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Competitive Equilibrium Cycles with Endogenous Labor

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Competitive equilibrium cycles with endogenous labor*

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Abstract: *In this paper we study a two-sector optimal growth model with elastic labor supply. We provide a complete characterization of the production possibility frontier in terms of the capital intensity difference accross sectors. We show that the modified golden rule is saddle-point stable when the investment good is capital intensive. On the other hand, to characterize stability with a capital intensive consumption good, we focus on either additively separable or homothetic preferences. In the first specification, we compute the critical values for the elasticities of intertemporal substitution in consumption and leisure in correspondence to which the modified golden rule undergoes flip bifurcations and endogenous business cycles occur. At the same time, we show that within a utility linear in leisure the modified golden rule is always saddle-point stable. In the second specification for preferences, we show that the local dynamic properties of the optimal path depend instead on the shares of consumption and leisure into total utility. We also compute the flip bifurcation values for these parameters and we prove that endogenous fluctuations are even more likely with homothetic preferences.*

Keywords: *Two-sector models, elastic labor supply, endogenous fluctuations, bifurcation.*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

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1 Introduction

The aim of this paper is to discuss the existence of endogenous fluctuations in a two-sector optimal growth model with leisure-dependent utility function (i.e. elastic labor supply). In particular we focus on the arbitrages between preferences and technologies. We consider two alternative standard specifications for the utility function: an additive separable formulation in which the stability properties of the optimal path will depend on the elasticities of intertemporal substitution in consumption and labor, and an homothetic formulation in which the dynamical system will depend instead on the shares of consumption and leisure into total utility.

Our main results are the following: First, and as in the standard formulation with inelastic labor supply, for any formulation for preferences, saddle-point stability holds when the investment good is capital intensive. Optimal oscillations then require a capital intensive consumption good.[1] Second, when the utility function is additively separable, for some given discount factor and under mild restrictions on the capital/labor ratio in the consumption good sector, we provide some conditions on the elasticities of intertemporal substitution for the existence of endogenous cycles. We show that optimal oscillations require the elasticity of consumption to be high enough while the elasticity of labor needs to be low enough. As a corollary we then derive the surprising result that an infinite elasticity of substitution in labor completely rules out the possibility of persistent cycles. Finally, when the utility function is homothetic, we prove that the existence of periodic cycles does not depend on the elasticitites of intertemporal susbtitution but on the shares of consumption and labor into total utility: Namely a high enough share of consumption is necessary. We show however that endogenous fluctuations require even less restrictions on the capital intensity difference accross sectors than in the additive separable case.

Since the seminal contribution of Benhabib and Nishimura [3], most of the papers on endogenous fluctuations in multisector optimal growth models deal with inelastic labor supply.[2] They may be classified into two subsets.[3] In a first one, we find the contributions in which a linear utility function is considered and the existence of persistent cycles is discussed with re-

[1] See Benhabib and Nishimura [4].

[2] Endogenous fluctuations are also exhibited in overlapping generations models. See for instance Grandmont [13] for a seminal contribution and Geanakoplos and Polemarchakis [12] for a survey.

[3] Optimal growth models with recursive preferences are also considered in the literature. See for instance Dana and Le Van [9].

spect to the discount rate.[4] Following the well-known Turnpike Theorem, endogenous fluctuations require the discount factor to be far enough from one, but arbitrarily small discounting may be compatible with optimal oscillations provided the technologies are adequately chosen.[5] By focussing only on the technological side, these analysis remain incomplete. The particular specification of the utility function prevents indeed to analyse the role of preferences on the occurrence of endogenous fluctuations. The second subset contains the contributions which deal with general reduced form formulations of multisector optimal growth models. In such a framework, endogenous cycles are obtained from a trade-off between the discount factor and the curvature of the indirect utility function.[6] As a consequence, it remains difficult to obtain precise conditions on the fundamentals.

Among the few exceptions which introduce endogenous labor supply, two papers desserve particular attention. Benhabib and Nishimura [5] consider a stochastic two-sector optimal growth model with elastic labor supply. They assume additively separable preferences which are also linear with respect to consumption. Their concern is about stochastic perturbations of endogenous cycles. The required conditions are still based on given technologies characterized by a capital intensive consumption good and some restriction of the discount factor. However, as we will show in the main text, the consideration of a linear utility function with respect to consumption implies that the elasticity of intertemporal substitution in labor does not influence the dynamical properties of the optimal path. In such a particular framework, Benhabib and Nishimura are actually not able to provide a clear picture of the role of labor arbitrages on the occurrence of business cycles.

A second exception is the recent paper of Drugeon [11] in which he provides a general framework to analyse the dynamical properties of equilibrium paths in two-sector growth models. Different specifications are considered and among them we find the optimal growth model with elastic labor supply and additively separable preferences.[7] Under standard homogeneity assumptions on technologies, he characterizes the production possibility frontier in terms of elasticities of factor substitution and shares of consumption, in-

[4]See for instance Baierl, Nishimura and Yano [1], Benhabib and Nishimura [3, 4], Boldrin and Deneckere [7].

[5]See Benhabib and Rustichini [6] and Venditti [25].

[6]See for instance Montrucchio [21, 22]. More complex optimal solutions such that chaotic paths are also exhibited from a similar trade-off (see Mitra and Sorger [20] as the most recent contribution).

[7]Drugeon also considers standard optimal growth models with inelastic labor supply, optimal growth models with two consumption goods, growth models with externalities, and endogenous growth models with cyclical paths.

vestment, wage and profits into national income. It follows that the local stability properties of the steady state are functions of these parameters together with the intertemporal elasticities of substitution in consumption and leisure. However, no precise conditions on these parameters for the occurrence of endogenous cycles are given since Drugeon is more concerned with methodological issues.

In the current paper, we use the methodology of Benhabib and Nishimura [4] to provide a precise characterization of the production possibility frontier in terms of the capital intensity difference between sectors, denoted b in the following. We first prove existence and uniqueness of the steady state for general standard preferences. Then we provide a complete analysis of the optimal dynamics in terms of b and, depending on the formulation of preferences, on the elasticities of intertemporal substitution in consumption and labor, or on the shares of consumption and leisure into total utility. We give precise conditions for the existence of optimal periodic cycles. As in the standard framework with inelastic labor supply, endogenous fluctuations require a capital intensive consumption good (i.e. $b < 0$). Considering then $b < 0$, our strategy is the following: for a given discount factor β, which may also be very close to one, we choose some technologies such that b satisfies mild additional conditions and then we provide restrictions on preferences, either on intertemporal substitution or on shares in total utility, such that there exist endogenous fluctuations. Proceeding that way, we are able to give a precise picture of the influence of the fundamentals on the dynamic properties of the optimal path.

The rest of the paper is organised as follows: In Section 2 we set up the basic model, we give necessary and sufficient conditions for the characterization of the optimal path, we prove existence and uniqueness of the steady state, we provide a characterization of the transformation frontier in terms of the technological parameters and we derive the characteristic polynomial. Section 3 contains the main results. In Section 4, we compare our results with related literature while concluding comments are given in Section 5. All of the proofs are gathered in a final Appendix.

2 The model

2.1 The basic structure

The basic model is a two-sector optimal growth model augmented to include endogenous labor supply. We assume that there are two commodities with one pure consumption good y_0 and one capital good y. Each good is produced with a standard constant returns to scale technology:

$$y_0 = f^0(k_0, l_0), \qquad y = f^1(k_1, l_1)$$

with $k_0 + k_1 \leq k$, k being the total stock of capital, and $l_0 + l_1 \leq \ell$, ℓ being the amount of labor. We will assume that $0 \leq \ell \leq \bar{\ell}$ with $\bar{\ell} > 0$ (possibly infinite) the agent's endowment of labor.

Assumption 1 . *Each production function $f^i : \mathbb{R}^2_+ \to \mathbb{R}_+$, $i = 0, 1$, is C^2, increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $f^i_1(0, x) = f^i_2(x, 0) = +\infty$, $f^i_1(+\infty, x) = f^i_2(x, +\infty) = 0$.*

For any given (k, y, ℓ), we define a temporary equilibrium by solving the following problem of optimal allocation of factors between the two sectors:

$$\begin{array}{ll} \max\limits_{k_0,k_1,l_0,l_1} & f^0(k_0, l_0) \\ s.t. & y \leq f^1(k_1, l_1) \\ & k_0 + k_1 \leq k \\ & l_0 + l_1 \leq \ell \\ & k_0, k_1, l_0, l_1 \geq 0 \end{array} \tag{1}$$

The associated Lagrangian is

$$L = f^0(k_0, l_0) + p[f^1(k_1, l_1) - y] + r[k - k_0 - k_1] + w[\ell - l_0 - l_1]$$

with p the price of the investment good, r the rental rate of capital and w the wage rate, all in terms of the price of the consumption good. Solving the associated first order conditions give optimal demand functions for capital and labor, namely $k_0(k, y, \ell)$, $l_0(k, y, \ell)$, $k_1(k, y, \ell)$ and $l_1(k, y, \ell)$. The resulting value function

$$T(k, y, \ell) = f^0(k_0(k, y, \ell), l_0(k, y, \ell))$$

is called the social production function and describes the frontier of the production possibility set. The constant returns to scale of technologies imply that $T(k, y, \ell)$ is concave non-strictly. We will assume in the following that $T(k, y, \ell)$ is at least C^2.

