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Sunspot Fluctuations in Two-Sector Economies with Heterogeneous Agents

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Sunspot fluctuations in two-sector economies with heterogeneous agents*

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Abstract: *We study a two-sector model with heterogeneous agents and borrowing constraint on labor income. We show that the relative capital intensity difference across sectors is crucial for the conditions required to get indeterminacy and endogenous fluctuations. When the consumption sector is significantly capital intensive, indeterminacy occurs with elasticities of capital-labor substitution which are in accordance with recent empirical estimates and with a large set of values for the elasticity of the offer curve. When the investment good is capital intensive, indeterminacy requires a quite low elasticity of substitution in the consumption good sector while this elasticity may remain close to unity for the investment good. In both cases, persistent endogenous cycles may also appear through flip or Hopf bifurcations.*

Keywords: *Heterogeneous agents, borrowing constraint, two-sector model, indeterminacy.*

Journal of Economic Literature Classification Numbers: C61, E32, E41.

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1 Introduction

We consider an infinite horizon model with heterogeneous agents and borrowing constraint on labor income in the spirit of Woodford [24] and Grandmont *et al.* [17]. Contrary to the initial aggregate formulation, we assume two different technologies producing a consumption good and an investment good, respectively. We then appraise the local stability properties of the economy as a function of technologies, i.e. the relative capital intensity difference across sectors and the elasticity of the interest rate, and preferences, i.e. the elasticity of the offer curve. Our aim is to give conditions for the existence of local indeterminacy when there are heterogeneous agents, some of them being financially constrained, i.e. being unable to borrow against labor income. From this point of view, our formulation is close to the model of Becker & Tsyganov [5] with the notable exception that we consider a monetary economy through a cash-in-advance constraint. The main result shows that when the consumption good is sufficiently capital intensive, local indeterminacy arises when the elasticities of capital-labor substitution in both sectors are slightly greater than unity and the elasticity of the offer curve is low enough. As recently shown in empirical analysis,[1] these conditions appear to be compatible with macroeconomic evidences.

It is now well known that in a wide class of macrodynamic models locally indeterminate equilibria and sunspot fluctuations easily arise. However, most of the conditions require the consideration of parameter values which do not fit standard empirical estimates. In one-sector models with homogeneous agents, either a large amount of increasing returns incompatible with the findings of Basu & Fernald [3], jointly with an unconventional slope for the labor demand, or lower externalities but with extremely large elasticities of intertemporal substitution in consumption and elasticities of labor supply, have to be considered.[2] In one-sector models with heterogeneous agents a very low elasticity of capital-labor substitution is required.[3] Finally, in two-sector models, local indeterminacy becomes compatible with arbitrarily low increasing or even constant social returns to scale, but requires a very high degree of intertemporal substitution in consumption.[4]

In this paper, our strategy consists in starting from a two-sector frame-

[1] [15] for estimations of the elasticity of capital-labor substitution and Takahashi *et al.* [23] for some evaluation of the capital intensity difference in the main developed countries.

[2] See Benhabib & Farmer [6], Farmer & Guo [16], Lloyd-Braga, Nourry & Venditti [18], Pintus [22].

[3] See Grandmont *et al.* [17].

[4] See, for instance, Benhabib & Nishimura [8], Nishimura & Venditti [20].

work but instead of assuming productive externalities, we consider two types of representative agents labeled, respectively, “capitalists” and “workers”, and some imperfection on the financial market. The agents are indeed distinguished with respect to their preferences, their degree of impatience and their ability to borrow on the credit market: “capitalists” are described by a logarithmic utility function, i.e. a unitary elasticity of intertemporal substitution in consumption. They do not work, consume from the returns on their savings and can borrow freely. On the contrary, “workers” are described by a general additively separable untility function defined over consumption and labor, which is characterized by a higher discount rate than the “capitalists”. They supply labor elastically and are subject every period to a liquidity (“cash-in-advance”) constraint, reflecting their difficulty to finance the consumption good out of their wage income. The economy then includes one consumption good, and two assets, outside money, which is in constant supply, and capital. Under these hypotheses, in a neighborhood of the monetary steady state, capitalists end up holding the whole capital stock and no money (they finance consumption and investment entirely out of capital income) whereas workers are forced to convert in money balances the whole amount of their wage bill.[5] It is worth recalling that our model is similar to the Becker and Tsyganov [5] two-sector model with heterogenous agents subject to borrowing constraints. The main differences lie on the facts that we consider a monetary economy with a segmented financial market and we allow for an elastic labor supply. The segmentation assumption allows to easily study the local stability properties of equilibria which are highly sensitive to the elasticity of the offer curve, i.e. the elasticity of the labor supply.

We show that when the consumption good sector is significantly capital intensive, indeterminacy occurs under mild conditions based on some elasticites of capital-labor substitution slightly larger than unity. While Cobb-Douglas technologies are widely used in growth theory, recent papers have questioned the empirical relevance of this specification. Duffy & Papageorgiou [15] for instance consider a panel of 82 countries over a 28-year period to estimate a CES production function specification. They find that for the entire sample of countries the assumption of unitary elasticity of substitution may be rejected. Moreover, dividing the sample of countries up into several subsamples, they find that capital and labor have an elasticity of substitution significantly greater than unity (*i.e.* contained in $[1.14, 3.24]$)

[5]As initially proved in Becker [4], the households with the lowest rate of discount, i.e. “capitalists”, own all the capital in the long-run.

in the richest group of countries while the elasticity is slightly less than unity in the poorest group of countries. The analysis of a CES example shows that our conditions fall into the estimates of Duffy & Papageorgiou [15]. We also prove that local indeterminacy is compatible with a large set of values for the elasticity of the offer curve. This implies that contrary to the two-sector models with productive externalities, sunspot fluctuations may arise while the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply remain low. In addition we show that in such a configuration, the change of stability occurs only through a flip bifurcation, giving rise to period-two cycles.

When the investment good is capital intensive we also show that local indeterminacy can arise, but requires a quite low elasticity of substitution in the consumption good sector (lower than the share of capital in total income), while this elasticity may remain close to unity for the investment good. Cycles may also appear through flip or Hopf bifurcations. Such a result contrasts with the conclusions obtained by Benhabib & Nishimura [7] in a standard two-sector optimal growth model. They show indeed that the existence of optimal cycles requires the consumption good to be more capital intensive than the investment good. We prove in the current framework that in presence of heterogeneous agents and borrowing constraint on labor income, persistent oscillations may also appear while the investment good is more capital intensive than the consumption good. However, they require the consideration of extreme parameters values.

The remainder of the paper is organized as follows. In section 2 we describe the behavior of the agents. Section 3 is devoted to the characterization of the technology while in Section 4 we introduce intertemporal equilibrium and prove the existence of a unique stationary solution. Section 5 presents the characteristic polynomial and the main arguments of the geometrical method used to study the local dynamics and bifurcations. Section 6 presents the main results. In Section 7 we calibrate the model using a CES specification. Section 8 discusses the plausibility of our results and provides economic intuitions. In Section 9 we conclude the paper. All the proofs are gathered in a final Appendix.

2 Agents and intertemporal optimization

The economy is populated by two types of infinite-lived agents, workers and capitalists, each of them identical within their own type. Workers consume, supply endogenously labor and are subject to a financial constraint preventing them from borrowing against current as well as future labor income. Capitalists conversely do not work and therefore are subject only to the budget constraint.

