EPEE
UNIVERSITE D'EVRY
VAL D'ESSONNE

DOCUMENT DE RECHERCHE

EPEE

CENTRE D'ETUDE DES POLITIQUES ECONOMIQUES DE L'UNIVERSITE D'EVRY

Preferences as Desire Fulfilment

Marc-Arthur DIAYE & Daniel SCHOCH

03 – 17

UNIVERSITE D'EVRY – VAL D'ESSONNE, 4 BD. FRANÇOIS MITTERRAND, 91025 EVRY CEDEX

Site Web : www.univ-evry.fr/EPEE

PREFERENCES AS DESIRE FULFILMENT

MARC-ARTHUR DIAYE AND DANIEL SCHOCH

Abstract. We consider preferences as fulfilment of cardinal conditional desires, which can be either positive or negative, or both. In contrast to the standard multiattributive approach, we do not presuppose the desires to be preferentially independent, but rather allow for conditional preference reversal. It is only assumed that the desires do not supervene on each other.

The relation between preferences and desires are analyzed. We formulate a representation theorem characterizing when a preference order is compatible with the logical structure of desires, and has an additive representation over the desires. It is unique relative to a given utility function representing the preferences. For cardinal preferences on a difference or ratio scale, this implies that the strength of desires is measured on a common ration scale.

1. Introduction

A utilitarian representation of conditional desires has been proposed by Lang [Lang (1996)]. It was first used by [Lang et al. (2002)] in the context of Qualitative Decision Theory, but differs from the classical multicriteria approach (e.g. see [Dubois et al. (2001)]) in that the criteria are conditional and they do not separate as product spaces. The formalism was later rediscovered in a different context by [Schoch (2000)].

Even since utilitarian representations of goals have been proposed for a certain time (e.g. [Doyle & Wellman (1991)], [Bacchus & Grove (1997)], for an update see [Doyle & McGeachie (2002)]), they stick to the paradigma of attribute factorizations. We believe that, in general, desires or goals cannot be identified with elementary attributes in product spaces. Many desires are conditional and thus depend on the fulfilment of others. Consider, for example, the two desires 'I would like to eat potatoes now' and 'If I am being served potatoes I strongly prefer them cooked (otherwise I prefer not to eat them at all and have something else)'. With respect to their product space, the two attributes $\{\text{potatoes}, \neg\text{potatoes}\}$ and $\{\text{cooked}, \neg\text{cooked}\}$ are not *preferentially independent* in the sense of [Keeney & Raiffa (1976)]: The preference for potatoes is reversed under the condition they are being served raw. The preferential entanglement of attributes reflects the *prima facie* character of desires. In contrast to the unconditional nature of the agent's preference for potatoes, there is no ceteris paribus condition which allows us to extract the desire from her preference structure. The alledged solution [Doyle & Wellman (1991), p. 701] of choosing

Date: December 22, 2003.

1991 *Mathematics Subject Classification.* 91B08, 91B16, 91B42, 91C05, 91C15.

Key words and phrases. Preferences, Desires, Utility.

We thank Jerome Lang and Christophe Gonzales for many helpful comments on this paper. We also thank for their comments the participants of seminars held at the university of Paris I Panthéon-Sorbonne and at the university of Evry.

This paper was written while Daniel SCHOCH was visiting the University of Evry.

logically dependend attributes seems alluring, but we believe it will not work out with more complicated examples, especially when multi-valued attributes are involved.

The model of Lang, van der Torre, and Weydert [Lang et al. (2002)], on the other hand, allows to represent conditional desires by propositions of any logical form and in arbitrary combinations. The purpose of this paper is to study the link between the utility representation of desires and cardinal preferences, and to state a necessary and sufficient condition under which the latter can be uniquely reduced to the former which is then serving as an explanatory foundation of cardinal preferences. It is organized as follows. In section 2, we set some basic definitions of qualitative decision theory and in section 3, we analyze the logical structure of a desire structure. Section 4 deals with the non-supervenience condition. This condition says that any desire introduces a partition in the information structure. Section 5 includes a representation theorem which completly characterizes the additive representation of a desire structure. In section 6, we show that Qualitative Decision Theory can be used in Choice Functions Theory to obtain a weaker definition of rationality which includes the Sen's [Sen (1993)] example. Finally section 7 concludes.

2. Desires and Utility

Let W be a set of possible worlds. We assign to it an algebra $\mathcal{L}$ of subsets of W representing the propositions, or the language, in which the content and the condition of the desire could be formulated. Within the model, one can differentiate between three types of desires

Loss Desires: A loss desire stands for the agent's preference to avoid a certain circumstance. An example for a loss desire is "I don't mind the color of my car, unless it is black, which I hate".

Win Desires: A win desire expresses the agent's positive acknowledgement upon an achievement without regret in its absence. The wish "If I eat fish I would like to have white Bourgogne wine with it (but I don't mind if wine is not available and I have just water)" represents a pure win desire.

Mixed Desires: Some desires combine both aspects. The above example "If I eat potatoes, I would like to have them cooked (I like potatoes, but I hate to be served raw ones)" is a mixed desire.

The logical structure of a conditional desire δ of the form "under condition B I desire A" (i.e. I wish that A comes true if B is true) is represented by a pair $\langle A, B\rangle$ with

$$A, B \subseteq W \tag{2.1}$$

to which we associate the type of the desire (loss, win, or mixed). The cardinal structure representing the *strength of the desire* is given by its **local utility function** $u : W \to \mathbb{R}$

$$u(\omega) = \begin{cases} -\alpha, & \text{if } \omega \in B \backslash A \\ 0, & \text{if } \omega \in W \backslash B \\ \beta, & \text{if } \omega \in A \cap B \end{cases} \tag{2.2}$$

with $\alpha, \beta \geq 0$, where a loss desire has $\alpha > 0$ and $\beta = 0$, while $\alpha = 0$ and $\beta > 0$ indicates a win desire, and both $\alpha > 0$ and $\beta > 0$ stands for a mixed desire. The form is intuitively appealing. Positive utility is assigned to desire fulfilment, and a punishment term in worlds violating the desire. When the condition of the desire fails, outside B, the local utility function behave neutral. This structure justifies to set

$$A \subseteq B \tag{2.3}$$

since substituting A by $A \cap B$ leaves the local utility function unchanged.

Let $\Delta = \{\langle A_i, B_i\rangle, i = 1, \ldots, d\}$ be a set of desires with local utility functions u_i, respectively. From the local utility functions we can in a natural way derive a preference order $\succsim$ on W by choosing an aggregation function ϕ to define the total utility function

$$U(\omega) = \phi(u_1(\omega), \ldots, u_d(\omega)),$$

which in our model is simply the sum

$$U(\omega) = \sum_{i=1}^{d} u_i(\omega), \tag{2.4}$$

by setting

$$\omega \succsim \omega' \Leftrightarrow U(\omega) \geq U(\omega'). \tag{2.5}$$

The coefficients of the local utility functions are treated cardinally relative to a given utility scale, which may be either cardinal or ordinal. It therefore seems natural to choose the straight sum as the aggregation function. Its local utility function contributes to the total sum only by adding a (non-negative) constant preserving the preference order.

The local utility function of each mixed desire can be written as a sum of local utility functions of a win and a loss desire. We call win and loss desires **elementary desires**, which are sufficient for cardinal representation, since (2.4) can always be decomposed into win and loss desires. The qualitative structure of desires can therefore best be represented by dividing the set Δ of desires into two non-disjoint subsets, the **positive desires** Δ^+ containing the win and mixed desires, and the **negative desires** Δ^- consisting of the loss desires and the mixed desires, too.

3. The Logical Structure of Desires

There is one general conceptual problem which appears whenever mixed desires or both win and loss desires are involved, the specific form of the aggregation formula (2.4) is very ambigous, since positive and negative terms interfere. If the range of U is finite, it could always be written as a step function

$$U(\omega) = \sum_{i=1}^{m} \gamma_i \chi_{S_i}(\omega), \tag{3.1}$$

where $\gamma_1, \ldots, \gamma_m$ are the nonzero elements of the range of U, $S_i = U^{-1}[\gamma_i]$, and χ_A is the characteristic function of A. This corresponds to an ad-hoc representation of unconditional win desires $\langle S_i, W\rangle$ for $\gamma_i > 0$, and loss desires $\langle W \backslash S_i, W\rangle$ for $\gamma_i < 0$. If the range of U is not finite, then U, which we assume to be a measurable, could at least be approximated by a sequence of step functions.