We also get from the first order conditions

$$\begin{aligned} r(k, y, \ell) &= f^0_1(k_0(k, y, \ell), l_0(k, y, \ell)) = pf^1_1(k_1(k, y, \ell), l_1(k, y, \ell)) \\ w(k, y, \ell) &= f^0_2(k_0(k, y, \ell), l_0(k, y, \ell)) = pf^1_2(k_1(k, y, \ell), l_1(k, y, \ell)) \\ p(k, y, \ell) &= \frac{f^0_1(k_0(k, y, \ell), l_0(k, y, \ell))}{f^1_1(k_1(k, y, \ell), l_1(k, y, \ell))} = \frac{f^0_2(k_0(k, y, \ell), l_0(k, y, \ell))}{f^1_2(k_1(k, y, \ell), l_1(k, y, \ell))} \end{aligned} \tag{2}$$

and it is easy to show that the rental rate of capital, the price of investment good and the wage rate satisfy

$$T_1(k, y, \ell) = r(k, y, \ell), \;\; T_2(k, y, \ell) = -p(k, y, \ell), \;\; T_3(k, y, \ell) = w(k, y, \ell) \quad (3)$$

Notice also that

$$w(k, y, \ell)\ell = T(k, y, \ell) - r(k, y, \ell)k + p(k, y, \ell)y \quad (4)$$

which implies the linear homogeneity of $T(k, y, \ell)$.

The economy is populated by a large number of identical infinitely-lived agents. We will assume without loss of generality that the total population is constant. The per-period utility function depends on consumption c and leisure $\mathcal{L} = \bar{\ell} - \ell$ and satisfies the following basic restrictions:

Assumption 2 . *$u(c, \mathcal{L})$ is C^2, increasing in each argument, concave and satisfies for any $x > 0$ the boundary conditions* $\lim_{c \to 0} u_1(c, x)/u_2(c, x) = +\infty$, $\lim_{\mathcal{L} \to 0} u_1(x, \mathcal{L})/u_2(x, \mathcal{L}) = 0$.[8]

We also introduce a standard normality assumption between consumption and leisure

Assumption 3 . *Consumption c and leisure $\mathcal{L}$ are normal goods.*

Notice that $T(k, y, \ell)$ gives the maximum production level of the consumption good which will be entirely consumed by the representative agent, i.e. $c_t = T(k_t, y_t, \ell_t)$.

The capital accumulation equation is

$$k_{t+1} = y_t + (1 - \delta)k_t \quad (5)$$

with $\delta \in [0, 1]$ the rate of depreciation of capital. The intertemporal maximisation program of the representative agent is thus as follows

$$\begin{array}{cl} \max\limits_{\{y_t, \ell_t, k_{t+1}\}_{t=0}^{+\infty}} & \sum\limits_{t=0}^{+\infty} \beta^t u(T(k_t, y_t, \ell_t), \bar{\ell} - \ell_t) \\ s.t. & k_{t+1} = y_t + (1 - \delta)k_t \\ & k_0 \text{ given} \end{array} \quad (6)$$

where $\beta \in (0, 1]$ denotes the discount factor.[9] Following Michel [18], we introduce the generalised Lagrangian at time $t \geq 0$

$$\mathcal{L}_t \;\; = u(T(k_t, y_t, \ell_t), \bar{\ell} - \ell_t) + \beta\lambda_{t+1}\Big[y_t + (1 - \delta)k_t\Big] - \lambda_t k_t$$

[8] These boundary conditions imply the Inada conditions.

[9] In the case $\beta = 1$, the infinite sum into the optimization program (6) may not converge. However, following Ramsey [23], we may consider the state of bliss as defined by

with λ_t the shadow price of capital k_t. Taking into account equations (3), the following first order conditions together with the transversality condition provide necessary and sufficient conditions for an optimal path

$$\begin{aligned} u_1(c_t,\bar{\ell}-\ell_t)w_t &= u_2(c_t,\bar{\ell}-\ell_t) && (7)\\ u_1(c_t,\bar{\ell}-\ell_t)r_t+\beta\lambda_{t+1}(1-\delta) &= \lambda_t && (8)\\ u_1(c_t,\bar{\ell}-\ell_t)p_t &= \beta\lambda_{t+1} && (9)\\ \lim_{t\to+\infty}\beta^t u_1(c_t,\bar{\ell}-\ell_t)p_t k_{t+1} &= 0 && (10)\end{aligned}$$

By manipulating equations (7)-(9), we easily obtain the following system of Euler equations

$$\begin{aligned} -u_1(c_t,\bar{\ell}-\ell_t)p_t+\beta u_1(c_{t+1},\bar{\ell}-\ell_{t+1})[r_{t+1}+(1-\delta)p_{t+1}] &= 0 && (11)\\ u_1(c_t,\bar{\ell}-\ell_t)w_t-u_2(c_t,\bar{\ell}-\ell_t) &= 0 && (12)\end{aligned}$$

It is then easy to conclude that this is an implicit system of two difference equations of order 1 in the capital stock k and the labor supply ℓ.

2.2 Steady state

A steady state is defined as $k_t=k^*$, $\ell_t=\ell^*$, $y_t=y^*=\delta k^*$, $c_t=c^*=T(k^*,\delta k^*,\ell^*)$, $p_t=p^*=-T_2(k^*,\delta k^*,\ell^*)$, $r_t=r^*=T_1(k^*,\delta k^*,\ell^*)$ and $w_t=w^*=T_3(k^*,\delta k^*,\ell^*)$ for all t. Recall now that $T(k,y,\ell)$ is a linear homogeneous function. This property is based on the fact that the capital and labor demand functions $k_0(k,y,\ell)$, $l_0(k,y,\ell)$, $k_1(k,y,\ell)$ and $l_1(k,y,\ell)$ are homogeneous of degree 0. Then denoting $\kappa=k/\ell$, a steady state (k^*,ℓ^*) may be also defined as a pair (κ^*,ℓ^*) solution of the following equations

$$-\frac{T_1(\kappa,\delta\kappa,1)}{T_2(\kappa,\delta\kappa,1)}=f_1^1(k_1(\kappa,\delta\kappa,1),l_1(\kappa,\delta\kappa,1))=\beta^{-1}-(1-\delta) \qquad (13)$$

$$u_1(\ell T(\kappa,\delta\kappa,1),\bar{\ell}-\ell)T_3(\kappa,\delta\kappa,1)=u_2(\ell T(\kappa,\delta\kappa,1),\bar{\ell}-\ell) \qquad (14)$$

As in the standard two-sector model with inelastic labor, we get the following result:

$$\bar{u}=\max_{c,\ell}\ u(c,\bar{\ell}-\ell) \quad s.t.\quad c+p\delta k=rk+w\ell$$

so that the infinite sum transformed as follows

$$\max\sum_{t=0}^{+\infty}\beta^t[u(c_t,\bar{\ell}-\ell_t)-\bar{u}]$$

will either converge toward a finite value when evaluated along the optimal path, or converge toward $-\infty$.

Proposition 1 . *Under Assumptions 1-3, there exists a unique steady state* (κ^*, ℓ^*) *solution of equations* $(13)-(14)$.[10]

A pair (k^*, ℓ^*) will be called the Modified Golden Rule. The stationary consumption is obtained from $c^* = T(k^*, \delta k^*, \ell^*)$.

2.3 The characteristic polynomial

In order to derive a tractable formulation for the degree 2 characteristic polynomial associated with the Euler equation, we need to compute the second derivatives of $T(k, y, \ell)$. As already mentioned above, we know that $T(k, y, \ell)$ is a concave function. It follows that:

$$T_{11}(k,y,\ell) = \frac{\partial r}{\partial k} \leq 0,\ T_{22}(k,y,\ell) = -\frac{\partial p}{\partial y} \leq 0,\ T_{33}(k,y,\ell) = \frac{\partial w}{\partial \ell} \leq 0$$

However the sign of the following cross derivatives is not obvious:

$$T_{12}(k,y,\ell) = \frac{\partial r}{\partial y} = T_{21}(k,y,\ell) = -\frac{\partial p}{\partial k}$$
$$T_{13}(k,y,\ell) = \frac{\partial r}{\partial \ell} = T_{31}(k,y,\ell) = \frac{\partial w}{\partial k}$$
$$T_{23}(k,y,\ell) = -\frac{\partial p}{\partial \ell} = T_{32}(k,y,\ell) = \frac{\partial w}{\partial y}$$

To study these derivatives we start from the homogeneity property of the production functions. We have:

$$\begin{array}{c} y_0 = k_0 f_1^0 + l_0 f_2^0 \\ y = k_1 f_1^1 + l_1 f_2^1 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} 1 = \frac{k_0}{y_0} r + \frac{l_0}{y_0} w \\ 1 = \frac{k_1}{y}\frac{r}{p} + \frac{l_1}{y}\frac{w}{p} \end{array}$$

We finally obtain:

$$\begin{pmatrix} w & r \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 1 & p \end{pmatrix} \tag{15}$$

with

$$a_{00} = l_0/y_0, \quad a_{10} = k_0/y_0, \quad a_{01} = l_1/y, \quad a_{11} = k_1/y$$

the capital and labor coefficients in each sector. Equation (15) gives the factor-price frontier and corresponds to the equality between price and cost. Differentiating this equation gives:

[10] Under the normality Assumption 3, uniqueness also holds in the one-sector optimal growth model with endogenous labor. In equation (13) the marginal productivity of capital is thus obtained from the aggregate technology and the proof of Proposition 1 applies. This result contrasts with the conclusions obtained by De Hek [10] in which multiple steady states are exhibited when normality between consumption and leisure is not considered.

$$\begin{pmatrix} dw & dr \end{pmatrix}\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} + \begin{pmatrix} w & r \end{pmatrix}\begin{pmatrix} da_{00} & da_{01} \\ da_{10} & da_{11} \end{pmatrix} = \begin{pmatrix} 0 & dp \end{pmatrix}$$

It can be easily shown that the envelope theorem implies

$$\begin{pmatrix} w & r \end{pmatrix}\begin{pmatrix} da_{00} & da_{01} \\ da_{10} & da_{11} \end{pmatrix} = 0$$

so that

$$\begin{pmatrix} dw & dr \end{pmatrix}\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 0 & dp \end{pmatrix}$$

