifurcation

[11] We cannot a prori distinguish between the transcritical, pitchfork or saddle-node bifurcations from the linearized difference equation. The consideration of non-linear terms is necessary. A discussion of this point will be made for a CES economy in Section 5 and will exhibit a transcritical bifurcation.

occurs at ϵ_f^F*.*

iv) If $\mathcal{N}(\epsilon_f) < 0$ *and* $\mathcal{P}_{-1}(\epsilon_f) > 0$ *for any* $\epsilon_f \in (0,1)$*,* $\lim_{\epsilon_f \to 0} D(\epsilon_f) = 0$ *and* $\lim_{\epsilon_f \to 1} D(\epsilon_f) = +\infty$*, there exists* $\epsilon_f^H \in (0,1)$ *such that the steady state is locally indeterminate for any* $\epsilon_f \in (0, \epsilon_f^H)$ *and locally unstable when* ϵ_f *belongs to a right neighbourhood of* ϵ_f^H*. A Hopf bifurcation occurs at* ϵ_f^H*.*

These results show that even if we impose a gross substitutability assumption, i.e. if the elasticities $\epsilon_{l,w}$, $\epsilon_{l,R}$, $\epsilon_{s,w}$ and $\epsilon_{s,R}$ are positive, local indeterminacy of an interior steady state is not ruled out. However, this result crucially depends on the existence of at least two non-trivial steady states. If uniqueness holds, gross substitutability implies local determinacy.

4.2 The case $\epsilon_{l,R} < 0$.

Assume now that the labor supply is a decreasing function of the interest factor. Under Assumption 2, $\mathcal{P}(0) < 0$ and the eigenvalues are real with opposite sign. We obtain:

Proposition 6 . *Under Assumptions 1-3 and* $\epsilon_{l,R} < 0$*, the steady state* $\bar{k}$ *is*

i) saddle-point stable if and only if $\Lambda_L\{\Lambda_L + 2[\epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R}]\} > 0$*;*

ii) locally indeterminate if and only if $\Lambda_L > 0 > \Lambda_L + 2[\epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R}]$*;*

iii) locally unstable if and only if $\Lambda_L + 2[\epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R}] > 0 > \Lambda_L$*.*

As in the previous subsection, if there exist n steady states which are ordered as $k_1 > k_2 > ... > k_n$, then the following characterization holds:

Proposition 7 . *Under Assumptions 1-3, let* $\epsilon_{l,R} < 0$*. Then the following cases hold:*

i) If $\Lambda_L + \epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R} > 0$*, all the steady states with an even index are locally unstable, whereas the steady states with an odd index are saddle-point stable or locally indeterminate. Moreover, the trivial steady state* $\hat{k} = 0$ *is locally unstable if* $\lim_{k \to 0} \phi'(k) < 0$*, and saddle-point stable or locally indeterminate if* $\lim_{k \to 0} \phi'(k) > 0$*.*

ii) If $\Lambda_L + \epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R} < 0$, all the steady states with an odd index are locally indeterminate, while the steady states with an even index are saddle-point stable or locally unstable. Moreover, the trivial steady state $\hat{k} = 0$ is locally indeterminate if $\lim_{k\to 0} \phi'(k) > 0$, and saddle-point stable or locally unstable if $\lim_{k\to 0} \phi'(k) < 0$.

Corollary 2 . *Under Assumptions 1-3, let $\epsilon_{l,R} < 0$. If there exits a unique non-trivial steady state $\bar{k} > 0$, it is saddle-point stable or locally indeterminate while the trivial steady state $\hat{k} = 0$ is locally unstable.*

When the labor supply is a decreasing function of the interest factor, if there exits a unique non-trivial steady state, indeterminacy is not ruled out. When compared to Proposition 4 and its Corollary, these results show that the gross substitutability axiom may preclude indeterminacy only when uniqueness of the steady state holds.

If we consider again that the inputs elasticity of substitution ς is constant, the local stability properties of a non-trivial steady state $\bar{k} > 0$ are summarized in the following:

Proposition 8 . *Under Assumptions 1-3, let $\epsilon_{l,R} < 0$ and the technology have a constant given elasticity of substitution $\varsigma \geq 0$. Then the following cases hold*

i) If $\mathcal{P}_{-1}(\epsilon_f) < 0$ for any $\epsilon_f \in (0,1)$, $\lim_{\epsilon_f\to 0} \mathcal{P}_1(\epsilon_f) = \epsilon_{l,R} - (\varsigma + \epsilon_{s,R}) < 0$ and $\lim_{\epsilon_f\to 1} \mathcal{P}_1(\epsilon_f) = \epsilon_{s,w} - (\varsigma + \epsilon_{l,w}) > 0$, there exists $\epsilon_f^T \in (0,1)$ such that the steady state is locally indeterminate for any $\epsilon_f \in (0, \epsilon_f^T)$ and saddle-point stable when ϵ_f belongs to a right neighbourhood of ϵ_f^T. An eigenvalue goes through one as ϵ_f crosses ϵ_f^T.

ii) If $\mathcal{P}_1(\epsilon_f) < 0$ for any $\epsilon_f \in (0,1)$, $\lim_{\epsilon_f\to 0} \mathcal{P}_{-1}(\epsilon_f) = \varsigma + \epsilon_{l,R} + \epsilon_{s,R} < 0$ and $\lim_{\epsilon_f\to 1} \mathcal{P}_{-1}(\epsilon_f) = \varsigma + \epsilon_{s,w} + \epsilon_{l,w} > 0$, there exists $\epsilon_f^F \in (0,1)$ such that the steady state is locally indeterminate for any $\epsilon_f \in (0, \epsilon_f^F)$ and saddle-point stable when ϵ_f belongs to a right neighbourhood of ϵ_f^F. A Flip bifurcation occurs at ϵ_f^T.

5 A CES economy

We will consider in this section an economy characterized by CES utility and production functions such that:[12]

$$\begin{array}{rcl} u(c,\mathcal{L},d) & = & (B/\alpha)\left[\sigma c^{-\rho}+\delta\mathcal{L}^{-\rho}+\beta d^{-\rho}\right]^{-\alpha/\rho} \\ f(k) & = & A\left[\theta k^{-\gamma}+1-\theta\right]^{-\alpha/\gamma} \end{array}$$

with $\sigma \geq 0$, $\delta, \beta > 0$, $\theta \in (0,1)$, $\alpha \leq 1$, $\rho, \gamma > -1$ and $A, B > 0$.[13] The intertemporal elasticity of substitution of the utility function is given by $\eta = 1/(1+\rho)$ while the factors elacticity of substitution is $\varsigma = 1/(1+\gamma)$. The share of capital in the total income is $\epsilon_f = \theta[\theta + (1-\theta)k^\gamma]^{-1}$. The degree of homogeneity α may be such that $\alpha = -\rho$ and the utility function is additively separable. However, and contrary to the formulation used by Cazzavillan and Pintus [6], the Arrow-Pratt index for consumptions and labor are not independent. The case $\sigma = 0$ will correspond to the Reichlin's model [22] in which the representative agent does not consume during his first period of life.

Taking into account the budget constraints of a representative consumer, the maximisation of his utility function with respect to consumptions and leisure gives the following saving and labor supply functions:

$$\begin{array}{rcl} s(w,R) & = & \dfrac{w^{\frac{1}{1+\rho}}(\beta R)^{\frac{1}{1+\rho}}}{\delta^{\frac{1}{1+\rho}}R + w^{\frac{-\rho}{1+\rho}}\left[\sigma^{\frac{1}{1+\rho}}R + (\beta R)^{\frac{1}{1+\rho}}\right]} \\ l(w,R) & = & \dfrac{\sigma^{\frac{1}{1+\rho}}R + (\beta R)^{\frac{1}{1+\rho}}}{w^{\frac{\rho}{1+\rho}}\delta^{\frac{1}{1+\rho}}R + \left[\sigma^{\frac{1}{1+\rho}}R + (\beta R)^{\frac{1}{1+\rho}}\right]} \end{array}$$

[12] CES functions do not satisfy the Inada conditions in Assumptions 1 and 3 but the optimization program (1) provides interior solutions for consumptions, saving and labor supply as illustrated in this example.

[13] We can chose $B = (\sigma + \delta + \beta)^{\rho/\alpha}$ so that σ, δ and β may be interpreted as weighting parameters.

with

$$
\begin{aligned}
w &= w(k) = A(1-\theta)\left[\theta k^{-\gamma} + 1 - \theta\right]^{-(1+\gamma)/\gamma} && (14)\\
R &= R(k) = A\theta k^{-1-\gamma}\left[\theta k^{-\gamma} + 1 - \theta\right]^{-(1+\gamma)/\gamma} && (15)
\end{aligned}
$$

We will study the dynamic behaviour of this economy using the geometric approach developped by Grandmont, Pintus and de Vilder [14] and used by Cazzavillan, Lloyd-Braga and Pintus [5] and Cazzavillan and Pintus [6]. We will consider the characteristic polynomial given with the elasticities of saving and labor supply, the share of capital in total income and the factors elasticity of substitution. We will use the normalisation procedure associated with the geometrical method which consists in finding some conditions on the parameters A, δ and β such that one steady state $\bar{k}$ is always equal to one. The objective is to study geometrically the stability properties of this steady state in a simple (Trace,Determinant) plan by varying the share of capital in total income and the factors elasticity of substitution.

