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Indeterminacy in a Cash-in-Advance Two-Sector Economy

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Indeterminacy in a Cash-in-Advance Two-Sector Economy[*]

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Abstract: *This paper aims to fill some theoretical gaps existing in literature between monetary indeterminate economies and multisector models. We carry out such a purpose by considering a two-sector infinite horizon economy with a partial cash-in-advance constraint on consumption expenditures. This formulation allows to consider a steady state velocity of money which is strictly greater than one and thus provides a more plausible framework than the standard formulation in which all the consumption purchases are paid cash. We prove that the steady state is bound to be indeterminate when the amplitude of the liquidity constraint is low enough and that a capital intensive investment good or a strongly capital intensive consumption good improve considerably the scope for indeterminacy. As a consequence, we show that without any restriction on the elasticity of intertemporal substitution in consumption, multiple equilibria may occur if the velocity of money is greater than a critical bound which is compatible with empirial estimates.*

Keywords: *CIA constraint, two-sector models, indeterminacy, sunspot equilibria.*

Journal of Economic Literature Classification Numbers: C61, E32, E41.

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1   Introduction

This paper aims to fill some theoretical gaps existing in literature between monetary indeterminate economies and multisector models. To this end, we consider an infinite horizon economy with representative agent in which consumption and investment goods are produced with two different constant returns to scale technologies. The demand of money is motivated by the introduction of a cash-in-advance (CIA) constraint. Yet, in contradiction with most of the related literature and in the spirit of Grandmont & Younès [20], we assume that only a share included between zero and one of consumption purchases must be financed out of money balances in the hand of the representative agent at the beginning of each period.[1] This assumption is motivated by the fact that the amplitude of the liquidity constraint is actually the inverse of the velocity of money which is known to be strictly greater than one. A fractional CIA constraint provides therefore a more plausible framework than the standard formulation in which all the consumption purchases are paid cash.[2]

It is well known that increasing returns can be an autonomous source for indeterminacy and that the required degrees of returns are much more empirically plausible in multisector models than those needed in one-sector ones.[3] Less convincing, however, is the capability of such economies to reproduce business cycles properties for non-controversial parameter configurations. Moreover, Herrendorf and Valentinyi [23] show that local indeterminacy in two-sector models with sector-specific externalities is not robust to the introduction of standard intertemporal capital adjustment costs.

One way which has been followed to avoid such unsatisfactory features is to account for some source of market imperfection other than increasing returns, as incomplete financial markets and liquidity constraints. However, this line of research has been applied almost exclusively to one-sector models. Examples of this kind date back at least to Woodford [35] financially

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[1] Observe that a similar partial CIA constraint on consumption purchases has been considered in the context of OLG models (see among others Hahn & Solow [21], Rochon [34], Polemarchakis & Rochon [33], Crettez, Michel & Wigniolle [12]). In such a framework where life-span is two periods, a partial constraint is indeed necessary to hold non-monetary assets and thus to accumulate capital. Notice also that indeterminacy in OLG models does not necessarily require the existence of financial constraints.

[2] See also Carlstrom and Fuerst [10] who consider a one-sector model with a fractional CIA constraint and transaction costs. Contrary to our framework, the share of consumption purchases financed with cash is endogenously chosen by households.

[3] See for instance Benhabib & Farmer [4], Benhabib & Nishimura [6] and Nishimura & Venditti [31].

constrained economy. More recent contributions in this direction include the Farmer [15], Matsuyama [27] and Carlstrom & Fuerst [11] money-in-the-utility-function (MIUF) economies, the Grandmont *et al.* [19] generalization of the original Woodford's model, the CIA economies of Woodford [36], Barinci & Chéron [2] and Bosi, Dufourt & Magris [8]. The main failure of this literature concerns its empirical plausibility, and more precisely the technological parametrization. This is, for example, the case of the Woodford [35] and Grandmont *et al.* [19] models, in which indeterminacy depends dramatically upon unrealistic arbitrarily low elasticities of factor substitution. We will show that our extension to multisector models provides a nice framework to deal with plausible parametrizations.

There is also another important justification to consider two-sector models which answers to a criticism often used against the standard formulation of CIA constraints in one-sector models. In such models there is indeed a unique market and a unique price for consumption and investment. The consumer could indeed avoid the opportunity cost of holding money by buying only the investment good and then decide to devote part of it to consumption. A CIA constraint on total expenditures is therefore more likely in a one-sector model, but local indeterminacy is then ruled out.[4] In a two-sector model such a problem does not appear. Consumption and investment goods are produced with different technologies and are characterized by different prices. It follows that they enter the budget constraint of the representative agent as distinct expenditures. Therefore, using a similar terminology as that adopted in Lucas & Stokey [26], we may easily define consumption as "cash good" and investment as "credit good".

Our main results are the following. Firstly, we find that a small amount of consumption purchases to be paid by cash holdings is sufficient to make equilibrium indeterminate. More precisely, the likelihood of indeterminacy and *sunspot* fluctuations improves as soon as the amplitude of the CIA constraint is decreased. Multiple equilibria become indeed compatible with any specification for the fundamentals. This conclusion is particularly important concerning the elasticity of intertemporal substitution in consumption. The literature does not provide a clear picture concerning the admissible values for this parameter. [5] We prove however that when the share of consumption purchases paid cash is low enough local indeterminacy is compatible

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[4]Using a CIA constraint on consumption and investment expenditures in a one-sector model, Abel [1] shows that the equilibrium is always locally unique.

[5]While many standard RBC models such that Hansen [22], King, Plosser & Rebelo [24] have assumed a relatively high value (i.e. around unity), recent empirical evidence suggests much smaller ones (See Campbell [9] and Kocherlakota [25]).

with any size of substitution effects. A similar conclusion has been initially reached in a one-sector framework by Bosi, Dufourt & Magris [8]. However, the values of the share associated with such a result appears to be too low to be compatible with admissible velocities of money. On the contrary, in our two-sector framework, we show that when the elasticities of capital/labor substitution fall into the recent estimates of Duffy & Papageorgiou [14], local indeterminacy is compatible with any size of substitution effects while the velocity of money remains in accordance with plausible values.

Secondly, we show that when the share of consumption expenditures paid by cash holdings is greater, indeterminacy requires low values of the intertemporal elasticity of substitution in consumption. We prove however that these values remain in accordance with the recent empirical estimates.[6]

Thirdly, we prove that, in contrast with one-sector models, when the investment good is capital intensive or when the consumption good is strongly capital intensive, the scope for local indeterminacy improves considerably. The first part of this conclusion is similar to the main result of Nishimura & Venditti [31] who shows that when intersectoral external effects are considered, local indeterminacy becomes compatible with a capital intensive investment good at the private level.[7]

Finally, we show that persistent endogenous fluctuations may arise under any configuration concerning the capital intensity difference.[8] In particular, we prove that contrary to a standard two-sector model without money in which a capital intensive consumption good is necessary,[9] endogenous cycles become compatible with a capital intensive investment good.

The remainder of the paper is organized as follows. In Section 2 we characterize the model by describing the properties of technology, the consumer's behavior as well as the intertemporal equilibrium, and by proving existence and uniqueness of the steady state. Section 3 includes the main results of the paper together with their economic interpretation and provides a specific application to a CES economy as well as some further comments on CIA constraint. Section 4 concludes the paper while all of the proofs are gathered in a final Appendix.