It is far from being easy to say what it means that a mixed desire is represented by some utility function U, not to speak of their comparison. [Lang et al. (2002)] have proposed that a *single* desire $\delta = \langle A, B\rangle$ is represented by a preference order $\succsim$ on W iff the set $B \backslash A$ is dominated by $A \cap B$ with respect to this order. This criterion seems to be adequate only if positive and negative desires do not interfer leading to preference reversal. In our former example, not having potatoes is not preferentially domineered by having them. As it will be expected by this insight, the representation theorems found in [Lang et al. (2002)] are restricted to the two cases of either pure win or pure loss desires. In contrast, we will adopt a holistic view declaring only of a complete set Δ to be representable by the preference order, which holds if and only if its utility function can be written in the form (2.4).

There are two major conceptual differences between norms and desires. Although, fulfilment of norms is desirable, the validity of a norm is independent of it. In a case of murder, even if the crime has irreversibly being commited, the norm not to do so would still be assumed to hold. In contrast to this, desires share with goals the principle of fulfilment expressed by (2.3), while a conditional norm of the type 'obligement towards A, given $\neg A$' would still make sense, at least in a Kripkean framework.

We call a desire $\delta = \langle A, B\rangle$ **consistent**, if and only if $A \cap B \neq \emptyset$, which under convention (2.3) is equivalent to $A \neq \emptyset$. Consistency also implies that the negative alternative can be avoided, $B \backslash A \neq W$.[1] Inconsistent desires are to have one's cake and eat it.[2] This should not be confused with the stronger notion of satisfiability of (consistent) wishes, which is judged relatively to the subset $W_0 \subseteq W$ of available alternative worlds, and holds, if and only if $W_0 \cap A \cap B \neq \emptyset$. Inconsistent conditional desires $\delta = \langle A, B\rangle$, $A \cap B \neq \emptyset$, are equivalent to consistent unconditional pure loss desires of the form $\delta' = \langle \neg B, W\rangle$ in the sense that they have a local utility function of the same form. We call a desire $\delta = \langle A, B\rangle$ **nondegenerated**, if and only if it is consistent and $B \backslash A \neq \emptyset$.

Secondly, the biconditional $O\left(.,.\right)$ representing qualitative conditional norms is generally assumed to be closed under logical consequence and conjunction stable, and follow the 'filter' axioms: $O\left(A|B\right) \wedge A$ implies $A' \Rightarrow O\left(A'|B\right)$ and $O\left(A|B\right) \wedge O\left(A'|B\right) \Rightarrow O\left(A \wedge A'|B\right)$, while $\neg O\left(\perp|B\right)$ with the contradictory proposition $\perp$ is supposed to hold for consistent B, and $O\left(.,.\right)$ is extensional in both arguments, allowing for substitution of equivalent propositions. In a qualitative context, a weaker norm like 'you should not kill' is therefore reducable to the stronger norm 'you should not injure', whenever killing implies injuring the victim. In a quantitative context, if strengths of norm violations are to be introduced, an injurance is definitely of minor severity than a murder. The same holds for quantitative conditional desires, where the desire 'I strongly prefer a Mercedes car' is to be differentiated from 'I would be happy to have a car at all'.

The notion of writing the logical structure of a desire as an ordered pair of sets can be justified as follows. Two desires $\delta = \langle A, B\rangle$ and $\delta' = \langle A', B'\rangle$ are called structurally equivalent, iff their utility functions differ only by coefficients, whenever they are relevant. More precisely,

[1] Since for $A, B \subseteq W$ we have $B \backslash A = W \Leftrightarrow B = W \wedge A = \varnothing$, and thus $A \cap B \neq \varnothing$ ensures $B \backslash A \neq W$.

[2] In deontic logic it is normally not assumed that for the dyadic deontic operator $O\left(A|B\right)$ representing a conditional *norm* we must have $A \wedge B$ consistent. We might insist that, even if a violation of a conditional obligation has already been committed, under this condition the norm not to do so might still hold. The consistency condition might constitute a conceptual difference between the logic of norms and of desires; the former which links real worlds to morally ideal worlds, the latter represents a decision theoretical concept.

Definition 1. *Two real-valued functions u and u' on W are called* ***structurally equivalent****, $u \approx u'$, iff there exists a bijective sign-preserving real-valued function φ with*

$$u(\omega) = \varphi(u'(\omega)) \text{ for all } \omega \in W.$$

Here, a real-valued function φ is called sign-preserving iff $\operatorname{sgn}(\varphi(x)) = \operatorname{sgn}(x)$ for all x, where, by convention, $\operatorname{sgn}(0) = 0$.

Structural equivalence implies ordinal equivalence for local utility functions in our sense (and is even equivalent to it for mixed desires), but obviously not in general, since the function φ is not required to be monotoneous. We obtain the following straightforward characterization:

Proposition 1. *Two local utility functions u and u' with parameters α, β and α', β' associated to desires $\delta = \langle A, B\rangle$ and $\delta' = \langle A', B'\rangle$, respectively, are equivalent iff the following two conditions hold:*
(i) $B' \backslash A' = B \backslash A$ or $((\alpha' = 0$ or $B' \backslash A' = \emptyset)$ and $(\alpha = 0$ or $B \backslash A = \emptyset))$
(ii) $A' \cap B' = A \cap B$ or $\left((\beta' = 0 \text{ or } A' \cap B' = \emptyset\right)$ and $(\beta = 0$ or $A \cap B = \emptyset))$
Moreover, if δ and δ' are nondegenerated mixed desires satisfying (2.3), the two conditions collapse to

$$B = B' \wedge A = A'.$$

Desires must be uniquely defined by the support of the local utility function. Positive desires should not occur under different conditions in the same desire structure. In these cases we assume that the weakest condition is chosen to represent the desire in the structure. Negative desires $\langle A, B\rangle$ are likewise characterised by $B \backslash A$. Moreover, no positive desire should be exactly outweighted by a corresponding negative desire for all possible worlds. These arguments motivate the conditions (i) (ii) and (iii) in the following definition.

Definition 2. *A pair $\mathcal{D} = \langle \Delta^+, \Delta^- \rangle$ is called a* ***desire structure*** *iff $\Delta = \Delta^+ \cup \Delta^-$ is a set of ordered pairs $\delta = \langle A, B\rangle$ with $A \subseteq B$ (2.3), each of which represents a non-degenerated desire, and*
(i) For all $\langle A, B\rangle, \langle A, B'\rangle \in \Delta^+$ there is $B = B'$.
(ii) For all $\langle A, B\rangle, \langle A', B'\rangle \in \Delta^-$, $B \backslash A = B' \backslash A'$ implies $A = A'$.
(iii) For all $\langle A, B\rangle \in \Delta^+$ and $\langle A', B'\rangle \in \Delta^-$ we have $A \neq B' \backslash A'$.
We call the elements of $\Delta^+ \backslash \Delta^-$ ***win desires****, $\Delta^- \backslash \Delta^+$ the* ***loss desires****, and $\Delta^+ \cap \Delta^-$ the* ***mixed desires****.*

Definition 3. *Let $\mathcal{D} = \langle \Delta^+, \Delta^- \rangle$ be a desire structure. Its* ***characteristic sets*** *are declared as follows,*

$$\begin{aligned} \Sigma \ &:= \Sigma^+ \cup \Sigma^-, \text{ where} \\ \Sigma^+ \ &:= \left\{ A | \langle A, B\rangle \in \Delta^+ \right\}, \ \Sigma^- := \left\{ B \backslash A | \langle A, B\rangle \in \Delta^- \right\}. \end{aligned}$$

Conditions (i) and (ii) of the definition of a desire structure (definition 2) ensure that each desire structure is associated to exactly one characteristic set. Clause (iii) ensures that each characteristic set can be associated to either a positive or a negative desire.

Remark 1. *The mapping from a desire structure to its characteristic sets is one-one. Moreover,*

$$\Sigma^+ \cap \Sigma^- = \emptyset. \tag{3.2}$$

Definition 4. *Let $\mathcal{D} = \langle \Delta^+, \Delta^- \rangle$ a finite desire structure, and $\succsim$ a relation on W. We say that $\succsim$ is **$\mathcal{D}$-representable** if and only if for the characteristic sets of $\Sigma = \{S_1, \ldots, S_d\}$, there are coefficients $\gamma_1, \ldots, \gamma_d$ with $\gamma_i > 0$ for $S_i \in \Sigma^+$, $\gamma_i < 0$ for $S_i \in \Sigma^-$, such that $\succsim$ is represented by the utility function*

$$U(\omega) = \sum_{i=1}^{d} \gamma_i \cdot \chi_{S_i}(\omega), \tag{3.3}$$

in the common sense

$$\forall \omega\omega' \in W : \ \omega \succsim \omega' \Leftrightarrow U(\omega) \geq U(\omega').$$

The qualitative desire structures induce the following natural dominance relation about the agent's desires:

Definition 5. *Let $\mathcal{D} = \langle \Delta^+, \Delta^- \rangle$ be a desire structure. World ω' is **weakly dominant** to ω, $\omega' \sqsupseteq \omega$, iff more gain or mixed desires are fulfilled in ω' than in ω, and less loss or mixed desires are unfulfilled.*

$$\omega' \sqsupseteq \omega \Leftrightarrow \begin{cases} \omega \notin B \backslash A \Rightarrow \omega' \notin B \backslash A & \text{for all } \langle A, B \rangle \in \Delta^- \text{ and} \\ \omega \in A \Rightarrow \omega' \in A & \text{for all } \langle A, B \rangle \in \Delta^+ \end{cases}.$$

From now on, $\succ$ and $\sim$ will denote the asymmetrical and the symmetrical parts of $\succsim$ respectively, and $\sqsupset$ and $\equiv$ stand for the corresponding parts of $\sqsupseteq$.