Assuming for the moment that $\bar{k} = 1$ is a steady state, let us first find some values for δ and β such that the elasticities of the saving and labor supply functions when evaluated at $\bar{k} = 1$ are not affected when the share of capital in total income and the factors elasticity of substitution are modified. Considering that $w(1) = A(1-\theta)$ and $R(1) = A\theta$, some tedious but straightforward computations show that if $\delta = [A(1-\theta)]^{-\rho}$ and $\beta = (A\theta)^{\rho}$, the elasticities of the saving and labor supply functions when $\bar{k} = 1$ are as follows:

$$
\begin{aligned}
\epsilon_{l,w} &= -\frac{\rho}{(1+\rho)(2+\sigma^{\frac{1}{1+\rho}})}\\
\epsilon_{l,R} &= \frac{\epsilon_{l,w}}{1+\sigma^{\frac{1}{1+\rho}}}\\
\epsilon_{s,R} &= \epsilon_{l,w}(1+\sigma^{\frac{1}{1+\rho}})\\
\epsilon_{s,w} &= -\frac{\epsilon_{l,w}}{\rho}[1+(1+\rho)(1+\sigma^{\frac{1}{1+\rho}})] > 0
\end{aligned}
$$

Based on these results, we may now study the existence and uniqueness of a stationary equilibrium. Equation (11) from Section 3 becomes

$$\phi(k) = (1+n)k - \frac{\frac{w}{R}(\beta R)^{\frac{1}{1+\rho}}}{\sigma^{\frac{1}{1+\rho}} + \beta^{\frac{1}{1+\rho}} R^{\frac{-\rho}{1+\rho}}}$$

Considering the values of the wage rate and the interest factor given in (14) and (15), we have $w/R = (1-\theta)k^{1+\gamma}/\theta$ and a steady state is finally a solution of the following equation

$$\phi_A(k) = k\left\{1+n - \frac{\frac{(1-\theta)k^\gamma}{\theta}\left[\beta A\theta[\theta+(1-\theta)k^\gamma]^{-(1+\gamma)/\gamma}\right]^{1/(1+\rho)}}{\sigma^{\frac{1}{1+\rho}} + \beta^{\frac{1}{1+\rho}}(A\theta)^{-\rho/(1+\rho)}[\theta+(1-\theta)k^\gamma]^{\rho(1+\gamma)/\gamma(1+\rho)}}\right\} = 0$$

Using $\beta = (A\theta)^\rho$ it is finally easy to show that $\bar{k} = 1$ is always a steady state, i.e. $\phi_A(1) = 0$, if and only if

$$A = A^* = (1+n)\frac{1+\sigma^{\frac{1}{1+\rho}}}{1-\theta} \tag{16}$$

After substitution of A^* into $\phi_A(k)$ we obtain:

$$\begin{aligned}\phi_{A^*}(k) &= (1+n)k\left\{1 - \frac{k^\gamma(1+\sigma^{\frac{1}{1+\rho}})}{\sigma^{\frac{1}{1+\rho}}[\theta+(1-\theta)k^\gamma]^{\frac{1+\gamma}{\gamma(1+\rho)}} + [\theta+(1-\theta)k^\gamma]^{\frac{1+\gamma}{\gamma}}}\right\} = 0 \\ &\equiv (1+n)k[1-\varphi(k)]\end{aligned}$$

so that the existence of other non-trivial steady states may be analysed from the following equation

$$\psi(k) \equiv 1 - \varphi(k) = 0$$

By definition we have $\psi(1) = 0$. Depending on the value of γ the properties of the function $\psi(k)$ are the following:

i) If $\gamma \leq 0$ then $\lim_{k\to 0}\psi(k) = -\infty$, $\lim_{k\to+\infty}\psi(k) = 1$ and $\psi'(k) > 0$ for any $k > 0$.

ii) If $\gamma > 0$ then $\psi(0) = 1$, $\lim_{k\to+\infty}\psi(k) = 1$ and there exists $\tilde{k} > 0$ such that $\psi'(k) \lesseqqgtr 0$ if and only if $k \lesseqqgtr \tilde{k}$. Note that $\tilde{k}$ may be less or greater than 1. Moreover depending on the value of $\rho + \theta$ and σ, we have the following configurations:

a) If $\rho+\theta < 0$ and $\sigma \geq [-\theta(1+\rho)/(\rho+\theta)]^{1+\rho}$, $\psi'(1) > 0$ for any $\gamma > 0$;

b) If $\rho + \theta < 0$ and $\sigma < [-\theta(1 + \rho)/(\rho + \theta)]^{1+\rho}$, or $\rho + \theta \geq 0$, there exists $\gamma^*(\theta) > 0$ such that $\psi'(1) \gtreqless 0$ if and only if $\gamma \lesseqgtr \gamma^*$.[14]

In case i) the function $\psi(k)$ is monotone increasing so that $\bar{k} = 1$ is the unique non-trivial steady state. In case ii), since $\bar{k} = 1$ is always a steady state and $\psi(0) = \lim_{k\to+\infty} \psi(k) = 1$, there generically exists a second non-trivial steady state as shown in the following Figure. Note also that in both cases $\phi_{A^*}(0) = 0$ so that the trivial steady state $\hat{k} = 0$ also exists.[15]

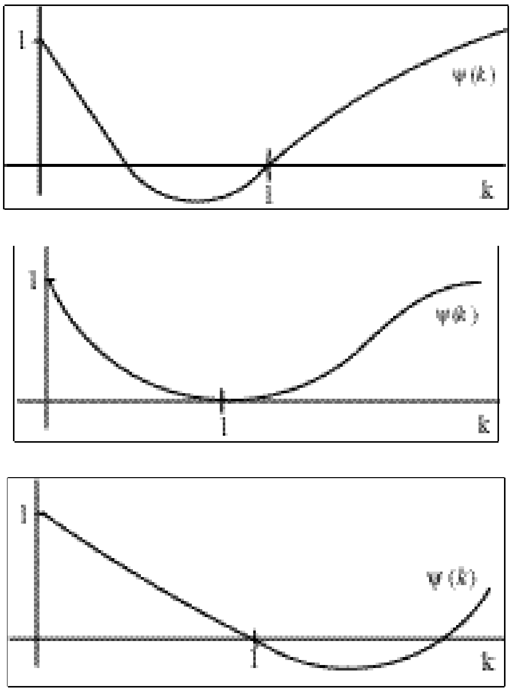

Figure 1: Multiple steady states when $\gamma > 0$.

[14] The exact value of $\gamma^*(\theta)$ is

$$\gamma^*(\theta) = \frac{(1-\theta)\left[1 + \sigma^{1/(1+\rho)} + \rho\right]}{\sigma^{1/(1+\rho)}(\rho+\theta) + \theta(1+\rho)}$$

[15] It can be shown that $\lim_{k\to 0} \phi_{A^*}(k) = 0$ when $\gamma \leq 0$.

We may thus summarize the results into the following proposition

Proposition 9 . *Consider the CES economy with $\delta = [A^*(1-\theta)]^{-\rho}$, $\beta = (A^*\theta)^\rho$ and A^* as defined in (16). There exists one trivial steady state $\hat{k} = 0$ and the following cases hold:*

i) If $\gamma \leq 0$, i.e. $\varsigma \geq 1$, $\bar{k} = 1$ is the unique non-trivial steady state.

ii) If $\gamma > 0$, i.e. $\varsigma < 1$, there exist two distinct non-trivial steady states, $\bar{k} = 1$ and $k^* > 0$. Moreover:

a) if $\rho + \theta < 0$ and $\sigma \geq [-\theta(1+\rho)/(\rho+\theta)]^{1+\rho}$, then $k^* \in (0,1)$ for any $\gamma > 0$;

b) if $\rho + \theta < 0$ and $\sigma < [-\theta(1+\rho)/(\rho+\theta)]^{1+\rho}$, or $\rho + \theta \geq 0$, then $k^* \in (0,1)$ when $\gamma < \gamma^*(\theta)$ and $k^* > 1$ when $\gamma > \gamma^*(\theta)$. $\bar{k} = 1$ and $k^* > 0$ coincide when $\gamma = \gamma^*(\theta)$.

In case ii)b), for any given $\gamma > 0$, i.e. $\varsigma < 1$, if solving the equation $\gamma = \gamma^*(\theta)$ with respect to θ gives a value $\theta^T \in (0,1)$, then the corresponding share of capital in total income evaluated at $\bar{k} = 1$, $\epsilon_f^T = \theta^T$, is a transcritical bifurcation value which involves an exchange of stability between the two stationary equilibria. The existence of an upper bound for the "weight" of the first period consumption suggests that the existence of a transcritical bifurcation depends on the parameter σ. A precise analysis of these phenomena will be conducted below.