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[6]See also Carlstrom and Fuerst [11], Farmer [16] and Woodford [36] for similar results in one-sector models.

[7]This conclusion differs from most of the contributions in the literature which deal with sector-specific externalities. Benhabib & Nishimura [6] show indeed that local indeterminacy requires a consumption good capital intensive at the private level.

[8]See Fukuda [15], Matsuyama [28] and Michener & Ravikumar [29] for related results on cyclic equilibria in one-sector models.

[9]See Benhabib & Nishimura [5] and more recently Mitra & Nishimura [30].

2 The model

The basic structure is a two-sector optimal growth model. We adapt it to a CIA constraint framework.

2.1 Technology

We assume that there are two commodities with one pure consumption good y_0 and one capital good y. The labor supply is assumed to be inelastic.[10] Total labor is normalised to one and each good is produced with a standard constant returns to scale technology:

$$y_0 = f^0(k_0, l_0), \quad y = f^1(k_1, l_1)$$

with $k_0 + k_1 \leq k$, k being the total stock of capital, and $l_0 + l_1 \leq 1$.

Assumption 1 . *Each production function $f^i : \mathbb{R}_+^2 \to \mathbb{R}_+$, $i = 0, 1$, is C^2, increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $f_1^i(0, x) = f_2^i(x, 0) = +\infty$, $f_1^i(+\infty, x) = f_2^i(x, +\infty) = 0$.*

For any given (k, y), we define a temporary equilibrium by solving the following problem of optimal allocation of factors between the two sectors:

$$
\begin{aligned}
\max_{k_0, k_1, l_0, l_1} \quad & f^0(k_0, l_0) \\
s.t. \quad & y \leq f^1(k_1, l_1) \\
& k_0 + k_1 \leq k \\
& l_0 + l_1 \leq 1 \\
& k_0, k_1, l_0, l_1 \geq 0
\end{aligned}
\tag{1}
$$

The associated Lagrangian is

$$L = f^0(k_0, l_0) + p[f^1(k_1, l_1) - y] + r[k - k_0 - k_1] + w[1 - l_0 - l_1]$$

with p the price of the investment good, r the rental rate of capital and w the wage rate, all in terms of the price of the consumption good. Solving the associated first order conditions give optimal input demand functions, namely $k_0(k, y)$, $l_0(k, y)$, $k_1(k, y)$ and $l_1(k, y)$. The resulting value function

$$T(k, y) = f^0(k_0(k, y), l_0(k, y))$$

is called the social production function and describes the frontier of the production possibility set. Constant returns to scale of technologies imply that $T(k, y)$ is concave non-strictly. We will assume in the following that $T(k, y)$ is at least C^2. Moreover it is easy to show that

$$r = T_1(k, y), \quad p = -T_2(k, y), \quad w = T(k, y) - rk + py$$

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[10] As shown by Bosi, Dufourt & Magris [8] in a one-sector model with partial CIA constraint and endogenous labor, the main results concerning the qualitative behavior of equilibrium paths do not depend on the elasticity of the labor supply.

2.2 Individual's problem

The economy is populated by a large number of identical infinite-lived agents. We assume without loss of generality that the population is constant. The per-period utility function satisfies the following basic restrictions:

Assumption 2 . $u(c)$ is C^2, such that for any $c > 0$, $u'(c) > 0$, $u''(c) < 0$ and satisfies $u'(0) = +\infty$, $u'(+\infty) = 0$.

At time $t \geq 0$, the representative agent is subject to the budget constraint

$$c_t + p_t[k_{t+1} - (1 - \delta)k_t] + q_t M_{t+1} = r_t k_t + w_t + q_t M_t + \tau_t \qquad (2)$$

where q denotes the price of money balances M in terms of the price of the consumption good and τ nominal lump-sum transfers issued by the government. A share $s \in (0, 1]$ of the purchases of the consumption good requires money balances accumulated in the previous period. This implies that agents must take into account the following CIA constraint

$$sc_t \leq q_t M_t \qquad (3)$$

The share s is obviously the inverse of the velocity of money denoted v.

The capital accumulation equation is

$$k_{t+1} = y_t + (1 - \delta)k_t \qquad (4)$$

with $\delta \in [0, 1]$ the rate of depreciation of capital. Finally, the intertemporal maximisation program of the representative agent is as follows

$$\max_{\{c_t, k_{t+1}, M_{t+1}\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t)$$
$$s.t. \qquad c_t + p_t[k_{t+1} - (1 - \delta)k_t] + q_t M_{t+1} = r_t k_t + w_t + q_t M_t + \tau_t$$
$$sc_t \leq q_t M_t$$
$$k_0, M_0 \text{ given}$$

where $\beta \in (0, 1]$ denotes the discount factor.

2.3 Intertemporal equilibrium

We will focus in the sequel on the case where

$$[r_{t+1} + (1 - \delta)p_{t+1}]/p_t > q_{t+1}/q_t \qquad (5)$$

holds at all dates, which means that the gross rate of return on capital is higher than the profitability of money holding. Under (5), constraint (3) binds. Government then follows a simple monetary rule: in each period it issues lump-sum transfers of money balances at the constant rate $\sigma - 1 > 0$,

so that in period t the supply of money M_t^s satisfies $M_t^s = \sigma^t M_0^s$, with $M_0^s = M_0$ the initial amount of money balances. Thus transfers are given by $\tau_t = (\sigma - 1)q_t M_t^s$. Denoting $m_t = q_t M_t$ the real money balances, the CIA constraint becomes $sc_t = m_t$ and we finally get

$$\sigma c_t \pi_{t+1} = c_{t+1} \tag{6}$$

with $\pi_t = q_t/q_{t-1}$. As shown in Appendix 5.1, a competitive equilibrium satisfies the Euler equation

$$\frac{p_{t-1}u'(c_t)c_t}{(1-s)p_{t-1}c_t + \sigma s c_{t-1}[r_t+(1-\delta)p_t]} = \frac{\beta u'(c_{t+1})c_{t+1}[r_{t+1}+(1-\delta)p_{t+1}]}{(1-s)p_t c_{t+1} + \sigma s c_t[r_{t+1}+(1-\delta)p_{t+1}]} \tag{7}$$

where $c_t = T(k_t, y_t)$, $r_t = T_1(k_t, y_t)$, $p_t = -T_2(k_t, y_t)$ and $y_t = k_{t+1} - (1-\delta)k_t$, together with the following transversality condition

$$\lim_{t \to +\infty} \beta^t u'(c_t)(p_t k_{t+1} + q_t M_{t+1}) = 0 \tag{8}$$

2.4 Steady-state

A steady state is defined as $k_t = k^*$, $y_t = y^* = \delta k^*$, $c_t = c^* = T(k^*, \delta k^*)$, $p_t = p^* = -T_2(k^*, \delta k^*)$, $r_t = r^* = T_1(k^*, \delta k^*)$ and $\pi_t = \pi^* = 1/\sigma$ for all t. From (7), the stationary level of capital k^* is obtained as a solution of

$$r/p = -T_1(k, \delta k)/T_2(k, \delta k) = \beta^{-1} - (1 - \delta) \tag{9}$$

The proof of Theorem 3.1 in Becker and Tsyganov [3] restricted to the case of homogeneous agents can be applied and gives:

Proposition 1 . *Under Assumptions 1-2, there exists a unique steady state k^* solution of equation* (9).