2.1 Capitalists

Capitalists maximize a logarithmic intertemporal utility function

$$\sum_{t=1}^{\infty} \beta^t \ln c_t^c$$

where c^c denotes their consumption and $\beta \in (0,1)$ their discount factor. Since capitalists do not earn any labor income, they are subject to the budget constraint

$$c_t^c + p_t \left[k_{t+1}^c - (1-\delta) k_t^c\right] + q_t M_{t+1}^c \leq r_t k_t^c + q_t M_t^c$$

where p stands for the price of investment, k for the capital, $\delta \in [0,1]$ for the depreciation rate of capital, M for the money balances, q for the price of money and r for the interest rate in terms of the consumption good. In addition, the usual non-negativity constraints $k_t^c \geq 0$ and $M_t^c \geq 0$, must be satisfied. The first order conditions state as follows

$$\begin{aligned} \frac{c_{t+1}^c}{c_t^c} &\geq \beta \frac{p_{t+1}(1-\delta)+r_{t+1}}{p_t} \\ \frac{c_{t+1}^c}{c_t^c} &\geq \beta \frac{q_{t+1}}{q_t} \end{aligned} \tag{1}$$

and hold with equality if $k_t^c > 0$ and $M_t^c > 0$. We shall focus in the sequel on the case where

$$\frac{p_{t+1}(1-\delta)+r_{t+1}}{p_t} > \frac{q_{t+1}}{q_t} \tag{2}$$

holds at all dates. The gross rate of return on capital is then higher than the profitability of money holding, and capitalists decide to hold only capital and no money.[6] Their optimal policy for all $t \geq 1$ takes the explicit form

$$k_{t+1}^c = \beta \left(1 - \delta + r_t/p_t\right) k_t^c \tag{3}$$

meanwhile consumption choice is given by $c_t^c = (1-\beta)\left[p_t(1-\delta) + r_t\right] k_t^c$.[7]

[6] According to Becker [4], the capital income distribution is determined in the long-run steady state by the lowest discount rate.

[7] A more general utility function for capitalists could be considered as in Barinci [1]. However, with non logarithmic preferences, the optimal policy cannot be explicitely computed and we have to deal with the Euler equation which defines an implicit 3-dimensional dynamical system. In order to reach more tractable conclusions we still consider the same assumption as in Woodford [24].

2.2 Workers

Workers maximize their lifetime utility function

$$\sum_{t=0}^{\infty} \gamma^t \left[u\left(c_t^w\right) - \gamma v\left(l_t\right) \right] \tag{4}$$

where c^w stands for consumption, u for the per-period utility of consumption, l for labor supply, v for the per-period disutility of labor and $\gamma \in (0,1)$ for the discount factor which also satisfies

$$\gamma < \beta \tag{5}$$

The functions u and v satisfy the following properties.

Assumption 1. *$u(c)$ and $v(l)$ are C^r, with r large enough, for, respectively, $c > 0$ and $0 \leq l < l^*$, where $l^* > 0$ is the (possibly infinite) workers' endowment of labor. They satisfy $u'(c) > 0$, $u''(c) < 0$, $v'(l) > 0$, $v''(l) > 0$ and $\lim_{l \to l^*} v'(l) = +\infty$. Consumption and leisure are assumed to be gross substitutes, i.e. $u'(c) + cu''(c) > 0$.*

Notice that gross substitutability is equivalent to assuming that the elasticity of intertemporal substitution in consumption, $\epsilon_c = -u'(c)/c''(c)c$, is larger than unity. This restriction implies that the labor supply is an increasing function of the real wage.

Workers are subject to the dynamic budget constraint

$$c_t^w + p_t \left[k_{t+1}^w - (1-\delta) k_t^w \right] + q_t M_{t+1}^w \leq w_t l_t + r_t k_t^w + q_t M_t^w \tag{6}$$

where all the variables are those introduced in the capitalists' budget constraint with the exception of the wage w in terms of units of consumption good. In addition, workers face the borrowing constraint

$$c_t^w + p_t \left[k_{t+1}^w - (1-\delta) k_t^w \right] \leq r_t k_t^w + q_t M_t^w \tag{7}$$

reflecting their difficulty to borrow against future labor income, and the non-negativity conditions on asset holding $k_t^w \geq 0$ and $M_t^w \geq 0$. Straightforward computations show that at the optimum $k_t^w = 0$ if and only if

$$u'\left(c_t^w\right) > \gamma u'\left(c_{t+1}^w\right) \tfrac{p_{t+1}(1-\delta) + r_{t+1}}{p_t} \tag{8}$$

We shall focus in the following on the case where (8) holds at all dates. Workers then choose not to hold capital $\left(k_t^w = 0\right)$. Moreover, we assume a constant money supply, *i.e.* $M_t = \bar{M} > 0$ for every t. Since capitalists do not hold any money we get $M_t^w = \bar{M}$ and the budget constraint implies

$$c_t^w = q_t \bar{M} \tag{9}$$

We then derive from the liquidity constraint that workers hold the quantity of money balances

$$q_t \bar{M} = w_t l_t \tag{10}$$

employers pay to them in exchange to their labor services at the end of each period. From (10) we obtain the following relationship between the growth factor of labor income and the growth factor of the money price:

$$\frac{w_t l_t}{w_{t+1} l_{t+1}} = \frac{q_t}{q_{t+1}} \tag{11}$$

Workers maximize (4) subject to (9) and (10). This yields, taking into account (11), the first order condition reflecting the standard trade-off between consumption and labor

$$U\left(c_{t+1}^w\right) = V\left(l_t\right) \tag{12}$$

with $U(c) \equiv cu'(c)$ and $V(l) \equiv lv'(l)$.

3 Technology

We assume that the consumption good Y^0 and the investment good Y^1 in each period t are produced by two different constant returns to scale technologies employing capital and labor as productive inputs:

$$Y^i = F^i\left(K^i, L^i\right)$$

$i = 0, 1$, where $\left(K^0, L^0\right)$ and $\left(K^1, L^1\right)$ denote the amount of inputs used respectively in consumption and investment sectors. At equilibrium we have:

$$\begin{aligned} K^0 + K^1 &= K = N^c k^c \\ L^0 + L^1 &= L = N^w l \end{aligned}$$

K and L stand for aggregate capital and labor, N^c and N^w denote respectively the number of capitalists and workers, k^c is the stock of capital held by the representative capitalist and l is the labor supply of each worker. All the previous variables can be normalized with respect to the size N^w of the labor force:

$$\begin{aligned} y^i &\equiv Y^i/N^w \\ k^i &\equiv K^i/N^w,\ l^i \equiv L^i/N^w \\ k &\equiv K/N^w = N^c k^c/N^w,\ l \equiv L/N^w \end{aligned}$$

$i = 0, 1$. Observe that at the equilibrium $k^0 + k^1 = k$ and $l^0 + l^1 = l$.

Without loss of generality assume a constant ratio n between capitalists and workers: $k = k^c N^c/N^w \equiv nk^c$. Homogeneity of production functions implies:

$$y^i = f^i\left(k^i, l^i\right)$$

where $f^i \equiv F^i/N^w$ is the per-worker production in sector $i = 0, 1$.

Assumption 2. *Each production function $f^i : R^2_+ \to R_+$, $i = 0, 1$, is C^r, with r large enough, increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $f^i_1(0, x) = f^i_2(x, 0) = +\infty$, $f^i_1(+\infty, x) = f^i_2(x, +\infty) = 0$.*

For any given $\left(k, y^1, l\right)$, we define a temporary equilibrium by solving the following problem of optimal allocation of factors between the two sectors:

$$\begin{array}{ll} \max\limits_{\{k^0,k^1,l^0,l^1\}} & f^0\left(k^0, l^0\right) \\ s.t. & y^1 \leq f^1\left(k^1, l^1\right) \\ & k^0 + k^1 \leq k \\ & l^0 + l^1 \leq l \\ & k^0, k^1, l^0, l^1 \geq 0 \end{array} \tag{13}$$

The associated Lagrangian is

$$L = f^0\left(k^0, l^0\right) + p\left[f^1\left(k^1, l^1\right) - y^1\right] + r\left[k - k^0 - k^1\right] + w\left[l - l^0 - l^1\right]$$

Solving the corresponding first order conditions give optimal demand functions for capital and labor, namely $k^0\left(k, y^1, l\right)$, $l^0\left(k, y^1, l\right)$, $k^1\left(k, y^1, l\right)$ and $l^1\left(k, y^1, l\right)$. The resulting value function

$$T\left(k, y^1, l\right) = f^0\left(k^0\left(k, y^1, l\right), l^0\left(k, y^1, l\right)\right)$$

is called the social production function and describes the frontier of the production possibility set. The constant returns to scale of technologies imply that $T\left(k, y^1, l\right)$ is also homogeneous of degree one and thus concave non-strictly. We will assume in the following that $T\left(k, y^1, l\right)$ is at least C^2.

