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On Outward-Looking Comparison Utility, Heterogeneous Preferences & the Third Dimension: A Geometric Perspective[*]

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[*]The idea that underlies this contribution was formulated by Jean-Michel Grandmont in the course of a conversation that took place by spring 1999. He then provided constant support and guidance to both of these authors and it is little to say that the contents of the contribution benefited from this authoritative knowledge. The usual disclaimer however applies.

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Abstract

The difficulties associated with the appraisal of the determinacy properties of a three-dimensional system are circumvented by the introduction of a new geometrical argument. It first brings about a complete and easy-to-use typology of the eigenvalues moduli in discrete time three-dimensional dynamical systems and then provides a new apparatus for assessing from a geometrical standpoint the emergence of local bifurcations. The argument is first illustrated in an environment where outward-looking comparison utility may question the determinacy of the steady state and subsequently used within a competitive monetary equilibrium with heterogeneous agents and financial constraints.

Keywords: Local indeterminacy, Three-Dimensional Dynamical Systems, Suboptimal Economies

JEL Classification: E32, O41, E52.

1 Introduction

The characterisation of the local behaviour of a three-dimensional discrete time dynamical system in the neighbourhood of a steady state position gives rise to a range of specific difficulties which are currently circumvented by the introduction of a new geometrical argument. It then provides a new apparatus for assessing from a geometrical standpoint the emergence of local bifurcations in three-dimensional dynamical systems. The argument is first illustrated in an environment where mimetic effects w.r.t. society's past consumption may question the local uniqueness of the equilibrium and subsequently used to study a model with heterogeneous agents and financial constraints.

As they reconsider a version of the Woodford [16] framework with heterogeneous agents and financial constraints that is amended to analyse the role of factors substitutability, Grandmont, Pintus & de Vilder [14] have come to introduce a tractable geometrical way of assessing local uniqueness or local indeterminacy for dynamical systems of order two. Their approach is based upon a graphical partition of the plane defined from the two coefficients of the second-order characteristic polynomial associated with the dynamical system in the neighbourhood of the steady state. Such a partition is then completed by drawing the critical loci associated to real and complex eigenvalues with unitary modulus, such loci featuring in turn boundaries between stability and unstability zonas. A given economy — a set of fundamental preferences and technological parameterisations — is thus to be understood as a point over that plane whilst the appraisal of its local dynamics summarised to the localisation of this point. Letting one of its building parameters vary gives rise to a family of economies featured through a curve over that plane whose localisation provided insights about the change undergone by the dynamical properties of the economy along the course of such a modification. The crux interest of this approach for economic theory stems from its explicit consideration of meaningful and generic concepts without having to resort to specific parametric formulations. As a matter of fact, its essential limit springs from its intimate link with a dimensionality of two for the dynamical system: higher orders formulations associated with, e.g., endogenous growth settings with heterogeneous capital goods, alternative formulations of intertemporal preferences with endogenous discounting, multi-country setups, could not be appraised by such a method and the assessment of their stability properties commonly remained extremely demanding. Anchoring the argument on formal developments that embed the approach of Grandmont, Pintus & de Vilder [14] as a particular case, the current contribution will nonetheless argue that the key-features that underlay the simplicity and the convenience of the two-dimensional analysis are entirely recovered in a three-dimensional dynamical system.

A noticeable methodological insight of the approach put forth by Grandmont, Pintus & de Vilder [14] sprung from the appraisal of unrestricted economic setpus through the reference to linear critical loci and basic notions of plane geometry. Two key difficulties however quickly emerge as being associated with the conceivability of such an approach for a three dimensional system. Firstly, the intricacies of three-dimensional graphs and the intrisic subtelities of the geometry of a three-dimensional space. Secondly, the uprise of a nonlinear critical locus that happens to describe the occurrence of complex eigenvalues with unitary modulus.

The first of these issues shall be circumvented by apprehending the original three-dimensional space — whose coordinates emerge from the three coefficients of the third-order characteristic polynomial — in terms of a collection of sections defined along a given coordinate and thus of two-dimensional planes.[1] Fortunately enough, this approach also recovers linear definitions for the critical loci and thus overcomes the second major difficulty of the appraisal of stability issues within a three-dimensional space. A direct byproduct states as the simplicity of the typologies it allows for the moduli of the eigenvalues and thus for the understanding of the boundaries between unstability and stability areas within a three-dimensional dynamical system. Another major advantage then emerges from the consideration of parameterised economies for which the current approach can anew boil the appraisal of the role of a fundamental preferences or technology parameter on the stability and bifurcation properties of the intrisically three-dimensional dynamical system down to a set of elementary notions of plane geometry — organised around the localisation of a curve featuring a family of economies.

The usefulness as well as the tractability of such an approach are first illustrated through the analysis of a variation on the basic Ramsey [15] model of capital accumulation, augmented by a Catching-up with the Joneses argument in the instantaneous utility of the consumers. A growing literature has indeed recently aimed at exploring the consequences of instantaneous utilities parameterised by a direct comparison of the individual consumption to a benchmark stock determined by the consumption of others. As a matter of fact, the literature distinguishes two forms of consumption benchmarks: the Catching-up with the Joneses which captures the influence of society's past consumption choices — vide Abel [1] —; the Keeping-up with the Joneses captures the influence of the society's current choices — — vide Gali [11].[2] The Catching-up with the Joneses form of the comparison utility has been recently introduced into the endogenous growth literature by Caroll, Overland & Weil [8] — — vide also Alonso-Carrera, Caballé & Raurich [2] and Alvarez-Cuadrado, Monteiro & Turnovsky [3]. For the current purpose, the noticeable conclusion reached by these authors is that the introduction of a lagged external consumption benchmark does not question the saddle-path stability property of the long-run equilibrium. The graphical methods developed in the first part of the article allow to evaluate whether and to what extent such a conclusion relies upon the functional specifications they employ and the parameterisations they retain. The main conclusions state as follows. As long as the influence of the lagged value of consumption remains moderate the steady state is saddle-point stable. Interestingly, a rise in this influence will translate into the possibility of a dramatic change under which the steady state becomes unstable and, in correlation, quasi-periodic orbits happen to emerge. This analysis is however supplemented by the contemplation of an alternative class of assumptions on instantaneous utility. As a matter of fact, while a Catching-up with the Joneses for instantaneous utility features a desire to be similar to others, the eventuality that consumers testimony of a desire

[1] This illuminating insight was made by J.M. Grandmont at C.R.E.S.T. by spring 1999 after the presentation of a preliminary contribution by the current authors that attempted at a general geometric picture of the local stability properties of three-dimensional dynamical systems.