The binding CIA constraint gives $sc^* = q_t M_t$. Therefore the government's monetary rule implies that there is no steady state for real money balances and their corresponding relative prices: M_t increases at rate σ and we get from (6) that q_t decreases at rate $1/\sigma$. Notice that steady state conditions $1 - \delta + (r/p) = \beta^{-1}$ and $q_{t+1}/q_t = 1/\sigma$ imply inequality (5) when evaluated at the steady state since $\beta < \sigma$. By continuity (5) will hold in a neighborhood of the steady state too.

3 Indeterminacy in a CIA two-sector economy

We introduce the following standard definition.

Definition 1 . *A steady state k^* is locally indeterminate if there exists $\epsilon > 0$ such that from any k_0 belonging to $(k^* - \epsilon, k^* + \epsilon)$ there are infinitely many equilibrium paths converging to the steady state.*

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At time $t \geq 0$, the Euler equation (7) contains one pre-determinate variable, k_t, and two forward variables, k_{t+1} and k_{t+2}. Therefore local indeterminacy will be obtained if at least two roots of the characteristic polynomial derived from the linearization of (7) around the steady-state are inside the unit circle.

Following Benhabib and Nishimura [5] our analysis will be based on the following relative capital intensity difference across sectors

$$b \equiv a_{01}\left(a_{11}/a_{01} - a_{10}/a_{00}\right) \tag{10}$$

with

$$a_{00} = l_0/y_0, \quad a_{10} = k_0/y_0, \quad a_{01} = l_1/y, \quad a_{11} = k_1/y$$

the capital and labor coefficients in each sector. It follows that b is positive (negative) if and only if the investment (consumption) good is capital intensive. Notice that the capital input coefficients when evaluated at the steady state are functions of the discount factor β and the rate of depreciation of capital δ. It follows that the capital intensity difference satisfies $b = b(\beta, \delta)$. We finally introduce the elasticity of intertemporal substitution in consumption evaluated at the steady state

$$\epsilon = -u'(c^*)/(u''(c^*)c^*) \tag{11}$$

and the following elasticities of the consumption good output and the interest rate with respect to the capital stock evaluated at the steady state

$$\varepsilon_c^* = T_1^* k^*/c^* > 0, \qquad \varepsilon_r^* = T_{11}^*/T_1^* < 0 \tag{12}$$

3.1 Main results

We now discuss the local determinacy of equilibria depending on the sign of the capital intensity difference between the two sectors b. We introduce the following critical values for the share s of the consumption good to be paid cash and the elasticity ϵ of intertemporal substitution in consumption:

$$\bar{s} = -\frac{\varepsilon_r^*\beta[1+(2-\delta)b]\big[\beta+b[1+(1-\delta)\beta]\big]}{\varepsilon_c^*4(1+\beta)\beta\theta^2\sigma - \varepsilon_r^*(\beta+\sigma)[1+(2-\delta)b]\big[\beta+b[1+(1-\delta)\beta]\big]} \tag{13}$$

$$\bar{\epsilon} = \frac{\varepsilon_c^*2(1+\beta)\beta\theta^2[(1-s)\beta+\sigma s]}{\varepsilon_r^*[(1-s)\beta-\sigma s][1+(2-\delta)b]\big[\beta+b[1+(1-\delta)\beta]\big] + \varepsilon_c^*4(1+\beta)\beta\theta^2\sigma s}$$

Building on these critical values we first get the following result for $b > 0$:

Theorem 1 . *Under Assumptions 1-2, let the investment good be capital intensive at the steady state. Then:*

i) when $s < \bar{s}$, the modified golden rule k^* is locally indeterminate for any $\epsilon \in (0, +\infty)$;

ii) when $s \geq \bar{s}$, the modified golden rule k^* is locally indeterminate for any $\epsilon \in (0, \bar{\epsilon})$ with $\lim_{s \to 1} \bar{\epsilon} \in (0, 1/2)$. Moreover, when ϵ goes through $\bar{\epsilon}$, the modified golden rule undergoes a flip bifurcation.

In Theorem 1-i), we show that if the share s of the consumption good to be paid cash is low enough, i.e. lower than a critical bound $\bar{s}$, local indeterminacy may appear for any specification of the intertemporal elasticity of substitution in consumption. Condition $s < \bar{s}$ is then equivalent to $v > \underline{v} = 1/\bar{s}$ with v the velocity of money. Equation (13) then shows that the critical bounds $\bar{s}$ and $\underline{v}$ depend on the capital intensity difference across sectors b, the elasticity of the consumption good output ε_c^* and the elasticity of the interest rate ε_r^* with respect to the capital stock. While ε_c^* depends mainly on first order elasticities such that the share of capital in total income, denoted $\kappa \in (0,1)$,[11] ε_r^* crucially depends on second order elasticities such that the elasticities of capital/labor substitution in each sector, denoted ς_i, $i = 0,1$. As shown in Drugeon [13] we have indeed the following expression for the elasticity of the interest rate:

$$\varepsilon_r^* = -\frac{py^1 wk(l^0)^2}{y^0(py^1 k^0 l^0 \varsigma_0 + y^0 k^1 l^1 \varsigma_1)} \qquad (14)$$

It follows that $\partial \varepsilon_r^* / \partial \varsigma_i > 0$. Moreover, as shown in the CES example provided in Section 3.3, b is also function of the elasticities of substitution. The sign of the derivative $\partial b / \partial \varsigma_i$ depends on whether b is positive or negative and on whether ς_i is greater or less than 1. Therefore, the total derivative of the bound $\bar{s}$ with respect to ς_i is given by

$$\frac{\partial \bar{s}}{\partial \varsigma_i} = \frac{\partial \bar{s}}{\partial \varepsilon_r^*}\frac{\partial \varepsilon_r^*}{\partial \varsigma_i} + \frac{\partial \bar{s}}{\partial \varepsilon_c^*}\frac{\partial \varepsilon_c^*}{\partial \varsigma_i} + \frac{\partial \bar{s}}{\partial b}\frac{\partial b}{\partial \varsigma_i} \qquad (15)$$

From (13) it is easy to get $\partial \bar{s}/\partial b > 0$ and

$$\begin{aligned}\partial \bar{s}/\partial \varepsilon_r^* &= -\varepsilon_c^* 4(1+\beta)(\beta\theta)^2\sigma\left[1 + (2-\delta)b\right]\left[\beta + b[1 + (1-\delta)\beta]\right]\Theta^{-2}\\ \partial \bar{s}/\partial \varepsilon_c^* &= \varepsilon_r^* 4(1+\beta)(\beta\theta)^2\sigma\left[1 + (2-\delta)b\right]\left[\beta + b[1 + (1-\delta)\beta]\right]\Theta^{-2}\end{aligned} \qquad (16)$$

with $\Theta = \varepsilon_c^* 4(1+\beta)\beta\theta^2\sigma - \varepsilon_r^*(\beta+\sigma)\left[1 + (2-\delta)b\right]\left[\beta + b[1 + (1-\delta)\beta]\right]$. The sign of the derivatives with respect to the elasticities ε_c^* and ε_r^* depends on the value of the capital intensity difference across sectors b. This fact is actually the main difference between one-sector and two-sector formulations. In a one-sector model, $b = 0$, $\varsigma_0 = \varsigma_1 = \varsigma$, equations (16) imply $\partial \bar{s}/\partial \varepsilon_c^* < 0$, $\partial \bar{s}/\partial \varepsilon_r^* < 0$, and it is easy to show from (14) that[12]

$$\varepsilon_r^* = -(1-\kappa)/\varsigma$$

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[11]Notice that, as in the CES case considered in Section 3.3, ε_c^* may also depend on the elasticities of capital/labor substitution. However this relationship is quite indirect and the derivative $\partial \varepsilon_c^* / \partial \varsigma_i$ remains small in absolute value.