It is easy to show from the first order conditions that the rental rate of capital, the price of investment good and the wage rate satisfy

$$\begin{array}{rcl} T_1\left(k, y^1, l\right) & = & r\left(k, y^1, l\right) \\ T_2\left(k, y^1, l\right) & = & -p\left(k, y^1, l\right) \\ T_3\left(k, y^1, l\right) & = & w\left(k, y^1, l\right) \end{array}$$

Concavity of $T\left(k, y^1, l\right)$ implies

$$T_{11}\left(k, y^1, l\right) \leq 0,\ T_{22}\left(k, y^1, l\right) \leq 0,\ T_{33}\left(k, y^1, l\right) \leq 0$$

However the signs of the cross derivatives are not obvious. Consider thus the relative capital intensity difference across sectors defined as follows

$$b \equiv a_{01}\left(\frac{a_{11}}{a_{01}} - \frac{a_{10}}{a_{00}}\right) \tag{14}$$

with

$$a_{00} \equiv l^0/y^0, \quad a_{10} \equiv k^0/y^0, \quad a_{01} \equiv l^1/y^1, \quad a_{11} \equiv k^1/y^1$$

the capital and labor coefficients in each sector. The sign of b is positive if and only if the investment good is capital intensive. It is shown in Bosi *et al.* [11] that

$$T_{12} = -T_{11}b,\ T_{31} = -T_{11}a \geq 0,\ T_{32} = T_{11}ab$$

with $a \equiv k^0/l^0 > 0$ the capital-labor ratio in the consumption good sector. It follows therefore that the sign of T_{12} and T_{32} crucially depends on the sign of the capital intensity difference across sectors b.[8] It is also easy to show that $T_{22}\left(k, y^1, l\right)$ and $T_{33}\left(k, y^1, l\right)$ may be written as

$$T_{22} = T_{11}b^2,\ T_{33} = T_{11}a^2$$

These expressions will be useful to study the dynamical properties of the equilibrium paths.

[8] As initially proved in a two-sector optimal growth model with inelastic labor by Benhabib & Nishimura [7], the cross derivative T_{12} is positive if and only if the investment good is capital intensive, *i.e.* $b > 0$.

4 Intertemporal equilibrium

Since within each type all agents are identical we can focus on symmetric equilibrium. Coupling the capital accumulation equation (3) and the workers' first order condition (12) with equilibrium conditions in factor markets and recalling that $c_{t+1}^w = w_t l_t$, we can introduce the intertemporal equilibrium with perfect foresight in terms of k and l. In each period t, k_t is a predetermined variable (in order to simplify notation, we will set $c = c^w$ and $k = k^c$) and $k_0 > 0$ is the stock of physical equipment available in period zero.

Definition 1. *For any given initial capital stock $k_0 > 0$, an intertemporal equilibrium with perfect foresight is a sequence $\{k_{t+1}, l_t\}_{t=0}^{\infty} > 0$ satisfying*

$$\begin{cases} k_{t+1} - \beta \left[1 - \delta - \frac{T_1(k_t, k_{t+1}-(1-\delta)k_t, l_t)}{T_2(k_t, k_{t+1}-(1-\delta)k_t, l_t)}\right] k_t = 0 \\ U\left(T_3\left(k_t, k_{t+1} - (1-\delta)\, k_t, l_t\right) l_t\right) - V\left(l_{t-1}\right) = 0 \end{cases} \tag{15}$$

together with the transversality condition[9]

$$\lim_{t \to +\infty} \beta^t (p_t/c_t) k_{t+1} = 0 \tag{16}$$

4.1 Steady state

Before going through the stability analysis of system (15), our first concern is to prove the existence of a stationary solution.

Definition 2. *An interior steady state equilibrium is a stationary sequence $\{k_{t+1}, l_t\}_{t=0}^{\infty} = \{k^*, l^*\}_{t=0}^{\infty} > 0$ that satisfies the two-dimensional system*

$$\begin{cases} -\frac{T_1(k,\delta k,l)}{T_2(k,\delta k,l)} = \beta^{-1} - (1-\delta) \\ U\left(T_3\left(k, \delta k, l\right) l\right) = V\left(l\right) \end{cases} \tag{17}$$

We can show that Assumptions 1 and 2 guarantee the existence and the uniqueness of the steady state.

Proposition 1. *Under Assumptions 1 and 2, there exists a unique steady state $(k^*, l^*) > 0$ solution of (17).*

It is easy to verify that at the steady state $1 - \delta + r/p = 1/\beta > 1$. Moreover, under Assumption (5) we get $1-\delta+r/p < 1/\gamma$. It follows therefore that conditions (2) and (8) are satisfied in a neighborhood of (k^*, l^*).

[9]It is proved in Michel [19] that equations (15) together with the transversality condition (16) provide necessary and sufficient conditions for an equilibrium path.

5 Characteristic polynomial and geometric method

Our aim consists now in analyzing the dynamics of system (15) around its stationary solution as well as along bifurcations. Let us introduce the expressions of the following elasticities all evaluated at the steady state: the elasticity of the interest rate

$$\varepsilon_r \equiv -\tfrac{T_{11}k}{T_1} \in (0,+\infty)$$

the elasticity of the real wage

$$\varepsilon_w \equiv -\tfrac{T_{33}l}{T_3} \in (0,+\infty)$$

and the elasticity of the offer curve $\lambda(l) \equiv U^{-1}(V(l))$

$$\varepsilon \equiv \tfrac{V'l}{U'c} \in (1,+\infty)$$

It is straightforward to show that the elasticity of the labor supply with respect to the real wage is equal to $\epsilon_{lw} = 1/(\varepsilon - 1)$. Notice that considering the elasticity of intertemporal substitution in consumption $\epsilon_c = -u'(c)/c''(c)c$ and the elasticity of the marginal disutility of labor $\epsilon_l = v'(l)/v''(l)l$, the elasticity ε may also be expressed as

$$\varepsilon = \frac{1+\frac{1}{\epsilon_l}}{1-\frac{1}{\epsilon_c}} \tag{18}$$

It follows that $\varepsilon = 1$, i.e. $\epsilon_{lw} = +\infty$, if and only if $\epsilon_l = +\infty$, while $\varepsilon = +\infty$, i.e. $\epsilon_{lw} = 0$, if and only if either $\epsilon_l = 0$ or $\epsilon_c = 1$. Therefore, the elasticity of the labor supply ϵ_{lw} may be equivalently appraised through ϵ_l.

Denoting $\theta \equiv \beta^{-1} - (1-\delta)$, let us also define the share of capital in total income $s \equiv rk/\left(T + py^1\right) \in (0,1)$ and the relative capital intensity across sectors $b \in (-\infty, 1/\theta)$,[10] again evaluated at the steady state. Linearizing system (15) around (k^*, l^*) yields to the following Proposition:

Proposition 2. *Under Assumptions 1 and 2, the characteristic polynomial is* $P(\lambda) = \lambda^2 - T\lambda + D$ *with*

$$T = 1 + D - (1-\varepsilon)\frac{\varepsilon_r \beta\theta(1-\theta b)(1-\delta b)}{1 - \varepsilon_r\left[(1-\delta b)^2 s/(1-s) + \beta\theta(1-\theta b)b\right]} \tag{19}$$

and

$$D = \varepsilon\frac{1 - \varepsilon_r\beta\theta(1-\theta b)\left[1 + (1-\delta)b\right]}{1 - \varepsilon_r\left[(1-\delta b)^2 s/(1-s) + \beta\theta(1-\theta b)b\right]} \tag{20}$$

[10] The fact that b must be lower than $1/\theta$ comes from the positivity constraint on the price of investment p (See Bosi *et al.* [11]).

As in Grandmont *et al.* [17], we study the variations of the trace T and the determinant D in the (T, D) plane as one of the parameters of interest is made to vary continuously in its admissible range. Notice indeed that both T and D are linear with respect to the elasticity of the offer curve ε. When the latter covers the interval $(1, +\infty)$, the locus of points $(T(\varepsilon), D(\varepsilon))$ consists in a half-line $\Delta(T)$ with slope

$$\psi = 1 - \frac{\varepsilon_r \beta\theta (1-\theta b)(1-\delta b)}{1-\varepsilon_r \beta\theta (1-\theta b) b} \tag{21}$$

Notice also that the origin (T_1, D_1) of $\Delta(T)$ lies on the line $T = 1 + D$. The method simply consists in locating $\Delta(T)$ in the plane (T, D), which means to study its origin and its slope.