[2] The Catching-up with the Joneses hypothesis is actually an external habit formation setup where the consumption benchmark is an externality. This contrasts with the internal habit formation setup where the reference is the consumer's own past consumption.

to be different from others cannot a priori be dismissed. Such an observation has led Dupor & Liu [10] to consider utilities displaying Running-away from the Joneses — — as a counterpart of Keeping-up with the Joneses. While, as mentioned, Catching-up with the Joneses has been the focus of some recent research in the growth literature, the Running-away from the Joneses counterpart has not, to the best knowledge of the authors, been yet explored. The definitely striking result that emerges from its consideration is that, in contradistinction with the Catching-up case, and even for moderate orders for the influence of the consumption benchmark remains, the desire to run-away from the Joneses gives rise, aside from deterministic quasi-periodic equilibria, to local indeterminacy, hence stochastic expectations-driven fluctuations.

This contribution is finally concerned with the properties of a monetary economy with heterogeneous households and financial constraints. The key role of heterogeneity in the emergence of endogenous fluctuations has been identified quite early with the contributions of Grandmont [12] in an overlapping generations environment. Alternative heterogeneity arguments that hinge upon more canonical infinitely-lived agents and borrowing constraints were then suggested by Bewley [7] and Woodford [16]. Recent developments have provided extensions of the Woodford [16] environment and raised the new dimensions it brings with respect to the standard Ramsey representative agent structure. In particular, Grandmont, Pintus & de Vilder [14] have relaxed the complementarity dimension of the technological set of Woodford [16] and emphasised the role of factors substitutability in the occurrence of endogenous fluctuations. Taking advantage of the recent literature on increasing returns which was motivated by the publication of Benhabib & Farmer [5], Cazzavillan, Lloyd-Braga & Pintus [9] have illustrated how the consideration of external effects could improve upon the close link between low orders for inputs substitutability and endogeneous cycles that was raised by Grandmont, Pintus & de Vilder [14]. Subsequently, Barinci [4] has relaxed another facet of the Woodford [16] framework, i.e., the postulate of an unitary elasticity of intertemporal substitutions for the capital-holder. It was notably shown therein that, even though a constant returns to scale asssumption was kept on the production sphere, low elasticities of substitution between capital and labor were not any longer required in order for endogenous cycles to occur. Unfortunately and as a direct byproduct of the analytical intricacy of the three-dimensional dynamical system under consideration, this finding was obtained in a knife-edge specification. The graphical methods will currently offer the opportunity to reconsider its robustness and, in addition, to reach clarified picture of the local dynamics. Succintly, a remarkable result formulates, having restricted labor supply to be sufficiently elastic, as the potential occurrence of two degrees of indeterminacy. Such an insight being further available under a Cobb-Douglas specification for the technology, it is unequivocally disconnected from the articulation between low orders for factors substitutability and local indeterminacy that was raised by earlier studies.

The geometrical techniques are introduced in Section 2. Section 3 deals with the comparison utility model, the heterogenous households one being examined in Section 4. Some technical details are provided in a final appendix.

2 A Geometric Argument for the Appraisal of Three-Dimensional Dynamical Systems

This section will first unveil the regards in which a sequence of simple geometric pictures underlies the typology of eigenvalues and the emergence of local bifurcations in discrete three-dimensional dynamical systems. It will thereafter detail the regards in which this provides an useful and easy-to-use apparatus for assessing the stability, uniqueness and bifurcations properties of a large class of parameterised economies.

2.1 A Geometric Picture for the Critical Loci

Letting the equilibrium dynamics of an economy be described by a system: $y_{t+1} = G(y_t)$, $y_t \in \mathbb{R}^3_+$, steady states equilibria are the roots of $\bar{y} - G(\bar{y}) = 0$. The characterisation of the local dynamics nearby a given steady equilibrium proceed from the appraisal of an associated linear map $z_{t+1} = \mathscr{J} z_t$ for $\mathscr{J} := DG(\bar{y})$ the Jacobian matrix of $G(\cdot)$ evaluated at $\bar{y}$ and $z_t := y_t - \bar{y}$ the deviation from the steady state. The eigenvalues of the matrix $\mathscr{J}$ are the zeroes of the following third degree polynomial:

$$(1)\quad \begin{aligned}\mathscr{P}(z) &= (z_1 - z)(z_2 - z)(z_3 - z)\\ &= -z^3 + \big(z_1 + z_2 + z_3\big)z^2 - \big(z_1 z_2 + z_1 z_3 + z_2 z_3\big)z + z_1 z_2 z_3\\ &= -z^3 + \mathscr{T} z^2 - \mathscr{M} z + \mathscr{D}\end{aligned}$$

for $\mathscr{T}, \mathscr{M}$ and $\mathscr{D}$ that denote the trace, the sum of the principal minors of order two and the determinant of the Jacobian matrix, respectively.