Eliminating dw we can solve this system to get

$$\frac{dp}{dr} = a_{01}\left(\frac{a_{11}}{a_{01}} - \frac{a_{10}}{a_{00}}\right) \equiv b \tag{16}$$

b is a relative capital intensity difference. The sign of b is thus positive if and only if the investment good is capital intensive. We can also solve the above system by eliminating dr and get

$$\frac{dw}{dp} = -\frac{a_{10}}{a_{00}}b^{-1} \equiv ab^{-1} \tag{17}$$

with $a = a_{10}/a_{00} = k_0/l_0 > 0$ the capital-labor ratio in the consumption good sector. Now consider the cross derivatives. We can write:

$$T_{12} = -\frac{\partial p}{\partial r}\frac{\partial r}{\partial k} = -T_{11}b \tag{18}$$

$$T_{31} = \frac{\partial w}{\partial p}\frac{\partial p}{\partial k} = -\frac{\partial w}{\partial p}T_{12} = -T_{11}a \geq 0 \tag{19}$$

$$T_{32} = \frac{\partial w}{\partial p}\frac{\partial p}{\partial y} = \frac{\partial w}{\partial p}\frac{\partial p}{\partial r}\frac{\partial r}{\partial y} = -\frac{\partial w}{\partial p}bT_{12} = T_{11}ab \tag{20}$$

As already shown by Benhabib and Nishimura [4], the sign of $T_{12}(k, y, \ell)$ is given by the sign of the relative capital intensity difference betwen the two sectors b. Notice also that $T_{22}(k, y, \ell)$ and $T_{33}(k, y, \ell)$ may be written as

$$T_{22} = -\frac{\partial p}{\partial r}\frac{\partial r}{\partial y} = T_{11}b^2 \tag{21}$$

$$T_{33} = \frac{\partial w}{\partial p}\frac{\partial p}{\partial \ell} = -\frac{\partial w}{\partial p}\frac{\partial w}{\partial y} = T_{11}a^2 \tag{22}$$

As shown in the previous subsection, the steady state (k^*, ℓ^*) is characterized by equations (13)-(14) which may be written as follows

$$-T_2^*\beta^{-1} = T_1^* - (1-\delta)T_2^* \tag{23}$$

$$u_1^*T_3^* = u_2^* \tag{24}$$

with $T_i^* = T_i(k^*, \delta k^*, \ell^*)$ and $u_i^* = u_i(c^*, \bar{\ell} - \ell^*)$. In the following we will also consider a second formulation for equation (23)

$$-T_2^* = \beta\theta T_1^* \tag{25}$$

with $\theta = [1 - \beta(1-\delta)]^{-1}$. The second derivatives of the functions $T(k, y, \ell)$ and $u(c, \bar{\ell} - \ell)$ will be evaluated at (k^*, ℓ^*) using the following notation: $T_{ij}^* = T_{ij}(k^*, \delta k^*, \ell^*)$ and $u_{ij}^* = u_{ij}(c^*, \bar{\ell} - \ell^*)$. Notice that the capital input coefficients when evaluated at the steady state are functions of the discount factor β and the rate of depreciation of capital δ. It follows that the capital intensity difference satisfies $b = b(\beta, \delta)$. We finally introduce the elasticities of substitution of consumption and leisure evaluated at the steady state[11]

$$\epsilon_{cc} = -\xi_{cc}^{-1} = -\left(u_{11}^* c^*/u_1^*\right)^{-1} > 0, \quad \epsilon_{\mathcal{L}c} = -\xi_{\mathcal{L}c}^{-1} = -\left(u_{21}^* c^*/u_2^*\right)^{-1} \tag{26}$$

$$\epsilon_{\mathcal{L}\mathcal{L}} = -\xi_{\mathcal{L}\mathcal{L}}^{-1} = -\left(u_{22}^* \mathcal{L}^*/u_2^*\right)^{-1} > 0, \quad \epsilon_{c\mathcal{L}} = -\xi_{c\mathcal{L}}^{-1} = -\left(u_{12}^* \mathcal{L}^*/u_1^*\right)^{-1} \tag{27}$$

and the following elasticities of the consumption good's output and the rental rate with respect to the capital stock, all evaluated at the steady state

$$\varepsilon_{ck} = T_1^* k^*/T^* > 0, \qquad \varepsilon_{rk} = T_{11}^* k^*/T_1^* < 0 \tag{28}$$

Using equations (18)-(28), total differenciation of the Euler equations (11)-(12) gives after tedious but straightforward computations:

$$\begin{aligned} -\mathcal{A}_1 dk_t + \mathcal{A}_2 d\ell_t + \mathcal{A}_3 dk_{t+1} + \beta\mathcal{A}_4 d\ell_{t+1} - \beta\mathcal{A}_1 dk_{t+2} &= 0 \qquad (29) \\ \mathcal{A}_4 dk_t + \mathcal{A}_5 d\ell_t + \mathcal{A}_2 dk_{t+1} &= 0 \qquad (30) \end{aligned}$$

with

$$\begin{aligned} \mathcal{A}_1 &= \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}} b[1 + (1-\delta)b] + \beta\theta^2\xi_{cc} \\ \mathcal{A}_2 &= \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}} ab - \beta\theta\tfrac{T_3^*}{T_1^*}(\xi_{cc} - \xi_{\mathcal{L}c}) \\ \mathcal{A}_3 &= \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}}\left[b^2 + \beta[1 + (1-\delta)b]^2\right] + \beta(1+\beta)\theta^2\xi_{cc} < 0 \\ \mathcal{A}_4 &= \theta\tfrac{T_3^*}{T_1^*}(\xi_{cc} - \xi_{\mathcal{L}c}) - \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}} a[1 + (1-\delta)b] \\ \mathcal{A}_5 &= \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}} a^2 + \left(\tfrac{T_3^*}{T_1^*}\right)^2 (\xi_{cc} - \xi_{\mathcal{L}c}) + \tfrac{T_3^*}{(T_1^*)^2}\tfrac{c^*}{\mathcal{L}^*}(\xi_{\mathcal{L}\mathcal{L}} - \xi_{c\mathcal{L}}) < 0 \end{aligned}$$

Solving equation (29) with respect to dk_{t+2} and substituting the result into equation (30) gives the following linear system

$$\begin{bmatrix} \beta\mathcal{A}_1\mathcal{A}_4 + \mathcal{A}_2\mathcal{A}_3 & \beta(\mathcal{A}_1\mathcal{A}_5 + \mathcal{A}_2\mathcal{A}_4) \\ \mathcal{A}_2 & 0 \end{bmatrix} \begin{bmatrix} dk_{t+1} \\ d\ell_{t+1} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1\mathcal{A}_2 & -\mathcal{A}_2^2 \\ -\mathcal{A}_4 & -\mathcal{A}_5 \end{bmatrix} \begin{bmatrix} dk_t \\ d\ell_t \end{bmatrix}$$

Assuming that the matrix on the left-hand-side is non singular, the following Lemma provides a nice formulation for the characteristic polynomial

[11] Notice that the normality Assumption 3 implies $\xi_{cc} - \xi_{\mathcal{L}c} < 0$ and $\xi_{\mathcal{L}\mathcal{L}} - \xi_{c\mathcal{L}} < 0$.

Lemma 1 . *Under Assumptions 1-3, let* $\mathcal{A}_2(\mathcal{A}_1\mathcal{A}_5 + \mathcal{A}_2\mathcal{A}_4) \neq 0$. *The characteristic polynomial is*

$$\mathcal{P}(\lambda) = \lambda^2 - \lambda\mathcal{T} + \mathcal{D} \tag{31}$$

with

$$\mathcal{D} = \beta^{-1} \ \ \textit{and} \ \ \mathcal{T} = \frac{\mathcal{A}_3\mathcal{A}_5 - \mathcal{A}_2^2 - \beta\mathcal{A}_4^2}{\beta(\mathcal{A}_1\mathcal{A}_5 + \mathcal{A}_2\mathcal{A}_4)}$$

3 Main results

As in the standard two-sector model with inelastic labor, we first prove that the saddle-point property holds when the representative consumer does not discount the future, i.e. $\beta = 1$. Notice however that we need to introduce an additional assumption concerning the curvature of the utility function around the steady state.[12]

Proposition 2 . *Under Assumptions 1-3, let* $\beta = 1$ *and the Hessian matrix of the utility function* $u(c, \mathcal{L})$ *be non singular when evaluated at the modified golden rule* (k^*, ℓ^*). *Then* (k^*, ℓ^*) *is saddle-point stable.*

Remark: We also show in Appendix 5.3 that under Assumptions 1-2, the characteristic roots are real for any $\beta \in (0, 1]$.

When $\beta \in (0, 1)$, we have to discuss the local stability properties of the modified golden rule depending on the sign of the capital intensity difference between the two sectors b, i.e. depending on whether the investment good is capital ($b > 0$) or labor ($b < 0$) intensive at the steady state. We first start with the case $b > 0$.

Proposition 3 . *Under Assumptions 1-3, if the investment good is capital intensive at the modified golden rule, the optimal path monotonically converges to the modified golden rule* (k^*, ℓ^*).

Remark: In a standard one-sector optimal growth model with endogenous labor, $b = 0$ and the proof of Proposition 3 also applies: under Assumption 3 the saddle-point property always holds. When normality between consumption and leisure is not considered, De Hek [10] shows on the contrary that the steady-state may become unstable and endogenous fluctuations may occur. Such a result requires however non-standard preferences.

[12] A similar result is mentioned as a remark in Benhabib and Nishimura [4] but no formal proof is available. Notice also that the authors only mention the case of an additively separable utility function which is linear with respect to consumption.