[12]In a one-sector model we have $p = 1$, $y^0 = y^1 = y$, $l^0 = l^1 = 1$, $k^0 = k^1 = k$, $\varsigma_0 = \varsigma_1 = \varsigma$ and $py^1 k^0 l^0 \varsigma_0 + y^0 k^1 l^1 \varsigma_1 = yk\varsigma$, so that (14) becomes $\varepsilon_r^* = -(1-\kappa)/\varsigma$.

Since $\partial \varepsilon_r^* / \partial \varsigma > 0$ and $\partial \varepsilon_c^* / \partial \varsigma$ remains low in general, it follows therefore from (15) that $\bar{s}$ is a decreasing function of ς. In order to be compatible with standard calibrations for quarterly velocity of money, we need to have $\bar{s} > 1/3$ which implies an extremely low elasticity of substitution as shown in Bosi, Dufourt and Magris [8].

On the contrary, as soon as $b \neq 0$, the total derivative (15) contains one additional term $\partial \varepsilon_r^* / \partial b > 0$. The size of this derivative depends on the value of the capital intensity difference across sectors b and may thus be quite important. In particular, when $b > 0$, the bound $\bar{s}$ may be an increasing function of ς_i if $\partial b / \partial \varsigma_i > 0$. In this case, $\bar{s}$ may be greater than $1/3$ for some realistic values of the elasticitites of substitution ς_0 and ς_1 as we will show in Section 3.3 below. It follows therefore that local indeterminacy becomes compatible with any value of the elasticity of intertemporal substitution in consumption while the velocity of money remains in accordance with empirical estimates.

Beside the use of money as a medium of exchange, increasing returns triggered by productive externalities are also an important source of sunspot fluctuations. When sector specific external effects are considered, Benhabib & Nishimura [6] show that local indeterminacy requires a consumption good capital intensive at the private level. On the contrary, with intersectoral externalities, Nishimura & Venditti [31] prove that a continuum of equilibria may arise when the investment good is capital intensive at the private level. Theorem 1 shows therefore that the arbitrages based on the holding of money may be in some sense equivalent to the capital/labor allocations between sectors influenced by intersectoral externalities.

We also prove in Theorem 1 that when the amount of money held by the representative agent is big enough, endogenous periodic cycles arise as soon as the intertemporal elasticity of substitution in consumption is not too low. In a two-sector optimal growth model without money, Benhabib & Nishimura [5] have established a similar result but under the condition that the consumption good is capital intensive.[13] When money is introduced as a medium of exchange, the capital/labor allocations also depend on monetary trade-off and persistent cycles become compatible with a capital intensive investment good.[14]

Consider now the converse capital intensity configuration with a consumption good capital intensive. We introduce the following additional re-

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[13]Theorem 1 is concerned with local bifurcations of periodic cycles. See also Mitra & Nishimura [30] for global results.

[14]A more detailled intuition is given in section 3.2 below.

striction:

Assumption 3 . $\dfrac{\varepsilon_r^*}{\theta^2 \varepsilon_c^*} b[1 + (1-\delta)b] - \beta < 0$

We obtain from this:

Theorem 2 . *Under Assumptions 1-3, let the consumption good be capital intensive at the steady state with $b \in (-\infty, -1/(2-\delta)) \cup (-\beta/[1+\beta(1-\delta)], 0)$. Then:*

i) when $s < \bar{s}$, the modified golden rule k^* is locally indeterminate for any $\epsilon \in (0, +\infty)$;

ii) when $s \geq \bar{s}$, the modified golden rule k^* is locally indeterminate for any $\epsilon \in (0, \bar{\epsilon})$ with $\lim_{s \to 1} \bar{\epsilon} \in (0, 1/2)$. Moreover, when ϵ goes through $\bar{\epsilon}$, k^* undergoes a flip bifurcation.

Theorem 2 shows that the sufficient conditions for the occurrence of local indeterminacy when the consumption good is capital intensive are more demanding than in the case with a labor intensive consumption good. Assumption 3 introduces a first set of restrictions for the value of the capital intensity difference. A second set of limitations which exclude intermediary values for b is introduced in the statement of Theorem 2. This shows that local indeterminacy is more likely to occur when the investment good is capital intensive. As a direct consequence of Theorem 2 simpler conditions are however obtained if the consumption good is strongly more capital intensive than the investment good.

Corollary 1 . *Under Assumptions 1-2, if $b < -1/(1-\delta)$ then:*

i) when $s < \bar{s}$, the modified golden rule k^* is locally indeterminate for any $\epsilon \in (0, +\infty)$;

ii) when $s > \bar{s}$, the modified golden rule k^* is locally indeterminate for any $\epsilon \in (0, \bar{\epsilon})$ with $\lim_{s \to 1} \bar{\epsilon} \in (0, 1/2)$. Moreover, when ϵ goes through $\bar{\epsilon}$, k^* undergoes a flip bifurcation.

As in Theorem 1, the main conclusions are given in part i) of Theorem 2 and Corollary 1. When the consumption good is capital intensive with $b \in (-\infty, -1/(2-\delta)) \cup (-\beta/[1+\beta(1-\delta)], 0)$, it follows from (15) and (16) that the bound $\bar{s}$ may be an increasing function of ς_i if $\partial b / \partial \varsigma_i > 0$. In this case again, $\bar{s}$ may be greater than $1/3$ for some realistic values of the elasticitites of substitution ς_0 and ς_1, and local indeterminacy becomes compatible with any value of the elasticity of intertemporal substitution in consumption while the velocity of money remains in accordance with empirical estimates.

If we consider finally the intermediary values of the capital intensity difference that were excluded in Theorem 2, we show that the occurrence of local indeterminacy is again easily obtained under a joint condition on the share s of consumption purchases paid cash and the intertemporal elasticity of substitution in consumption.

Corollary 2 . *Under Assumptions 1-3, let $b \in (-1/(2 - \delta), -\beta/[1 + \beta(1 - \delta)])$ and $s \leq \beta/(\beta + \sigma)$. Then the modified golden rule k^* is locally indeterminate for any $\epsilon \in (0, \bar{\epsilon})$. Moreover, when ϵ goes through $\bar{\epsilon}$, the modified golden rule undergoes a flip bifurcation.*

Considering some calibrations compatible with quarterly data, we have $\beta = 0.98$ and $\sigma = 1.015$ so that the condition on the share of consumption purchases paid cash becomes $s \leq \beta/(\beta + \sigma) \approx 0.49$. It follows that local indeterminacy will arise when the velocity of money v is greater than 2.035, a value in accordance with empirical estimates.