The locus such that the coefficients $\mathscr{T}, \mathscr{M}, \mathscr{D}$ satisfy $\mathscr{P}(+1) = 0$ — generically, a saddle-node bifurcation[3] will occur in its neighbourhood and the uniqueness properties of the steady state will be lost — is a plane whose characteristic equation is given by :

$$(2)\quad -1 + \mathscr{T} - \mathscr{M} + \mathscr{D} = 0.$$

Similarly, the locus such that the coefficients $\mathscr{T}, \mathscr{M}, \mathscr{D}$ satisfy $\mathscr{P}(-1) = 0$ — generically, a flip bifurcation[4] will occur in its neighbourhood and two-period cycles will emerge — is a plane whose characteristic equation is given by:

$$(3)\quad 1 + \mathscr{T} + \mathscr{M} + \mathscr{D} = 0.$$

Lastly, when a pair of nonreal characteristic roots exhibiting an unitary norm emerges, the outstanding eigenvalue, e.g., z_3, sumarises to the product of the eigenvalues $\mathscr{D}$. The latter thus becomes a characteristic root, i.e., $\mathscr{P}(\mathscr{D}) = 0$. Solving, the characteristic polynomial restates as $\mathscr{P}(z) = (\mathscr{D} - z)\big[z^2 - (\mathscr{T} - \mathscr{D})z + \mathscr{M} - (\mathscr{T} - \mathscr{D})\mathscr{D}\big]$. A standard analysis of the two-degree polynomial in square brackets then indicates that the locus of coefficients $\mathscr{T}, \mathscr{M}$ and $\mathscr{D}$ such that two roots are complex conjugate with unitary modulus is a regulated surface —

[3] Vide Grandmont [12].

[4] Vide Grandmont [12].

generically, a Poincaré-Hopf bifurcation will occur and quasi-periodic equilibria will emerge in its neighbourhood — delimited by $|\mathscr{T} - \mathscr{D}| < 2$ and defined from

$$(4) \quad \mathscr{M} - 1 - (\mathscr{T} - \mathscr{D})\mathscr{D} = 0.$$

The central difficulty in the appraisal of this ultimate locus in a three-dimensional graph stems from its nonlinear shape that in turn results from the appearance of the quadratic expression $(\mathscr{T} - \mathscr{D})\mathscr{D}$ in its definition. Interestingly, both of them are circumvented upon the fixation of the coefficient $\mathscr{D}$. An analysis with a strong two-dimensional flavour — any of the aforementioned critical loci can anew be represented through a straight-line — being then conceivable in the space of the two outstanding coefficients $\mathscr{T}$ and $\mathscr{M}$. More explicitly and

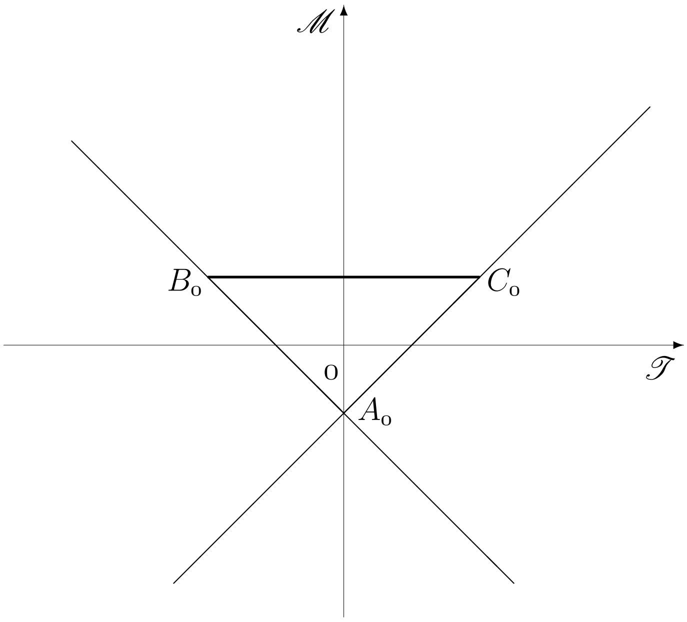

Figure 1: Benchmark case $\mathscr{D} = 0$.

first introducing the benchmark case $\mathscr{D} = 0$ on Figure 1, the set of coefficients $(\mathscr{T}, \mathscr{M})$ such that $\mathscr{P}(+1) = 0$ and $\mathscr{P}(-1) = 0$ respectively boil down to the lines $(A_0 C_0)$ and $(A_0 B_0)$ — the index 0 refers to the value of the parameter $\mathscr{D}$ under which the whole picture is drawn — while the corresponding set for two nonreal eigenvalues with unitary norm is depicted by the horizonal segment $[B_0 C_0]$. This gives rise to a construction familiar from the two-dimensional analysis, namely the triangle $(A_0 B_0 C_0)$ defined by $|\mathscr{T}| < |1 + \mathscr{M}|$ and $|\mathscr{M}| < 1$.

As $\mathscr{D}$ is increased over $\mathbb{R}_+$ and as illustrated on Figure 2, the slopes of $(A_\mathscr{D} C_\mathscr{D})$ and $(A_\mathscr{D} B_\mathscr{D})$ are unmodified while the segment $[B_\mathscr{D} C_\mathscr{D}]$, of slope $\mathscr{D}$, essentially follows a translated counter-clockwise rotation.