As in the two-sector model with inelastic labor, Proposition 3 shows that when the investment good is capital intensive the optimal path monotonically converges to the modified golden rule. The same result has been obtained independently by Drugeon [11]. The existence of endogenous fluctuations thus requires a capital intensive consumption good. The intuition for this result, which is found in Benhabib and Nishimura [4], comes from the Rybczinsky and Stolper-Samuelson effects. It may be summarised as follows. Consider an instantaneous increase in the capital stock k_t. This results in two opposing forces:

- Since the consumption good is more capital intensive than the investment good, the trade-off in production becomes more favorable to the consumption good. Moreover, the Rybczinsky theorem implies a decrease of the output of the capital good y_t. This tends to lower the investment and the capital stock in the next period k_{t+1}.

- In the next period the decrease of k_{t+1} implies again through the Rybczinsky effect an increase of the output of the capital good y_{t+1}. This mechanism is explained by the fact that the decrease of k_{t+1} improves the trade-off in production in favor of the investment good which is relatively less intensive in capital. Therefore this tends to increase the investment and the capital stock in period $t+2$, k_{t+2}. Notice also that the rise of y_{t+1} implies a decrease of the rental rate w_{t+1} and through the Stolper-Samuelson effect an increase of the price p_{t+1}.

So far the discussion concerns the existence of oscillations but saddle-point stability may still hold. Persistent cycles require more conditions. For cycles to be sustained, the oscillations in relative prices must not indeed present intertemporal arbitrage opportunities. For instance, possible gains from postponing consumption from periods when the marginal rate of transformation between consumption and investment is high to periods when it is low must not be worth it. We need therefore to introduce in the discussion the properties of preferences. In order to simplify the analysis, we will respectively consider in the following two standard formulations: an additively separable utility function and an homogeneous utility function.

Considering equations (13)-(14), and as shown in the proof of Proposition 1, the steady-state (κ^*, ℓ^*) is defined as follows: the capital/labor ratio $\kappa^* = k^*/\ell^*$ is only determined by the technological characteristics, the discount factor β and the rate of depreciation of capital δ. The properties of preferences only influence the stationary value of labor ℓ^*. Then for some given technologies and a given pair (β, δ), the value of κ^* is fixed and it is easy to consider variations of preferences in order to study the occurrence of bifurcations. This will be our strategy of proof in the sequel of the paper.

3.1 The case of additively separable preferences

The assumption of an additively separable utility function is used for instance in a stochastic version of this model by Benhabib and Nishimura [5]. They also assume that the utility function is linear with respect to consumption. We will consider in the following a general formulation for the utility function and prove that in this case the elasticity of intertemporal substitution in labor/leisure has a great influence on the local stability properties of the steady state. Recall from equations (26)-(27) that the elasticities of intertemporal subsitution in consumption and leisure are

$$\epsilon_{cc} = -\xi_{cc}^{-1} \in [0, +\infty], \ \epsilon_{\mathcal{L}\mathcal{L}} = -\xi_{\mathcal{L}\mathcal{L}}^{-1} \in [0, +\infty]$$

We introduce the following critical value for the elasticity of intertemporal substitution in consumption that will be useful to prove the existence of endogenous fluctuations:

$$\bar{\epsilon}_{cc} \equiv \frac{\varepsilon_{ck} 2\beta(1+\beta)\theta^2}{\varepsilon_{rk}[1+(2-\delta)b][b+\beta[1+(1-\delta)b]]} \tag{32}$$

Proposition 4 . *Under Assumptions 1-3, let $u(c, \mathcal{L})$ be additively separable, $\beta \in (0,1)$ and the consumption good be capital intensive at the modified golden rule (k^*, ℓ^*). The following cases hold:*

i) *When $b \in (-\infty, -1/[2-\delta]) \cup (-\beta/[1+\beta(1-\delta)], 0)$, (k^*, ℓ^*) is saddle-point-stable for all $\epsilon_{cc}, \epsilon_{\mathcal{L}\mathcal{L}} > 0$.*

ii) *When $b \in (-1/[2-\delta], -\beta/[1+\beta(1-\delta)])$, for any given $\epsilon_{cc} \leq \bar{\epsilon}_{cc}$, (k^*, ℓ^*) is saddle-point-stable for all $\epsilon_{\mathcal{L}\mathcal{L}} > 0$.*

iii) *When $b \in (-1/[2-\delta], -\beta/[1+\beta(1-\delta)])$, for any given $\epsilon_{cc} > \bar{\epsilon}_{cc}$, there exists $\bar{\epsilon}_{\mathcal{L}\mathcal{L}} \in (0, +\infty)$ such that (k^*, ℓ^*) is saddle-point-stable if and only if $\epsilon_{\mathcal{L}\mathcal{L}} \in (\bar{\epsilon}_{\mathcal{L}\mathcal{L}}, +\infty)$. Moreover, when $\epsilon_{\mathcal{L}\mathcal{L}}$ crosses $\bar{\epsilon}_{\mathcal{L}\mathcal{L}}$ from above, (k^*, ℓ^*) undergoes a flip bifurcation.*

Proposition 4 shows that the existence of endogenous fluctuations is based on some restrictions on the values of the elasticities of intertemporal substitution in consumption and leisure. Considering cases ii) and iii) shows that the elasticity of consumption needs to be high enough while the elasticity of labor needs to be low enough. This confirms the standard intuition in optimal growth models that the concavity of the utility function restricts the possibility of cycles.[13] However our results show that this intuition only applies to the concavity with respect to consumption. Endogenous fluctuations indeed require some concavity of preferences with respect to labor. It

[13] See for instance Magill [21] and Rockafellar [24].

is indeed possible to derive from this the surprising strong result that saddle-point stability is obtained without additional restriction when the elasticity of intertemporal substitution in leisure is close to infinity, i.e. when the utility function is nearly linear in leisure.

Corollary 1 . *Under Assumptions 1-3, let* $u(c, \mathcal{L})$ *be additively separable and linear with respect to leisure. Assume also that the consumption good is capital intensive at the modified golden rule* (k^*, ℓ^*). *Then* (k^*, ℓ^*) *is saddle-point stable for all* $\epsilon_{cc} \in (0, +\infty)$.

This Corollary confirms a result that is already suggested in Proposition 4: when the consumption good is capital intensive, the capital/labor allocations between sectors generate some oscillations in relative prices that will present intertemporal arbitrages opportunities. The presence of a strong intertemporal elasticity of substitution in labor allows the representative agent to obtain some gains from postponing leisure from periods when the productivity of labor is high to periods when it is low. This optimal policy smooths the intertemporal allocations of leisure and consumption and thus prevents the existence of persistent cycles.[14] Notice that this result does not directly depend on the value of the discount factor, i.e. on the degree of myopia of the representative agent. Everything happens as if leisure is definitively more important than consumption.

If we assume as in Benhabib and Nishimura [5] that the utility function is additively separable and linear with respect to consumption, leisure arbitrages no longer influence the local stability properties of the steady state.[15] Moreover the existence of endogenous fluctuations requires stronger restrictions on the discount factor.

Corollary 2 . *Under Assumptions 1-3, let* $u(c, \mathcal{L})$ *be additively separable and linear with respect to consumption. Assume also that the consumption good is capital intensive at the modified golden rule* (k^*, ℓ^*). *The following results hold:*

i) (k^*, ℓ^*) *is saddle-point stable if and only if* $b(\beta, \delta) \in (-\infty, -1/[2-\delta]) \cup (-\beta/[1+\beta(1-\delta)], 0)$.

ii) *If there is some* $\beta^* \in (0,1)$ *such that* $b(\beta^*, \delta) \in (-1/[2-\delta], -\beta^*/[1+$

[14] In a simple one-sector stochastic growth model with shocks to technology, Hansen [14] considers a similar linear formulation of preferences with respect to leisure through the assumption of indivisible labor. In such a framework business cycles fluctuations are based on technological disturbances.

[15] This result has already been pointed out by Drugeon [11] though his concern was more methodological.

$\beta^*(1-\delta)]$), *then there exists* $\bar{\beta} \in (0,1)$ *such that, when* β *crosses* $\bar{\beta}$ *from above,* (k^*, ℓ^*) *undergoes a flip bifurcation.*

When the intertemporal elasticity of substitution in consumption is infinite, we find the same intuition as in the standard two-sector optimal growth model with inelastic labor considered by Benhabib and Nishimura [4]: for cycles to be sustained, the oscillations in relative prices must not present intertemporal arbitrages opportunities. This implies the existence of a minimum degree of myopia for the representative agent.

3.2 The case of homogeneous preferences

When the utility function is linear homogeneous, it is useful to define the shares of consumption and leisure into total utility at the steady state as follows

$$\sigma_c = u_1^* c^*/u^* > 0, \text{ and } \sigma_{\mathcal{L}} = u_2^* \mathcal{L}^*/u^* > 0$$

Linear homogeneity of $u(c, \mathcal{L})$ then implies $\sigma_c + \sigma_{\mathcal{L}} = 1$.