Specifically, if T and D lie in the interior of the triangle ABC depicted in Fig. 1, the stationary solution is stable (namely a sink), hence locally indeterminate. In the opposite case, it is locally determinate: it is either a saddle when $|T| > |1 + D|$, or a source otherwise.

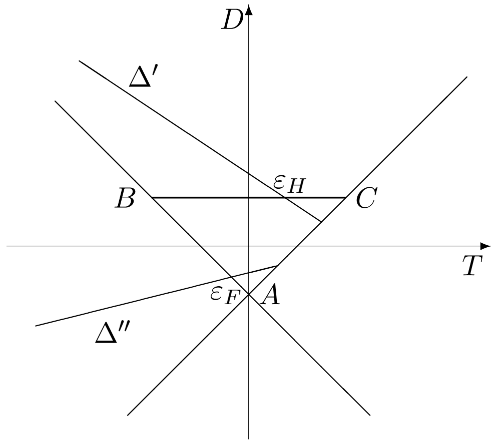

Figure 1: Geometrical analysis.

This geometrical method may also be exploited to characterize bifurcations. Indeed, as it is shown in Fig. 1, when $\Delta(T)$ goes through the line $D = -T-1$ (at $\varepsilon = \varepsilon_F$) one eigenvalue is equal to -1 and we get a flip bifurcation. When $\Delta(T)$ intersects the interior of the segment BC (at $\varepsilon = \varepsilon_H$) the modulus of the complex conjugate eigenvalues is one and the system undergoes a Hopf bifurcation.[11] As shown in Proposition 2 , for given values of β, δ and thus θ, the half-line $\Delta(T)$ depends on the technological parameters:

[11]Notice that, in our system, the uniqueness of the steady state rules out the occurrence of transcritical bifurcations.

the share of total income s, the capital intensity difference across sectors b, the elasticity of the interest rate ε_r and the elasticity of the real wage ε_w. It is easy to show that all these parameters are linked through the following relationship:

$$\varepsilon_w = \varepsilon_r (1 - \delta b)^2 \frac{s}{1-s}$$

Thus, for some fixed value of s, we can vary independently both ε_r and b. Our aim is now to characterize the origin $(T_1, D_1) \equiv (T(1), D(1))$, the slope ψ and the endpoint $(T(\infty), D(\infty))$ of the half-line $\Delta(T)$ when b and ε_r are made to vary within their domain of definition. By observing that expressions (19) and (20) are first-order polynomials in ε_r, it is easier to proceed by first fixing the value of b and then by considering variations of ε_r. By repeating this procedure with different values of b, we will be able to appraise the whole evolution of the local dynamics and bifurcations. Of course, b must fall within the range compatible with positive prices. As it is shown in details in Bosi *et al.* [11], this requires $b < 1/\theta$. The following Lemma shows that two types of geometrical configurations, associated with different properties of the slope ψ, may be exhibited:

Lemma 1. *Under Assumptions 1 and 2, the following properties hold:*
i) The slope satisfies $\psi(\varepsilon_r) \in (1 - \delta + 1/b, 1)$ *for any* $\varepsilon_r > 0$;
ii) $\lim_{\varepsilon \to +\infty} D(\varepsilon) = +(-)\infty$ *if and only if* $D_1 > (<) 0$.

The first part of Lemma 1 implies that if $b < -1/(1-\delta)$, the slope $\psi(\varepsilon_r)$ is included in the interval $(0, 1)$ for any $\varepsilon_r \geq 0$. On the contrary, if $b \in (-1/(1-\delta), 1/\theta)$, the slope may be positive or negative depending on the value of ε_r. More precisely, it can be shown that for large values of ε_r the slope ψ is negative, while it is positive when ε_r admits lower values. The second part emphasizes the fact that the localization of the endpoint $(T(\infty), D(\infty))$ of the half-line $\Delta(T)$ depends upon the value of its initial point (T_1, D_1). Specifically, as shown in Fig. 1, when $D_1 \in (0, 1)$, then $D(\infty) = +\infty$ and we will be able to find conditions to get a Δ-line as Δ'. In such a case, the steady state is locally indeterminate when $\varepsilon \in (1, \varepsilon_H)$ and a Hopf bifurcation occurs when ε crosses ε_H. Similarly, when $D_1 \in (-1, 0)$, then $D(\infty) = -\infty$ and we will be able to find conditions to get a Δ-line as Δ''. In such a case, the steady state is locally indeterminate when $\varepsilon \in (1, \varepsilon_F)$ and a flip bifurcation occurs when ε crosses ε_F.

6 Main results

We now study the local dynamics and bifurcations of the system in each of the two cases exhibited in Lemma 1, namely $b < -1/(1-\delta)$ and $b \in (-1/(1-\delta), 1/\theta)$.

6.1 $b < -1/(1-\delta)$

Since $b < -1/(1-\delta)$, Lemma 1 implies that the slope ψ is positive and less than one. We know from equation (20) in Proposition 2 that $D(\varepsilon)$ is a linear function of ε. To get local indeterminacy, we then need to find conditions for $D_1 \in (-1, 1)$. To this end, exploiting Lemma 1 allows to show that there exist some critical values for, respectively, the share of capital in total income s^* and the elasticity of interest rate ε_r^*, such that if $s \leq s^*$ or $\varepsilon_r \in (0, \varepsilon_r^*)$, the slope of $\Delta(T)$ is positive and lower than one and either $D_1 > 0$ (and $\lim_{\varepsilon \to +\infty} D(\varepsilon) = +\infty$) or $D_1 < -1$ (and $\lim_{\varepsilon \to +\infty} D(\varepsilon) = -\infty$). As a consequence $\Delta(T)$ remains in the saddle point region and the steady state is always locally determinate. Conversely, when $s > s^*$ and $\varepsilon_r > \varepsilon_r^*$, as it is shown in Fig. 2, one has $D_1 \in (-1, 0)$ and $\lim_{\varepsilon \to +\infty} D(\varepsilon) = -\infty$. It follows that for low elasticities of the offer curve ε, the half-line $\Delta(T)$ crosses the interior of the triangle ABC and therefore the steady state is locally indeterminate. Then $\Delta(T)$ intersects the line $D = -T - 1$ at $\varepsilon = \varepsilon_F$ and a flip bifurcation generically occurs. Eventually, for $\varepsilon > \varepsilon_F$ the steady state becomes a saddle, thus locally determinate.

All this is summarized in the following Proposition:

Proposition 3. *Let $b < -1/(1-\delta)$ and Assumptions 1-2 hold. Then there exist $s^* \in (0, 1)$ and $\varepsilon_r^* > 0$, such that:*

(i) If $s \leq s^*$ or $\varepsilon_r \in (0, \varepsilon_r^*)$, then the steady state is a saddle (locally determinate) for all $\varepsilon > 0$.

(ii) If $s > s^*$ and $\varepsilon_r > \varepsilon_r^*$, then there exists $\varepsilon_F > 1$ such that the steady state is a sink (locally indeterminate) when $\varepsilon \in (1, \varepsilon_F)$ and a saddle when $\varepsilon > \varepsilon_F$. A flip bifurcation generically occurs at $\varepsilon = \varepsilon_F$.