As a matter of fact, the parameterised coordinates of $A_\mathscr{D}$, $B_\mathscr{D}$ and $C_\mathscr{D}$ respectively derive from the solving of (2) and (3), (3) and (4), (2) and (4). They list as:

$$(5) \quad \begin{aligned} (\mathscr{T}_{A_\mathscr{D}}, \mathscr{M}_{A_\mathscr{D}}) &= (-\mathscr{D}, -1), \\ (\mathscr{T}_{B_\mathscr{D}}, \mathscr{M}_{B_\mathscr{D}}) &= (-2 + \mathscr{D}, 1 - 2\mathscr{D}), \\ (\mathscr{T}_{C_\mathscr{D}}, \mathscr{M}_{C_\mathscr{D}}) &= (2 + \mathscr{D}, 1 + 2\mathscr{D}). \end{aligned}$$

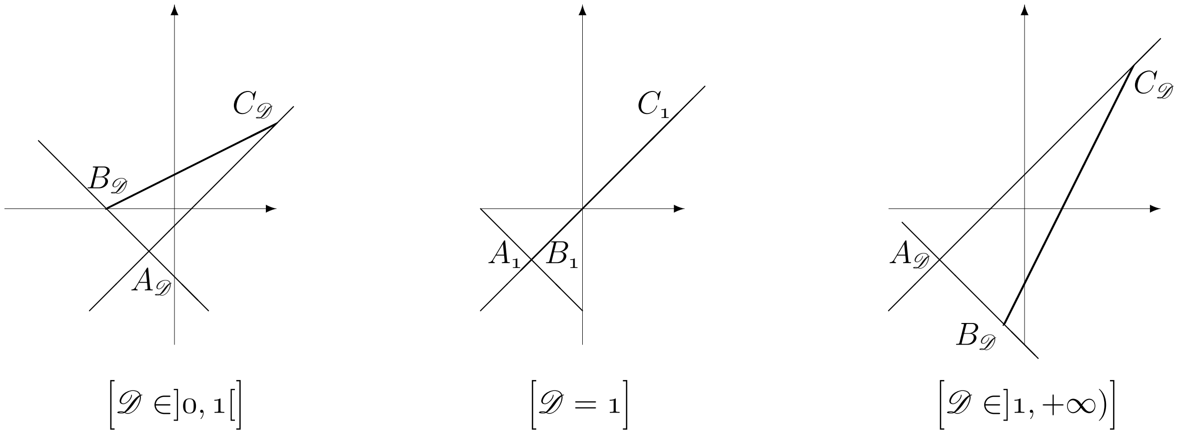

Figure 2: Translated counterclockwise rotation as $\mathscr{D}$ is increased over $\mathbb{R}_+$.

It is worth emphasising that on Figure 2, the Poincaré-Hopf and the saddle-node critical loci coincide and merge for $\mathscr{D} = 1$ in the sense that $A_1 = B_1$. A counterpart scenario is available on Figure 3 where negative values are considered for $\mathscr{D}$. Similarly, the Poincaré-Hopf and the flip critical loci coincide and merge for $\mathscr{D} = -1$ in the sense that $A_{-1} = C_{-1}$. These mergers imply that the definition of the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ is modified as $|\mathscr{D}|$ goes through one: for $|\mathscr{D}| < 1$ it is available as $|\mathscr{T} + \mathscr{D}| < 1 + \mathscr{M}$ and $\mathscr{M} < 1 + (\mathscr{T} - \mathscr{D})\mathscr{D}$; for $|\mathscr{D}| > 1$, it is changed to $|1 + \mathscr{M}| < \mathscr{T} + \mathscr{D}$ and $\mathscr{M} > 1 + (\mathscr{T} - \mathscr{D})\mathscr{D}$.

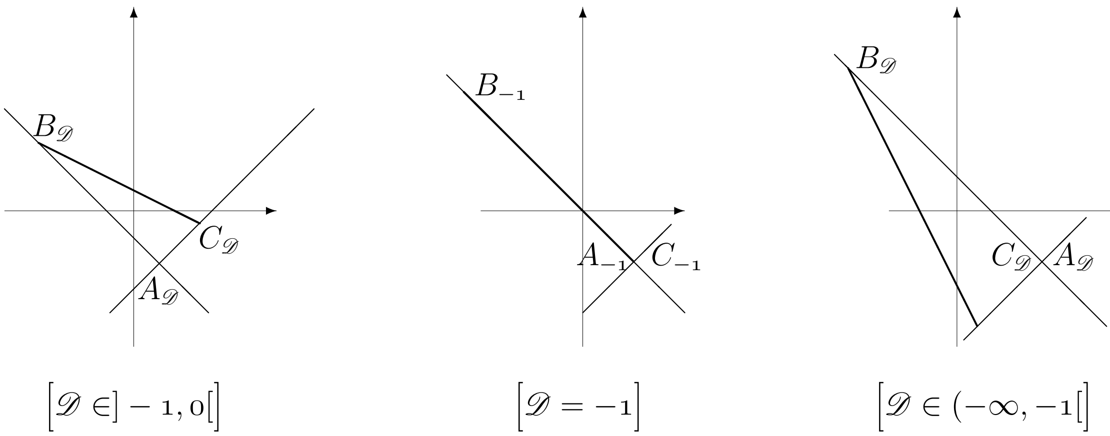

Figure 3: Translated clockwise rotation as $\mathscr{D}$ is decreased over $\mathbb{R}_-$.

Figures 1, 2 and 3 illustrate how such a three-dimensional parameterised construction, still organised around the same features, essentially the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$, keeps on proceeding from the same lines as the traditional two-dimensional one, but to the qualification that the slope of the Hopf segment $[B_{\mathscr{D}}C_{\mathscr{D}}]$, being given by $\mathscr{D}$, will vary accordingly.

2.2 A Typology of the Eigenvalues

In order to reach the essence of the argument about the cardinality of stable eigenvalues, consider Figure 1 and the basic configuration for which $\mathscr{D} = 0$. As at least one eigenvalue is nil and hence stable, the analysis essentially boils down to the standard two-dimension argument. An economy within the triangle $(A_0B_0C_0)$ — this means for values of $\mathscr{T}$ and $\mathscr{M}$ that remain close to zero — displays three eigenvalues with modulus inside the unit circle. Consider then an upward perturbation on Figure 1. The system will cross the segment $[B_0C_0]$: this implies that the modulus of the complex eigenvalues gets out of the unit circle and there only remains a unique eigenvalue with norm less than one. When one, after a rightward or a leftward perturbation, leaves the origin stability area by crossing (A_0C_0) or (A_0B_0), the position of a unique eigenvalue with respect to the unit circle will be modified and the system falls in an area with two stable eigenvalues. Finally, a downward perturbation from any of these areas will lead the system within an area that exhibits one modulus within the unit circle, hence an unstable steady state.