In the following Proposition we will consider variations of the share of consumption into total utility. Before stating the Proposition, we introduce the following critical values for σ_c:

$$\underline{\sigma}_c \equiv 1 + \frac{1-s}{2s\beta\theta} \frac{b + \beta[1 + (1-\delta)b]}{1 - \delta b} \tag{33}$$

$$\bar{\sigma}_c \equiv 1 + \frac{1-s}{s(1+\beta)\theta} \frac{1 + (2-\delta)b}{1 - \delta b} \tag{34}$$

Proposition 5 . *Under Assumptions 1-3, let* $u(c, \mathcal{L})$ *be homogeneous of degree one,* $\beta \in (0,1)$ *and the consumption good be capital intensive at the modified golden rule* (k^*, ℓ^*)*. The following cases hold:*

i) *When* $b \in (-\beta/[1 + \beta(1-\delta)], 0)$, (k^*, ℓ^*) *is saddle-point stable;*

ii) *When* $b \in (-1/[2-\delta], -\beta/[1+\beta(1-\delta)])$, (k^*, ℓ^*) *is saddle-point stable if and only if* $\sigma_c \in (0, \underline{\sigma}_c)$*. Moreover, when* σ_c *crosses* $\underline{\sigma}_c$ *from below,* (k^*, ℓ^*) *undergoes a flip bifurcation;*

iii) *When* $b < -1/[2-\delta]$, (k^*, ℓ^*) *is saddle-point stable if and only if* $\sigma_c \in (0, \underline{\sigma}_c) \cup (\bar{\sigma}_c, 1)$*. Moreover, when* σ_c *crosses* $\underline{\sigma}_c$ *from below, or* σ_c *crosses* $\bar{\sigma}_c$ *from above,* (k^*, ℓ^*) *undergoes a flip bifurcation.*

Remark: Notice that the local stability properties of the steady-state and thus the possible existence of endogenous fluctuations do not depend on the elasticities of substitution in consumption and leisure.

As in the additive separable case, Proposition 5 exhibits a kind of substitutability between consumption and leisure concerning the existence of endogenous fluctuations. Notice also that the interval of values for the capital intensity difference which are compatible with endogenous fluctuations is much more important than in the additive separable case. Proposition shows indeed that cycles may exist even if $b < -1/[2-\delta]$. From this point of view, it also appears that depending on the values of b, the kind of substitutability between consumption and leisure is different. If $b \in (-1/[2-\delta], -\beta/[1+\beta(1-\delta)])$, endogenous fluctuations require a high relative share of consumption into total utility and thus a low relative share of leisure. This reinforces the conclusions derived in the additive separable case in which persistent cycles require strong substitution of consumption and low substitution of leisure. On the contrary, if $b < -1/[2-\delta]$, endogenous fluctuations are ruled out when the relative share of consumption is too high or too low. Intermediary values for σ_c are indeed required.

To find an intuition for this result we need to study carefully the properties of the homogeneous utility function. Notice that the marginal utility of leisure is an inceasing function of consumption, i.e. $u_{12} > 0$. It follows that from the point of view of marginal utility, consumption and leisure are substitute: an increase of consumption implies an increase of the marginal utility of leisure, so that for a given level of utility, the representative agent will decrease his leisure and thus increase his labor supply. Then the oscillations of the consumption good output that come from the Rybczinsky theorem affect more the labor supply than in the case of separable preferences.[16] Notice also that a rise of the total labor supply implies, all other things equal, an increase of the labor allocation in the investment good sector which is more labor intensive. This decreases the labor supply in the consumption good sector so that the capital intensity difference between the two sectors is increased. The oscillations of the consumption good output then become more important. As a result, the conjunction of these two effects provide more opportunities for persistent oscillations.

3.3 Comparison with related literature

The main comparison has to be made with the paper by Benhabib and Nishimura [4] in which a two-sector optimal growth model with inelastic labor is considered. In such a formulation, $u(c, \mathcal{L}) = \mathcal{U}(c)$ and the necessary

[16] A mechanism based on similar properties of preferences is provided by Michel and Venditti [19]. They consider a one-sector overlapping generations model with an altruistic representative agent having non-separable preferences over his life-cycle. They prove that contrary to the separable case, when the utility function is characterized by a positive cross derivative ($u_{12} > 0$), endogenous cycles are compatible with operative bequests.

condition for an optimal path is given by the Euler equation (11) with $\ell_t = \bar{\ell}$ for all $t \geq 0$. Proposition 1 still applies for the existence of a unique steady state. The linearised Euler equation (29) is then considered with $\mathcal{A}_2 = \mathcal{A}_4 = 0$ and it is easy to derive the following:

$$\begin{aligned}
\mathcal{P}(0) &= \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}} b[1+(1-\delta)b] - \tfrac{1}{\epsilon_{cc}}\beta\theta^2 \\
\mathcal{P}(1) &= -\tfrac{\varepsilon_{rk}}{\varepsilon_{ck}}(1-\delta b)(\beta - \theta^{-1}b) > 0 \\
\mathcal{P}(-1) &= \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}}[1+(2-\delta)b]\Big[b+\beta[1+(1-\delta)b]\Big] - \tfrac{2}{\epsilon_{cc}}\beta(1+\beta)\theta^2
\end{aligned}$$

In order to discuss the existence of endogenous cycles we will assume that the consumption good is capital intensive at the modified golden rule $(k^*, \bar{\ell})$, i.e. $b < 0$. Following the same arguments as in the proof of Proposition 4, it is then easy to prove:

Result 1: When $b \in (-\infty, -1/[2-\delta]) \cup (-\beta/[1+\beta(1-\delta)], 0)$, $(k^*, \bar{\ell})$ is saddle-point stable.

Result 2: When $b \in (-1/[2-\delta], -\beta/[1+\beta(1-\delta)])$, $(k^*, \bar{\ell})$ is saddle-point stable if and only if $\epsilon_{cc} \in (0, \bar{\epsilon}_{cc})$ with $\bar{\epsilon}_{cc}$ as defined by equation (32). Moreover, when ϵ_{cc} crosses $\bar{\epsilon}_{cc}$ from below, $(k^*, \bar{\ell})$ undergoes a flip bifurcation.

Result 3: When $\mathcal{U}(c) = c$, $(k^*, \bar{\ell})$ is saddle-point stable if and only if $b(\beta, \delta) \in (-\infty, -1/[2-\delta]) \cup (-\beta/[1+\beta(1-\delta)], 0)$. Moreover, if there is some $\beta^* \in (0,1)$ such that $b(\beta^*, \delta) \in (-1/[2-\delta], -\beta^*/[1+\beta^*(1-\delta)])$, then there exists $\bar{\beta} \in (0,1)$ such that, when β crosses $\bar{\beta}$ from above, $(k^*, \bar{\ell})$ undergoes a flip bifurcation.

Notice that while Benhabib and Nishimura [4] consider a general reduced form model with one state variable, specific results concerning two-sector optimal growth models are given only under the assumption of a linear utility function. This is precisely Result 3. Even if the intuition that the concavity of preferences decreases the possibility of endogenous fluctuations was suggested by the indirect utility formulation of Benhabib and Nishimura [4], Results 1 and 2 as stated above were not previously available in the literature.

Result 2 may be directly compared with Proposition 4. It corresponds to the particular case of an elasticity of intertemporal substitution in labor equal to zero. Endogenous fluctuations then require enough intertemporal substitution in consumption. Result 3 is similar to Corollary 2. It confirms the conclusion that when the utility function is linear with respect to consumption, leisure/labor arbitrages do not play any role on the stability properties of the optimal path. The specification used by Benhabib and Nishimura [5] does not provide therefore an accurate framework to study the effects of labor arbitrages on the distributions of the capital stock, investment, consumption and employment. Finally, Proposition 4 and Corollary 1 show that the standard two-sector optimal growth model with inelastic labor supply is actually a very particular case. As soon as the elasticity of intertemporal substitution in labor is high enough, endogenous cycles are ruled out and saddle-point stability holds.

When compared with the contributions which deal with general reduced from optimal growth models,[17] our results provide a very clear picture of the role of preferences. As it was shown by Venditti [26], curvatures of the indirect utility function may be easily linked with some characteristics of technologies such that the returns to scale. However, up to our knowledge, there is no result available in the literature which provides links between these curvatures and some characteristics of preferences.

We may finally compare our conclusions with the contribution of Benhabib and Rustichini [6]. They show that small discounting is compatible with endogenous cycles provided the coefficients of the Cobb-Douglas technologies are adequately chosen.[18] Their linear specification for the utility function prevents to characterize the effects of preferences. Our results show that periodic cycles are also compatible with small discounting when the technologies and the characteristics of preferences (either the elasticities of intertemporal substitution or the shares of consumption and leisure into total utility) are adequately chosen. We thus consider additional degrees of freedom to improve the plausibility of endogenous fluctuations.

4 Conclusion

In this paper we consider a two-sector optimal growth model with elastic labor supply. Contrary to most of the contributions available in the literature, we assume a general utility function for the representative agent. The technological side of the model described by the production possibility frontier is completely characterized in terms of the capital intensity difference accross sectors, b, and the elasticity of the rental rate, ε_{rk}. We prove existence and uniqueness of the steady state. Then we provide a dynamical analysis based on the fundamentals. We first show that when the investment good is capital intensive, i.e. $b \geq 0$, the steady state is always saddle-point stable. When the consumption good is capital intensive, i.e. $b < 0$, we consider two standard specifications for preferences. Considering an additive sepa-

[17] See for instance Montrucchio [21, 22].

[18] See also Venditti [25] for similar conclusions in a more general framework.

rable utility function, we show that the local dynamic properties depend on the elasticities of intertemporal substitution in consumption and labor. We then prove that, provided the parameter b satisfies mild restrictions, the occurrence of two-period cycles through a flip bifurcation requires the elasticity of intertemporal substitution in consumption to be high enough while the elasticity of labor needs to be low enough. As a corollary, we then obtain the surprising strong result that when the elasticity of intertemporal substitution in labor is infinite, endogenous fluctuations are ruled out.

When an homothetic utility function is considered, we first show that the local dynamic properties depend instead on the shares of consumption and leisure into total utility. Then we prove that conditions on these shares similar to the one exhibited with the elasticities of intertemporal susbtitution explains the existence of periodic cycles. Namely, the share of consumption needs to be high enough. We show however that endogenous fluctuations are compatible with a wider set of values for the parameter b than in the case of separable preferences. All there results therefore give a precise picture of the role of preferences on the occurrence of business cycles.