We will now study the stability properties of the steady state $\bar{k} = 1$. Let us introduce the following notations for the trace T and the determinant D:

$$\begin{aligned} T(\sigma, \epsilon_f, \varsigma) &= \frac{\varsigma + \epsilon_f \epsilon_{l,w} + (1-\epsilon_f)\epsilon_{s,R}}{(1-\epsilon_f)\epsilon_{l,R}} \\ D(\sigma, \epsilon_f) &= \frac{\epsilon_f \epsilon_{s,w}}{(1-\epsilon_f)\epsilon_{l,R}} \end{aligned}$$

It is important to point out that the local stability properties of the steady state will not depend on the degree of homegeneity α of the utility function. This means in particular that our results will also cover the case of an additively separable function, i.e. $\alpha = -\rho$. This result is easily explained by the fact that in an overlapping generations model, utility is ordinal and preferences are described up to a monotone increasing transformation.

Note that the following ratio of derivatives

$$\frac{\partial D(\sigma,\epsilon_f)/\partial\epsilon_f}{\partial T(\sigma,\epsilon_f,\varsigma)/\partial\epsilon_f} \equiv p(\sigma,\varsigma) = \frac{\epsilon_{s,w}}{\epsilon_{l,w}+\varsigma}$$

does not depend on ϵ_f. It follows that $p(\sigma,\varsigma)$ corresponds to the slope of a line Δ in the (Trace,Determinant) plan which gives a linear relationship between T and D. Substituting the expressions of the saving and labor supply elasticities into the trace, the determinant and the slope of Δ gives

$$\begin{aligned}
T(\sigma,\epsilon_f,\varsigma) &= \frac{\varsigma(1+\sigma^{\frac{1}{1+\rho}}) + \epsilon_{l,w}(1+\sigma^{\frac{1}{1+\rho}})[\epsilon_f + (1-\epsilon_f)(1+\sigma^{\frac{1}{1+\rho}})]}{(1-\epsilon_f)\epsilon_{l,w}} \\
D(\sigma,\epsilon_f) &= -\frac{\epsilon_f}{\rho(1-\epsilon_f)}(1+\sigma^{\frac{1}{1+\rho}})[1+(1+\rho)(1+\sigma^{\frac{1}{1+\rho}})] \\
p(\sigma,\varsigma) &= -\frac{\epsilon_{l,w}[1+(1+\rho)(1+\sigma^{\frac{1}{1+\rho}})]}{\rho(\epsilon_{l,w}+\varsigma)}
\end{aligned}$$

Note that $D(\sigma,0)=0$, $T(\sigma,0,\varsigma) = (1+\sigma^{\frac{1}{1+\rho}})[\varsigma+\epsilon_{l,w}(1+\sigma^{\frac{1}{1+\rho}})]/\epsilon_{l,w}$. Moreover $\lim_{\epsilon_f\to 1} D(\sigma,\epsilon_f) = \pm\infty$ if and only if $\rho \lesseqgtr 0$ and $\lim_{\epsilon_f\to 1} T(\sigma,\epsilon_f,\varsigma) = \pm\infty$ depending on the sign of ρ and the value of ς. Therefore varying the share of capital in total income ϵ_f from 0 to 1 describes a half-line Δ that starts on the axis $D=0$ and whose limit also depends on the sign of ρ and the value of ς. The half-line Δ is defined by the equation

$$\begin{aligned}
&D = p(\sigma,\varsigma)T - p(\sigma,\varsigma)T(\sigma,0,\varsigma) \\
&\Leftrightarrow \\
&D = [1+(1+\rho)(1+\sigma^{\frac{1}{1+\rho}})]\left\{-\frac{\epsilon_{l,w}}{\rho(\epsilon_{l,w}+\varsigma)}T + (1+\sigma^{\frac{1}{1+\rho}})\frac{\varsigma+\epsilon_{l,w}(1+\sigma^{\frac{1}{1+\rho}})}{\rho(\epsilon_{l,w}+\varsigma)}\right\}
\end{aligned}$$

We will divide the discussion into two subcases depending on the sign of the parameter ρ, i.e. on the value of the intertemporal elasticity of substitution η.[16] Since the sign of the elasticity of the labor supply with respect to the

[16]When $\rho = 0$, i.e. $\eta = 1$, the utility function is Cobb-Douglas and it is easy to show

interest factor $\epsilon_{l,R}$ is determined by the sign of ρ, this distinction will echoe the one considered in the general analysis of Section 4.

5.1 The case $\eta > 1$.

When $\rho \in (-1, 0)$, the saving is increasing with respect to the interest factor and $\epsilon_{l,w}, \epsilon_{l,R}, \epsilon_{s,R} > 0$. We also have the following results:

- $p(\sigma, \varsigma) > 0$ for any $(\sigma, \varsigma) \in \mathbb{R}_+^2$ with $p(\sigma, 0) > 1$, and $\lim_{\varsigma \to +\infty} p(\sigma, \varsigma) = 0$.

- $T(\sigma, 0, 0) = (1 + \sigma^{\frac{1}{1+\rho}})^2$, $lim_{\varsigma \to +\infty} T(\sigma, 0, \varsigma) = +\infty = lim_{\epsilon_f \to 1} T(\sigma, \epsilon_f, \varsigma)$ for any $(\sigma, \varsigma) \in \mathbb{R}_+^2$.

- $D(\sigma, 0) = 0$ and $lim_{\epsilon_f \to 1} D(\sigma, \epsilon_f) + \infty$ for any $\sigma \geq 0$.

Note that $T(0, 0, 0) = 1$. In the (T, D) plan, since the slope is always positive, a necessary condition for the existence of local indeterminacy of the steady state is $T(\sigma, 0, 0) < 2$, i.e. $\sigma < \bar{\sigma}(\rho) \equiv (\sqrt{2} - 1)^{1+\rho} < 1$, with $\lim_{\rho \to -1} \bar{\sigma}(\rho) = 1$. We may however find a more accurate upper bound for σ. Since the slope is maximal when $\varsigma = 0$, the upper bound is defined as the solution of the following system:

$$\begin{array}{rcl} T(\sigma, \epsilon_f, 0) & = & 2 \\ D(\sigma, \epsilon_f) & = & 1 \end{array}$$

From the second equation we derive

$$\epsilon_f^H = \frac{-\rho}{(1 + \sigma^{\frac{1}{1+\rho}})[1 + (1+\rho)(1 + \sigma^{\frac{1}{1+\rho}})] - \rho} \quad (17)$$

After substitution into the first equation and denoting $x = 1 + \sigma^{\frac{1}{1+\rho}}$, we find that the upper bound for σ is obtained from the following degree 3 polynomial

$$h(x) = (1 + \rho)x^3 + x^2 - 2(1 + \rho)x - 2 - \rho = 0 \quad (18)$$

Since we have $h(0) < 0$ and $h(\sqrt{2}) > 0$, we have proved that there exists $\sigma^*(\rho) \in (0, \bar{\sigma}(\rho))$ which satisfies $h(\sigma^*(\rho)) = 0$ and such that the steady state cannot be locally indeterminate as soon as $\sigma \geq \sigma^*(\rho)$.

---

that the labor supply is constant for any wage rate and interest factor. The dynamical system becomes one-dimensional and the stability analysis is the same as in the standard Diamond model.

Considering the value ϵ_f^H such that $D(\sigma, \epsilon_f^H) = 1$ defined above, we may now find for any given $\rho \in (-1, 0)$ and $\sigma \in [0, \sigma^*(\rho))$, the value of ς, denoted $\bar{\varsigma}$, such that $T(\sigma, \epsilon_f^H, \varsigma) = 2$, i.e such that the steady state cannot be locally indeterminate for any $\varsigma \geq \bar{\varsigma}$. Some simple computations give[17]

$$\bar{\varsigma} = \epsilon_{l,w} \frac{[1 + (1+\rho)(1 + \sigma^{\frac{1}{1+\rho}})][2 - (1 + \sigma^{\frac{1}{1+\rho}})^2] + \rho}{(1 + \sigma^{\frac{1}{1+\rho}})[1 + (1+\rho)(1 + \sigma^{\frac{1}{1+\rho}})] - \rho} > 0 \qquad (19)$$

For any given $\sigma \in [0, \sigma^*(\rho))$ we may finally find the value of ς, denoted ς_1, such that $p(\sigma, \varsigma) = 1$. We easily obtain

$$\varsigma_1 = 1$$

Since $\bar{\varsigma}$ is close to zero when σ is close to $\sigma^*(\rho)$, and $\bar{\varsigma} = -\rho/2 < 1$ when $\sigma = 0$, we thus conclude that for any $\sigma \in [0, \sigma^*(\rho))$, $\bar{\varsigma} < \varsigma_1 = 1$. We may summarize all the results by the following figures before stating a proposition.