Proposition 3 shows that local indeterminacy requires a large enough share of capital in total income, a large enough elasticity of interest rate and a low enough elasticity of the offer curve. We know that in the one-sector model studied by Grandmont *et al.* [17] a necessary condition to get indeterminacy is an elasticity of capital-labor substitution lower than the share of capital. In order to understand the implications of our results in terms of the elasticity of capital-labor substitution, we need to know how to

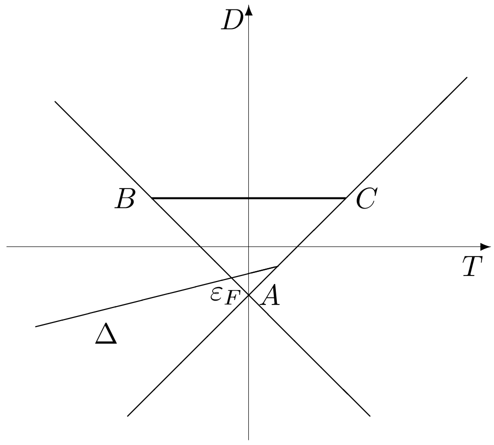

Figure 2: $b < -1/(1-\delta)$ with $s > s^*$ and $\varepsilon_r > \varepsilon_r^*$.

interpret the elasticity of interest rate in terms of the elasticity of capital-labor substitution. The main difference with the one-sector formulation then appears: in a two-sector model, each technology is characterized by some particular substitutability properties and we need to take into account two distinct elasticities of capital-labor substitution. Denoting σ_i the elasticity of sector $i = 0, 1$ and using the recent contribution of Drugeon [14] we have the following expression for the elasticity of the interest rate:

$$\varepsilon_r = \frac{py^1 wk \left(l^0\right)^2}{y^0 \left(py^1 k^0 l^0 \sigma_0 + y^0 k^1 l^1 \sigma_1\right)} \tag{22}$$

In the particular case with identical elasticities across sectors $\sigma_0 = \sigma_1 = \sigma$, we find as in the one-sector model that the elasticity of the interest rate will be high enough provided σ is low enough.[12] However, as soon as there are some asymmetries between sectors, larger elasticities of capital-labor substitution may become compatible with a large elasticity of the interest rate. Indeed ε_r also depends on the factors and output prices, the amount of factors used in each sectors, and the production levels. More precise results remain difficult to obtain at this point of the analysis since the capital intensity difference b, the outputs and the amounts of capital and labor used in each sector are functions of the elasticities of capital-labor substitution σ_0 and σ_1. Numerical simulations in a CES economy will give additional

[12] Notice that in a one-sector model we have $p = 1$, $y^0 = y^1 = y$, $l^0 = l^1 = 1$, $k^0 = k^1 = k$, $\sigma_0 = \sigma_1 = \sigma$ and $py^1k^0l^0\sigma_0 + y^0k^1l^1\sigma_1 = yk\sigma$, so that equation (22) becomes $\varepsilon_r = (1-s)/\sigma$.

conclusions in Section 8. In particular, it will be shown that contrary to the one-sector formulation, indeterminacy is possible even under elasticities of capital-labor substitution larger than the share of capital in total income which are consistent with recent empirical estimates.

Consider now the condition on the elaticity of the offer curve. As shown by (18), ε is defined from the elasticity of intertemporal substitution in consumption ϵ_c and the elasticity of the marginal disutility of labor ϵ_l. The restriction $\varepsilon \in (1, \varepsilon_F)$ in Proposition 3 can thus be stated as follows

$$\epsilon_c \left[1 + \epsilon_l(1 - \varepsilon_F)\right] < -\epsilon_l \varepsilon_F \tag{23}$$

and to be satisfied it requires

$$1 + \epsilon_l(1 - \varepsilon_F) < 0 \quad \Leftrightarrow \quad \epsilon_l > \tfrac{1}{\varepsilon_F - 1} \equiv \underline{\epsilon}_l \tag{24}$$

i.e. a large enough elasticity of the labor supply. Condition (23) then becomes

$$\epsilon_c > \tfrac{\epsilon_l \varepsilon_F}{\epsilon_l(\varepsilon_F - 1) - 1} \equiv \underline{\epsilon}_c \tag{25}$$

i.e. a large enough elasticity of intertemporal substitution in consumption. However, depending on the value of the critical bound ε_F, local indeterminacy may be compatible with low values for these elasticities. In particular, straightforward computations show that for some finite value of ϵ_l satisfying (24), the lower bound on ϵ_c as given by (25) may remain close to 1.

6.2 $b \in \left(-1/\left(1-\delta\right), 1/\theta\right)$

As shown in Lemma 1 and contrary to the previous configuration, if $b \in \left(-1/\left(1-\delta\right), 1/\theta\right)$, the slope ψ may be positive or negative depending on the value of ε_r. Recall from equation (20) that $D(\varepsilon)$ is a linear function of ε. To get local indeterminacy, we then need to find conditions for $D_1 \in (-1, 1)$. In order to obtain results compatible with the one-sector case studied in Grandmont *et al.* [17], *i.e.* with $b = 0$, we have to introduce a mild restriction which requires a sufficiently low depreciation rate of capital.

Assumption 3. $\beta\theta\left(1-s\right)/s < 1$

Clearly, in a two-sector framework, with $b \neq 0$, Assumption 3 is not sufficient to get $D_1 \in (-1, 1)$. We need to introduce additional restrictions on the value of the elasticity of the interest rate. Indeed there exists a critical value $\varepsilon_r^A > 0$, such that $D_1 \in (-1, 1)$ when $\varepsilon_r > \varepsilon_r^A$. In order to distinguish two cases depending on the type of bifurcation that may occur, we introduce a second critical value $\varepsilon_r^B > \varepsilon_r^A$, such that when $\varepsilon_r = \varepsilon_r^B$, the corresponding Δ-line starts from an initial point with $D_1 \in (-1, 1)$ and goes through B as shown in Fig. 3.

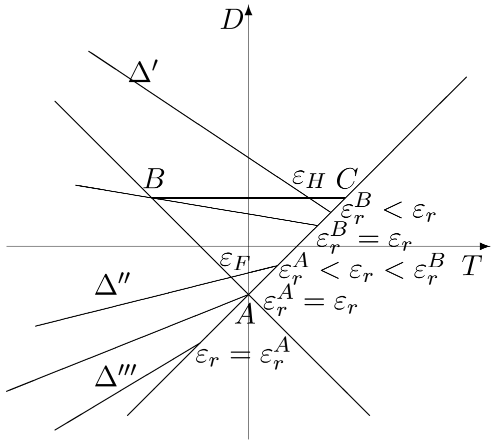

Figure 3: $b \in (-1/(1-\delta), 1/\theta)$

Roughly speaking, as long as ε_r increases, the half-line $\Delta(T)$ undergoes a clockwise rotation and its origin moves upward along the line $D = T - 1$, starting from the point C in correspondence to which the slope ψ is equal to one. Indeed it is easy to notice that D_1 is a monotonic increasing function of ε_r with a discontinuity for some finite critical value $\bar{\varepsilon}_r$. In particular $D_1 \in (1, +\infty)$ when $\varepsilon_r \in (0, \bar{\varepsilon}_r)$ and $D_1 \in (-\infty, 1)$ when $\varepsilon_r > \bar{\varepsilon}_r$. In order to observe a non-empty intersection of $\Delta(T)$ with the triangle ABC and then in order to get indeterminacy, it must be $\varepsilon_r > \varepsilon_r^A \, (> \bar{\varepsilon}_r)$: indeed, at $\varepsilon_r = \varepsilon_r^A$, the origin of $\Delta(T)$ is A. $\Delta(T)$ goes through B when $\varepsilon_r = \varepsilon_r^B$ and, therefore, we need $\varepsilon_r > \varepsilon_r^B$ in order to get a Hopf bifurcation. Of course, for $\varepsilon_r \in (\varepsilon_r^A, \varepsilon_r^B)$ the half-line intersects the line $D = -T - 1$ and a flip bifurcation generically occurs when $\varepsilon = \varepsilon_F$. Fig. 3 captures the two possible configurations to get indeterminacy.

Proposition 4. *Let $-1/(1-\delta) < b < 1/\theta$ and Assumptions 1-3 hold. Then there exist $\varepsilon_r^A > 0$ and $\varepsilon_r^B > \varepsilon_r^A$ such that:*

(i) If $\varepsilon_r \in (0, \varepsilon_r^A)$, then the steady state is a saddle (locally determinate) for all $\varepsilon > 0$.

(ii) If $\varepsilon_r \in (\varepsilon_r^A, \varepsilon_r^B)$, then there exists $\varepsilon_F > 1$ such that the steady state is a sink (locally indeterminate) when $\varepsilon \in (1, \varepsilon_F)$ and a saddle (locally determinate) when $\varepsilon > \varepsilon_F$. A flip bifurcation generically occurs at $\varepsilon = \varepsilon_F$.

(iii) If $\varepsilon_r \in (\varepsilon_r^B, +\infty)$, then there exists $\varepsilon_H > 1$ such that the steady

state is a sink (locally indeterminate) when $\varepsilon \in (1, \varepsilon_H)$ and a source (locally determinate) when $\varepsilon > \varepsilon_H$. A Hopf bifurcation generically occurs at $\varepsilon = \varepsilon_H$.