5 Appendix

5.1 Proof of Proposition 1

Consider in first step equation (13). Notice that the steady state value for κ only depends on the characteristics of the technologies and is independent from the per-period utility function. Moreover, equation (13) is equivalent to the equation which defines the stationary capital stock of a two-sector optimal growth model with inelastic labor.[19] The proof of Theorem 3.1 in Becker and Tsyganov [2] restricted to the case of one homogeneous agent applies so that there exists one unique κ^* solution of (13).

Consider now equation (14) evaluated at κ^*. We get:

$$T_3(\kappa^*, \delta\kappa^*, 1) = \frac{u_2(\ell T(\kappa^*, \delta\kappa^*, 1), \bar{\ell} - \ell)}{u_1(\ell T(\kappa^*, \delta\kappa^*, 1), \bar{\ell} - \ell)} \equiv h(\ell)$$

The function $h(\ell)$ is defined over $(0, \bar{\ell})$ and satisfies

$$h'(\ell) = -\frac{T(u_{11}w - u_{12}) + u_{22} - u_{21}w}{u_1}$$

The normality Assumption 3 implies that for any $(c, \mathcal{L}) > 0$, when considering the equilibrium wage rate $w = T_3(k, y, \ell)$, $u(c, \mathcal{L})$ satisfies $u_{11}(c, \mathcal{L})w -$

[19] See for instance Bosi, Magris and Venditti [8].

$u_{12}(c,\mathcal{L}) < 0$ and $u_{22}(c,\mathcal{L}) - u_{21}(c,\mathcal{L})w < 0$. It follows that $h'(\ell) > 0$. This monotonicity property together with the boundary conditions in Assumption 2 finally garantee the existence and uniqueness of a solution $\ell^* \in (0,\bar{\ell})$ of equation (14). $\square$

5.2 Proof of Lemma 1

The determinant of the matrix on the left-hand-side of the linearized dynamical system is equal to $-\beta\mathcal{A}_2(\mathcal{A}_1\mathcal{A}_5 + \mathcal{A}_2\mathcal{A}_4)$ and is assumed to be non zero. The linearized dynamical system then becomes

$$\begin{bmatrix} dk_{t+1} \\ d\ell_{t+1} \end{bmatrix} = \begin{bmatrix} -\frac{\mathcal{A}_4}{\mathcal{A}_2} & -\frac{\mathcal{A}_5}{\mathcal{A}_2} \\ \frac{\mathcal{A}_1\mathcal{A}_2^2+\mathcal{A}_4(\beta\mathcal{A}_1\mathcal{A}_4+\mathcal{A}_2\mathcal{A}_3)}{\beta\mathcal{A}_2(\mathcal{A}_1\mathcal{A}_5+\mathcal{A}_2\mathcal{A}_4)} & -\frac{\mathcal{A}_2^3-\mathcal{A}_5(\beta\mathcal{A}_1\mathcal{A}_4+\mathcal{A}_2\mathcal{A}_3)}{\beta\mathcal{A}_2(\mathcal{A}_1\mathcal{A}_5+\mathcal{A}_2\mathcal{A}_4)} \end{bmatrix} \begin{bmatrix} dk_t \\ d\ell_t \end{bmatrix}$$

Straightforward computations give the result. $\square$

5.3 Proof of Proposition 2

The intertemporal maximisation program of the representative agent (6) may be considered as two distinct optimization programs: a static one and a dynamic one. Consider the following static problem:

$$\begin{aligned} U(k_t, y_t) = \ & \max_{\ell_t} \quad u(T(k_t, y_t, \ell_t), \bar{\ell} - \ell_t) \\ & s.t. \quad T(k_t, y_t, \ell_t) + p_t y_t = r_t k_t + w_t \ell_t \end{aligned}$$

Denoting $\mu \geq 0$ the Lagrange multiplier, the first order condition is

$$u_1(c_t, \bar{\ell} - \ell_t)T_3(k_t, y_t, \ell_t) - u_2(c_t, \bar{\ell} - \ell_t) + \mu[w_t - T_3(k_t, y_t, \ell_t)] = 0$$

Since $T_3(k_t, y_t, \ell_t) = w_t$ we find the same necessary condition for an optimal leisure choice as equation (12). Under Assumptions (1)-(2), this optimization program is concave and it follows from Theorem 19.2, p. 282, in Madden [16] that the value function $U(k_t, y_t)$ is concave. We may then define the indirect utility function as $V(k_t, k_{t+1}) = U(k_t, k_{t+1} - (1-\delta)k_t)$ and the dynamic optimization program becomes:

$$\begin{aligned} & \max_{\{k_t\}_{t=0}^{+\infty}} \quad \sum_{t=0}^{+\infty} \beta^t V(k_t, k_{t+1}) \\ & s.t. \quad (k_t, k_{t+1}) \in \mathcal{D} \\ & \qquad\quad k_0 \text{ given} \end{aligned}$$

with

$$\mathcal{D} = \left\{ (k_t, k_{t+1}) \in \mathbb{R}_+^2 / (1-\delta)k_t \leq k_{t+1} \leq f^1(k_t, \bar{\ell}) + (1-\delta)k_t \right\}$$

The Euler equation for an interior optimal path is then given by the following difference equation of order two

$$V_2(k_t, k_{t+1}) + \beta V_1(k_{t+1}, k_{t+2}) = 0$$

This equation is equivalent to the system (11)-(12). The steady state k^* is thus given by Proposition 1. The characteristic polynomial is then

$$P(\lambda) = \lambda^2 \beta V_{12}^* + \lambda(\beta V_{11}^* + V_{22}^*) + V_{12}^* = 0$$

with $V_{ij}^* = V_{ij}(k^*, k^*)$. It is easy to show that the discriminant is

$$\Delta = (\beta V_{11}^* + V_{22}^* + 2\sqrt{\beta} V_{12}^*)(\beta V_{11}^* + V_{22}^* - 2\sqrt{\beta} V_{12}^*)$$

Under Assumptions 1-2, the Hessian matrix of $V(k_t, k_{t+1})$ is negative semi-definite so that $\Delta \geq 0$ and the characteristic roots are real. Following Levhari and Liviatan [15], when $\beta = 1$, it is easy to show as in the standard two-sector optimal growth model with inelastic labor that if the Hessian matrix of $V(k_t, k_{t+1})$ is non-singular at the steady state, the characteristic roots cannot be equal to 1 or -1. Moreover, since the product of the roots is equal to 1, the steady state is necessarily saddle-point stable. The proof is completed by the fact that the condition on the Hessian matrix of $V(k_t, k_{t+1})$ holds if the Hessian matrix of $u(c, \mathcal{L})$ is itself non singular at the steady state. □

5.4 Proof of Proposition 3

Consider the characteristic polynomial in Lemma 1 which may be rewritten as follows:

$$\mathcal{P}(\lambda) = \lambda^2 \beta[\mathcal{A}_1\mathcal{A}_5 + \mathcal{A}_2\mathcal{A}_4] - \lambda[\mathcal{A}_3\mathcal{A}_5 - \mathcal{A}_2^2 - \beta\mathcal{A}_4^2] + \mathcal{A}_1\mathcal{A}_5 + \mathcal{A}_2\mathcal{A}_4 \quad (35)$$

When $\lambda = 1$ and $\lambda = 0$, we easily get:

$$\begin{aligned} \mathcal{P}(1) &= (\mathcal{A}_2 + \beta\mathcal{A}_4)(\mathcal{A}_2 + \mathcal{A}_4) + \mathcal{A}_5[(1+\beta)\mathcal{A}_1 - \mathcal{A}_3] \\ \mathcal{P}(0) &= \mathcal{A}_1\mathcal{A}_5 + \mathcal{A}_2\mathcal{A}_4 \end{aligned}$$

Tedious but straightforward computations give

$$(\mathcal{A}_2 + \beta\mathcal{A}_4)(\mathcal{A}_2 + \mathcal{A}_4) + \mathcal{A}_5[(1+\beta)\mathcal{A}_1 - \mathcal{A}_3] = -\tfrac{\varepsilon_{rk}}{\varepsilon_{ck}}(\beta - \theta^{-1}b)\tfrac{T_3^*}{T_1^*}$$
$$\times \left\{ \left[a(1-\beta)\theta + (1-\delta b)\tfrac{T_3^*}{T_1^*} \right] (\xi_{cc} - \xi_{\mathcal{L}c}) + (1-\delta b)\tfrac{1}{T_1^*}\tfrac{c^*}{\mathcal{L}^*}(\xi_{\mathcal{L}\mathcal{L}} - \xi_{c\mathcal{L}}) \right\}$$

and

$$\begin{aligned}\mathcal{A}_1\mathcal{A}_5+\mathcal{A}_2\mathcal{A}_4 &= \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}}b[1+(1-\delta)b]\tfrac{T_3^*}{(T_1^*)^2}\Big\{T_3^*(\xi_{cc}-\xi_{\mathcal{L}c})+\tfrac{c^*}{\mathcal{L}^*}(\xi_{\mathcal{L}\mathcal{L}}-\xi_{c\mathcal{L}})\Big\}\\ &+ \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}}a^2\beta\theta^2\xi_{cc}+\tfrac{\varepsilon_{rk}}{\varepsilon_{ck}}a\theta\tfrac{T_3^*}{T_1^*}(\xi_{cc}-\xi_{\mathcal{L}c})\Big[b+\beta[1+(1-\delta)b]\Big]\\ &+ \beta\theta^2\tfrac{T_3^*}{(T_1^*)^2}\Big\{T_3^*\xi_{\mathcal{L}c}(\xi_{cc}-\xi_{\mathcal{L}c})+\tfrac{c^*}{\mathcal{L}^*}\xi_{cc}(\xi_{\mathcal{L}\mathcal{L}}-\xi_{c\mathcal{L}})\Big\}\end{aligned}$$

Concavity of the utility function implies

$$T_3^*\xi_{\mathcal{L}c}(\xi_{cc}-\xi_{\mathcal{L}c})+\frac{c^*}{\mathcal{L}^*}\xi_{cc}(\xi_{\mathcal{L}\mathcal{L}}-\xi_{c\mathcal{L}})=\frac{c^{*2}}{u_1^*u_2^*}[u_{11}^*u_{22}^*-u_{12}^{*2}]\geq 0$$

Moreover the normality assumption together with $\varepsilon_{rk}^*/\varepsilon_{ck}^*<0$ imply that if the investment good is capital intensive ($b>0$), $\mathcal{P}(0)$ is strictly positive.