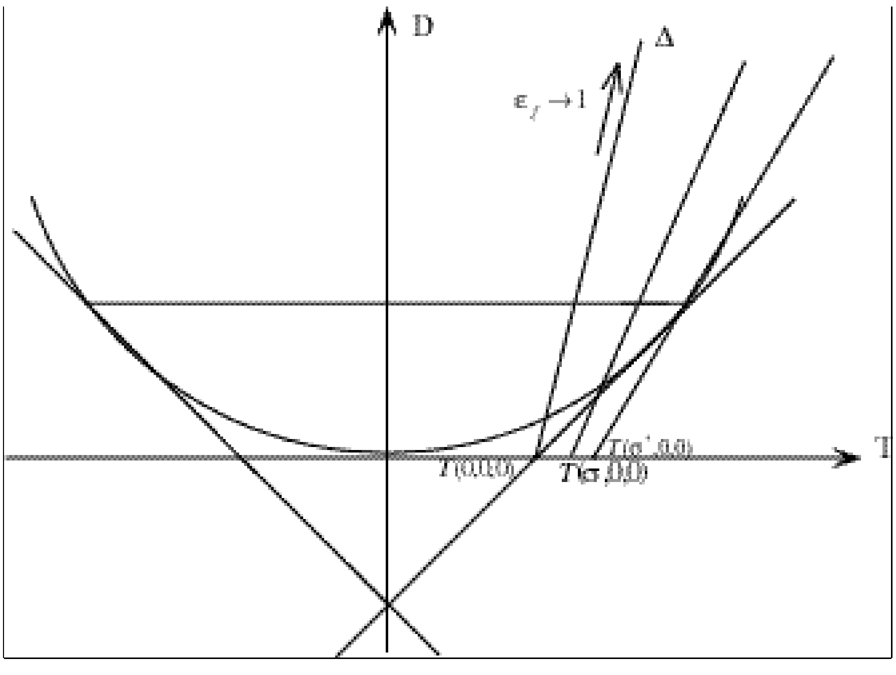

Figure 2: Existence of the upper bound σ^*.

[17] The numerator of $\bar{\varsigma}$ is equal to the polynomial $h(1 + \sigma^{\frac{1}{1+\rho}})$ so that $\bar{\varsigma} = 0$ when $\sigma = \sigma^*(\rho)$.

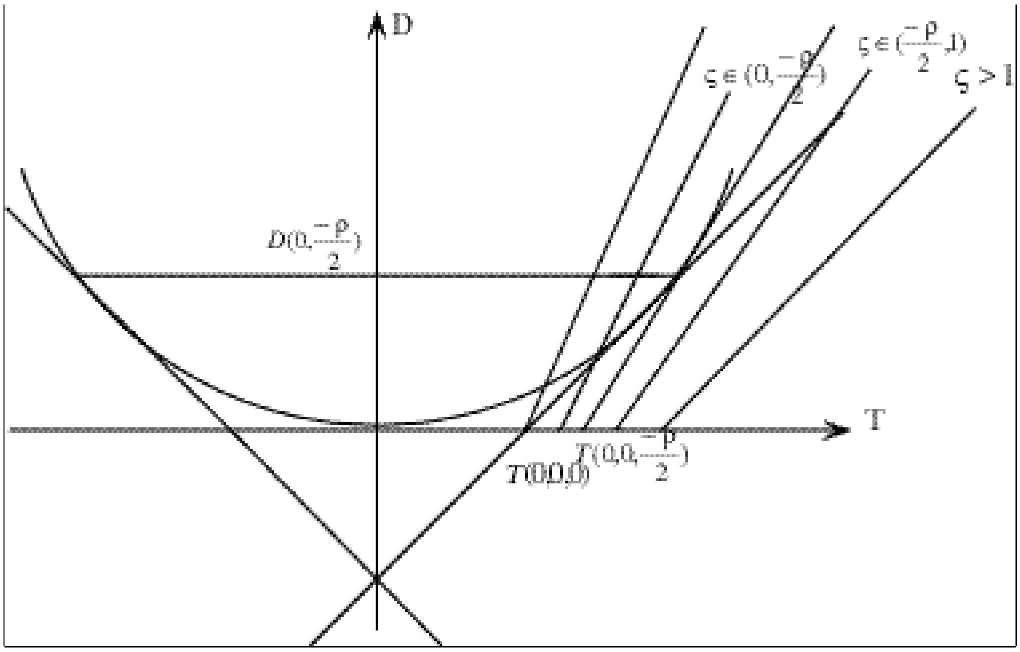

Figure 3: Local indeterminacy and bifurcations when $\sigma = 0$.

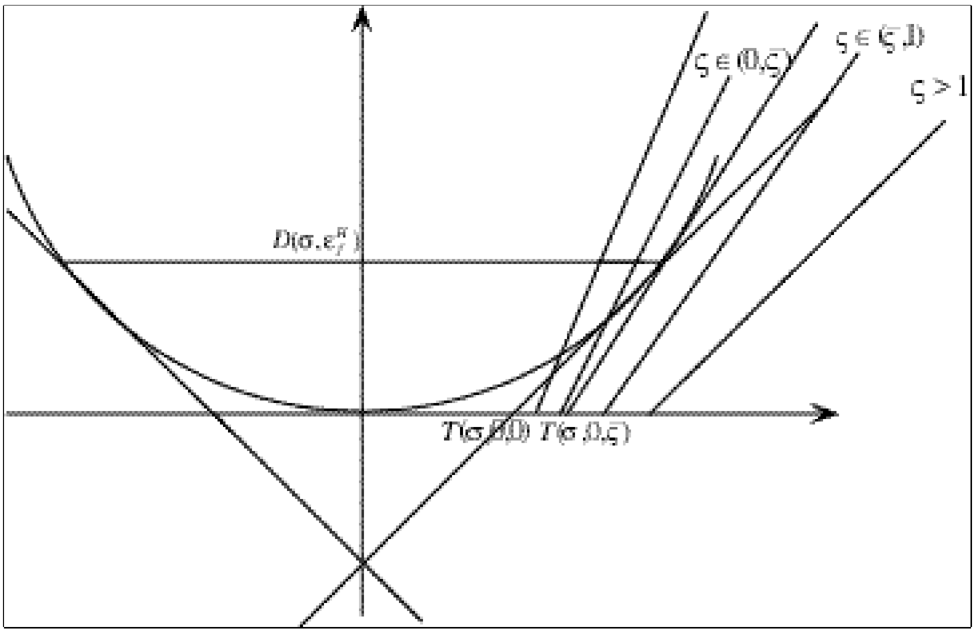

Figure 4: Local indeterminacy and bifurcations when $\sigma \in (0, \sigma^*)$.

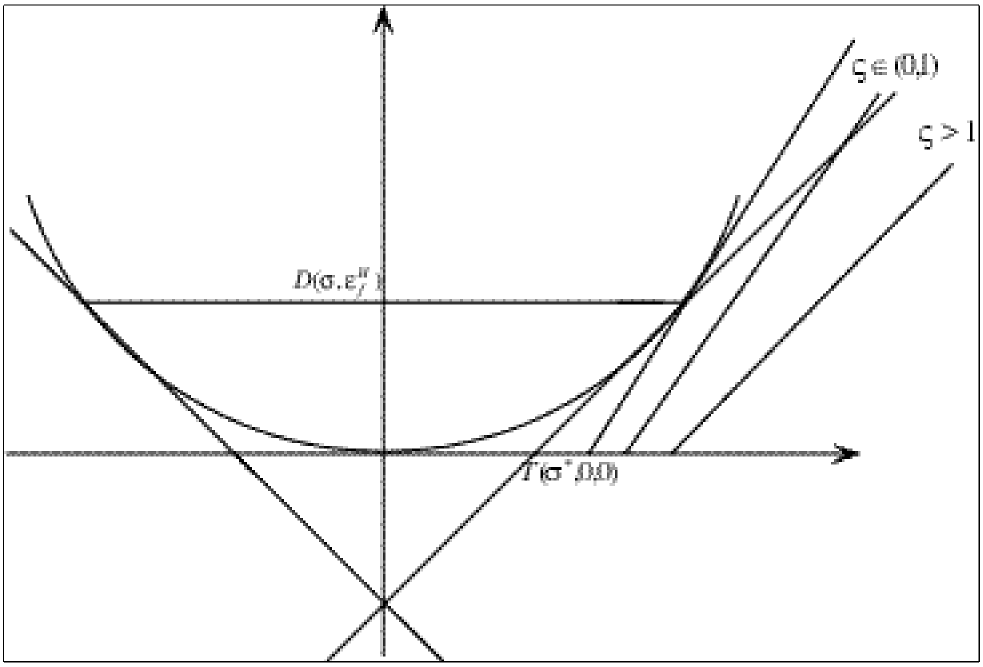

Figure 5: Saddle-point stability and bifurcations when $\sigma \geq \sigma^*$.