3.2 Economic intuitions

Despite the complicated three-dimensional system describing intertemporal equilibria, a direct inspection of the Euler equation (7) may be of some help in interpreting our results. To get an idea of the mechanism at work, let us assume for sake of simplicity a logarithmic instantaneous utility function and, taking into account equilibrium money market condition, let us rewrite the Euler equation (7) as

$$\sigma = \beta \frac{si_t + 1 - s}{s + (1 - s)\, i_{t+1}^{-1}} \tag{17}$$

with

$$i_t \equiv [r_t + (1 - \delta)\, p_t]/(\pi_t p_{t-1}) \tag{18}$$

the nominal interest factor. Following the logic of Benhabib & Nishimura [6], let us suppose that the system is at time t at its steady state equilibrium and let us try to construct an alternative equilibrium path that does not violate the transversality condition. For this purpose, assume that agents collectively revise their expectations in reaction to a given sunspot signal and come to believe that the nominal interest factor will undergo a depreciation. It follows that to re-establish (17), foregoing nominal interest factor i_t must be driven up. Yet, the magnitude of the required appreciation will depend crucially on the amplitude of the liquidity constraint s. If the latter is close to one, a slight increase in i_t would be sufficient, its weight in the right-hand side of (17) being large relatively to that of i_{t+1}. It follows that the steady state will be unstable in forward dynamics, violating thus the transversality

condition. If, conversely, s is relatively small, the instantaneous appreciation of i_t must be strong enough to compensate the expected depreciation of i_{t+1}, generating a convergent, although oscillatory, forward dynamics which will move the system back to its stationary solution.

We must show now that the above mechanism leading to indeterminacy is self-fulfilling, i.e. that agents' revised behavior is consistent with a lower expected nominal interest rate and with an over-reaction of the instantaneous one. To this end, observe first that such a belief will induce agents to accumulate money balances at a faster rate and divert foregoing GDP from consumption to investment in order to take advantage from tomorrow cheaper consumption. Yet, the effects of a larger future capital stock on the real interest rate and on the price of investment will depend crucially upon the relative capital intensity difference between the two sectors, namely b.

Assume first a capital intensive investment sector, i.e. $b > 0$. Under this configuration, it is easy to show that the increased accumulation of capital stock in period t will entail an appreciation of the real interest rate r_t, meanwhile it will induce a depreciation of r_{t+1}, p_t and p_{t+1}. Indeed, the following expressions evaluated around the steady state are easily derived by a direct application of the Stolper-Samuelson and the Rybczynski theorems:[15]

$$\frac{\partial r_t}{\partial y_t} = -T_{11}b, \quad \frac{\partial r_{t+1}}{\partial y_t} = T_{11}\left[1 + (1-\delta)b\right]$$
$$\frac{\partial p_t}{\partial y_t} = T_{11}b, \quad \frac{\partial p_{t+1}}{\partial y_t} = T_{11}b\left[1 + (1-\delta)b\right] \tag{19}$$

Therefore it is easy to prove, again by inspecting equation (18), that the depreciation of i_{t+1} and the over-reaction of i_t are consistent with the agents' revised beliefs. More specifically, the higher b and $|T_{11}|$ (the magnitude of the latter depending on the elasticity of the real interest rate ε_r^*) the higher the apreciation of i_t induced by a higher given expected nominal interest rate. In other words, larger values of b and ε_r^* improve the scope for indeterminacy.

The case with a consumption sector capital intensive is apparently less straightforward. Indeed, when $b < 0$, if on the one hand increasing investment at time t entails a decrease in r_t and an increase in p_t, the effects it produces on r_{t+1} and p_{t+1} depend upon the magnitude of b compared to $-1/(1-\delta)$, as expressions in (19) show. If $b < -1/(1-\delta)$, then r_{t+1} increases meanwhile p_{t+1} decreases. However, as one can easily verify, the overall impact on i_{t+1} is negative and thus agents' revised expectations are self-fulfilling. Finally, when $-1/(1-\delta) < b < 0$, one has that r_{t+1} goes down meanwhile p_{t+1} is driven up. This case is similar to the previous

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[15]Notice that p_{t-1} is pre-determined in period t and is not affected by any change in foregoing agents' consumption-investment arbitrage.

one, although in order to explain it following the same logic as before, some additional conditions are needed, as it is shown in Theorem 2.

3.3 A CES economy

To provide some quantitative insights of the plausibility of indeterminacy in our model, we consider the classical example of a CES economy. We retain the following functional forms

$$u(c) = c^{1-1/\epsilon}/(1-1/\epsilon)$$

with $\epsilon > 0$ for preferences and

$$f^0(k^0, l^0) = \left[\alpha_{10}(k^0)^{(\varsigma_0-1)/\varsigma_0} + \alpha_{00}(l^0)^{(\varsigma_0-1)/\varsigma_0}\right]^{\varsigma_0/(\varsigma_0-1)}$$

$$f^1(k^1, l^1) = \left[\alpha_{11}(k^1)^{(\varsigma_1-1)/\varsigma_1} + \alpha_{01}(l^1)^{(\varsigma_1-1)/\varsigma_1}\right]^{\varsigma_1/(\varsigma_1-1)}$$

with $\alpha_{00} + \alpha_{10} = \alpha_{01} + \alpha_{11} = 1$, $\varsigma_0, \varsigma_1 > 0$ for technologies. It follows obviously that ϵ is the elasticity of intertemporal substitution in consumption while ς_0 and ς_1 are the elasticities of capital/labor substitution. Considering the recent contribution of Nishimura and Venditti [32], tedious but straightforward computations give

$$b = (\beta\theta\alpha_{11})^{\varsigma_1}\left[1 - \left(\frac{\alpha_{10}\alpha_{01}}{\alpha_{00}\alpha_{11}}\right)^{\varsigma_0}\Sigma^{\varsigma_0-\varsigma_1}\right]$$

$$\varepsilon_r^* = -\frac{\alpha_{00}\left(\frac{\alpha_{10}\alpha_{01}}{\alpha_{00}\alpha_{11}}\right)^{1-\varsigma_0}(\beta\theta\alpha_{11})^{1-\varsigma_1}\Sigma^{1-\varsigma_0}}{\varsigma_0\varsigma_1\alpha_{01}\left[\alpha_{10}+\alpha_{00}\left(\frac{\alpha_{10}\alpha_{01}}{\alpha_{00}\alpha_{11}}\right)^{1-\varsigma_0}\Sigma^{1-\varsigma_0}\right]\Omega}$$

$$\varepsilon_c^* = \alpha_{10}\left[\alpha_{10} + \alpha_{00}\left(\frac{\alpha_{10}\alpha_{01}}{\alpha_{00}\alpha_{11}}\right)^{1-\varsigma_0}\Sigma^{1-\varsigma_0}\right]^{-1}\left[1 - \delta(\beta\theta\alpha_{11})^{\varsigma_1}\right]^{-1}$$

with

$$\Sigma = \left(\frac{(\beta\theta\alpha_{11})^{1-\varsigma_1}-\alpha_{11}}{\alpha_{01}}\right)^{\frac{1}{1-\varsigma_1}}$$

$$\Omega = \frac{1}{\varsigma_1} + \frac{\varsigma_1-\varsigma_0}{\varsigma_0\varsigma_1}\delta(\beta\theta\alpha_{11})^{\varsigma_1} + \frac{\alpha_{11}\left[\frac{1}{\varsigma_1}-\delta(\beta\theta\alpha_{11})^{\varsigma_1}\left[\frac{1}{\varsigma_1}-\frac{1}{\varsigma_0}\left(\frac{\alpha_{10}\alpha_{01}}{\alpha_{00}\alpha_{11}}\right)^{\varsigma_0}\Sigma^{\varsigma_0-\varsigma_1}\right]\right]}{\alpha_{01}\Sigma^{1-\varsigma_1}}$$

As shown in Nishimura and Venditti [32], with CES technologies interior solutions require a restriction on the elasticity of capital/labor substitution in the investment good sector, namely

$$(\beta\theta\alpha_{11})^{1-\varsigma_1} > \alpha_{11} \qquad (20)$$

This will define in general an upper bound for ς_1.[16]

We calibrate the structural parameters in a standard way compatible with quarterly data. We choose indeed $\beta = 0.98$, $\delta = 0.025$, $\sigma = 1.015$.