From equation (15), we get $wa_{01}+ra_{11}=p$. When evaluated at the steady state, the Euler equation (11) rewrites as $p=\beta\theta r$. We then obtain after substitution in the previous equation

$$\beta r(\theta-\beta^{-1}a_{11})=wa_{01}>0$$

Prices positivity implies $\theta-\beta^{-1}a_{11}>0$. From equation (17), we observe that

$$\beta-\theta^{-1}b=\frac{a_{00}(\beta-\theta^{-1}a_{11})+\theta^{-1}a_{10}a_{01}}{a_{00}}>0$$

Then we have $b<\beta\theta$, which entails $b<1/\delta$. Summing up, we obtain $\beta-\theta^{-1}b>0$, $1-\delta b>0$. Then from the normality assumption and $\varepsilon_{rk}^*/\varepsilon_{ck}^*<0$, $\mathcal{P}(1)$ is strictly negative. We conclude that $\mathcal{P}(1)/\mathcal{P}(0)<0$ and there exists one characteristic root into $(0,1)$ while the other is greater than 1. □

5.5 Proof of Proposition 4

Lemma 1 implies that the characteristic roots have the same sign. Moreover, we know from the proof of Proposition 3 that $\mathcal{P}(1)<0$. Then the saddle-point property will be obtained either with $\mathcal{P}(0)>0$, or with $\mathcal{P}(0)<0$ and $\mathcal{P}(-1)>0$.

Consider equation (4) which can be written as follows

$$T(k,y,\ell)+p(k,y,\ell)y=r(k,y,\ell)k+w(k,y,\ell)\ell$$

We may then define the share of capital income in the gross national product evaluated at the steady state as

$$s=r^*k^*/(T^*+p^*y^*)$$

and it follows easily that

$$\frac{1-s}{s}=\frac{w^*\ell^*}{r^*k^*}=\frac{T_3^*\ell^*}{T_1^*k^*}$$

From the homogeneity of the social production function and equations (18)-(19) we get

$$\begin{aligned} 0 &= T_{11}^* k^* + T_{12}^* \delta k^* + T_{13}^* \ell^* \\ &= T_{11}^* k^* [1 - \delta b + a\ell^*/k^*] \\ \Leftrightarrow \quad \ell^*/k^* &= (1-\delta b)/a \end{aligned}$$

It follows finally

$$\frac{T_3^*}{T_1^*} = \frac{1-s}{s} \frac{a}{1-\delta b} \tag{36}$$

Now consider the characteristic polynomial as given by equation (35). Using the computations given in the proof of Proposition 3 and the above results we get

$$\begin{aligned} \mathcal{P}(0) = & -\tfrac{\varepsilon_{rk}}{\varepsilon_{ck}} \tfrac{a^2}{\epsilon_{cc}} \left[\tfrac{1-s}{s} \tfrac{b}{1-\delta b} + \beta\theta \right] \left[\theta + \tfrac{1-s}{s} \tfrac{1+(1-\delta)b}{1-\delta b} \right] \\ & - \tfrac{T_3^*}{(T_1^*)^2} \tfrac{c^*}{\mathcal{L}^*} \tfrac{1}{\epsilon_{\mathcal{L}\mathcal{L}}} \left[\tfrac{\varepsilon_{rk}}{\varepsilon_{ck}} b[1+(1-\delta)b] - \tfrac{1}{\epsilon_{cc}} \beta\theta^2 \right] \end{aligned}$$

$$\begin{aligned} \mathcal{P}(-1) = & -\tfrac{\varepsilon_{rk}}{\varepsilon_{ck}} \tfrac{a^2}{\epsilon_{cc}} \left[\tfrac{1-s}{s} \tfrac{1+(2-\delta)b}{1-\delta b} + (1+\beta)\theta \right] \left[\tfrac{1-s}{s} \tfrac{b+\beta[1+(1-\delta)b]}{1-\delta b} + 2\beta\theta \right] \\ & - \tfrac{T_3^*}{(T_1^*)^2} \tfrac{c^*}{\mathcal{L}^*} \tfrac{1}{\epsilon_{\mathcal{L}\mathcal{L}}} \left\{ \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}} [1+(2-\delta)b] \Big[b + \beta[1+(1-\delta)b] \Big] - \tfrac{2}{\epsilon_{cc}} \beta(1+\beta)\theta^2 \right\} \end{aligned}$$

Notice that from equation (25) we get $(1-s)/s\beta\theta = w^*\ell^*/p^*k^*$ and thus

$$\frac{1-s}{s} \frac{b}{1-\delta b} + \beta\theta = \beta\theta \left(\frac{w^* b}{p^* a} + 1 \right)$$

From equation (15) we finally get

$$\frac{1-s}{s} \frac{b}{1-\delta b} + \beta\theta = \beta\theta \frac{a_{11}}{p^* a_{10}} > 0 \tag{37}$$

Notice also that

$$\frac{1-s}{s} \frac{1+(1-\delta)b}{1-\delta b} + \theta \geq \frac{1-s}{s} \left[\frac{b}{1-\delta b} + 1 \right] + \beta\theta > 0$$

It follows that if $b < -1/[1-\delta]$, $\mathcal{P}(0) > 0$ and the steady state is saddle-point stable.

Consider now the case $b \in (-1/[1-\delta], -1/[2-\delta]) \cup (-\beta/[1+\beta(1-\delta)], 0)$. Since $b > -1/[1-\delta]$, it is easy to show that

$$\tfrac{1-s}{s} \tfrac{1+(2-\delta)b}{1-\delta b} + (1+\beta)\theta > \tfrac{1-s}{s} \tfrac{b+\beta[1+(1-\delta)b]}{1-\delta b} + 2\beta\theta > \tfrac{1-s}{s} \tfrac{b}{1-\delta b} + \beta\theta > 0$$

Then $\mathcal{P}(-1) > 0$ and $\mathcal{P}(0)$ may be positive or negative depending on the values of ϵ_{cc} and $\epsilon_{\mathcal{L}\mathcal{L}}$. In both cases the steady state is saddle-point stable. This proves case i).

Assume finally that $b \in (-1/[2-\delta], -\beta/[1+\beta(1-\delta)])$. The term between brackets on the second line of the expression of $\mathcal{P}(0)$ is positive if and only if $\epsilon_{cc} > \underline{\epsilon}_{cc}$ with

$$\underline{\epsilon}_{cc} \equiv \frac{\varepsilon_{ck}\beta\theta^2}{\varepsilon_{rk} b[1+(1-\delta)b]} > 0 \tag{38}$$

Consider now the term between brackets on the second line in the expression of $\mathcal{P}(-1)$. It will be positive if and only if $\epsilon_{cc} > \bar{\epsilon}_{cc}$, with $\bar{\epsilon}_{cc} > 0$ as defined by equation (32), and $\mathcal{P}(-1) > 0$ when $\bar{\epsilon}_{cc} \geq \epsilon_{cc}$. It is easy to show that $\bar{\epsilon}_{cc} > \underline{\epsilon}_{cc}$. Then we have shown that if $\epsilon_{cc} \leq \underline{\epsilon}_{cc}$, $\mathcal{P}(0) > 0$. Moreover, when $\bar{\epsilon}_{cc} \geq \epsilon_{cc} > \underline{\epsilon}_{cc}$, independently from the sign of $\mathcal{P}(0)$, we have $\mathcal{P}(-1) > 0$ while $\mathcal{P}(1) < 0$. It follows that when $\bar{\epsilon}_{cc} \geq \epsilon_{cc}$, the steady state is saddle-point stable for any $\epsilon_{\mathcal{LL}} > 0$ and case ii) is proved.

Consider finally case iii) with $\epsilon_{cc} > \bar{\epsilon}_{cc}$. We have $\lim_{\epsilon_{\mathcal{LL}} \to 0} \mathcal{P}(-1) = -\infty$ while $\lim_{\epsilon_{\mathcal{LL}} \to \infty} \mathcal{P}(-1) > 0$. Therefore, there exists $\bar{\epsilon}_{\mathcal{LL}} \in (0, +\infty)$ such that $\mathcal{P}(-1) < 0$ for any $\epsilon_{\mathcal{LL}} \in (0, \bar{\epsilon}_{\mathcal{LL}})$. Notice now that by definition

$$\begin{aligned} \mathcal{P}(-1) &= (1+\beta)P(0) - \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}}\tfrac{1}{\epsilon_{cc}}\left\{\beta\left[a\theta + \tfrac{T_3^*}{T_1^*}[1+(1-\delta)b]\right]^2 + \left[\beta a\theta + \tfrac{T_3^*}{T_1^*}b\right]^2\right\} \\ &\quad - \tfrac{T_3^*}{(T_1^*)^2}\tfrac{c^*}{\mathcal{L}^*}\tfrac{1}{\epsilon_{\mathcal{LL}}}\left\{\tfrac{\varepsilon_{rk}}{\varepsilon_{ck}}\left[b^2 + \beta[1+(1-\delta)b]^2\right] - \tfrac{1}{\epsilon_{cc}}\beta(1+\beta)\theta^2\right\} \end{aligned}$$