Definition 6. *Let $\mathcal{D}$ be a desire structure and $\sqsupseteq$ its associated dominance relation. A weak order (complete and transitive) $\succsim$ on W is said to obey the condition of **weak monotonicity** iff for each $w, w' \in W$*

$$\omega' \sqsupseteq \omega \Rightarrow \omega' \succsim \omega. \tag{3.4}$$

*It satisfies the condition of **strong monotonicity** iff it satisfies weak monotonicity and for each $w, w' \in W$*

$$\omega' \sqsupset \omega \Rightarrow \omega' \succ \omega. \tag{3.5}$$

The following fact is easily to be concluded from (3.3).

Proposition 2. *If $\mathcal{D}$ is a desire structure and $\succsim$ a $\mathcal{D}$-representable weak ordering on W, then $\succsim$ satisfies the strong monotonicity condition.*

4. The Non-Supervenience Condition

The following section introduces the concept of supervenience, which we will need to impose restrictions on the logical structure of desires. A property or a proposition S is said to supervene on a set Σ of properties or propositions if and only if from a complete information on Σ it is determined whether S holds or not. Supervenience is weaker than definability, since the function obtaining the truth value of S from those of Σ is unknown. It is only assumed to exist. In certain special languages, one can infer that supervenience is equivalent to an adjunction of definitions.[3] We will assume that a feasible set of desires is independent in the sense that no supervenience relation holds among them: Any desire introduces

[3]Let N be a set of propositions. A predicate D is definable in N if and only if there is an open formula φx such that $N \models \forall x \, (Dx \Leftrightarrow \varphi x)$. Within monadic first-order predicate logic, supervenience has been shown to be equivalent to

$$N \models \vee_{i=1}^{n} \forall x \, (Dx \Leftrightarrow \varphi_i x)$$

for some open formulas $\varphi_1, \ldots, \varphi_n$.

a new partition in the information structure generated by the characteristic sets. In other words, for each desire δ there is a set $C \subseteq W$ on which, in the sense of a *ceteris-paribus condition*, all other desires are equally satisfied, and which is subdivided by the characteristic set of the desire into two nonempty subsets. This seems to be a natural condition, which also allows for hierachically ordered desires, as in the potatoe example above.

Assume an agent has a preference order representable by a desire structure with characteristic sets Σ. Then the following equivalence relation represents the *information structure* on W given by her desires.

Definition 7. *Let $\Sigma \subseteq \wp(W)$. We define an equivalence relation $=_\Sigma$ on W by*

$$\forall \omega, \omega' \in W, \quad \omega =_\Sigma \omega' \Leftrightarrow \forall S \in \Sigma \ \ (\omega \in S \Leftrightarrow \omega' \in S)\,.$$

Indeed, the subject is indifferent to all $=_\Sigma$-equivalent states, since $=_\Sigma$ is a subrelation of $\equiv$.

Remark 2.

$$(=_\Sigma) \subseteq (\equiv)$$

Remark 3.

$$\Sigma' \subseteq \Sigma \Rightarrow (=_\Sigma) \subseteq (=_{\Sigma'})\,.$$

Definition 8. *Let $\Sigma \subseteq \wp(W)$. We say that $S \subseteq W$* ***supervenes*** *on Σ, in symbols $S \mid \Sigma$, iff*

$$\forall \omega, \omega' \in W: \ \omega =_\Sigma \omega' \Rightarrow (\omega \in S \Leftrightarrow \omega' \in S)\,.$$

The following three remarks follow immediately from the definition.

Remark 4. *$S \mid \Sigma$ iff S can be written as a sum of equivalence classes from $W/=_\Sigma$ iff for all $F \in W/=_\Sigma: F \subseteq S \vee F \cap S = \emptyset$.*

Remark 5.

$$S \in \Sigma \Rightarrow S \mid \Sigma.$$

Remark 6.

$$\Sigma' \subseteq \Sigma \wedge S \mid \Sigma' \Rightarrow S \mid \Sigma.$$

Supervenience and measurability by a set-algebra are closely related, as the following lemma shows. It emphasizes again the connection between supervenience and the information structure given by Σ.

Proposition 3. *Let $\Sigma \subseteq \wp(W)$. If S is measurable with respect to the smallest (σ-) algebra containing Σ and W, $S \in \boldsymbol{\sigma}(\Sigma \cup \{W\})$, then S supervenes on Σ. Moreover, if Σ is finite, then the two statements are equivalent.*

Corollary 1. *(to the proof)*

$$S \mid \Sigma \Leftrightarrow S \mid \boldsymbol{\sigma}(\Sigma \cup \{W\})$$

The rest of the section contains material needed for the proof section and can be skipped at first reading. The following proposition gives an important information about the number of elementary Σ-supervenient sets.

Definition 9. *The granularity of a set of sets $\Sigma \subseteq \wp(W)$ is defined as the number of equivalence classes*

$$\operatorname{gran}(\Sigma) := \#\left(W/=_{\Sigma}\right).$$

Proposition 4. *Let $\Sigma \subseteq \wp(W)$ be finite such that $S \mid \Sigma \backslash \{S\}$ is false for all $S \in \Sigma$. Then*

$$\operatorname{gran}(\Sigma) \geq \#\Sigma + 1.$$

Morevover, the number of worlds exceeds the number of desires.

5. A Representation Theorem

We now impose a non-supervenience condition on desire structures which will be a prerequisite for our representation theorem and for the relative uniqueness of the representation.

Definition 10. *A desire structure $\mathcal{D}$ with characteristic sets Σ is called **non-supervenient**, iff no characteristic set supervenes on the others,*

$$\textit{not}\,(S \mid \Sigma \backslash \{S\}) \text{ for all } S \in \Sigma. \tag{5.1}$$

Let us now deal with the representational problem. Fix a desire structure $\mathcal{D} = \langle \Delta^{+},\ \Delta^{-} \rangle$ with its associated set of characteristic sets Σ.

Definition 11. *For $p \geq 2$ we define a binary relation Λ_p over W^p,*

$$(\omega_1, \ldots, \omega_p)\ \Lambda_p\ \left(\omega_1', \ldots, \omega_p'\right),$$

with $\omega_j,\ \omega_j' \in W$, $j = 1 \ldots p$, by

$$\#\{j : \omega_j \in S\} = \#\left\{j : \omega_j' \in S\right\}$$

for each $S \in \Sigma$.

It is easy to see that Λ_p is an *equivalence relation.* In order to achieve an additive representation, we introduce the cancellation conditions C_p by Tversky [Tversky (1964)] and Fishburn [Fishburn (1970)],[Fishburn (2001)]. See also Wakker [Wakker (1989)].

Condition 1 (C_p)**.** *For $\omega_1, \ldots, \omega_p,\ \omega_1', \ldots, \omega_p' \in W$:*
$(\omega_1, \ldots, \omega_p)\ \Lambda_p\ \left(\omega_1', \ldots, \omega_p'\right)$, $\omega_k' \succsim \omega_k$ for all $k = 1$ to $p-1$
is equivalent to not$(\omega_p' \succ \omega_p)$.

It is easy to see that :

Remark 7. *If Conditions C_2 and C_3 are satisfied then $\succsim$ is a weak order.*

Theorem 1 (The main theorem)**.** *Let $\mathcal{D} = \langle \Delta^{+},\ \Delta^{-} \rangle$ be a finite non-supervenient desire structure, and let $\Sigma = \{S_1, \ldots, S_d\}$ be its associated characteristic sets. The following two conditions are equivalent for a relation $\succsim$ on W.*

1. *Condition C_p is satisfied for $p = 2, 3, 4, \ldots$ and $\succsim$ satisfies strong monotonicity.*

(2) $\succsim$ *is a* $\mathcal{D}$*-representable weak order, i.e. there exists* $\gamma_1, \ldots, \gamma_d$ *with* $\gamma_i > 0$ *for* $S_i \in \Sigma^+$, $\gamma_i < 0$ *for* $S_i \in \Sigma^-$ *such that for all* $\omega,\ \omega' \in W$

$$\omega \succsim \omega' \Leftrightarrow U(\omega) \geq U(\omega'), \tag{5.2}$$

where

$$U(\omega) = \sum_{i=1}^{d} \gamma_i \cdot \chi_{S_i}(\omega). \tag{5.3}$$

Moreover, for each utility function U *of the form (5.3) representing* $\succsim$ *in the sense of (5.2), the coefficients* γ_i *are uniquely defined, and there are always more constraints on* U *than those needed to determine the coefficients.*

The uniqueness property is especially useful in case $\succsim$ is stemming from a richer framework ensuring cardinal utilities, which is unique up to a positive affine transformation. Then the coefficients are unique up to a common positive factor. The last statement is to show that under the condition of non-supervenience, desire theory has an empirical content: For a given preference order $\succsim$ on W and a given non-supervenient desire structure $\mathcal{D}$, the question whether $\succsim$ is $\mathcal{D}$-representable is non-trivial.