As illustrated by Figures 2 and 4, the earlier assessment on stable eigenvalues — the number between parenthesis on these figures — will remain unmodified until the attainment of $\mathscr{D} = 1$ and the already mentioned occurrence of $A_1 = C_1$. All these conclusions being recovered from forthright applications of the Implicit Function Theorem, a detailed argument being provided in Appendix 1.

For $\mathscr{D} > 1$ and again on Figure 4, the typology of the stable eigenvalues is drastically modified since the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ now delimits an area without any stable eigenvalue. To perceive

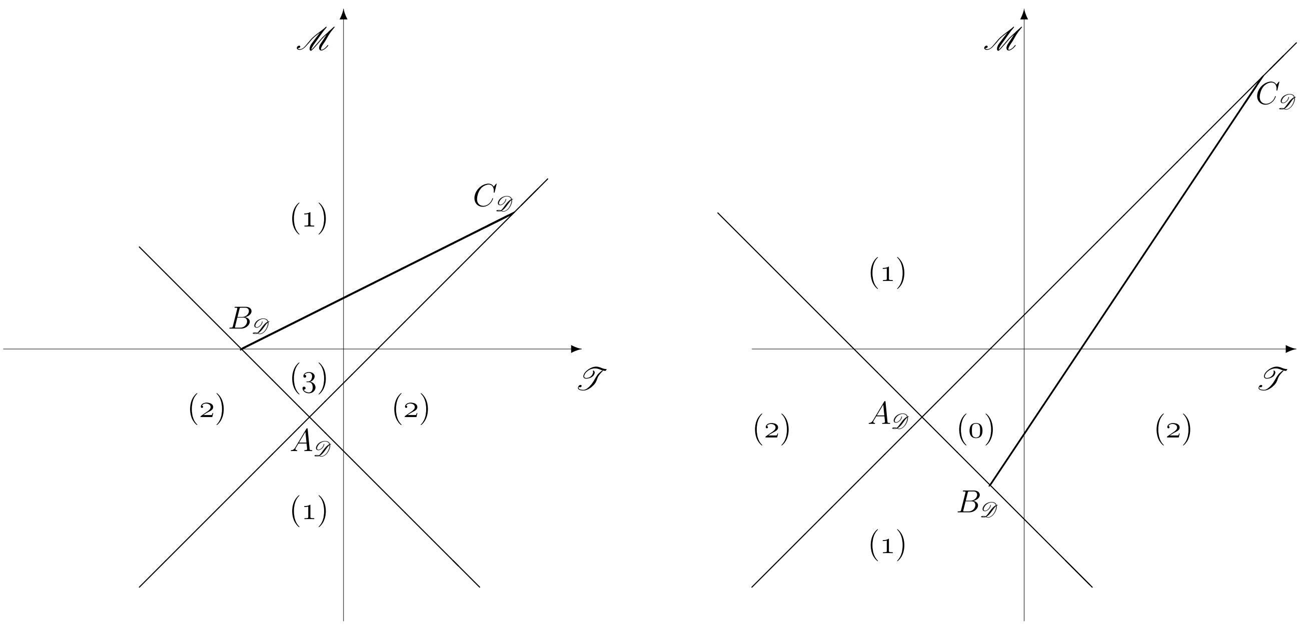

Figure 4: Typologies for $\mathscr{D} \in]0, 1[$ and $\mathscr{D} > 1$

this, first consider the Poincaré-Hopf segment $[B_{\mathscr{D}}C_{\mathscr{D}}]$ and recall that, by definition, it is associated with a pair of complex eigenvalues with unitary norm plus a real eigenvalue given by $\mathscr{D}$, hence greater than $+1$ within the current configuration. Then assuming an upward perturbation of the coefficient $\mathscr{M}$ and again from the implicit function argument developed in Appendix 1, this results in a modulus of the complex eigenvalues that gets out of the unit circle. Hence, in deep contradiction with the case $\mathscr{D} \in]0, 1[$, the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ is now associated with an area with no modulus inside the unit circle and thus corresponds to the occurrence of an unstable steady state. The number of stable eigenvalues for the outstanding areas can be inferred from the same lines of reasoning used for the case $\mathscr{D} \in]0, 1[$.

As this is also illustrated on Figure 5, a mirror argument can be completed when $\mathscr{D}$ is decreased starting from zero: a position within the triangle $(A_\mathscr{D} B_\mathscr{D} C_\mathscr{D})$ corresponds to three stable eigenvalues and the features of the outstanding areas can be assessed from the same reasoning as above. A decrease in $\mathscr{D}$ now entails a rotation around $C_\mathscr{D}$ that takes place until the threshold value of $\mathscr{D} = -1$. When the latter is attained, the Poincaré-Hopf and the flip loci coincide and the positions of three moduli with respect to the unit circle are modified. For $\mathscr{D} < -1$, one obtains the exact mirror picture of the configuration described above for $\mathscr{D} > 1$ with a location inside the triangle $(A_\mathscr{D} B_\mathscr{D} C_\mathscr{D})$ that corresponds to no modulus inside the unit circle.