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[16]Notice that with a Cobb-Douglas technology, $\varsigma_1 = 1$ and condition (20) always holds.

It follows that $\theta \approx 22.47$, $1/(\delta - 1) \approx -1.025$, $1/(\delta - 2) \approx -0.506$ and $-\beta/[1 + \beta(1 - \delta)] \approx -0.501$.[17] The literature does not provide a clear picture concerning the admissible values for the elasticity of intertemporal substitution in consumption. While many standard RBC models such that Hansen [22] or King, Plosser & Rebelo [24] have assumed a relatively high value (i.e. around unity), recent empirical estimates taken from Campbell [9] and Kocherlakota [25] suggest the following plausible interval $\epsilon \in (0.2, 0.6)$.

Interpreting money to mean the monetary base, a standard calibration for quarterly velocity is $v = 3$.[18] Since we consider a CIA constraint only on consumption expenditures, $1/s$ gives a lower approximation for v. We need therefore to introduce a correction based on the consumption/income ratio. We will then consider the following interval $s \in (2/7, 0.48)$ which corresponds to $v \in (2.08, 3.5)$.

In the *RBC* literature, Cobb-Douglas technologies are usually considered and standard calibrations are based on capital shares in the consumption and investment good sectors which are such that $\alpha_{10}, \alpha_{11} \in (0.2, 0.6)$. However there is no clear conclusion on the sign of the capital intensity difference. For instance, Benhabib, Perli & Sakellaris [7] assume that the investment good is capital intensive while the opposite capital intensity configuration is considered by Benhabib & Nishimura [6]. Moreover recent papers have questioned the empirical relevance of Cobb-Douglas technologies. Duffy & Papageorgiou [14] for instance consider a panel of 82 countries over a 28-year period to estimate a CES production function specification. They find that for the entire sample of countries the assumption of unitary elasticity of substitution is rejected. Moreover, dividing the sample of countries up into several subsamples, they find that capital and labor have an elasticity of substitution significantly greater than unity (i.e. contained in [1.14, 3.24]) in the richest group of countries. We will therefore give in the following numerical illustrations for different configurations in terms of capital intensities and factor substitution. Since the estimates provided by Duffy & Papageorgiou [14] have been derived from a one-sector model, we will consider that only one elasticity of substitution belongs to the above interval while the other remains close to one.

As benchmark cases, we first consider two particular specifications. In a one-sector framework with $\varsigma_0 = \varsigma_1$ and $b = 0$, it clearly appears as shown in Bosi, Dufourt and Magris [8] that the critical bound $\bar{s}$ for the share of

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[17]The interval $(-1/(2 - \delta), -\beta/[1 + \beta(1 - \delta)]) \approx (-0.506, -0.501)$ is so small that we will not give illustration for Corollary 2.

[18]See for instance Carlstrom & Fuerst [11].

14

consumption expenditures paid by cash holdings is very close to zero when Cobb-Douglas technologies are considered. Actually, $\bar{s}$ may be greater than 2/7 if the elasticity of capital/labor substitution is very close to zero, a case which is rejected by empirical evidences. It follows that local indeterminacy compatible with realistic values for the velocity of money requires further restrictions on the elasticity of intertemporal substitution in consumption. When $\alpha_{10} = \alpha_{11} \in (0.2, 0.6)$ and $s \in (2/7, 0.48)$ we get indeed $\bar{\epsilon} \in (1.02, 1.7)$.

Similar results hold in a two-sector framework with $b \neq 0$ and Cobb-Douglas technologies. When $\alpha_{10}, \alpha_{11} \in (0.2, 0.6)$, we get $b \in (-22.02, 11.01)$ and $\bar{s} \in (0.1 \ 10^{-3}, 0.22)$. Obviously these bounds are associated with unrealistically high velocities of money. However, when $s \in (2/7, 0.48)$ the critical value of the elasticity of intertemporal substitution in consumption below which local indeterminacy holds belongs to the interval $\bar{\epsilon} \in (1.02, 4.19)$.

We will now consider a general two-sector economy with CES technologies having asymmetric elasticities of capital/labor substitution. As in Benhabib, Perli and Sakellaris [7], assume first that the investment good is capital intensive. Considering $\alpha_{10} = 0.2$ and $\alpha_{11} = 0.4$, condition (20) implies $\varsigma_1 < 1.421$. When ς_1 is close to one, numerical simulations show that the bound $\bar{s}$ is decreasing with respect to ς_0. On the contrary, when ς_0 is close to one, $\bar{s}$ is increasing with respect to $\varsigma_1 \in (0, 1.421)$. Considering therefore $\varsigma_0 = 1$, $\varsigma_1 = 1.38$ we get $\bar{s} \approx 0.36$ and $\bar{\upsilon} \approx 2.778$. We thus show that when the velocity of money satisfies $\upsilon > 2.778$, local indeterminacy is compatible with any size of substitution effects.

Assume now as in Benhabib & Nishimura [6] that the consumption good is capital intensive. Considering $\alpha_0 = 0.52$ and $\alpha_1 = 0.22$, condition (20) implies $\varsigma_1 < 1.959$. When ς_0 is close to one, numerical simulations show that the bound $\bar{s}$ is a non-monotonous function of ς_1 which cannot reach realistic values over the interval $\varsigma_1 \in (0, 1.959)$. On the contrary, when ς_1 is close to one, $\bar{s}$ is a non-monotonous function of ς_0 which converges to 0.41 when ς_0 converges to the upper bound 3.24 given in Duffy & Papageorgiou [14]. Considering therefore $\varsigma_1 = 1$, $\varsigma_0 = 1.8$ we get $b < -1/(1-\delta)$, $\bar{s} \approx 0.348$ and $\bar{\upsilon} \approx 2.874$. We thus show again that when the velocity of money belongs to a plausible interval, local indeterminacy is compatible with any size of substitution effects.

3.4 Further remarks on CIA constraint

One criticism often addressed towards CIA models - encompassing our partial specification as well - rests upon the zero nominal interest rate money demand elasticity such models entail. The main argument runs as follows:

since the nominal interest rate represents the opportunity cost of money holding, any increase in the former should produce a decrease in the amount of real balances agents are willing to hold. However, within CIA formulations, the demand of real balances is not sensible to the nominal interest rate once consumption and income have been fixed. To better understand this point, let us recall to mind that a binding liquidity constraint yields the following equation

$$q_t M_t = s c_t \qquad (21)$$

where M_t denotes nominal balances accumulated in period $t-1$ and used in period t to buy consumption good, q_t being the price of money. By simple inspection of (21), it is immediate to verify that, given the level of consumption, the amount of real balances $q_t M_t$ is not affected by any change in the nominal interest rate.[19] This is actually the argument that several authors (see, among the others, Carlstrom and Fuerst [11]) exploit to criticize the use of CIA constraint and to recommend instead the study of more general formulations in which real balances enter the utility function. Such a device allows indeed to re-establish a more realistic non-zero money demand elasticity.