Qualitative Decision Theory suggests a definition of *rationality.* Let us first define the following binary relation induced by a desire structure $\mathcal{D}$.

Definition 12. *A Desire structure* $\mathcal{D}$ *induces a binary relation over* W *denoted* $R_{\mathcal{D}}$ *and defined by :* $\forall\ \omega, \omega' \in W,\ \omega R_{\mathcal{D}} \omega'$ *if there exists a desire* $\delta = \langle A, B \rangle \in \mathcal{D}$ *such that* $\omega \in A$ *and* $\omega' \in B$.

Our assumption is that $R_{\mathcal{D}}$ the *binary relation induced by a desire structure* $\mathcal{D}$ is not necessarily the agent's true preference over W (except when $R_{\mathcal{D}}$ is a weak order) which is assumed to be always a weak order. That is we want to extend $R_{\mathcal{D}}$ to a weak order called *latent weak order preference* and denoted $\succsim_{\mathcal{D}}$. However we want this latent preference to respect the first condition of compatibility in the sense of Roberts [Roberts (1971)][4] between a binary relation and a weak order:

$$\succsim_{\mathcal{D}} \subseteq^{5} R_{\mathcal{D}} \cup J_{R_D} \tag{5.4}$$

where $J_{R_D} = \left\{(\omega, \omega') \in W^2 : (\omega', \omega) \notin R_{\mathcal{D}} \text{ and } (\omega, \omega') \notin R_{\mathcal{D}}\right\}$ is the incomparability relation w.r.t. $R_{\mathcal{D}}$. For instance if $R_{\mathcal{D}}$ is acyclic, we can take the following weak order:

(5.5)

$\succsim_{\mathcal{D}} = T^u_{R^*} + \left(J_{T^u_{R^*}} \cap T^l_{R^*}\right) + A$ with:

$R^* = R_{\mathcal{D}} \cup J_{R_D}$

$T^u_{R^*}$ is defined by : $\forall\ \omega, \omega' \in W,\ \omega T^u_{R^*} \omega' \Leftrightarrow \{b \in W : bR^*\omega\} \subseteq \{b \in W : bR^*\omega'\}$

$J_{T^u_{R^*}}$ is the incomparability relation w.r.t. $T^u_{R^*}$

$T^l_{R^*}$ is defined by : $\forall\ \omega, \omega' \in W,\ \omega T^l_{R^*} \omega' \Leftrightarrow \{b \in W : \omega' R^* b\} \subseteq \{b \in W : \omega R^* b\}$

A is constructed from $J_{T^u_{R^*}} \cap J_{T^l_{R^*}}$ by orienting the edges of the graph representing $T^u_{R^*} + \left(J_{T^u_{R^*}} \cap T^l_{R^*}\right) + J_{T^u_{R^*}} \cap J_{T^l_{R^*}}$ such as to preserve the transitivity of $T^u_{R^*} + \left(J_{T^u_{R^*}} \cap T^l_{R^*}\right)$.

[4] See also [Monjardet (1978)] or [Diaye (1999)].

[5] By duality, we have $P_{R_{\mathcal{D}}} \subseteq \succ_{\mathcal{D}}$ where $P_{R_{\mathcal{D}}}$ and $\succ_{\mathcal{D}}$ are the asymmetric components of $R_{\mathcal{D}}$ and $\succsim_{\mathcal{D}}$ respectively.

$T^u_{R^*}$ and $T^l_{R^*}$ are partial weak orders *(reflexive and transitive)* and are called *upper section partial weak order* and *lower section partial weak order associated with* R^* respectively.

Example 1. *$W = \{a,b,c,d\}$, $\mathcal{D} = \{\langle\{a,b\},\{a,b,c\}\rangle, \langle\{a\},\{a,c,d\}\rangle\}$.*
The binary relation induced by the desire structure is

$$R_{\mathcal{D}} =^6 \{(a,a),(b,b),(a,b),(b,a),(a,d),(a,c),(b,c)\}$$

This binary relation is neither transitive (indeed there is no condition in $\mathcal{D}$, including both b and d) nor complete. We can extend it to the weak order $\succsim_{\mathcal{D}}= T^u_{R^*} + \left(J_{T^u_{R^*}} \cap T^l_{R^*}\right) + A$ which is here equal to $T^u_{R^*}$.

$$\succsim_{\mathcal{D}}= \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,d),(a,c),(b,c),(b,d),(d,c),(c,d)\}$$

Definition 13. *Let W be a set of possible worlds, and $\mathcal{D}$ be the agent's desire structure. The agent is rational w.r.t. $\mathcal{D}$ (or is $\mathcal{D}$-rational) if :*
(i) $\mathcal{D}$ is independent.
(ii) There exists $\succsim_{\mathcal{D}}$ a weak order over W, which respects (5.4) and is $\mathcal{D}$-representable.

Remark that *all these weak orders are equivalent from $\mathcal{D}$-rationality standpoint.* Let us illustrate the concept of $\mathcal{D}$-rationality with the following example.

6. Example with Choice Functions Theory

Let W be a finite set of objects and $P(W)$ be the power set. A *Domain of Choice* F is a subset of $P(W)\backslash\emptyset$. A *Choice Function* C is a function defined from F to $P(W)$ with the condition that $C(S) \subseteq S$. A choice function is *univalent* if $\forall\ S \in F,\ \#C(S) = 1$. For simplicity purpose, we will restrict ourselves to the class of *univalent* choice functions whose domains of choice are closed under $\cup$ and $\cap$. Finally we will remove[7], without loss of generality, from the domain of choice F the single element sets $\{x\}$.

$C(S)$ is usually interpreted as the agent's choice over a choice set S. We will interpret $C(S)$ as a *conditional choice*. That is we interpret $\langle C(S), S\rangle$ as a *conditional desire* : 'I desire $C(S)$ if S happens'. $\langle C(S), S\rangle$ is of course not degenerated in the sense defined in section 3. The domain of choice F and the choice function C induce a *desire structure* $\mathcal{D}$ in the sense of definition 2, constructed by the following way:

RULE OF CONSTRUCTION:

(1) Set $\Delta = \{\langle C(S), S\rangle : S \in F\}$
(2) $\mathcal{D} = \langle \Delta^+, \Delta^-\rangle$ where

$\Delta^- = \{\langle C(S), S\rangle : \exists\ S' \in F,\ S' \subset S\ \ s.t.\ \exists\ x \in C(S) \cap S'\ and\ \ x \notin C(S')\}$
$\Delta^+ = \Delta\backslash\Delta^-$.

(3) If there exist: $\langle C(S), S\rangle$, $\langle C(S'), S'\rangle \in \Delta^+$ with $C(S) = C(S')$ or $\langle C(S), S\rangle$, $\langle C(S'), S'\rangle \in \Delta^-$ with $S\backslash C(S) = S'\backslash C(S')$ or $\langle C(S), S\rangle \in \Delta^+$, $\langle C(S'), S'\rangle \in \Delta^-$ with $C(S) = S'\backslash C(S')$ then remove from $\mathcal{D}$ the one with the smallest condition (in term of $\subset$).

[6] $(x,y) \in R_{\mathcal{D}} \Leftrightarrow xR_{\mathcal{D}}y$
[7] The main reason is that when C is univalent, then trivially $C(\{x\}) = \{x\}$.

END.

Thus a desire $\langle C(S), S\rangle$ is a loss desire (see step 2) if[8] there exists another desire $\langle C(S'), S'\rangle$ whose condition S' is included in S and such that the element x desired by the agent under S (and belonging to S') is not desired by him under $S' : S' \subset S,\ C(S) = \{x\},\ x \in S',\ C(S') = \{\overline{x}\}$ and $\overline{x} \neq x$. Finally let us remark that the condition specified in Δ^- is a violation of the so-called Condition α ([Chernoff (1954)]):

$$\forall\ S, S' \in F,\ S' \subset S \Rightarrow C(S) \cap S' \subset C(S')$$

According to our above definition 13, C is $\mathcal{D}$-rational if the associated desire structure $\mathcal{D}$ is independent and there exists a weak order over W, which respects (5.4) and is $\mathcal{D}$-representable.