As in Proposition 3 local indeterminacy occurs for high enough elasticities of the interest rate. However there is no restriction on the share of capital in total income. As previously, we have to understand what are the restrictions on the elasticities of capital-labor substitution in both sectors for which the elasticity of the interest rate is large. This point will be discussed in the next section. However, one may right now expect that the conclusions in terms of indeterminacy will be less positive than in the previous case. We assume indeed that $b \in (-1/(1-\delta), 1/\theta)$, i.e. either the consumption good is weakly capital intensive, or the investment good is capital intensive. Starting from the results obtained in the one-sector formulation, i.e. with $b = 0$, a simple intuition suggests that the occurrence of local indeterminacy will require low elasticities of capital-labor substitution, at least in one of the two sectors. This conclusions will be confirmed in the next section.

Notice finally that Proposition 4 shows that cycles may still appear through flip or Hopf bifurcations. Such a result contrasts with the conclusions obtained by Benhabib & Nishimura [7] in a standard two-sector optimal growth model. They show indeed that the existence of optimal cycles do require the consumption good to be more capital intensive than the investment good. In our case, persistent oscillations may also appear while the investment good is more capital intensive than the consumption good. As in the one-sector formulation, this result is essentially based on the presence of money through the cash-in-advance constraint affecting "workers".[13]

7 A CES economy

To provide some quantitative insights of the plausibility of indeterminacy in our model, we consider the classical example of a CES economy. We retain the following functional forms

$$u(c^w) = (c^w)^{1-\eta_1}/(1-\eta_1), \quad v(l) = l^{1+\eta_2}/(1+\eta_2) \tag{26}$$

with $\eta_1 \in (0,1)$, $\eta_2 \geq 0$ for preferences and

$$f^0(k^0, l^0) = \left[\alpha_{10}(k^0)^{-\rho_0} + \alpha_{00}(l^0)^{-\rho_0}\right]^{-1/\rho_0}$$
$$f^1(k^1, l^1) = \left[\alpha_{11}(k^1)^{-\rho_1} + \alpha_{01}(l^1)^{-\rho_1}\right]^{-1/\rho_1}$$

[13] See for instance Barinci & Chéron [2], Bosi & Magris [10], Bosi *et al.* [12] and Woodford [25] for indeterminacy results in one or two-sector models with CIA constraint.

with $\alpha_{00} + \alpha_{10} = \alpha_{01} + \alpha_{11} = 1$, $\rho_0, \rho_1 > -1$ for technologies. It follows obviously that the elasticity of the offer curve is $\varepsilon = (1 + \eta_2)/(1 - \eta_1) \geq 1$ while the elasticities of capital-labor substitution are $\sigma_i = 1/(1+\rho_i)$, $i = 0, 1$. Considering the recent contribution of Nishimura & Venditti [21], tedious but straightforward computations give

$$\begin{aligned}
\kappa^* &= \frac{\left(\frac{\alpha_{10}\alpha_{01}}{\alpha_{00}\alpha_{11}}\right)^{\frac{1}{1+\rho_0}} \Sigma^{1+\rho_1}}{1-\delta b} \\
b &= \left(\frac{\alpha_{11}}{\theta}\right)^{\frac{1}{1+\rho_1}} \left[1 - \left(\frac{\alpha_{10}\alpha_{01}}{\alpha_{00}\alpha_{11}}\right)^{\frac{1}{1+\rho_0}} \Sigma^{\rho_1-\rho_0}\right] \\
r &= \alpha_{10} \left[\alpha_{10} + \alpha_{00} \left(\frac{\alpha_{10}\alpha_{01}}{\alpha_{00}\alpha_{11}}\right)^{\frac{\rho_0}{1+\rho_0}} \Sigma^{\rho_0(1+\rho_1)}\right]^{-\frac{1+\rho_0}{\rho_0}} \\
s &= \left[1 + \frac{\alpha_{01}}{\alpha_{11}} \left(\frac{\alpha_{10}\alpha_{01}}{\alpha_{00}\alpha_{11}}\right)^{\frac{-1}{1+\rho_0}} \Sigma^{\rho_0(1+\rho_1)}(1 - \delta b)\right]^{-1} \\
\varepsilon_r &= \frac{(1+\rho_0)(1+\rho_1)\left(\frac{r}{\alpha_{10}}\right)^{\frac{\rho_0}{1+\rho_0}} \frac{\alpha_{00}}{\alpha_{01}} \left(\frac{\alpha_{10}\alpha_{01}}{\alpha_{00}\alpha_{11}}\right)^{\frac{\rho_0}{1+\rho_0}} \left(\frac{\alpha_{11}}{\theta}\right)^{\frac{\rho_1}{1+\rho_1}} \Sigma^{\rho_0-\rho_1}}{\Omega}
\end{aligned}$$

with

$$\begin{aligned}
\Sigma &= \left(\frac{\left(\frac{\alpha_{11}}{\theta}\right)^{\frac{\rho_1}{1+\rho_1}} - \alpha_{11}}{\alpha_{01}}\right)^{\frac{1}{\rho_1(1+\rho_0)}} \\
\Omega &= 1 + \rho_1 + (\rho_0 - \rho_1)\delta \left(\frac{\alpha_{11}}{\theta}\right)^{\frac{1}{1+\rho_1}} + \frac{\alpha_{11}}{\alpha_{01}\Sigma^{\rho_1(1+\rho_0)}} \left[1 - \delta \left(\frac{\alpha_{11}}{\theta}\right)^{\frac{1}{1+\rho_1}}\right] \\
&\times \frac{1+\rho_1+(\rho_0-\rho_1)\delta\kappa^*\left(\frac{\alpha_{11}}{\theta}\right)^{\frac{1}{1+\rho_1}} \Sigma^{-(1+\rho_0)}}{1-\delta\kappa^*\left(\frac{\alpha_{11}}{\theta}\right)^{\frac{1}{1+\rho_1}} \Sigma^{-(1+\rho_0)}}
\end{aligned}$$

We calibrate the structural parameters in a standard way compatible with quarterly data by choosing $\beta = 0.99$ and $\delta = 0.025$. Assumption 3 is then obviously satisfied and it follows that the bounds for the capital intensity difference b are $-1/(1 - \delta) \approx -1.025$ and $1/\theta \approx 28.489$. In the *RBC* literature, Cobb-Douglas technologies are usually considered and standard calibrations are based on capital shares in the consumption and investment good sectors which are such that $\alpha_{10}, \alpha_{11} \in (0.2, 0.6)$. However there is no clear conclusion on the sign of the capital intensity difference. For instance, Benhabib *et al.* [9] assume that the investment good is capital intensive while the opposite capital intensity configuration is considered by Benhabib & Nishimura [8]. We give in the following numerical illustrations for both cases. Notice that with CES technologies, the capital shares now also depends on the parameters ρ_i as this clearly appears in the formulation of the capital intensity difference b stated above.

7.1 $b < -1/(1-\delta)$

Let us first consider a capital intensive consumption good with $\alpha_{00} = 0.4$, $\alpha_{11} = 0.6$. Assuming that $\rho_0 \in [-0.75, -0.65]$ and $\rho_1 \in [-0.14, -0.07]$, the associated relative capital intensity difference satisfies $b < -1/(1-\delta)$ as in section 6.1. The corresponding elasticities of substitution in each sector are thus $\sigma_0 \in [2.86, 4]$ and $\sigma_1 \in [1.075, 1.16]$. They correspond to the estimates for the richest group of countries provided in Duffy & Papageorgiou [15]. Computations then show that the steady state is locally indeterminate for all values of the elasticity of the offer curve such that $\varepsilon \in (1, \varepsilon_F)$ with $\varepsilon_F \in (2.06, 14.33)$ depending on the values of ρ_i considered. Moreover when ε crosses ε_F from below a flip bifurcation occurs and period-two cycles exist. More precisely if we set $\rho_0 = -0.65$ (*i.e.* $\sigma_0 = 2.86$) and $\rho_1 = -0.14$ (*i.e.* $\sigma_1 = 1.16$) we get $\varepsilon_F \approx 14.02$. It is worthwhile remarking that within such an example indeterminacy arises in correspondence to an investment good technology exhibiting an elasticity of factor substitution very close to the unitary Cobb-Douglas case, condition known for eliminating such a phenomenon in the one-sector framework. Notice also that the lower bound on ϵ_l as defined by (24) is $\underline{\epsilon_l} \approx 7.68\%$. It follows that if, in accordance with stylized facts,[14] we choose low values for the elasticity of the labor supply, i.e. for instance $\epsilon_l \in (0.1, 4)$, the lower bound on ϵ_c given by (25) is $\underline{\epsilon_c} \in (1.098, 4.64)$. Therefore, contrary to two-sector models with productive externalities, local indeterminacy remains compatible with a small elasticity of intertemporal substitution in consumption and a low elasticity of the labor supply.