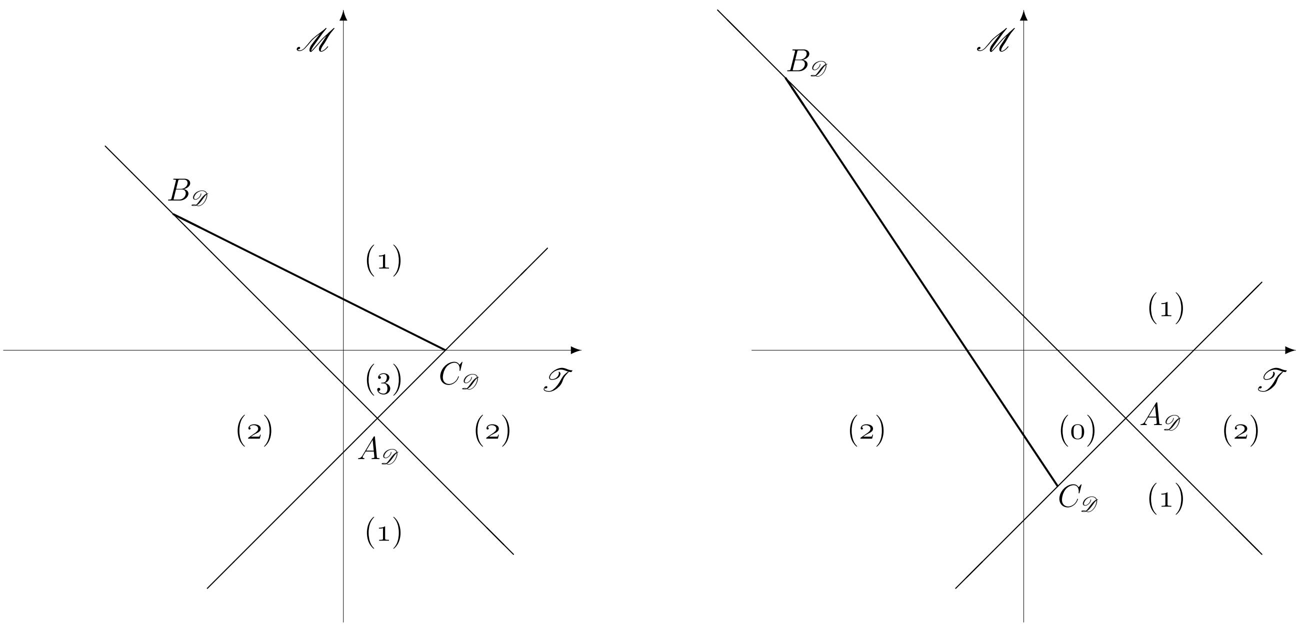

Figure 5: Typologies for $\mathscr{D} \in] -1, 0[$ and $\mathscr{D} < -1$

As this soon will become clear, this collection of figures introduces an alternative benchmark for analysing the stability properties of parameterised economies that entails the treatment of a three-dimensional dynamical system.

2.3 Parameterised Economies

This subsection shall argue that Figures 4-5 provide a suitable device for undertaking a sensivity analysis in actual parameterised economies. Consider indeed a set of economies indexed by a parameter $\eta \in]0, +\infty[$ and further assume that the coefficient $\mathscr{D}$ is independent of that parameter. A set of parameterised economies is then symbolised by a parameterised family of triples $(\mathscr{T}(\eta), \mathscr{M}(\eta), \mathscr{D})$.

The spanning of its interval by η generates a parameterised curve $\Delta(\eta)$ that originates from $\Delta(0)$. This curve becomes a parameterised half-line in the plane $(\mathscr{T}, \mathscr{M})$ that is associated with a given value of $\mathscr{D}$ as soon as $\Delta' := \mathscr{M}'(\eta)/\mathscr{T}'(\eta)$ is a constant, an assumption that is currently retained.[5] It clearly appears that whenever $\mathscr{D}$ is independent of η, the key features that underly the simplicity of the two-dimensional analysis, i.e., an argument conducted over a given plane partitioned by linear boundaries, is recovered in spite of the extra dimension of the embedding environment.

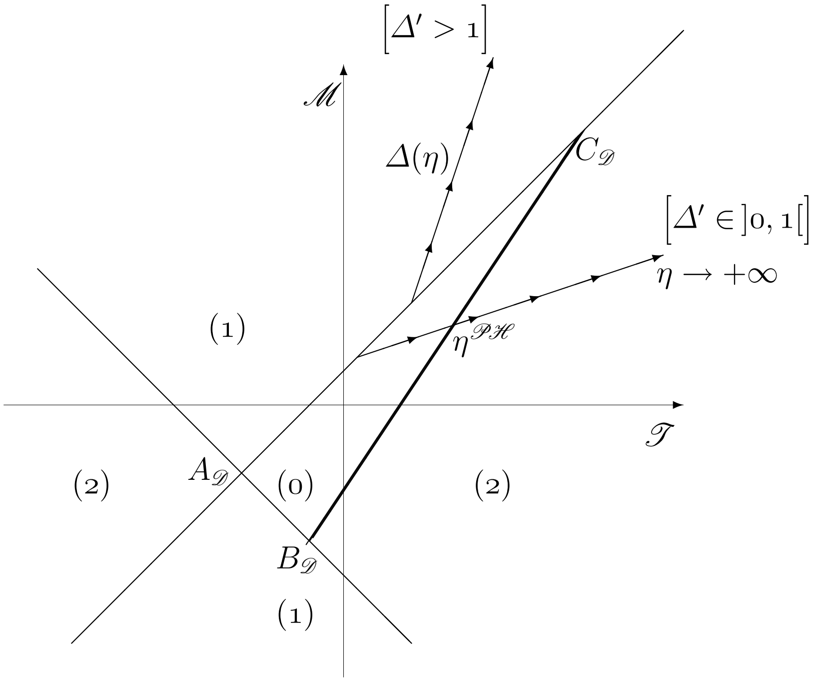

Figure 6: Basic Bifurcation Scheme for a Parameterised Economy with $\mathscr{D} > 1$