Let us now recall the current definition of rational choice used in choice function theory.

Definition 14 (Richter 1971)**.** *Let $C : F \to P(W)$ be a choice function. C is Richter-rational* [9] *if there exists a binary relation Q over W such that:*

$$\forall\ S \in F,\ C(S) = \{x \in S : xQy, \forall\ y \in S\}$$

Remark 8. *Let $C : F \to P(W)$ be a univalent choice function with F closed under $\cup$ and $\cap$. Let $\mathcal{D}$ be the desire structure constructed using the Rule of Construction. Then C is Richter-rational iff $\Sigma = \Sigma^+$. Indeed* [Moulin (1985)] *has proved the equivalence between Richter-rationality and Condition α for univalent choice functions over abstract domains of choice. Moulin's result holds, of course, in our case where F is closed under $\cup$ and $\cap$.*

Remark 9. *When C is Richter-rational then $R_{\mathcal{D}}$ is actually the so-called revealed preference relation and is rationally equivalent to any weak order $\succsim_{\mathcal{D}}$ which respects (5.4) and is $\mathcal{D}$-representable.*

The following result shows that $\mathcal{D}$-rationality is weaker than Richter-Rationality.

Proposition 5. *Let $C : F \to P(W)$ be a univalent choice function with F closed under $\cup$ and $\cap$. Let $\mathcal{D}$ be the desire structure constructed using the Rule of Construction.*
(1) *implies* (2).

(1) *C is Richter-Rational (or equivalently $\Sigma = \Sigma^+$).*
(2) *C is $\mathcal{D}$-rational.*

Example 2. *Let $W = \{x, y, z, a\}$ and $C\{x, y\} = C\{x, a\} = \{x\}$; $C\{\{a, y\} = \{a\}$; $C\{a, x, y\} = \{x\}$.*
It is easy to see that Condition α is fulfilled, therefore $\Delta^- = \emptyset$.
Using the rule of construction we get the following desire structure :

$$\mathcal{D} = \{\langle \{x\}, \{x, y, a\}\rangle, \langle \{a\}, \{y, a\}\rangle\}$$

All these desires are win desires and the associated set of characteristic sets is :

$$\Sigma = \Sigma^+ = \{\{x\}, \{a\}\}$$

---

[8] This interpretation is due to the fact that C is univalent.

[9] This definition is known under the name of G-rationality. There exists another concept called M-rationality, however they are equivalent in duality.

C is Richter-Rational. $\mathcal{D}$ is independent and the binary relation $R_{\mathcal{D}}$ is $\{(x,x),(a,a),(x,y),(x,a),(a,y)\}$.
Let $\succsim_{\mathcal{D}}$ be the upper section weak order associated with $R_{\mathcal{D}} \cup J_{R_{\mathcal{D}}}$:

$x \succ_{\mathcal{D}} a \succ_{\mathcal{D}} y \sim_{\mathcal{D}} z$
where $\succ_{\mathcal{D}}$ and $\sim_{\mathcal{D}}$ are the asymmetric and symmetric parts of $\succsim_{\mathcal{D}}$ respectively.

$\succsim_{\mathcal{D}}$ is $\mathcal{D}$-representable with $U(x) = \gamma_1$, $U(a) = \gamma_2$, $U(y) = 0$, $U(z) = 0$ and $\gamma_1 > \gamma_2 > 0$. Of course $\succsim_{\mathcal{D}}$ is rationally equivalent to $R_{\mathcal{D}}$.

However the converse ((2) $\Rightarrow$ (1)) is not true as shown by the below example in which the individual in the Sen's example ([Sen (1993)]) is $\mathcal{D}$-rational but not Richter-rational.

Example 3. *Let $W = \{x,y,z\}$ with $x =$"go home", $y =$"tea", $z =$"cocaine". $C\{x,y\} = \{y\}$; $C\{x,y,z\} = \{x\}$. $\delta_1 = \langle\{y\},\{x,y\}\rangle$ is a win desire while $\delta_2 = \langle\{x\},\{x,y,z\}\rangle$ is a loss desire. Therefore $\Sigma = \Sigma^+ \cup \Sigma^-$, where $\Sigma^+ = \{\{y\}\}$, $\Sigma^- = \{\{y,z\}\}$ and $R_{\mathcal{D}} = \{(x,y),(y,x),(x,z)\}$. C is not Richter-Rational since Condition α is violated, however C is $\mathcal{D}$-rational. Indeed $\mathcal{D}$ is independent and the following complete order $\succsim_{\mathcal{D}} = y \succ_{\mathcal{D}} x \succ_{\mathcal{D}} z$, is $\mathcal{D}$-representable with $U(x) = 0$, $U(y) = \gamma_1 + \gamma_2$, $U(z) = \gamma_2$, and $\gamma_1 > 0$, $\gamma_2 < 0$, $\gamma_1 + \gamma_2 > 0$.*

$\mathcal{D}$-rationality is also weaker than the concept of *sub-rationality by a weak order* ([Fishburn (1976)], [Deb (1983)], [Moulin (1985)]) since $\mathcal{D}$-rationality allows for the violation of the Fishburn's *Partial Congruence Axiom* (which is required for sub-rationality by a weak order when the domain of choice is selective) and for the violation of Deb's *Axiom* α^* (which is required for sub-rationality by a weak order when the domain of choice is abstract). It is also weaker than the concept of *pseudo-rationality by some linear orders* because it allows for the violation of Condition α while, according to [Aizerman & Malishevski (1981)], pseudo-rationality by some linear orders requires its respect.

7. Conclusion

The formalism of Qualitative Decision Theory has been designed to model goals for automatic decisions. It is, however, suitable for analysing preferences in terms of the motives of the agent, which we called his or her desires. Applied to choice functions theory, this leads to generalised form of rationality beyond the standard paradigm of Richter rationality, which corresponds to the revealed preferences approach. We are able to cope with the Sen's example of menu-dependent preference reversal.

APPENDIX: PROOFS

Proof of proposition 1. The first part follows directly from the definition. For the proof the second part observe that for consistent nondegenerated mixed desires the two conditions (i) and (ii) are equivalent to $B'\backslash A' = B\backslash A$ and $A' \cap B' = A \cap B$. It follows

$$B = (A \cap B) \cup (B\backslash A) = (A' \cap B') \cup (B'\backslash A') = B',$$

and further by (2.3) we obtain $A = A'$. □

Proof of proposition 3. Let $\Sigma' := \boldsymbol{\sigma}(\Sigma \cup \{W\})$. We have to show that $(=_\Sigma) = (=_{\Sigma'})$, then the first proposition follows from remark 5. By remark 3, $(=_{\Sigma'}) \subseteq (=_\Sigma)$. For the conversion we only have to show that for all $x, y \in W$ (i) if $x \in S_i \Leftrightarrow y \in S_i$ holds for each $i \in I$, then $x \in \bigcup_{i\in I} S_i \Leftrightarrow y \in \bigcup_{i\in I} S_i$, and (ii) $x \in S \Leftrightarrow y \in S$ implies $x \in W \backslash S \Leftrightarrow y \in W \backslash S$. The latter is trivial, so let $x \in S_i \Leftrightarrow y \in S_i$ hold for each $i \in I$. Define

$$S := \bigcup_{i\in I} S_i.$$

If there is an $i \in I$ with $x \in S_i$, then both $x, y \in S_i \subseteq S$. Otherwise, neither x nor y is in one of the S_i, thus $x, y \notin S$. Thus whenever $\Sigma_1 \subseteq \Sigma_2$, and Σ_2 differs from Σ_1 only by unions and complements of sets, then $(=_{\Sigma_1}) \subseteq (=_{\Sigma_2})$. Since clearly $W \mid \Sigma$, we have completed the proof of $(=_\Sigma) = (=_{\Sigma'})$.

For the proof of the second proposition, let Σ be finite. In this case, $\Sigma' := \boldsymbol{\sigma}(\Sigma \cup \{W\})$ is also finite and thus atomic. Let $A_1, \ldots, A_n$ be its atoms. Then $S \in \Sigma'$ iff S can be written as a union of atoms. We have to show that the atoms coincide with $W/ =_\Sigma$, then the proposition follows from remark 4. Each element of $W/ =_\Sigma$ is either an intersection of a 'maximally consistent' subset of Σ, or $W \backslash \bigcup \Sigma$. In either case, it is an atom of Σ'. Conversely, each atom supervenes on Σ, and, since it is minimal w.r.t. this property, it must be in $W/ =_\Sigma$. □

In order to give a proof of theorem 4, we will need the following three lemmata.