Proposition 10 . *Consider the CES economy and the values of* ϵ_f^H *and* $\bar{\varsigma}$ *respectively defined by equations (17) and (19). Assume that* $\eta > 1$*, i.e.* $\rho \in (-1, 0)$*. The following cases hold:*

(1) *Let* $\sigma = 0$*.*

i) If $\varsigma = 0$*, the steady state is locally indeterminate for* $\epsilon_f \in (0, -\rho/2)$ *and locally unstable for* $\epsilon_f > -\rho/2$*. A Hopf bifurcation occurs at* $\epsilon_f^H = -\rho/2$*.*

ii) If $\varsigma \in (0, -\rho/2)$*, there exists* $\epsilon_f^T \in (0, 1)$ *such that the steady state is saddle-point stable when* $\epsilon_f \in (0, \epsilon_f^T)$*, locally indeterminate when* $\epsilon_f \in (\epsilon_f^T, -\rho/2)$ *and locally unstable when* $\epsilon_f > -\rho/2$*. A (reverse) transcritical bifurcation occurs at* ϵ_f^T*, a Hopf bifurcation at* $\epsilon_f^H = -\rho/2$*.*

iii) If $\varsigma = -\rho/2$*, the steady state is saddle-point stable when* $\epsilon_f \in (0, -\rho/2)$ *and locally unstable when* $\epsilon_f > -\rho/2$*. A transcritical-Hopf bifurcation occurs at* $\epsilon_f^H = -\rho/2$*.*

iv) If $\varsigma \in (\bar{\varsigma}, 1)$*, there exists* $\epsilon_f^T \in (0, 1)$ *such that the steady state is saddle-point stable for* $\epsilon_f \in (0, \epsilon_f^T)$ *and locally unstable for* $\epsilon_f > \epsilon_f^T$*. A* *transcritical bifurcation occurs at* ϵ_f^T.

v) If $\varsigma \geq 1$, *the steady state is saddle-point stable for all* $\epsilon_f \in (0, 1)$.

(2) *Let* $\sigma \in (0, \sigma^*(\rho))$ *with* $\sigma^*(\rho)$ *defined by equation (18).*

i) If $\varsigma \in [0, \bar{\varsigma})$, *there exists* $\epsilon_f^T \in (0, 1)$ *such that the steady state is saddle-point stable when* $\epsilon_f \in (0, \epsilon_f^T)$, *locally indeterminate when* $\epsilon_f \in (\epsilon_f^T, \epsilon_f^H)$ *and locally unstable when* $\epsilon_f > \epsilon_f^H$. *A (reverse) transcritical bifurcation occurs at* ϵ_f^T, *a Hopf bifurcation at* ϵ_f^H.

ii) If $\varsigma = \bar{\varsigma}$, *the steady state is saddle-point stable for* $\epsilon_f \in (0, \epsilon_f^H)$ *and locally unstable for* $\epsilon_f > \epsilon_f^H$. *A transcritical-Hopf bifurcation occurs at* ϵ_f^H.

iii) If $\varsigma \in (\bar{\varsigma}, 1)$, *there exists* $\epsilon_f^T \in (0, 1)$ *such that the steady state is saddle-point stable when* $\epsilon_f \in (0, \epsilon_f^T)$ *and locally unstable for* $\epsilon_f > \epsilon_f^T$. *A transcritical bifurcation occurs at* ϵ_f^T.

iv) If $\varsigma \geq 1$, *the steady state is saddle-point stable for all* $\epsilon_f \in (0, 1)$.

(3) *Let* $\sigma \geq \sigma^*(\rho)$.[18]

i) If $\varsigma \in [0, 1)$, *there exists* $\epsilon_f^T \in (0, 1)$ *such that the steady state is saddle-point stable when* $\epsilon_f \in (0, \epsilon_f^T)$ *and locally unstable for* $\epsilon_f > \epsilon_f^T$. *A transcritical bifurcation occurs at* ϵ_f^T.

ii) If $\varsigma \geq 1$, *the steady state is saddle-point stable for all* $\epsilon_f \in (0, 1)$.

This Proposition shows in particular how the Cobb-Douglas formulation for the technology is a specific case since local indeterminacy and cycles cannot occur when $\varsigma = 1$. Note also that in cases (1)-iii) and (2)-ii), a co-dimension 2 Bogdanov-Takens bifurcation occurs.[19]

It is well-known from the earlier contribution by Reichlin [22] that under the assumption of additive separability for the utility function, a Hopf bifurcation is only possible if the inputs elasticity of substitution is lower than the share of capital in total income. This result, initially obtained in a

[18] Note that if $\sigma = \sigma^*(\rho)$ and $\varsigma = 0$, then a transcritical-Hopf bifurcation occurs at ϵ_f^H as in cases (1)-iii) and (2)-ii). However this parameter configuration is too specific to be mentioned in Proposition 10.

[19] See Kuznetsov [16].

model without consumption during the agent's first period of life, has been extended to models with consumptions in both periods and still additively separable utility functions by Lloyd-Braga [17] and Cazzavillan and Pintus [6].[20] In our CES economy, we find a similar result since it is easy to show that as soon as $\sigma \in (0, \sigma^*(\rho))$, the critical value for the inputs elasticity of substitution above which local indeterminacy and Hopf bifurcation no longer exist is less than the Hopf biurcation value for the share of capital in total income, i.e. $\epsilon_f^H > \bar{\varsigma}$. Note however that when $\sigma = 0$ the Bogdanov-Takens bifurcation appears for $\epsilon_f^H = \bar{\varsigma} = -\rho/2$.

Cazzavillan and Pintus [6] also provide some interesting conclusions concerning the occurrence of local indeterminacy and endogenous fluctuations. They first prove that the range of values of the elasticity of capital-labor substitution consistent with multiple equilibrium paths and cycles is narrowed when the model is extended to include first period consumption. They also prove that when preferences over consumption are identical, equilibria are always locally unique.

Proposition 10 shows that their first conclusion is even more drastic in a CES economy. Local indeterminacy and Hopf bifurcation do not occur anymore as soon as the weight σ of first period consumption in the utility function is higher than $\sigma^*(\rho)$ with $\sigma^*(\rho) < (\sqrt{2} - 1)^{1+\rho} < 1$. Proposition 10 seems also to prove that their second conclusion actually heavily rely on the "weight" of the first period consumption which they assume to be equal to 1. Denoting our utility function as

$$u(c, \mathcal{L}, d) = (B\alpha) \left[\sigma u_1(c) + \delta u_2(\mathcal{L}) + \beta u_3(d)\right]^{-\alpha/\rho}$$

and assuming additive separability, i.e. $\alpha = -\rho$, we have $u_i(x) = x^{-\rho}$ for all $i = 1, 2, 3$ and local indeterminacy may still occur as soon as $\sigma \in [0, \sigma^*(\rho))$.

[20] Lloyd-Braga [17] however proves that a Hopf bifurcation will occur with some inputs elasticity of substitution higher than the capital share of output when the model is amended to include imperfect competition and increasing retruns to scale internal to the firm. The same result is proved to hold also when increasing returns to scale are generated by external effects in production (See Cazzavillan and Pintus [6]).

When dealing with the empirical plausibility of our results, we may calibrate the model so that the Hopf bifurcation value ϵ_f^H for the share of capital in total output is equal to 1/3. Let also $\rho = -0.74$ so that the intertemporal elasticity of the utility function is $\eta \approx 3.85$. It follows that the corresponding upper bound for σ is $\sigma^*(-0.74) \approx 0.66$ and the value of σ derived from the solving of $\epsilon_f^H = 1/3$ is $\sigma \approx 0.6$. The upper bound for the factors elasticity of substitution above which local indeterminacy is precluded is finally $\bar{\varsigma} \approx 0.24$. It follows that for any $\varsigma \in [0, 0.24)$ the steady state is locally indeterminate when the share of capital ϵ_f is in a left neighbourhood of 1/3 and, if the Hopf bifurcation is supercritical, an indeterminate quasi-periodic cycles exists when ϵ_f is in a right neighbourhood of 1/3.

5.2 The case $\eta < 1$.

When $\rho > 0$, the saving is decreasing with respect to the interest factor and $\epsilon_{l,w}, \epsilon_{l,R}, \epsilon_{s,R} < 0$. We have the following results:

- For any given $\sigma \geq 0$, there exists $\varsigma_\infty = -\epsilon_{l,w} = \rho/[(1+\rho)(2+\sigma^{1/(1+\rho)})] < 1$ such that $p(\sigma, \varsigma_\infty) = \infty$. Therefore $p(\sigma, \varsigma) < 0$ when $\varsigma \in [0, \varsigma_\infty)$ and $p(\sigma, \varsigma) > 0$ when $\varsigma > \varsigma_\infty$. Moreover $p(\sigma, 0) < -1$ and $\lim_{\varsigma \to +\infty} p(\sigma, \varsigma) = 0$ for any $\sigma \geq 0$.

- $T(\sigma, 0, 0) = (1 + \sigma^{\frac{1}{1+\rho}})^2$, $T(\sigma, 0, \varsigma_\infty) = \sigma^{\frac{1}{1+\rho}}(1 + \sigma^{\frac{1}{1+\rho}})$,[21] $lim_{\varsigma \to +\infty} T(\sigma, 0, \varsigma) = -\infty$ and $lim_{\sigma \to +\infty} T(\sigma, 0, \varsigma) = +\infty$. Moreover, $lim_{\epsilon_f \to 1} T(\sigma, \epsilon_f, \varsigma) = +\infty$ if $\varsigma \in [0, \varsigma_\infty)$ and $lim_{\epsilon_f \to 1} T(\sigma, \epsilon_f, \varsigma) = -\infty$ if $\varsigma > \varsigma_\infty$.

- $D(\sigma, 0) = 0$ and $lim_{\epsilon_f \to 1} D(\sigma, \epsilon_f) - \infty$ for any $\sigma \geq 0$.