As an illustration of the bifurcation scheme and of the way it articulates with the determinacy issue, consider a configuration $\mathscr{D} > 1$ and an economy with one predetermined variable, the interval for the bifurcation parameter η being $[0, +\infty)$. As this has, e.g., been examplified by Grandmont, Pintus & de Vilder [14], it may be the case that $\Delta(0)$ belongs to the saddle-node line $(A_{\mathscr{D}}C_{\mathscr{D}})$ on Figure 8, hence illustrating the uniqueness of the steady state for interior values of the bifurcation parameter. By construction, $\mathscr{D}$ being independent of the bifurcation parameter η, the economy will stay of the same Figure when the latter spans its interval. Assume, e.g., that the slope Δ' is positive and further let $\mathscr{M}'(\eta) > 0$. If $\Delta' < 1$, the economy will cross the Poincaré-Hopf segment $[B_{\mathscr{D}}C_{\mathscr{D}}]$ for a given critical value of the bifurcation parameter denoted as $\eta^{\mathscr{PH}}$. For $\eta \in]0, \eta^{\mathscr{PH}}[$, it will assume no eigenvalue into the unit circle and will hence be locally unstable. As it crosses the segment $[B_{\mathscr{D}}C_{\mathscr{D}}]$, two — conjugate — eigenvalues will become stable and for $\eta \in]\eta^{\mathscr{PH}}, +\infty[$, the equilibrium will be locally indeterminate with a degree of indeterminacy equal to 1. In opposition to this, if $\Delta' > 1$, no bifurcation occurs and the economy ends in an area with a single stable root. It is thus locally determinate. It immediately follows from Figure 6 that no flip bifurcation would happen in any of these two configurations.

To sum up, the drawing of the parameterised half-line $\Delta(\eta)$ over the plane $(\mathscr{T}, \mathscr{M})$ allows for

[5]For clarification purposes, it may be noticed that while $\Delta(\eta)$ results in a half-line when η spans its interval and both $\mathscr{T}(\cdot)$ and $\mathscr{M}(\cdot)$ are linear functions of η, it can be a segment when η spans its interval and both $\mathscr{T}(\cdot)$ and $\mathscr{M}(\cdot)$ are nonlinear functions of η while $\mathscr{M}'(\eta)/\mathscr{T}'(\eta)$ is a constant.

a comprehensive appraisal of the local dynamics of the economy. As this shall be extensively illustrated though the subsequent sections, its localisation commonly starts from the one of its origin benchmark point $\Delta(0)$ and is usually completed by the identification of the coordinates of its generating directional vector. Although these hints may admittedly display a impressionist flavour at that stage, the forthcoming examples shall establish that they do provide a remarkably easy-to-use set of tools for the understanding of otherwise complex settings.

3 Catching-up with the Joneses / Running-away from the Joneses

3.1 The setup

This section will consider a variation of the neo-classical growth model that is augmented by an outward-looking specification of intertemporal preferences for the representative individuals; the level of utility derived from a given amount of instantaneous consumption is thus assumed to exhibit an extra dependency with respect to previous society consumption standards. More explicitly, these preferences state as

$$(6) \quad \sum_{t=0}^{\infty} \beta^t u\big(c_t; C_{t-1}\big),$$

for $\beta \in]0,1]$, c_t and C_{t-1} respectively their positive rate of marginal impatience, their consumption at date $t \geq 0$ and the average consumption across all consumers at the previous date, $u(\cdot; C)$ being a continuous concave instantaneous utility function which maps $\mathbb{R}_+$ into $\mathbb{R}$, is of class C^k, $k \geq 3$, over $\mathbb{R}_+^*$ and satisfies the Inada conditions at the origin. Further, $u'_c > 0$, $u''_{cc} < 0$ for any $c \in \mathbb{R}_+^*$. Besides, the Catching-up with the Joneses dimension of this formulation is ensured by letting $u''_{cC} > 0$ prevail, i.e., mimetism effects with respect to earlier consumption standards. In opposition to this, a Running-away from the Joneses dimension is ensured by the converse occurrence of $u''_{cC} < 0$ that translates the potential from a repulsive dimension from previous consumption standards.

The representative individual maximises his intertemporal utility function (6) subject to

$$(7) \quad K_{t+1} = F\big(K_t, L_t\big)\big(\bar{K}_t\big)^{\gamma} - c_t,$$

for an aggregate production function $F\big(K_t, L_t\big)\big(\bar{K}_t\big)^{\gamma}$, $\gamma \geq 0$ such that $F(\cdot,\cdot)$ is a continuous concave function which maps $\mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$, is of class C^3 over $\mathbb{R}_+^* \times \mathbb{R}_+^*$, homogeneous of degree one and satisfies the Inada conditions, the capital stock having been assumed to fully depreciate at each period of time. Finally, the component $\bar{K}_t$ features an external effect of the Romer type. Letting $L_t f\big(k_t\big) := L_t F\big(K_t/L_t, 1\big)$, $k_t := K_t/L_t$ and $L_t = 1$, an intertemporal competitive equilibrium with externalities for which $c_t = C_t$ and $K_t = \bar{K}_t$ at any $t \geq 0$ is described by the holding of:

$$
\begin{aligned}
(8) \quad u_c'(c_t;x_t) - \beta f'(k_t)k_t^{\gamma}u_c'(c_{t+1};x_{t+1}) &= 0, \\
k_{t+1} - f(k_t)k_t^{\gamma} + c_t &= 0, \\
x_{t+1} - c_t &= 0,
\end{aligned}
$$