Lemma 1. *For each $S \subseteq W$ and each $\Sigma \subseteq \wp(W)$ we have*

$$S \mid \Sigma \Leftrightarrow (=_\Sigma) = \left(=_{\Sigma \cup \{S\}}\right).$$

Proof. "$\Rightarrow$": The direction "$\supseteq$" follows directly from remark 3. In order to show "$\subseteq$" let there be $x =_\Sigma y$, then by assumption $x \in S \Leftrightarrow y \in S$, which completes $x =_{\Sigma \cup \{S\}} y$.

"$\Leftarrow$": Assume $\neg S \mid \Sigma$. Then there are $x, y \in W$ with $x =_\Sigma y$ but $\neg(x \in S \Leftrightarrow y \in S)$. But this implies $\neg\left(x =_{\Sigma \cup \{S\}} y\right)$, and $(=_\Sigma) \neq \left(=_{\Sigma \cup \{S\}}\right)$. □

Lemma 2. *Let E and E' be equivalence relations on W with $E \subseteq E'$. Then*

$$\begin{array}{ll} (i) & \#(W/E') \leq \#(W/E), \\ (ii) & \#(W/E') = \#(W/E) \Leftrightarrow E = E', \end{array}$$

where (ii) holds only for finite cardinal numbers.

Proof. (i) We have to define an embedding of the equivalence classes of E' into those of E'. Let there be $F \in W/E'$ an arbitrary equivalence class from E'. We define a mapping $h : W/E' \to W/E$ by selecting an arbitrary $x_F \in F$ and letting $h(F)$ be the equivalence class of x_F in W/E. For $y \in h(F)$ we have $x_F E y$ and thus $x_F E' y$, or $y \in F$. This shows $h(F) \subseteq F$. It follows that the mapping h is injective. Two different equivalence classes $F, G \in W/E'$, $F \neq G$ are always disjoint and thus $h(F) \cap h(G) \subseteq F \cap G = \emptyset$, which establishes $h(F) \neq h(G)$.

(ii) Only "$\Rightarrow$" remains to show. Let there be $E \subsetneq E'$, then there are $x, y \in W$ with $xE'y$ and $\neg xEy$. In other words, their equivalence classes coincide in W/E', but are disjoint in W/E. Let F denote the common equivalence class of x and y in W/E', and H_x, H_y the disjoint equivalence classes in W/E. Since $E \subseteq E'$, we find $H_x, H_y \subseteq F$. Construct h as above, then it must be $h(F) \neq H_x$ or $h(F) \neq H_y$. Since as shown before, for any $G \in W/E'$, $G \neq F$, we have $h(G) \cap F = \emptyset$, either

H_x or H_y are not in the range of h. If W/E is finite, this proves $\#(W/E') < \#(W/E)$. $\square$

Lemma 3. *For* $\Sigma, \Sigma' \subseteq \wp(W)$

$$\begin{array}{ll} \textit{(i)} & \Sigma' \subseteq \Sigma \Rightarrow \operatorname{gran}(\Sigma') \leq \operatorname{gran}(\Sigma). \\ \textit{(ii)} & \operatorname{gran}(\Sigma \cup \{S\}) = \operatorname{gran}(\Sigma) \Leftrightarrow S \mid \Sigma. \end{array}$$

Proof. (i): With remark 3 we conclude $(=_\Sigma) \subseteq (=_{\Sigma'})$ and further with lemma 2 (i) $\operatorname{gran}(\Sigma') \leq \operatorname{gran}(\Sigma)$.

(ii): With lemma 2 (ii) $\operatorname{gran}(\Sigma \cup \{S\}) = \operatorname{gran}(\Sigma)$ is equivalent to $(=_\Sigma) = (=_{\Sigma \cup \{S\}})$, and with lemma 1 equivalent to $S \mid \Sigma$. $\square$

We are now able to complete the proof of proposition 4.

Proof of proposition 4. Induction over $n := \#\Sigma$.

$n = 0$: It follows $\Sigma = \emptyset$ and thus $x =_\Sigma y$ for all $x, y \in W$. Thus $\operatorname{gran}(\Sigma) = 1$.

$n - 1 \to n$: Since $\Sigma \neq \emptyset$ let $S \in \Sigma$ and $\Sigma' := \Sigma \setminus \{S\}$. By assumption, $\neg S \mid \Sigma'$, and by inductive assumption

$$\operatorname{gran}(\Sigma') \geq \#\Sigma' + 1 = n - 1 + 1 = n.$$

With lemma 3 (i) we find $\operatorname{gran}(\Sigma) \geq \operatorname{gran}(\Sigma')$, and by lemma 3 (ii) from $\neg S \mid \Sigma'$ even $\operatorname{gran}(\Sigma) > \operatorname{gran}(\Sigma')$, establishing

$$\operatorname{gran}(\Sigma) \geq n + 1 = \#\Sigma + 1,$$

which has to be shown. Since $\#\Sigma = \#\Delta$ by remark 1, and $\#W \geq \operatorname{gran}(\Sigma)$, the number of worlds exceeds the number of desires. $\square$

We now want to prove the theorem 1.

Proof of the main theorem. Let $\mathcal{D} = \langle \Delta^+, \Delta^- \rangle$ be a finite non-supervenient desire structure with characteristic sets $\Sigma = \{S_1, \ldots, S_d\}$. If $\succsim$ is a $\mathcal{D}$-representable weak order in the sense of condition 2.), then by the additive representation theorem the cancellation conditions C_p hold for $p = 2, 3, 4, \ldots$, and strong monotonicity follows from proposition 2.

Existence: In order to show the converse, assume that $\succsim$ is a relation on W satisfying strong monotonicity and the cancellation conditions C_p, $p = 2, 3, 4, \ldots$. Then by remark 7, it is a weak order.

We define the following function $\pi : W \to \prod_{i=1}^{d} Z_i$, with $Z_i := \{0, 1\}$, by

$$\pi_i(\omega) = \begin{cases} 1, \text{ if } \omega \in S_i, \\ 0 \text{ otherwise}, \end{cases} = \chi_{S_i}(\omega). \tag{7.1}$$

By (3.2), this function is well-defined.

Let $Z = \pi(W)$ be the domain of π. By the monotonicity condition (3.4) we find

$$\forall \omega,\ \omega' \in W\colon\ \omega =_\Sigma \omega' \Rightarrow \omega \sim \omega'.$$

This can be rewritten as

$$\forall \omega,\ \omega' \in W\colon\ \pi(\omega) = \pi(\omega') \Rightarrow \omega \sim \omega'.$$

Therefore, $\succsim$ *induces* a binary relation, denoted $\succsim_Z$, over the set Z, in the natural way,

$$\forall\ \omega, \omega' \in W,\ \omega \succsim \omega' \Leftrightarrow \pi(\omega) \succsim_Z \pi(\omega'), \tag{7.2}$$

such that is there is an order-isomorphism between $(W, \succsim)$ and $(Z, \succsim_Z)$.

The aim of the proof is now to find an additive utility representation ϕ of $(Z, \succsim_Z)$, such that the the composition of two functions ϕ and π

$$U : (W, \succsim) \underset{\pi}{\longrightarrow} (Z, \succsim_Z) \underset{\phi}{\longrightarrow} (\mathbb{R}, \geq),$$

$$\omega \mapsto \phi(\pi(\omega)),$$

serves as a $\mathcal{D}$-representation of $\succsim$ in the sense of definition 4. We can therefore analyze our problem as a problem of constructing additive utilities with finite sets.

Using 7.1 we can write Condition C_p in terms of elements of Z. Then according to the additive representation theorem, C_p is satisfied for $p = 2, 3, ...$ if and only if there exist d real-valued functions $m_1, \ldots, m_d$ such that

$$\phi'(z) = \sum_{i=1}^{d} m_i(z_i)$$

represents $\succsim_Z$,

$$\forall\ z,\ z' \in Z : \quad z \succsim_Z z' \Leftrightarrow \phi'(z) \geq \phi'(z').$$

Renormalizing ϕ' by subtracting a constant

$$b := \sum_{i=1}^{d} m_i(0)$$

gives a representation of $\succsim_Z$ by

$$\phi(z) := \phi'(z) - b = \sum_{i=1}^{d} [m_i(z_i) - m_i(0)] = \sum_{i=1}^{d} \tilde{m}_i(z_i),$$

with

$$\tilde{m}_i(z_i) := m_i(z_i) - m_i(0).$$

Since $\tilde{m}_i(0) = 0$ and $z_i \in \{0, 1\}$, we find by the definition of π

$$\tilde{m}_i(\pi_i(\omega)) = \tilde{m}_i(1) \cdot \chi_{S_i}(\omega).$$

Together with 7.2, this establishes that $\succsim$ is represented by

$$U(\omega) = \phi(\pi(\omega)) = \sum_{i=1}^{d} \tilde{m}_i(\pi_i(\omega)) = \sum_{i=1}^{d} \gamma_i \cdot \chi_{S_i}(\omega), \tag{7.3}$$

with $\gamma_i := \tilde{m}_i(1)$. This is already the desired form of the utility function.