7.2 $b \in (0, 1/\theta)$

Let us consider now a capital intensive investment good with $\alpha_{00} = 0.42$, $\alpha_{11} = 0.32$. Assuming that $\rho_0, \rho_1 > 0$, the associated relative capital intensity difference across sectors satisfies $b \in (0, 1/\theta)$ as in Section 6.2. If we set $\rho_0 \in [11, 15]$ and $\rho_1 \in [0.012, 0.2]$, which corresponds to elasticities of substitution such that $\sigma_0 \in [0.0625, 0.083]$ and $\sigma_1 \in [0.83, 0.988]$, b is high enough to get local indeterminacy for any $\varepsilon \in (1, \bar{\varepsilon})$ with $\bar{\varepsilon} \in (1.023, 1.077)$. Notice that $\bar{\varepsilon}$ is a bifurcation value which may correspond to a flip or a Hopf

[14] See Blundell & McCurdy [13].

bifurcation depending on the values of ρ_i. More precisely if we set $\rho_0 = 15$ (*i.e.* $\sigma_0 = 0.0625$) and $\rho_1 = 0.2$ (*i.e.* $\sigma_1 = 0.83$) we get $\bar{\varepsilon} = \varepsilon_H \approx 1.059$ and a Hopf bifurcation occurs when ε crosses ε_H from below. On the contrary, if we set $\rho_0 = 11$ (*i.e.* $\sigma_0 = 0.083$) and $\rho_1 = 0.012$ (*i.e.* $\sigma_1 = 0.988$) we get $\bar{\varepsilon} = \varepsilon_F \approx 1.024$ and a flip bifurcation occurs when ε crosses ε_F from below. These results show that while the elasticity of capital-labor substitution in the investment good sector remains in accordance with the estimates for the poorest group of countries provided in Duffy & Papageorgiou [15], this is not the case for the consumption good sector. Local indeterminacy requires extremely low substitutability in that sector. Notice also that contrary to the configuration with a capital intensive consumption good, the elasticity of the offer curve needs to be very close to one. This means that we have to consider values for the elasticity of the labor supply close to $+\infty$ as in one-sector models with productive externalities.[15]

8 Plausibility of sunspots and economic intuitions

The numerical simulations clearly show that local indeterminacy arises with plausible values for the elasticities of capital-labor substitution, the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply when the consumption good is sufficiently capital-intensive. This configuration appears to be consistent with national accounting data of the main developed countries over the last three decades. Indeed in a recent contribution, Takahashi *et al.* [23] aggregate sectoral data in order to get a two-sector representation of the Japanese, U.S. and German economies from 1955 to 2000. They find that, while the U.S. and German economies are characterized by a capital-intensive aggregate consumption good sector over the whole period, the Japanese economy experiences a capital-intensity reversal in 1975, the aggregate consumption good sector being labor-intensive before and capital-intensive since then. These findings may be explained by the fact that, within developed countries, consumption goods with an increasing amount of technological content have become a growth engine. Our results then provide a theoretical background to explain why developed countries are characterized by a strong macroeconomic volatility based on expectations-driven fluctuations.

It remains now to understand the economic mechanisms at the core of these results. Actually, if the consumption good is sufficiently capital intensive, there exist in our model two main forces from which endogenous

[15]See Benhabib & Farmer [6], Pintus [22].

fluctuations originate: a pure technological mechanism based on factor allocations across sector, a monetary mechanism based on the cash-in-advance constraint. As initially shown in Benhabib and Nishimura [7], the technological mechanism refers to the Rybczinsky effect. Starting from one equilibrium paths, consider an instantaneous increase in the capital stock k_t. This results in two opposing forces:

- Since the consumption good is more capital intensive than the investment good, the trade-off in production becomes more favorable to the consumption good. Moreover, the Rybczinsky effect implies a decrease of the output of the capital good y_t. This tends to lower the investment and the capital stock in the next period k_{t+1}.

- In the next period the decrease of k_{t+1} implies again through the Rybczinsky effect an increase of the output of the capital good y_{t+1}. This mechanism is explained by the fact that the decrease of k_{t+1} improves the trade-off in production in favor of the investment good which is relatively less intensive in capital. Therefore this tends to increase the investment and the capital stock in period $t+2$, k_{t+2}.

The monetary mechanism, as shown for instance in Bosi *et al.* [12], is based on the agents' expectations. Let us consider again the CES instantaneous utility function as given in (26). Equations (1) and (8) then become respectively

$$\frac{c^c_{t+1}}{c^c_t} = \beta i_{t+1}, \quad \left(\frac{c^w_{t+1}}{c^w_t}\right)^{\eta_1} > \gamma i_{t+1} \tag{27}$$

with

$$i_{t+1} \equiv [r_{t+1} + (1-\delta)\, p_{t+1}]/p_t$$

the nominal interest factor. Starting from one equilibrium path, let us try to construct an alternative equilibrium. For this purpose, assume that agents collectively revise their expectations in reaction to a given sunspot signal and come to believe that the nominal interest factor will undergo an appreciation. It follows that to re-establish (27), future consumptions c^c_{t+1} and c^w_{t+1} must be driven up. When mixed with the technological effect, this expectation becomes self-fulfilling and a new equilibrium path can be defined.

The simultaneous consideration of technological and monetary effects therefore generates sunspot fluctuations while the elasticities of capital-labor substitution, the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply may be fixed at some standard values. However, all this story crucially depends on the assumption of a consumption good which is sufficiently capital intensive. If on the contrary, either the consumption good is weakly capital intensive or the investment good is capital intensive, the technological effect does not occur as in one-sector models. In order to generate sunspot fluctuations, the monetary mechanism therefore requires extreme parameterizations for the structural elasticities, i.e. low elasticities of capital-labor substitution (lower than the share of capital), or large elasticities of intertemporal substitution in consumption.

9 Concluding remarks

In this paper we consider a two-sector infinite horizon model with heterogeneous agents and borrowing constraint on labor income. We show that the relative capital intensity across sectors plays a relevant role with respect to the emergence of indeterminacy and deterministic as well as sunspot fluctuations. The most important result is obtained when the consumption sector is significantly more capital intensive than the investment sector. Indeed local indeterminacy comes about for elasticities of capital-labor substitution slightly greater than unity and for a broad range of values for the elasticity of the offer curve which are compatible with plausible values for the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply.

These conditions appear to be consistent with recent macroeconomic empirical evidences. On the one side, concerning the capital intensity difference, Takahashi *et al.* [23] find that over the last three decades the main OECD countries are characterized by a capital-intensive consumption good sector. On the other side, concerning the substitutability properties, Duffy & Papageorgiou [15] show that capital and labor have an elasticity of substitution greater than unity in the richest group of countries.