For any given $\sigma \geq 0$, we may find the value of ς, denoted $\hat{\varsigma}$, such that the line Λ goes through the point $(T, D) = (0, -1)$. We need to solve the following system:

$$\begin{aligned} T(\sigma, \epsilon_f, 0) &= 0 \\ D(\sigma, \epsilon_f) &= -1 \end{aligned}$$

[21] Note that $T(0, 0, \varsigma_\infty) = 0$.

From the second equation we get

$$\epsilon_f^{FT} = \frac{\rho}{(1+\sigma^{\frac{1}{1+\rho}})[1+(1+\rho)(1+\sigma^{\frac{1}{1+\rho}})]+\rho} \tag{20}$$

After substitution into the first equation we find

$$\hat{\varsigma} = -\epsilon_{l,w} \frac{(1+\sigma^{\frac{1}{1+\rho}})^2[1+(1+\rho)(1+\sigma^{\frac{1}{1+\rho}})]+\rho}{(1+\sigma^{\frac{1}{1+\rho}})[1+(1+\rho)(1+\sigma^{\frac{1}{1+\rho}})]+\rho} > 0 \tag{21}$$

We may also compute the value of ς, denoted ς_T, such that $T(\sigma, 0, \varsigma) = 1$. We derive:

$$\varsigma_T = \frac{\rho\left[(1+\sigma^{\frac{1}{1+\rho}})^2 - 1\right]}{(1+\rho)\left[(1+\sigma^{\frac{1}{1+\rho}})(2+\sigma^{\frac{1}{1+\rho}})\right]} < 1 \tag{22}$$

It is easy to show that $\varsigma_T < \hat{\varsigma}$ and $\varsigma_\infty < \hat{\varsigma}$, and the following geometrical configuration holds. Local indeterminacy of the steady state $\bar{k} = 1$ cannot therefore occur after a transcritical bifurcation.

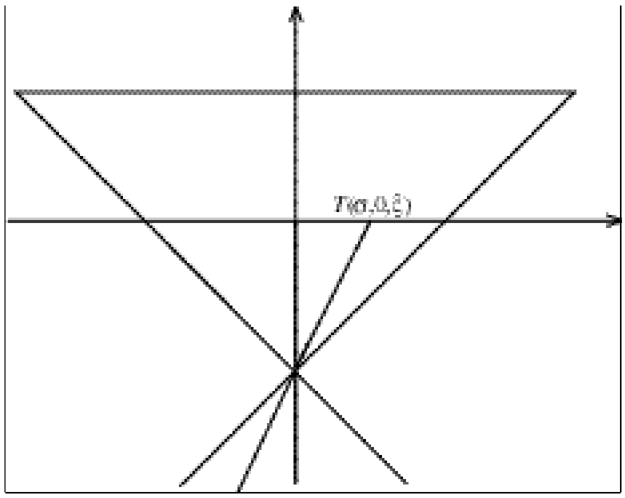

Figure 6: A co-dimension 2 bifurcation when $\sigma > 0$.

We need finally to compute the value of ς, denoted ς_F, such that $T(\sigma, 0, \varsigma) = -1$. We easily obtain:

$$\varsigma_F = \frac{\rho\left[1+(1+\sigma^{\frac{1}{1+\rho}})^2\right]}{(1+\rho)\left[1+\sigma^{\frac{1}{1+\rho}}+(1+\sigma^{\frac{1}{1+\rho}})^2\right]} < 1 \tag{23}$$

We may summarize all the results by the following figures before stating a proposition.

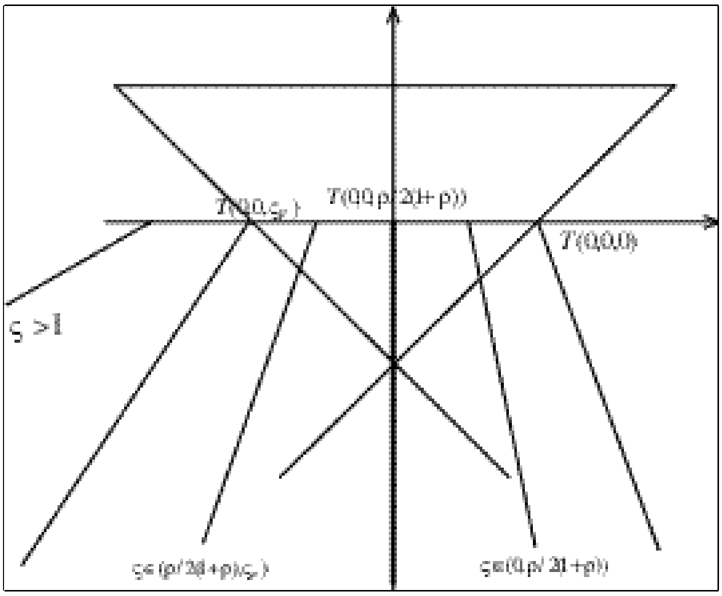

Figure 7: Local indeterminacy and bifurcations when $\sigma = 0$.

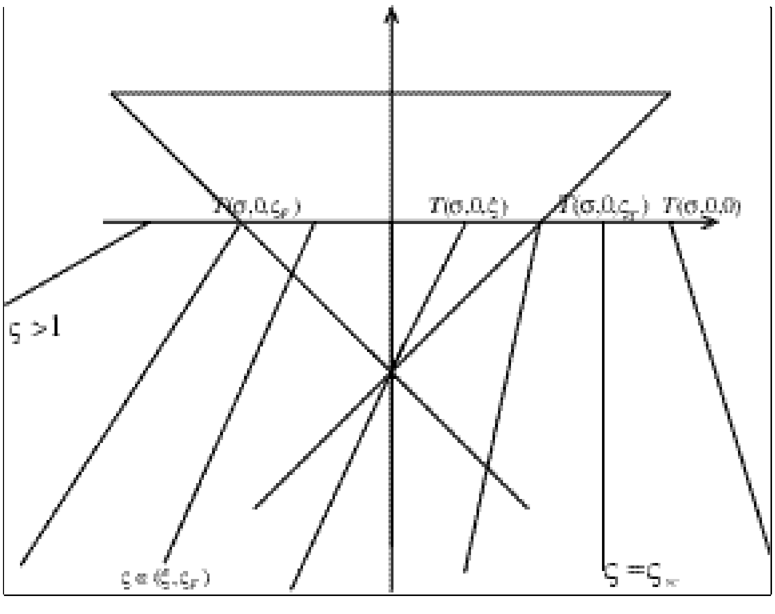

Figure 8: Local indeterminacy and bifurcations when $\sigma > 0$ and $\varsigma_\infty < \varsigma_T$.

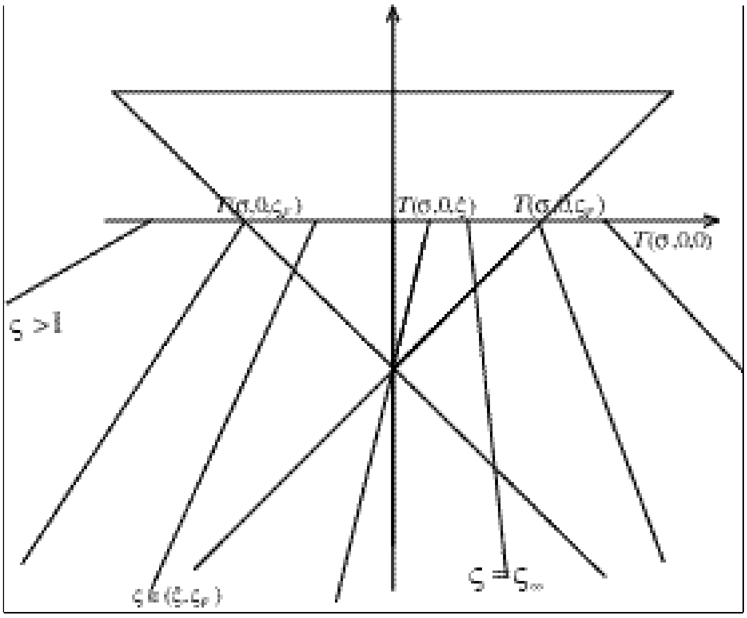

Figure 9: Local indeterminacy and bifurcations when $\sigma = 0$ and $\varsigma_\infty > \varsigma_T$.