It remains to be shown that $\gamma_i > 0$ for $S_i \in \Sigma^+$, and $\gamma_i < 0$ for $S_i \in \Sigma^-$ for all $i = 1, \ldots, d$. Fix any i between 1 and d. Since the desire structure is non-supervenient, $S_i \mid \Sigma'$ can not be true for $\Sigma' := \Sigma \setminus \{S_i\}$. Thus by definition, there exist $\omega, \omega' \in W$ with $\omega =_{\Sigma'} \omega'$ and $\omega \in S_i \not\Leftrightarrow \omega' \in S_i$. Without loss of generality, assume that $\omega \in S_i$ and $\omega' \notin S_i$. Since $\omega \in S_j \iff \omega' \in S_j$ for all $j \neq i$, the additive representation (7.3) gives us

$$U(\omega) \gtreqless U(\omega') \Leftrightarrow \gamma_i \gtreqless 0.$$

Now, if $S_i \in \Sigma^+$, then obviously $\omega \sqsupset \omega'$, and strong monotonicity entails $\gamma_i > 0$. Analogously, for $S_i \in \Sigma^-$ we get $\gamma_i < 0$.

Uniqueness: The question of uniqueness of the coefficients γ_i can only be characterized in abstract terms, although it depends on the sets only. We define the **rank** of the collection, $\mathrm{rk}(\Sigma)$, as follows. Let $=_\Sigma$ denote the equivalence relation from definition 7. Let $W/ =_\Sigma = \{F_1, \dots, F_m\}$ be the quotient sets and $x_1 \in F_1, \dots, x_m \in F_m$ representative elements. Define an $m \times d$-matrix M by

$$(M)_{ij} = \begin{cases} 1, & \text{if } F_i \subseteq S_j \\ 0 & \text{else} \end{cases} = \chi_{S_j}(x_i) \tag{7.4}$$

for $i = 1, \dots, m$, and $j = 1, \dots, d$. We call M the **characteristic matrix** of Σ. Then define $\mathrm{rk}(\Sigma)$ as the rank of the characteristic matrix M.

With the help of the vectors $\vec{\gamma} = (\gamma_1, \dots, \gamma_d)$ and $\vec{y} = (U(x_1), \dots, U(x_m))$ one can formulate the problem in algebraic terms as the uniqueness of the solution of the linear equation

$$M\vec{\gamma} = \vec{y}. \tag{7.5}$$

The solution $\vec{\gamma}$ given $\vec{y}$ is unique if and only if $\mathrm{rk}(\Sigma) = d$, which depends on the logical structure of Σ only. Assume now that Σ satisfies the non-supervenience condition (5.1). Then proposition 4 implies that

$$m > d, \tag{7.6}$$

which means that the equational system contains even more equations than necessary. It has the form

$$\begin{array}{cc} & \\ x_1 \in & F_1 \\ \vdots & \vdots \\ \vdots & \vdots \\ x_m \in & F_m \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & \cdots & S_d \end{array} \\ \begin{pmatrix} M_{11} & \cdots & M_{1d} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{md} \end{pmatrix} \end{array} \begin{array}{c} \vec{\gamma} \\ \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_d \end{pmatrix} \end{array} = \begin{array}{c} \vec{y} \\ \begin{pmatrix} U(x_1) \\ \vdots \\ U(x_m) \end{pmatrix} \end{array}.$$

In order to ensure that the matrix has full rank, it is sufficient to show that the column vectors are linearly independent. Consider the k-th column vector. Since $S_k \mid \Sigma \setminus \{S_k\}$ is false, by definition in remark 4, there are two (disjoint) equivalence classes F_i and F_j of $=_\Sigma$, which are undistinguishable by $\Sigma \setminus \{S_k\}$ (a more detailed argument can be extracted from the proof of lemma 2 (ii)). As a consequence, $M_{il} = M_{jl}$ for all $l \neq k$, the column vectors except for the k-th coincide on the i-th and j-th position and so does every linear combination of them. On the other hand, the k-th column vector has different values at these positions. Since this holds for all k, the column vectors are linearly independent and the matrix has full rank. Thus, the solution $\vec{\gamma}$ given the vector $\vec{y}$ of the domain of U is unique for each utility function U representing $\succsim$ in the common sense. Moreover, from (7.6) we see that there are $m - d > 0$ more constraints on U than needed to determine the coefficients. $\square$

Finally let us prove proposition 5.

Proof. Since $\#S = 1, \forall S \in \Sigma$ and $S \cap S' = \emptyset, \forall S, S' \in \Sigma,\ S \neq S'$, then we have: $\#W \geq \#\Sigma$. Finally since C is Richter-rational and F is closed under $\cup$ and $\cap$, then it is impossible to have $\#W = \#\Sigma$. Thus $\#W > \#\Sigma$.

Since $\#W > \#\Sigma$ and $\#S = 1, \forall S \in \Sigma$ and $S \cap S' = \emptyset, \forall S, S' \in \Sigma,\ S \neq S'$, then $\mathcal{D}$ the associated desire structure is independent. To complete the proof, we have to show that there exists a $\mathcal{D}$-representable weak order that respects (5.4). Whatever the candidate functional U , we have:

$$U(x_i) = \gamma_i \text{ if } x_i \in \bigcup_{k=1}^{d} S_k \text{ , and}$$
$$U(x_i) = 0 \text{ if } x_i \in W \setminus \bigcup_{k=1}^{d} S_k \text{ , where } \#\Sigma = d.$$

Moreover an element $x_i \in W \setminus \bigcup_{k=1}^{d} S_k$ for the following two reasons:

$$x_i \notin A \text{ for any } \langle A, B\rangle \in \mathcal{D} \text{ with } x_i \in B \tag{7.7}$$

$$x_i \notin B \text{ , } \forall \ \langle A, B\rangle \in \mathcal{D} \tag{7.8}$$

The main difficulty in constructing a $\mathcal{D}$-representable weak order $\succsim_{\mathcal{D}}$ is that it should be such that:

$$\begin{aligned} &x \succ_{\mathcal{D}} y,\ \forall\ y \in W \setminus \bigcup_{k=1}^{d} S_k \ \ \textit{and}\ \forall\ x \in \bigcup_{k=1}^{d} S_k \\ &x \sim_{\mathcal{D}} y,\ \forall\ x, y \in W \setminus \bigcup_{k=1}^{d} S_k \end{aligned} \tag{7.9}$$

Let us show that the weak order (see 5.5) $\succsim_{\mathcal{D}} = T^u_{R^*} + \left(J_{T^u_{R^*}} \cap T^l_{R^*}\right) + A$ respects (5.4) and is $\mathcal{D}$-representable. Indeed since C is Richter-rational, univalent and its domain of choice is closed under $\cup$ and $\cap$ then $R_{\mathcal{D}}$ is antisymmetric and transitive. $R^* = R_{\mathcal{D}} \cup J_{R_{\mathcal{D}}}$ is therefore quasi-transitive[10]. Hence (5.4) is fulfilled. Let x be an element such that (7.7) or (7.8) then $\{y \in W : yR^*x\} = W$, therefore (7.9) is fulfilled. Finally for any $x_i,\ x_j \in \bigcup_{k=1}^{d} S_k$, $i \neq j$, set $\gamma_i > \gamma_j$ if $x_i \succ_{\mathcal{D}} x_j$. The resulting functional is a $\mathcal{D}$-representation of $\succsim_{\mathcal{D}} = T^u_{R^*} + \left(J_{T^u_{R^*}} \cap T^l_{R^*}\right) + A$. $\square$

References

[Aizerman & Malishevski (1981)] Aizerman, Mark A. and Malishevski Andrei V. 'General Theory of Best Variants Choice: Some Aspects'. *IEEE Trans. on. Aut. Control* **26**, pp. 1030-1040.

[Bacchus & Grove (1997)] Bacchus, Fahiem and Grove, Adam. *Independence and Qualitative Decision Theory.* In: AAAI Spring Symposium on Qualitative Preferences in Deliberation and Practical Reasoning, 1997.

[Chernoff (1954)] Chernoff, Herman. 'Rational Selection of Decision Functions'. *Econometrica* **22**, pp. 423-443.

[Deb (1983)] Deb, Rajat. 'Binariness and Rational Choice'. *Mathematical Social Sciences* **5**, pp. 97-105.

[Diaye (1999)] Diaye, Marc-Arthur. 'Variable Intervals Model'. *Mathematical Social Sciences* **38**, pp. 21-33.

[Doyle & Wellman (1991)] Doyle, Jon and Wellman, Michael P. 'Preferential Semantics for Goals'. In: *Proceedings of the Ninth National Conference on Artificial Inteligence (AAAAI-91)*, pp. 698-703, 1991.

[10] Its asymmetric part is transitive.