If $\epsilon_{\mathcal{LL}} = \hat{\epsilon}_{\mathcal{LL}}$, then $\mathcal{P}(0) = 0$ and we get after simplifications $\mathcal{P}(-1) > 0$. It follows that $\hat{\epsilon}_{\mathcal{LL}} > \bar{\epsilon}_{\mathcal{LL}}$ and we have $\mathcal{P}(-1) > 0$ as soon as $\epsilon_{\mathcal{LL}} > \bar{\epsilon}_{\mathcal{LL}}$ while $\mathcal{P}(1) < 0$. Combining these results we conclude that the saddle-point property holds when $\epsilon_{\mathcal{LL}} \in (\bar{\epsilon}_{\mathcal{LL}}, +\infty)$. It is finally immediate to verify that when $\epsilon_{\mathcal{LL}}$ crosses $\bar{\epsilon}_{\mathcal{LL}}$ from above, one characteristic root goes through -1 and the system undergoes a flip bifurcation. □

5.6 Proof of Corollary 2

i) Using the computations given in the proof of Proposition 4 we get

$$\frac{\mathcal{P}(-1)}{\mathcal{P}(0)} = \frac{[1+(2-\delta)b][b+\beta[1+(1-\delta)b]]}{b[1+(1-\delta)b]} < 0$$

if and only if $b \in (-1/[1-\delta], -1/[2-\delta]) \cup (-\beta/[1+\beta(1-\delta)], 0)$.

ii) Assume that there is some $\beta^* \in (0,1)$ such that $b(\beta^*, \delta) \in (-1/[2-\delta], -\beta^*/[1+\beta^*(1-\delta)])$. Then from Proposition 2 and a continuity argument, there exists $\bar{\beta} \in (0,1)$ such that $\mathcal{P}(-1)/\mathcal{P}(0) = 0$ when $\beta = \bar{\beta}$ and $\mathcal{P}(-1)/\mathcal{P}(0) > 0$ when β is in a left neighbourhood of $\bar{\beta}$. It follows that when $\beta = \bar{\beta}$, the system undergoes a flip bifurcation. □

5.7 Proof of Proposition 5

From equation (23) and the linear homogeneity of $u(c, \mathcal{L})$ we get

$$u^* = u_1^*(c^* + \mathcal{L}^* T_3^*), \; u_{12}^* = -u_{11}^*(c^*/\mathcal{L}^*) = -u_{22}^*(\mathcal{L}^*/c^*)$$

We easily derive from this

$$\xi_{cc} - \xi_{\mathcal{L}c} = \xi_{\mathcal{L}\mathcal{L}} - \xi_{c\mathcal{L}} = \xi_{cc}(1-\sigma_c)^{-1}$$

Now consider the characteristic polynomial as given by equation (35) and the expression (36) in the proof of Proposition 4. Using the computations given in the proof of Proposition 3 and the above results we get

$$\mathcal{P}(0) = \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}} \xi_{cc} a^2 (1-\sigma_c)^{-2} \left[\tfrac{1-s}{s}\tfrac{b}{1-\delta b} + (1-\sigma_c)\beta\theta\right]\left[\tfrac{1-s}{s}\tfrac{1+(1-\delta)b}{1-\delta b} + (1-\sigma_c)\theta\right]$$

$$\begin{aligned}\mathcal{P}(-1) = & \tfrac{\varepsilon_{rk}}{\varepsilon_{ck}} \xi_{cc} a^2 (1-\sigma_c)^{-2} \left[\tfrac{1-s}{s}\tfrac{1+(2-\delta)b}{1-\delta b} + (1-\sigma_c)(1+\beta)\theta\right] \\ & \times \left[\tfrac{1-s}{s}\tfrac{b+\beta[1+(1-\delta)b]}{1-\delta b} + (1-\sigma_c)2\beta\theta\right]\end{aligned}$$

In order to study the local stability properties of the steady state we need to check the sign of the expressions between brackets into $\mathcal{P}(0)$ and $\mathcal{P}(-1)$. We have

$$\begin{aligned}
\tfrac{1-s}{s}\tfrac{b}{1-\delta b} + (1-\sigma_c)\beta\theta > 0 \quad &\Leftrightarrow \quad 1-\sigma_c > -\tfrac{1-s}{s\beta\theta}\tfrac{b}{1-\delta b} \equiv \mathcal{B}_1 \\
\tfrac{1-s}{s}\tfrac{1+(1-\delta)b}{1-\delta b} + (1-\sigma_c)\theta > 0 \quad &\Leftrightarrow \quad 1-\sigma_c > -\tfrac{1-s}{s\theta}\tfrac{1+(1-\delta)b}{1-\delta b} \equiv \mathcal{B}_2 \\
\tfrac{1-s}{s}\tfrac{1+(2-\delta)b}{1-\delta b} + (1-\sigma_c)(1+\beta)\theta > 0 \quad &\Leftrightarrow \quad 1-\sigma_c > -\tfrac{1-s}{s(1+\beta)\theta}\tfrac{1+(2-\delta)b}{1-\delta b} \equiv \mathcal{B}_3 \\
\tfrac{1-s}{s}\tfrac{b+\beta[1+(1-\delta)b]}{1-\delta b} + (1-\sigma_c)2\beta\theta > 0 \quad &\Leftrightarrow \quad 1-\sigma_c > -\tfrac{1-s}{2s\beta\theta}\tfrac{b+\beta[1+(1-\delta)b]}{1-\delta b} \equiv \mathcal{B}_4
\end{aligned}$$

Some straightforward computations show that $\mathcal{B}_1 > \mathcal{B}_4 > \mathcal{B}_3 > \mathcal{B}_2$. From equation (37) in the proof of Proposition 4 we get

$$\frac{1-s}{s}\frac{b}{1-\delta b} + (1-\sigma_c)\beta\theta = \beta\theta\left(\frac{a_{11}}{p^* a_{10}} - \sigma_c\right)$$

Using again equation (15) it is then easy to show that since $b < 0$, we have $a_{11}/p^* a_{10} < 1$ so that the above expression may be positive or negative. This implies also that $\mathcal{B}_1 < 1$ and thus $1 > \mathcal{B}_1 > \mathcal{B}_4 > \mathcal{B}_3 > \mathcal{B}_2$.

We have now to discuss the local stability properties as a function of b. Lemma 1 implies that the characteristic roots have the same sign. Moreover, we know from the proof of Proposition 3 that $\mathcal{P}(1) < 0$. Then the saddle-point property will be obtained either with $\mathcal{P}(0) > 0$, or with $\mathcal{P}(0) < 0$ and $\mathcal{P}(-1) > 0$.

Assume first that $b \in (-\beta/[1+\beta(1-\delta)], 0)$. Then $0 > \mathcal{B}_4 > \mathcal{B}_3 > \mathcal{B}_2$ and $\mathcal{P}(-1) > 0$ while $\mathcal{P}(0)$ may be positive or negative depending on the value of σ_c. The steady state is then saddle-point stable.

If $b \in (-1/[2-\delta], -\beta/[1+\beta(1-\delta)])$, we have $0 > \mathcal{B}_3 > \mathcal{B}_2$. It follows that $\mathcal{P}(-1) > 0$ if and only if $\sigma_c \in (0, \underline{\sigma}_c)$ with $\underline{\sigma}_c$ as defined by (33), while $\mathcal{P}(0) < 0$ if and only if $\sigma_c \in (\tilde{\sigma}_c, 1)$ with

$$\tilde{\sigma}_c \equiv 1 + \frac{1-s}{s\beta\theta}\frac{b}{1-\delta b} \tag{39}$$

From $\mathcal{B}_1 > \mathcal{B}_4$ we derive $\underline{\sigma}_c > \tilde{\sigma}_c$ and the steady state is saddle-point stable if and only if $\sigma_c \in (0, \underline{\sigma}_c)$. It is finally immediate to verify that when σ_c crosses $\underline{\sigma}_c$ from below, one characteristic root goes through -1 and the system undergoes a flip bifurcation.

Assume now that $b \in (-1/[1-\delta], -1/[2-\delta])$. Then $0 > \mathcal{B}_2$ and $\mathcal{P}(-1) > 0$ if and only if $\sigma_c \in (0, \underline{\sigma}_c) \cup (\bar{\sigma}_c, 1)$ with $\bar{\sigma}_c$ as defined by equation (34), while $\mathcal{P}(0) < 0$ if and only if $\sigma_c \in (\tilde{\sigma}_c, 1)$. The steady state is thus saddle-point stable if and only if $\sigma_c \in (0, \underline{\sigma}_c) \cup (\bar{\sigma}_c, 1)$. It is finally immediate to verify that when σ_c crosses $\underline{\sigma}_c$ from below, or σ_c crosses $\bar{\sigma}_c$ from above, one characteristic root goes through -1 and the system undergoes a flip bifurcation.

Finally, when $b < -1/[1-\delta]$, we have $\mathcal{P}(-1) > 0$ if and only if $\sigma_c \in (0, \underline{\sigma}_c) \cup (\bar{\sigma}_c, 1)$ while $\mathcal{P}(0) < 0$ if and only if $\sigma_c \in (\tilde{\sigma}_c, \hat{\sigma}_c)$ with

$$\hat{\sigma}_c \equiv 1 + \frac{1-s}{s\theta}\frac{1+(1-\delta)b}{1-\delta b} > \bar{\sigma}_c \tag{40}$$

The steady state is thus saddle-point stable if and only if $\sigma_c \in (0, \underline{\sigma}_c) \cup (\bar{\sigma}_c, 1)$. It is finally immediate to verify that when σ_c crosses $\underline{\sigma}_c$ from below, or σ_c crosses $\bar{\sigma}_c$ from above, one characteristic root goes through -1 and the system undergoes a flip bifurcation. □

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