Proposition 11 . *Consider the CES economy and the values of ϵ_f^{FT}, $\hat{\varsigma}$, ς_T and ς_F respectively defined by equations (20), (21), (23) and (22). Assume that $\eta < 1$, i.e. $\rho > 0$. The following cases hold:*

(1) *Let $\sigma = 0$.*

i) If $\varsigma = 0$, there exists $\epsilon_f^T \in (0,1)$ such that the steady state is saddle-point stable when $\epsilon_f \in (0, \epsilon_f^T)$ and locally unstable when $\epsilon_f > \epsilon_f^T$. A transcritical bifurcation occurs at ϵ_f^T.

ii) If $\varsigma \in (0, \rho/[2(1+\rho)])$, there exist $\epsilon_f^T \in (0,1)$ and $\epsilon_f^F \in (0,1)$ such that the steady state is locally indeterminate when $\epsilon_f \in (0, \epsilon_f^T)$, saddle-point stable when $\epsilon_f \in (\epsilon_f^T, \epsilon_f^F)$ and locally unstable when $\epsilon_f > \epsilon_f^F$. A transcritical bifurcation occurs at ϵ_f^T, a Flip at ϵ_f^F.

iii) If $\varsigma = \rho/[2(1+\rho)]$, the steady state is locally indeterminate for $\epsilon_f \in (0, \epsilon_f^{FT})$ and locally unstable for $\epsilon_f > \epsilon_f^{FT}$. A transcritical-Flip bifurcation occurs at ϵ_f^{FT}.

iv) If $\varsigma \in (\rho/[2(1+\rho)], \varsigma_F)$, there exist $\epsilon_f^F \in (0,1)$ and $\epsilon_f^T \in (0,1)$ such that the steady state is locally indeterminate when $\epsilon_f \in (0, \epsilon_f^F)$, saddle-point *stable when* $\epsilon_f \in (\epsilon_f^F, \epsilon_f^T)$ *and locally unstable when* $\epsilon_f > \epsilon_f^T$*. A Flip bifurcation occurs at* ϵ_f^F*, a transcritical at* ϵ_f^T*.*

v) If $\varsigma \in (\varsigma_F, 1)$*, there exists* $\epsilon_f^T \in (0,1)$ *such that the steady state is saddle-point stable when* $\epsilon_f \in (0, \epsilon_f^T)$ *and locally unstable when* $\epsilon_f > \epsilon_f^T$*. A transcritical bifurcation occurs at* ϵ_f^T*.*

vi) If $\varsigma \geq 1$*, the steady state is saddle-point stable for any* $\epsilon_f \in (0,1)$*.*

(2) *Let* $\sigma > 0$*.*

i) If $\varsigma \in [0, \varsigma_T]$*, there exists* $\epsilon_f^F \in (0,1)$ *such that the steady state is saddle-point stable when* $\epsilon_f \in (0, \epsilon_f^F)$ *and locally unstable when* $\epsilon_f > \epsilon_f^F$*. A Flip bifurcation occurs at* ϵ_f^F*.*

ii) If $\varsigma \in (\varsigma_T, \hat{\varsigma})$*, there exist* $\epsilon_f^T \in (0,1)$ *and* $\epsilon_f^F \in (0,1)$ *such that the steady state is locally indeterminate when* $\epsilon_f \in (0, \epsilon_f^T)$*, saddle-point stable when* $\epsilon_f \in (\epsilon_f^T, \epsilon_f^F)$ *and locally unstable when* $\epsilon_f > \epsilon_f^F$*. A transcritical bifurcation occurs at* ϵ_f^T*, a Flip at* ϵ_f^F*.*

iii) If $\varsigma = \hat{\varsigma}$*, the steady state is locally indeterminate for* $\epsilon_f \in (0, \epsilon_f^{FT})$ *and locally unstable for* $\epsilon_f > \epsilon_f^{FT}$*. A transcritical-Flip bifurcation occurs at* ϵ_f^{FT}*.*

iv) If $\varsigma \in (\hat{\varsigma}, \varsigma_F)$*, there exist* $\epsilon_f^F \in (0,1)$ *and* $\epsilon_f^T \in (0,1)$ *such that the steady state is locally indeterminate when* $\epsilon_f \in (0, \epsilon_f^F)$*, saddle-point stable when* $\epsilon_f \in (\epsilon_f^F, \epsilon_f^T)$ *and locally unstable when* $\epsilon_f > \epsilon_f^T$*. A Flip bifurcation occurs at* ϵ_f^F*, a transcritical at* ϵ_f^T*.*

v) If $\varsigma \in [\varsigma_F, 1)$*, there exists* $\epsilon_f^T \in (0,1)$ *such that the steady state is saddle-point stable when* $\epsilon_f \in (0, \epsilon_f^T)$ *and locally unstable when* $\epsilon_f > \epsilon_f^T$*. A trnascritical bifurcation occurs at* ϵ_f^T*.*

vi) If $\varsigma \geq 1$*, the steady state is saddle-point stable for any* $\epsilon_f \in (0,1)$*.*

This Proposition confirms that the Cobb-Douglas formulation for the technology is a singular case since local indeterminacy and cycles cannot occur when $\varsigma = 1$. We have thus proved the following Corollary:

Corollary 3 . *In a CES economy, as soon as the inputs elasticity of substitution is greater than or equal to 1, the steady state is saddle-point stable.*

Note also that in cases (1)-iii) and (2)-iii) a co-dimension 2 bifurcation, which has not yet been treated in the mathematical literature, occurs.[22]

6 Concluding comments

We have studied an overlapping generations model with production and endogenous labor supply. We are able to fully characterize the local dynamics generated by the model. We derive these results in the context of fairly broad class of overlapping generations economies because we do not consider any separability of the utility function. We also show that, when the production function elasticity increases, and we consider a normalisation such that the steady state remains unchanged, then some bifurcations are liable to occur.

7 Appendix

Proof of Proposition 1:
A steady state is a solution of $(1+n)k = s(w(k), R(k))/l(w(k), R(k))$. As agents cannot save more than their labor income $w(k)$, for all $k > 0$ we have $0 < s(w(k), R(k))/l(w(k), R(k)) \leq w(k)$. If $w(0) = 0$ then $\lim_{k\to 0} s(w(k), R(k))/l(w(k), R(k)) = 0$ and $\hat{k} = 0$ is a steady state. Now it is proved in de la Croix and Michel [7] that $w(0) = f(0)$. The result therefore follows from Assumption 3.

$\square$

Proof of Proposition 2:
A steady state $\bar{k}$ satisfies $\phi(\bar{k}) = 0$. As agents cannot save more than their labor income $w(k)$, for all $k > 0$ we have $\phi(k) \geq (1+n)k - w(k)$. Moreover, $\lim_{k\to+\infty} (1+n)k - w(k) = \lim_{k\to+\infty} k\left(1 + n - \frac{w(k)}{k}\right) = +\infty$ since $lim_{k\to+\infty} \frac{f(k)-kf'(k)}{k} = 0$.[23] It follows that $\lim_{k\to+\infty} \phi(k) = +\infty$.

[22] See Kuznetsov [16].

[23] Under assumption 3, this property is proved in de la Croix and Michel [7].

i) As $\phi(k)$ is a $\mathbf{C}^1$ function for $k > 0$, then if $\lim_{k\to 0} \phi'(k) < 0$, there exists $\bar{k}$ such that $\phi(\bar{k}) = 0$. If there exist $\bar{k}_1, \cdots, \bar{k}_n$ capital-labor ratio such that $\phi(\bar{k}_i) = 0$, $i = 1, \cdots, n$, then n is an even number only if there is a value $\bar{k}_j$, $j \in \{1, n\}$, such that $\phi(\bar{k}_j) = 0$ and $\phi'(\bar{k}_j) = 0$. This situation is not robust to any small change in the parameters, and we will thus refer to it as "non generic".

ii) If now $\lim_{k\to 0} \phi'(k) > 0$, then, using the previous argument, the number n of steady states is generically even and may be 0.

□

Proof of Proposition 3:

Under Assumptions 1-3 and $\epsilon_{l,R} > 0$, we have $\lim_{\lambda\to\pm\infty} \mathcal{P}(\lambda) = +\infty$ and $\mathcal{P}(0) > 0$. Denoting $\Lambda_L = \Lambda_D + \epsilon_f \epsilon_{l,w} - (1 - \epsilon_f)\epsilon_{l,R}$ with Λ_D given by equation (13), we have:

$$
\begin{aligned}
\mathcal{P}(\lambda) &= \lambda^2 - \lambda \frac{\Lambda_L + \epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R}}{(1-\epsilon_f)\epsilon_{l,R}} + \frac{\epsilon_f \epsilon_{s,w}}{(1-\epsilon_f)} \\
\mathcal{P}(1) &= -\frac{\Lambda_L}{(1-\epsilon_f)\epsilon_{l,R}} \\
\mathcal{P}(-1) &= \frac{\Lambda_L + 2[\epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R}]}{(1-\epsilon_f)\epsilon_{l,R}}
\end{aligned}
$$

Assume first that $\Lambda_L + \epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R} > 0$. Then $\mathcal{P}(\lambda) > 0$ for any $\lambda \leq 0$. The condition $P(1) < 0$, i.e. $\Lambda_L > 0$, is thus a necessary and sufficient condition for the steady state to be saddle-point stable. As soon as $P(1) > 0$, i.e. $\Lambda_L < 0$, the eigenvalues are real or complex. In this case, the steady state is locally indeterminate if and only if the product of eigenvalues, i.e. $P(0) = \epsilon_f \epsilon_{s,w}/[(1-\epsilon_f)\epsilon_{l,R}]$ is less than one, and locally unstable if and only if the $P(0)$ is greater than one.

Assume now that $\Lambda_L + \epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R} < 0$. Then $\mathcal{P}(\lambda) > 0$ for any $\lambda \geq 0$. The condition $P(-1) < 0$, i.e. $\Lambda_L + 2[\epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R}] < 0$, is thus a necessary and sufficient condition for the steady state to be saddle-point stable. As soon as $P(-1) > 0$, i.e. $\Lambda_L + 2[\epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R}] > 0$, the eigenvalues are real or complex. The local indeterminacy or local unstability of the steady state depends on whether $P(0) \lesseqgtr 1$.