[Doyle & McGeachie (2002)] Doyle, Jon and McGeachie, Michael. *Utility Functions for Ceteris Paribus Preferences.* Unpublished Draft, www.csc.ncsu.edu/faculty/doyle/publications/uc02.pdf, 2002.

[Dubois et al. (2001)] Dubois, Didier, Fargier, Helene, Perny, Patrice and Prade, Henry. 'Towards a Qualitative Multicriteria Decision Theory'. In: *Proceeding of Eurofuse Workshop on Preference Modelling and Applications*, Granada, Spain, 2001. (www.irit.fr/recherches/RPDMP/CDGEP.shtml)

[Fishburn (1970)] Fishburn, Peter C. *Utility Theory for Decision Making.* John Wiley and Sons, New York, 1970.

[Fishburn (1976)] Fishburn, Peter C. 'Representable Choice Functions'. *Econometrica* **44**, pp. 1033-1043.

[Fishburn (2001)] Fishburn, Peter C. 'Cancellation Conditions for Finite Two-Dimensional Additive Measurement'. *Journal of Mathematical Psychology* **45**, pp. 2-26, 2001.

[Keeney & Raiffa (1976)] Keeney, Ralph E. and Raiffa, Howard. *Decision with multiple Objectives: Preferences and Value Tradeoffs.* John Wiley and Sons, New York, 1976.

[Lang (1996)] Lang, Jérôme. 'Conditional Desires and Utilities - an Alternative Approach to Qualitative Decision Theory'. In: *Proceedings of the Twelfth European Conference on Artificial Intelligence (ECAI '96).* pp. 318-322, 1996.

[Lang et al. (2002)] Lang, Jérôme, van der Torre, Leendert, and Weydert, Emil, 'Utilitarian Desires'. *Autonomous Agents and Multi-Agent Systems* **5**, pp. 329-363, 2002.

[Monjardet (1978)] Monjardet, Bernard. 'Axiomatiques et propriétés des quasi-ordres'. *Mathématiques et Sciences Humaines* **63**, pp. 51-82.

[Moulin (1985)] Moulin, Hervé. 'Choice Functions over a Finite Set: A Summary'. *Social Choice and Welfare* **2**, pp. 147-160.

[Richter (1971)] Richter, Marcel K. *Rational Choice.* In: J.S. Chipman, M.K. Richter and H. Sonnenschein eds., 'Preference, Utility and Demand'. Harcourt Brace Jovanovich, New York, 1971.

[Roberts (1971)] Roberts, Fred 'On the Compatibility between a Graph and a Simple Order'. *Journal of Combinatorial Theory* **11**, pp. 28-38.

[Schoch (2000)] Schoch, Daniel. 'A Fuzzy Measure for Explanatory Coherence'. *Synthese* **122**, pp. 291-311, 2000.

[Sen (1993)] Sen, Amartya K. 'Internal Consistency of Choice'. *Econometrica* **61**, pp. 495-522.

[van der Torre & Weydert (2001)] van der Torre, Leendert, Weydert, Emil. 'Parameters for utilitarian desires in a qualitative decision theory', *Applied Intelligence* **14**, pp. 285-301, 2001.

[Tversky (1964)] Tversky, Amos. 'Finite additive structures'. *Michigan Mathematical Psychology Program*, MMPP 64-6. University of Michigan, Ann Arbor.

[Wakker (1989)] Wakker, Peter. *Additive Representation of Preferences, A New Foundation of Decision Analysis.* Kluwer Akademic Publishers, Dordrecht.

Department of Economics, University of Evry (EPEE)
E-mail address: marc-arthur.diaye@eco.univ-evry.fr
URL: http://www.univ-evry.fr/PagesHtml/laboratoires/Epee/EPEE.html

Philosophical Institute, University of Saarbrücken
E-mail address: d.schoch@mx.uni-saarland.de
URL: http://www.uni-sb.de/fak3/fr31/html/per_dsch.htm

DOCUMENTS DE RECHERCHE EPEE

2003

0301 Basic Income/ Minimum Wage Schedule and the Occurrence of Inactivity Trap: Some Evidence on the French Labor Market
Thierry LAURENT & Yannick L'HORTY

0302 Exonérations ciblées de cotisations sociales et évolution de l'emploi : un bilan à partir des études disponibles
Philippe DE VREYER

0303 Incorporating Labour Market Frictions into an Optimizing-Based Monetary Policy Model
Stéphane MOYEN & Jean-Guillaume SAHUC

0304 Indeterminacy in a Cash-in-Advance Two-Sector Economy
Stefano BOSI, Francesco MAGRIS & Alain VENDITTI

0305 Competitive Equilibrium Cycles with Endogenous Labor
Stefano BOSI, Francesco MAGRIS & Alain VENDITTI

0306 Robust European Monetary Policy
Jean-Guillaume SAHUC

0307 Reducing Working Time and Inequality: What Has the French 35-Hour Work Week Experience Taught Us?
Fabrice GILLES & Yannick L'HORTY

0308 The Impact of Growth, Labour Cost and Working Time on Employment: Lessons from the French Experience
Yannick L'HORTY & Christophe RAULT

0309 Inflation, Minimum Wage and Other Wages: an Econometric Study on French Macroeconomic Data
Yannick L'HORTY & Christophe RAULT

0310 Exogeneity in Vector Error Correction Models with Purely Exogenous Long-Run Paths
Jacqueline PRADEL & Christophe RAULT

0311 Retraite choisie et réduction des déficits : quelles surcotes proposer?
Jean-Olivier HAIRAULT, François LANGOT & Thepthida SOPRASEUTH

0312 Indeterminacy in a Two-Sector Finance Constrained Economy
Stefano BOSI, Francesco MAGRIS & Alain VENDITTI

0313 La nouvelle économie irlandaise
Nathalie GREENAN & Yannick L'HORTY

0314 Pace versus Type: The Effect of Economic Growth on Unemployment and Wage Patterns
Martine CARRE & David DROUOT

0315 Limited Participation and Exchange Rate Dynamics: Does Theory Meet the Data?
Frédéric KARAME, Lise PATUREAU & Thepthida SOPRASEUTH

0316 Increasing returns, Elasticity of Intertemporal Substitution and Indeterminacy in a Cash-in-Advance Economy

	Jean-Paul BARINCI
0317	**Preferences as Desire Fulfilment**
	Marc-Arthur DIAYE & Daniel SCHOCH

2002

0201	**Inflation, salaires et SMIC: quelles relations?**
	Yannick L'HORTY & Christophe RAULT
0202	**Le paradoxe de la productivité**
	Nathalie GREENAN & Yannick L'HORTY
0203	**35 heures et inégalités**
	Fabrice GILLES & Yannick L'HORTY
0204	**Droits connexes, transferts sociaux locaux et retour à l'emploi**
	Denis ANNE & Yannick L'HORTY
0205	**Animal Spirits with Arbitrarily Small Market Imperfection**
	Stefano BOSI, Frédéric DUFOURT & Francesco MAGRIS
0206	**Actualité du protectionnisme : l'exemple des importations américaines d'acier**
	Anne HANAUT
0207	**The Fragility of the Fiscal Theory of Price Determination**
	Gaetano BLOISE
0208	**Pervasiveness of Sunspot Equilibria**
	Stefano BOSI & Francesco MAGRIS
0209	**Du côté de l'offre, du côté de la demande : quelques interrogations sur la politique française en direction des moins qualifiés**
	Denis FOUGERE, Yannick L'HORTY & Pierre MORIN
0210	**A "Hybrid" Monetary Policy Model: Evidence from the Euro Area**
	Jean-Guillaume SAHUC
0211	**An Overlapping Generations Model with Endogenous Labor Supply: a Dynamic Analysis**
	Carine NOURRY & Alain VENDITTI
0212	**Rhythm versus Nature of Technological Change**
	Martine CARRE & David DROUOT
0213	**Revisiting the "Making Work Pay" Issue: Static vs Dynamic Inactivity Trap on the Labor Market**
	Thierry LAURENT & Yannick L'HORTY
0214	**Déqualification, employabilité et transitions sur le marché du travail : une analyse dynamique des incitations à la reprise d'emploi**
	Thierry LAURENT, Yannick L'HORTY, Patrick MAILLE & Jean-François OUVRARD
0215	**Privatization and Investment: Crowding-Out Effect vs Financial Diversification**
	Guillaume GIRMENS & Michel GUILLARD

0216	**Taxation of Savings Products: An International Comparison**
	Thierry LAURENT & Yannick L'HORTY
0217	**Liquidity Constraints, Heterogeneous Households and Sunspots Fluctuations**
	Jean-Paul BARINCI, Arnaud CHERON & François LANGOT
0218	**Influence of Parameter Estimation Uncertainty on the European Central Banker Behavior: an Extension**
	Jean-Guillaume SAHUC

Les documents de recherche des années 1998-2003 sont disponibles sur www.univ-evry.fr/EPEE