10 Appendix

10.1 Proof of Proposition 1

Recalling that $T(k, \delta k, l)$ is homogeneous of degree one, the two equations in (17) can be rewritten as

$$\begin{array}{rcl} -\frac{T_1(\kappa,\delta\kappa,1)}{T_2(\kappa,\delta\kappa,1)} & = & \beta^{-1} - (1-\delta) \\ U\left(T_3\left(\kappa, \delta\kappa, 1\right) l\right) & = & V\left(l\right) \end{array} \tag{28}$$

with $\kappa \equiv k/l$. Consider the first equation in (28): this is equivalent to the equation defining the stationary capital stock of a two-sector optimal growth model with inelastic labor supply. Then, the proof of Theorem 3.1

in Becker and Tsyganov [5] applies and there exists a unique solution κ^* of the first equation of (28). Consider now the second equation in (28) evaluated at κ^* and, in view of the definition of U and V, rewrite it as $T_3(\kappa^*, \delta\kappa^*, 1)\, u'(T_3(\kappa^*, \delta\kappa^*, 1)\, l) = v'(l)$. Then, under Assumption 1, it is immediate to verify that such an equation possesses a unique solution l^*. $\square$

10.2 Proof of Proposition 2

By linearizing system (15), after straightforward although tedious computations, we obtain the following expression for the Jacobian J^*:

$$J^* = \begin{bmatrix} -b\varepsilon_w & a(1-\varepsilon_w) \\ 1 - b\vartheta\varepsilon_r & -a\vartheta\varepsilon_r \end{bmatrix}^{-1} \begin{bmatrix} -[1+(1-\delta)b]\,\varepsilon_w & a\varepsilon \\ 1-[1+(1-\delta)b]\,\vartheta\varepsilon_r & 0 \end{bmatrix} \tag{29}$$

with $\vartheta \equiv \beta\theta(1-\theta b)$, from which we get the expressions for the trace T and the determinant D. $\square$

10.3 Proof of Lemma 1

To prove the first part of the Lemma, it is sufficient to look at expression (21). Point ii) follows directly from a check of (20). $\square$

10.4 Proof of Proposition 3

Assume that $b < -1/(1-\delta)$. When ε_r moves from zero to $+\infty$, ψ decreases continuously from one to $1-\delta+1/b \in (0, 1-\delta)$. This in particular means that $\psi \in (0,1)$ for all $\varepsilon_r \geq 0$. We also notice that from (20) with $\varepsilon = 1$, $T_1 = 1 + D_1$ so that the origin of Δ belongs to $D = T - 1$.

Now, let us define the following useful formulas

$$\begin{aligned} z &\equiv \tfrac{1-\delta b}{1-\theta b}\tfrac{1}{\beta\theta}\tfrac{s}{1-s} \\ z_1 &\equiv \tfrac{1-\varepsilon_r\beta\theta(1-\theta b)b}{\varepsilon_r\beta\theta(1-\theta b)(1-\delta b)} > 1 \\ z_2 &\equiv \tfrac{2-\varepsilon_r\beta\theta(1-\theta b)[1+(2-\delta)b]}{\varepsilon_r\beta\theta(1-\theta b)(1-\delta b)} > z_1 \end{aligned}$$

Then one easily verifies that $\partial D_1/\partial\varepsilon_r < 0$ if and only if $z < 1$. Still, in the interval under study for b, straightforward computations show that:

- when $z < 1$ then $D_1 \in (0,1)$ for every ε_r and $D(+\infty) = +\infty$,
- when $1 < z < z_1$ then $D_1 > 1$ and $D(+\infty) = +\infty$,
- when $z_1 < z < z_2$ then $D_1 < -1$ and $D(+\infty) = -\infty$,

- when $z > z_2$, then $D_1 \in (-1, 0)$ and $D(+\infty) = -\infty$.

Recall that, since $b < -1/(1-\delta)$, the slope ψ is positive and less than one. Therefore, we face two possible subcases:

(i) if $z < z_2$, then the Δ-line does not cross the triangle ABC and the steady state is a saddle;

(ii) if $z > z_2$, then Δ-line crosses the triangle ABC and, as shown in Fig. 2, there exists $\varepsilon_F > 1$ such that the steady state is a sink when $\varepsilon \in (1, \varepsilon_F)$ and a saddle when $\varepsilon > \varepsilon_F$.

One could easily prove that $z \leq z_2$ if $s \leq s^*$ with

$$s^* \equiv \frac{\beta\theta(1-\theta b)[1+(2-\delta)b]}{\beta\theta(1-\theta b)[1+(2-\delta)b]-(1-\delta b)^2} \in (0,1)$$

and that when $s > s^*$ then $z < z_2$ if and only if $\varepsilon_r \in (0, \varepsilon_r^*)$, with

$$\varepsilon_r^* \equiv \frac{2(1-s)}{s(1-\delta b)^2+(1-s)\beta\theta(1-\theta b)[1+(2-\delta)b]} > 0$$

□

10.5 Proof of Proposition 4

As shown in Lemma 1, if $b \in (-1/(1-\delta), 1/\theta)$, the slope ψ may be positive or negative depending on the value of ε_r. Recall from equation (20) that $D(\varepsilon)$ is a linear function of ε. To get local indeterminacy, we then need to find conditions for $D_1 \in (-1, 1)$. Firstly, there exists a critical value $\varepsilon_r^A > 0$, defined as the solution of $D_1 = -1$, such that $D_1 \in (-1, 1)$ when $\varepsilon_r > \varepsilon_r^A$. Secondly, there exists a second critical value ε_r^B, such that when $\varepsilon_r = \varepsilon_r^B$, the corresponding Δ-line starts from an initial point with $D_1 \in (-1, 1)$ and goes through B as shown in Fig. 3. Tedious but straightforward computations give the expressions for ε_r^A and ε_r^B as

$$\begin{aligned} \varepsilon_r^A &\equiv \frac{2}{a_1+a_2} \\ \varepsilon_r^B &\equiv 2\frac{\sqrt{(a_2-a_0)(a_2-a_1)}-a_1-a_2}{(a_1-a_0)(a_1-a_2)+4a_1a_2} \end{aligned}$$

where

$$\begin{aligned} a_0 &\equiv \beta\theta(1-\theta b)\, b \\ a_1 &\equiv \beta\theta(1-\theta b)\left[b+(1-\delta b)\right] \\ a_2 &\equiv \beta\theta(1-\theta b)\left[b+(1-\delta b)\frac{1-\delta b}{1-\theta b}\frac{1}{\beta\theta}\frac{s}{1-s}\right] \end{aligned}$$

Then, we easily prove that under Assumption 3, $\varepsilon_r^A < \varepsilon_r^B$ and jointly with inequality $b < 1/\theta$ we get

$$\frac{1-\delta b}{1-\theta b}\frac{1}{\beta\theta}\frac{s}{1-s} > 1$$

Then $0 < 1/a_2 < \varepsilon_r^A < 1/a_1 < \varepsilon_r^B$, since ε_r^A is the harmonic mean of $1/a_1$ and $1/a_2$.

Focus, now, on the origin of the half-line Δ. We notice that from (20) with $\varepsilon = 1$ that

$$D_1 = \frac{1-a_1\varepsilon_r}{1-a_2\varepsilon_r}, \quad T_1 = 1 + D_1$$

Moreover, this origin always moves upward, since $\partial D_1/\partial\varepsilon_r > 0$. More precisely (T_1, D_1) goes from $(1, 2)$ to $(+\infty, +\infty)$ when ε_r moves from 0 to $1/a_2$, and from $(-\infty, -\infty)$ to $(0, -1)$ when ε_r moves from $1/a_2$ to ε_r^A. Eventually, when ε_r goes from ε_r^A to $+\infty$, then the origin moves from $(0, -1)$ to $(1 + a_1/a_2, a_1/a_2)$.

Finally, focus on the slope of the half-line Δ. In the configuration under study, one also has that $\partial\psi/\partial\varepsilon_r < 0$. More precisely, when ε_r moves from zero to $+\infty$, the half-line rotates in a clockwise sense: its slope ψ decreases from 1 to $1 - \delta + 1/b$ and can be negative if $b \in (-1/(1-\delta), 0)$.

We are, now, able to conclude about the local stability. When $\varepsilon_r \in (0, \varepsilon_r^A)$, Δ lies entirely in the cone $1 \leq D \leq T - 1$ or in the cone $T - 1 \leq D \leq -1$. Then the steady state is a saddle (locally determinate) for all $\varepsilon > 0$. If $\varepsilon_r \in (\varepsilon_r^A, \varepsilon_r^B)$, as it is depicted in Fig. 3, the half-line Δ starts from the segment AC and crosses the segment AB. It follows that there exists $\varepsilon_F > 1$ such that the steady state is a sink when $\varepsilon \in (1, \varepsilon_F)$ and a saddle when $\varepsilon > \varepsilon_F$, and that a flip bifurcation generically occurs at $\varepsilon = \varepsilon_F$. As it shown in Fig. 3, when $\varepsilon_r > \varepsilon_r^B$ the half-line Δ, still starts from AC, but, now, crosses the segment BC: then there exists an elasticity of the offer curve $\varepsilon_H > 1$ such that the steady state is a sink when $\varepsilon \in (1, \varepsilon_H)$ and a source when $\varepsilon > \varepsilon_H$. It follows that a Hopf bifurcation generically occurs at $\varepsilon = \varepsilon_H$. □

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