□

Proof of Proposition 4.
Let $\epsilon_{l,R} > 0$ and assume that the steady states are ordered as $k_1 > k_2 > ... > k_n$. Since $\lim_{k \to +\infty} \phi(k) = +\infty$, we generically have $\phi'(k_1) > 0$. Then, generically, the steady states $\bar{k}_1, \bar{k}_3, \bar{k}_5, \dots$ are such that $\phi'(\bar{k}) > 0$, whereas the steady states $\bar{k}_2, \bar{k}_4, \bar{k}_6, \dots$ are such that $\phi'(\bar{k}) < 0$. Straightforward computations give $\phi'(k) = \Lambda_L$. Since $\Lambda_L > 0$ implies $\Lambda_L + \epsilon_f \epsilon_{s,w} + (1-\epsilon_f)\epsilon_{l,R} > 0$, Proposition 3 shows that the steady states $\bar{k}_1, \bar{k}_3, \bar{k}_5, \dots$ are saddle-point stable. On the contrary the steady states $\bar{k}_2, \bar{k}_4, \bar{k}_6, \dots$ are necessarily locally indeterminate or locally unstable. Finally, the trivial steady state $\hat{k} = 0$ will have an odd index when $\lim_{k \to 0} \phi'(k) > 0$ and the closest non-trivial steady state will have an even index. $\hat{k} = 0$ is therefore necessarily saddle-point stable. Similarly $\hat{k} = 0$ has an even index when $\lim_{k \to 0} \phi'(k) < 0$ and the closest non-trivial steady state has an odd index. $\hat{k} = 0$ is therefore necessarily locally indeterminate or locally unstable.

□

Proof of Proposition 5:
The notations used in the Proposition are $\mathcal{N}(\epsilon_f)$ for the numerator of the trace $T(\epsilon_f)$, $\mathcal{P}_1(\epsilon_f)$ for the numerator of $\mathcal{P}(1)$ and $\mathcal{P}_{-1}(\epsilon_f)$ for the numerator of $\mathcal{P}(-1)$. Since $\epsilon_{l,R} > 0$, cases i) and ii) consider two positive eigenvalues with $\mathcal{P}(\lambda) > 0$ for any $\lambda \leq 0$, while cases iii) and iv) consider negative eigenvalues with $\mathcal{P}(\lambda) > 0$ for any $\lambda \geq 0$.

i) If $\lim_{\epsilon_f \to 0} \mathcal{P}_1(\epsilon_f) = \epsilon_{l,R} - (\varsigma + \epsilon_{s,R}) < 0$ and $\lim_{\epsilon_f \to 1} \mathcal{P}_1(\epsilon_f) = \epsilon_{s,w} - (\varsigma + \epsilon_{l,w}) > 0$, there exists $\epsilon_f^T \in (0,1)$ such that $\mathcal{P}(1) < 0$ for any $\epsilon_f \in (0, \epsilon_f^T)$ and $\mathcal{P}(1) > 0$ when $\epsilon_f > \epsilon_f^T$. The steady state is thus saddle-point stable for $\epsilon_f \in (0, \epsilon_f^T)$, one eigenvalue goes through 1 when ϵ_f crosses ϵ_f^T, and it becomes locally indeterminate when ϵ_f belongs to a tight neighbourhood of ϵ_f^T. Note that by continuity when $\epsilon_f > \epsilon_f^T$ but close enough to ϵ_f^T, $D(\epsilon_f) < 1$.

ii) If $\mathcal{P}_1(\epsilon_f) > 0$ for any $\epsilon_f \in (0,1)$, the eigenvalues are either real or complex with modulus less than 1, or complex with modulus greater than

1. If moreover, $\lim_{\epsilon_f \to 0} D(\epsilon_f) = 0$ and $\lim_{\epsilon_f \to 1} D(\epsilon_f) = +\infty$, there exists a Hopf bifurcation value $\epsilon_f^H \in (0,1)$ such that the steady state is locally indeterminate when $\epsilon_f \in (0, \epsilon_f^H)$ and locally unstable when ϵ_f belongs to a right neighbourhood of ϵ_f^H.

The proof for cases iii) and iv) is similar.

□

Proof of Proposition 6:
Under Assumptions 1-3 and $\epsilon_{l,R} < 0$, we have $\lim_{\lambda \to \pm\infty} \mathcal{P}(\lambda) = +\infty$ and $\mathcal{P}(0) < 0$. Therefore, the steady state will be locally indeterminate if and only if $\mathcal{P}(1)$ and $\mathcal{P}(-1)$ are positive, saddle-point stable if and only if the product $\mathcal{P}(1)\mathcal{P}(1)$ is negative and locally unstable if and only if $\mathcal{P}(1)$ and $\mathcal{P}(-1)$ are negative. Using the notations of Proposition 3 the rest of the proof is obvious.

□

Proof of Proposition 7.
Let $\epsilon_{l,R} > 0$ and assume that the steady states are ordered as $k_1 > k_2 > ... > k_n$. Since $\lim_{k \to +\infty} \phi(k) = +\infty$, we generically have $\phi'(k_1) > 0$. Then, generically, the steady states $\bar{k}_1, \bar{k}_3, \bar{k}_5, \ldots$ are such that $\phi'(\bar{k}) > 0$, whereas the steady states $\bar{k}_2, \bar{k}_4, \bar{k}_6, \ldots$ are such that $\phi'(\bar{k}) < 0$.

i) Assume first that $\Lambda_L + \epsilon_f \epsilon_{s,w} + (1 - \epsilon_f)\epsilon_{l,R} > 0$. This implies that $\mathcal{P}(1) > \mathcal{P}(-1)$. Recalling that $\phi'(k) = \Lambda_L$, and from the notations of Proposition 3, $\Lambda_L < 0$ implies $0 > \mathcal{P}(1) > \mathcal{P}(-1)$ and Proposition 6 shows that the steady states $\bar{k}_2, \bar{k}_4, \bar{k}_6, \ldots$ are locally unstable. On the contrary, if $\Lambda_L > 0$, the steady states $\bar{k}_1, \bar{k}_3, \bar{k}_5, \ldots$ are necessarily locally indeterminate or saddle-point stable. Finally, the trivial steady state $\hat{k} = 0$ will have an even index when $\lim_{k \to 0} \phi'(k) < 0$ and the closest non-trivial steady state will have an odd index. $\hat{k} = 0$ is therefore necessarily locally unstable. Similarly $\hat{k} = 0$ has an even index when $\lim_{k \to 0} \phi'(k) > 0$ and the closest non-trivial steady state has an odd index. $\hat{k} = 0$ is therefore necessarily saddle-point stable or locally indeterminate.

ii) Assume now that $\Lambda_L + \epsilon_f \epsilon_{s,w} + (1 - \epsilon_f)\epsilon_{l,R} < 0$. This implies that $\mathcal{P}(1) < \mathcal{P}(-1)$. Similarly, $\Lambda_L > 0$ implies $0 < \mathcal{P}(1) < \mathcal{P}(-1)$ and Proposition

6 shows that the steady states $\bar{k}_1, \bar{k}_3, \bar{k}_5, \ldots$ are locally indeterminate. Therefore, when $\Lambda_L < 0$, the steady states $\bar{k}_2, \bar{k}_4, \bar{k}_6, \ldots$ are necessarily saddle-point stable or locally unstable. Finally, the trivial steady state $\hat{k} = 0$ will have an odd index when $\lim_{k \to 0} \phi'(k) > 0$ and the closest non-trivial steady state will have an even index. $\hat{k} = 0$ is therefore necessarily locally indeterminate. Similarly $\hat{k} = 0$ has an odd index when $\lim_{k \to 0} \phi'(k) > 0$ and the closest non-trivial steady state has an even index. $\hat{k} = 0$ is therefore necessarily saddle-point stable or locally unstable.

□

Proof of Proposition 8:
Since $\epsilon_{l,R} < 0$, the eigenvalues are real with opposite sign.

i) If $\mathcal{P}_{-1}(\epsilon_f) < 0$ for any $\epsilon_f \in (0, 1)$, one eigenvalue is negative and greater than -1 for any $\epsilon_f \in (0, 1)$. If moreover $\lim_{\epsilon_f \to 0} \mathcal{P}_1(\epsilon_f) = \epsilon_{l,R} - (\varsigma + \epsilon_{s,R}) < 0$ and $\lim_{\epsilon_f \to 1} \mathcal{P}_1(\epsilon_f) = \epsilon_{s,w} - (\varsigma + \epsilon_{l,w}) > 0$, there exists $\epsilon_f^T \in (0, 1)$ such that $\mathcal{P}(1) > 0$ for any $\epsilon_f \in (0, \epsilon_f^T)$, $\mathcal{P}(1) = 0$ when $\epsilon_f = \epsilon_f^T$ and $\mathcal{P}(1) < 0$ when ϵ_f is in a right neighbourhood of ϵ_f^T.

ii) The proof is similar.

□

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