However, such a way of proceeding can be also subject to some criticism. The demand of real balances, according to this procedure, is indeed obtained by multiplying money balances at their price prevailing in the period in which the latter are used to buy consumption. However, the demand of nominal balances is actually formulated one period before, namely in $t-1$. It follows that a more correct definition of the demand of real balances would require to deflate the amount of nominal balances at the price prevailing at the same moment in which agents decide to buy them: in other words, it would correspond to $q_{t-1} M_t$. Now, let us rewrite equation (21) as

$$q_{t-1} M_t = s c_t q_{t-1}/q_t. \qquad (22)$$

Since the nominal interest rate i_t is defined by $(1 + i_t) = R_t q_{t-1}/q_t$ where R stands for the real interest factor, equation (22) can be easily rearranged in order to get

$$q_{t-1} M_t = (1 + i_t) s c_t / R_t. \qquad (23)$$

From (23) one has that, fixing consumption and capital intensity and as a consequence the real interest factor, the demand of real balances is increasing in the nominal interest rate. Of course, such a feature seems again to be at

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[19]The nominal interest rate influences the demand of real balances only indirectly through the level of consumption: the higher the nominal interest rate, the more expensive consumption, thus the lower the amount of the latter agents will decide to buy and, as a consequence, the level of real balances they chose to hold.

odds with the idea that real balances are decreasing in the cost of money. But such a result is only apparently counter-factual and is shared also by MIUF models, provided money and consumption are gross complements and balances entering utility are those individuals do hold at the outset of each period. Such a point is clearly stressed in Feenstra [17], in which the functional equivalence between MIUF models and transaction costs ones is established exactly in these terms. The interpretation is rather simple: as soon as money is held principally in view of its liquidity services, in response to a larger cost of money, agents will increase their demand of real balances in order to make it still possible to finance a given amount of consumption. It follows that a demand of money increasing in the nominal interest rate is perfectly consistent.

4   Concluding comments

In this paper we present a two-sector growth model with representative agent and partial CIA constraint on consumption expenditures. Namely, we adopt two different constant returns to scale technologies producing, respectively, a pure consumption good and a pure investment good. At the same time we assume that only a share of foregoing consumption purchases must be financed out of money balances held at the beginning of each period. This assumption is justified by the fact that the velocity of money is greater than one and thus provides a more plausible framework than the standard formulation in which all consumption purchases are paid cash.

Our main conclusions may be summarized as follows: multiple equilibria come about when the degree of market imperfection - namely the share of consumption expenditures to be paid cash - decreases, in contradiction to most studies, e.g. models with increasing returns, where the scope for indeterminacy improves as soon as the market distortion is made larger and larger. The existence of a threshold for the amplitude of the CIA constraint below which the steady state is bound to be indeterminate whatever the specification of preferences reinforces this result. We prove indeed that as soon as the elasticities of capital/labor substitution fall into the recent estimates of Duffy & Papageorgiou [14], multiple equilibria occur without any restriction on the elasticity of intertemporal substitution if the velocity of money is greater than a critical bound which is compatible with empirial estimates. We have therefore shown that technology plays a crucial role in establishing the likelihood of indeterminacy.

5 Appendix

5.1 The Euler equation

From the intertemporal maximisation program of the representative agent we derive the generalized Lagrangian

$$\mathcal{L} = \sum_{t=0}^{+\infty} \beta^t u(c_t) + \sum_{t=0}^{+\infty} \beta^t \mu_t [q_t M_t - s c_t]$$
$$+ \sum_{t=0}^{+\infty} \beta^t \lambda_t [r_t k_t + w_t + q_t M_t + \tau_t - c_t - p_t [k_{t+1} - (1-\delta)k_t] - q_t M_{t+1}]$$

with λ and μ the non-negative Lagrange multipliers . If equation (5) holds, constraint (3) binds and the first order conditions are easily derived as

$$
\begin{aligned}
u'(c_t) &= \lambda_t + s\mu_t \\
\lambda_t p_t &= \beta \lambda_{t+1} [r_{t+1} + (1-\delta)p_{t+1}] \\
\lambda_t q_t &= \beta q_{t+1} (\lambda_{t+1} + \mu_{t+1})
\end{aligned}
$$

By manipulating these equations, and after substitution of $\sigma c_t \pi_{t+1} = c_{t+1}$, we easily obtain the Euler equation (7).

5.2 Proof of Theorem 1

Denoting $T_i^* = T_i(k^*, \delta k^*)$, equation (9) may be written as

$$-T_2^* \beta^{-1} = T_1^* - (1-\delta)T_2^* \quad \Leftrightarrow \quad -T_2^* = \beta \theta T_1^* \tag{24}$$

with $\theta = [1 - \beta(1-\delta)]^{-1}$. Denoting $T_{ij}^* = T_{ij}(k^*, \delta k^*)$, we easily get

$$T_{12}^* = -T_{11}^* b, \quad T_{22}^* = T_{11}^* b^2 < 0 \tag{25}$$

Using the elasticities (11)-(12) together with equations (24)-(25), tedious but straightforward computations give the following characteristic polynomial:

$$
\begin{aligned}
\mathcal{P}(\lambda) = &-\lambda^3 \Big[(1-s)\frac{\varepsilon_r^*}{\theta^2 \varepsilon_c^*} b[1 + (1-\delta)b] - \frac{1}{\epsilon}[(1-s)\beta + \sigma s(1-\epsilon)] \Big] \\
&+ \frac{\lambda^2}{\beta} \Big[\frac{\varepsilon_r^*}{\theta^2 \varepsilon_c^*} \Big[(1-s)(b^2 + \beta) - \sigma s b \\
&- b(1-\delta)\Big(b[\sigma s - (1-s)\beta(1-\delta)] - 2(1-s)\beta \Big) \Big] \\
&+ \frac{1}{\epsilon}\Big(\sigma s[(1+\beta)(\epsilon - 1) + \beta\epsilon] - (1+\beta)(1-s)\beta \Big) \Big] \\
&+ \frac{\lambda}{\beta^2} \Big[\frac{\varepsilon_r^*}{\theta^2 \varepsilon_c^*} \Big[\sigma s(b^2 + \beta) - (1-s)\beta b \\
&- \beta b(1-\delta)\Big(b[1 - s - \sigma s(1-\delta)] - 2\sigma s \Big) \Big] \\
&+ \frac{\beta}{\epsilon}\Big((1-s)\beta + \sigma s[1 - \epsilon - (1+\beta)\epsilon] \Big) \Big] - \frac{\sigma s}{\beta^2}\Big[\frac{\varepsilon_r^*}{\theta^2 \varepsilon_c^*} b[1 + (1-\delta)b] - \beta \Big] = 0
\end{aligned}
$$

18

We first provide a Lemma which will be useful to prove Theorem 1:

Lemma 1 . *Under Assumptions 1-2, $\mathcal{P}(1) < 0$.*

Proof: The characteristic polynomial evaluated at $\lambda = 1$ gives

$$\mathcal{P}(1) = \frac{\varepsilon_r^*}{(\beta\theta)^2 \varepsilon_c^*} \left[(1-s)\beta + \sigma s\right] (1 - \delta b) \left(\beta - \theta^{-1} b\right)$$

From the homogeneity property of the production functions we may derive the following factor-price frontier which corresponds to the equality between price and cost

$$\begin{pmatrix} w & r \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 1 & p \end{pmatrix}$$

We then get $wa_{01} + ra_{11} = p$. When evaluated at the steady state, the Euler equation rewrites as $p = \beta\theta r$. We obtain after substitution in the previous equation

$$\beta r(\theta - \beta^{-1} a_{11}) = wa_{01} > 0$$

Prices positivity implies $\theta - \beta^{-1} a_{11} > 0$. From equation (10), notice that

$$\beta - \theta^{-1} b = \frac{a_{00}(\beta - \theta^{-1} a_{11}) + \theta^{-1} a_{10} a_{01}}{a_{00}} > 0$$

Then we have $b < \beta\theta$, which entails $b < 1/\delta$. It follows from $\varepsilon_r^*/\varepsilon_c^* < 0$ that $\mathcal{P}(1) < 0$. $\qquad\square$

    Proof of Theorem 1: Lemma 1 shows that $\mathcal{P}(1) < 0$. The characteristic polynomial evaluated at $\lambda = 0$ and $\lambda = -1$ gives

$$\mathcal{P}(0) = -\frac{\sigma s}{\beta^2} \left[\frac{\varepsilon_r^*}{\theta^2 \varepsilon_c^*} b[1 + (1-\delta)b] - \beta\right]$$

$$\mathcal{P}(-1) = \frac{\varepsilon_r^*}{(\beta\theta)^2 \varepsilon_c^*} \left[(1-s)\beta - \sigma s\right] \left[1 + (2-\delta)b\right] \left[\beta + b[1 + (1-\delta)\beta]\right]$$
$$- 2\left(1 + \beta^{-1}\right) \left[\frac{(1-s)\beta + \sigma s}{\epsilon} - 2\sigma s\right]$$

If $b > 0$, we get $\mathcal{P}(0) > 0$. Local indeterminacy will be obtained if $\mathcal{P}(-1) < 0$. Notice that

$$\frac{\partial \mathcal{P}(-1)}{\partial \epsilon} = 2(1 + \beta^{-1})\frac{(1-s)\beta + \sigma s}{\epsilon^2} > 0$$

Moreover we have

$$\lim_{\epsilon \to 0} \mathcal{P}(-1) = -\infty$$
$$\lim_{\epsilon \to +\infty} \mathcal{P}(-1) = \frac{\varepsilon_r^*}{(\beta\theta)^2 \varepsilon_c^*} \left[(1-s)\beta - \sigma s\right] \left[1 + (2-\delta)b\right] \left[\beta + b[1 + (1-\delta)\beta]\right]$$
$$+ 4(1 + \beta^{-1})\sigma s$$

Since $\varepsilon_r^*/\varepsilon_c^* < 0$, we get $\lim_{\epsilon \to +\infty} \mathcal{P}(-1) \leq 0$ if and only if $s \leq \bar{s}$. From the monotonicity of $\mathcal{P}(-1)$ with respect to ϵ, we conclude that when $s < \bar{s}$ local indeterminacy holds for any $\epsilon \in (0, +\infty)$.

When $s \geq \bar{s}$, we get $\mathcal{P}(-1) \leq 0$ if and only if $\epsilon \leq \bar{\epsilon}$. The steady state is thus locally indeterminate for any $\epsilon \in (0, \bar{\epsilon})$. We also have $\lim_{s \to 1} \bar{\epsilon} \in (0, 1/2)$. Finally, it is immediate to verify that when $\epsilon = \bar{\epsilon}$, one characteristic root goes through -1 and the system undergoes a flip bifurcation. $\qquad\square$

5.3   Proof of Theorem 2

Lemma 1 implies $\mathcal{P}(1) < 0$. Under Assumption 3, $\mathcal{P}(0) > 0$. Local indeterminacy is obtained if $\mathcal{P}(-1) < 0$. We may rewrite $\mathcal{P}(-1)$ as follows:

$$\mathcal{P}(-1) = \tfrac{\varepsilon_r^*}{(\beta\theta)^2 \varepsilon_c^*} \left[(1-s)\beta - \sigma s\right] AB - 2\left(1 + \beta^{-1}\right)\left[\tfrac{(1-s)\beta + \sigma s}{\epsilon} - 2\sigma s\right]$$

with $A = 1 + (2 - \delta)b$ and $B = \beta + b[1 + \beta(1 - \delta)]$. Consider the first part of the right-hand-side of $\mathcal{P}(-1)$. We easily verify that $A > 0$ if and only if $b > -1/(2 - \delta) = b_2^*$ and $B > 0$ if and only if $b > -\beta/[1 + \beta(1 - \delta)] = b_1^*$ with $b_1^* > b_2^*$. It follows that $AB > 0$ if $b \in (-\infty, b_2^*) \cup (b_1^*, 0)$.
Taking now into account the second part of the right-hand-side of $\mathcal{P}(-1)$ and following the same argument as in the proof of Proposition 1, we conclude that if $b \in (-\infty, b_2^*) \cup (b_1^*, 0)$, $\mathcal{P}(-1)$ will be negative when $s < \bar{s}$ no matter what the value of ϵ is.

Assuming finally that $s \geq \bar{s}$, the same argument as in the proof of Theorem 1 applies.

5.4   Proof of Corollary 2

If $b \in (-1/(2 - \delta), -\beta/[1 + \beta(1 - \delta)])$ we get $AB < 0$. It follows that $\lim_{\epsilon \to 0} \mathcal{P}(-1) = -\infty$ while

$$\lim_{\epsilon \to +\infty} \mathcal{P}(-1) = \tfrac{\varepsilon_r^*}{(\beta\theta)^2 \varepsilon_c^*} \left[(1-s)\beta - \sigma s\right] AB + 4\sigma s \left(1 + \beta^{-1}\right)$$

Under $s \leq \beta/(\beta + \sigma)$ we get $\lim_{\epsilon \to +\infty} \mathcal{P}(-1) > 0$ and $\mathcal{P}(-1) < 0$ for any $\epsilon \in (0, \bar{\epsilon})$. The rest of the proof easily follows.

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22

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2003

0301 **Basic Income/ Minimum Wage Schedule and the Occurrence of Inactivity Trap: Some Evidence on the French Labor Market**
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Philippe DE VREYER

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Stéphane MOYEN & Jean-Guillaume SAHUC

0304 **Indeterminacy in a Cash-in-Advance Two-Sector Economy**
Stefano BOSI, Francesco MAGRIS & Alain VENDITTI

0305 **Competitive Equilibrium Cycles with Endogenous Labor**
Stefano BOSI, Francesco MAGRIS & Alain VENDITTI

0306 **Robust European Monetary Policy**
Jean-Guillaume SAHUC

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Fabrice GILLES & Yannick L'HORTY

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Jacqueline PRADEL & Christophe RAULT

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Jean-Olivier HAIRAULT, François LANGOT & Thepthida SOPRASEUTH

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2002

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	Gaetano BLOISE
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