From Appendix 2, a linearisation around the steady state gives the following expressions for the coefficients of the characteristic polynomial:

$$
\begin{aligned}
(9) \quad \mathscr{T} &= 1 + \frac{\eta^{\mathscr{E}}}{\eta} - \frac{1}{\eta}\left(\gamma - \frac{1-s}{\sigma}\right)\left(\frac{1}{\beta s} - 1\right) + \frac{\gamma + s}{\beta s}, \\
\mathscr{M} &= \left(1 + \frac{\eta^{\mathscr{E}}}{\eta}\right)\frac{\gamma+s}{\beta s} + \frac{\eta^{\mathscr{E}}}{\eta}, \\
\mathscr{D} &= \frac{\gamma+s}{\beta s}\frac{\eta^{\mathscr{E}}}{\eta},
\end{aligned}
$$

for

$$
\sigma := \frac{F_K' F_L'}{F F_{KL}''}, \qquad 1 - s := \frac{F_L' L}{F}, \qquad s := \frac{F_K' K}{F},
$$

$$
\eta := -\frac{u_{cc}''(c_t, C_{t-1})c_t}{u_c'(c_t, C_{t-1})}, \qquad \eta^{\mathscr{E}} := \frac{u_{cC}''(c_t, C_{t-1})C_{t-1}}{u_c'(c_t, C_{t-1})}.
$$

that respectively depict the elasticity of substitution between capital and labour, the share of labour and the share of capital, the intertemporal elasticity of substitution in consumption and an outward-looking comparison utility coefficient defined from the marginal utility on consumption.

At that stage and for illustrative purpose, a multiplicative C.E.S.-type instantaneous utility, namely

$$
(10) \quad u(c_t; C_{t-1}) = \left[c_t/(C_{t-1})^{\alpha}\right]^{1-\sigma^c}/(1-\sigma^c)
$$

boils the coefficient $\eta^{\mathscr{E}}/\eta$ down to $(\sigma^c - 1)\alpha/\sigma^c$ while η would simplify to σ^c. In order to provide a similarly simple benchmark, the coefficient $\eta^{\mathscr{E}}/\eta$ shall henceforward uniformly be referred to as ς, Catching-up with the Joneses and Running-away from the Joneses configurations being thus respectively circumscribed by $\varsigma > 0$ and $\varsigma < 0$.

The coefficients of the characteristic polynomial would thence reformulate to:

$$
\begin{aligned}
(11) \quad \mathscr{T} &= 1 + \varsigma - \frac{1}{\eta}\left(\gamma - \frac{1-s}{\sigma}\right)\left(\frac{1}{\beta s} - 1\right) + \frac{\gamma+s}{\beta s}, \\
\mathscr{M} &= (1+\varsigma)\frac{\gamma+s}{\beta s} + \varsigma, \\
\mathscr{D} &= \frac{\gamma+s}{\beta s}\varsigma.
\end{aligned}
$$

3.2 The analysis

It is noticed that $\mathscr{D}$ does not depend on the elasticity of substitution between capital and labour, i.e., σ. In accordance with the earlier approach and selecting $1/\sigma$ as the tuning parameter, such a feature will bring about the possibility of a purely geometric argument for appraising the dynamical properties of the model.

The current setting is moreover slightly particular on a formal basis. As a matter of fact, as

$$(12)\quad \mathscr{T}' = \frac{(1-s)}{\eta}\left(\frac{1}{\beta s} - 1\right) > 0,$$
$$\mathscr{M}' = 0,$$

the parameterised curve $\Delta(1/\sigma)$ simplifies to a straight-line with a slope given by $\Delta' := \mathscr{M}'/\mathscr{T}' = 0$ and an origin provided by $\Delta(0) = (\mathscr{T}(0), \mathscr{M})$.

It is further remarked that

$$-1 + \mathscr{T}(1/\sigma) - \mathscr{M} + \mathscr{D} = -\frac{1}{\eta}\left[\gamma - \frac{(1-s)}{\sigma}\right]\left(\frac{1}{\beta s} - 1\right),$$

whence

$$(13)\quad -1 + \mathscr{T}(0) - \mathscr{M} + \mathscr{D} = -\frac{1}{\eta}\gamma\left(\frac{1}{\beta s} - 1\right) \leq 0.$$

This means that $\Delta(0)$ will locate on the L.H.S. of the critical line $(A_{\mathscr{D}}C_{\mathscr{D}})$ —— it is worth noticing that the benchmark case without productive externality $\gamma = 0$ would have brought $\Delta(0)$ to $(A_{\mathscr{D}}C_{\mathscr{D}})$.

Finally, notice that, from (5), the comparison with the ordinate of $\mathscr{M}_{C_{\mathscr{D}}}$ will detail as

$$(14)\quad \mathscr{M} - 2\mathscr{D} - 1 = (\varsigma - 1)\left[1 - \frac{\gamma + s}{\beta s}\right]$$

and hence only depends, the component between square brackets being unambiguously negative, upon $\varsigma \gtreqless 1$.

On a methodological basis, the subsequent argument shall hence first be organised around the values of ς that emerges as the most significant determinant of $\mathscr{D}$ and thus of the plane $(\mathscr{T}, \mathscr{M})$ over which the analysis is to be completed. A second step then consists in locating $\Delta(1/\sigma)$ upon that plane. As the slope of $\Delta(1/\sigma)$ is invariantly nil, a localisation simplifies to place $\Delta(0)$, i.e., $\mathscr{M}$ and $\mathscr{T}(0)$ — these two latter coefficients can be considered in a separate way due to the independency of $\mathscr{M}$ with respect to the values of η.

3.2.1 Catching-up with the Joneses: $\varsigma > 0$

In that case and from (11), $\mathscr{D} > 0$. Assuming further that $0 < \varsigma < \beta s/(\gamma + s)$, $\mathscr{D} < 1$. Moreover and from (14), as $\mathscr{M} - 2\mathscr{D} - 1 > 0$, $\Delta(1/\sigma)$ is unambiguously located above the Poincaré-Hopf segment $[B_{\mathscr{D}}C_{\mathscr{D}}]$ on Figure 7. It is also observed from the same Figure that it now only remains to precisely locate $\Delta(0)$ with respect to the critical boundaries in order to complete the analysis of the current case. For convenience, this value being strongly dependent of the one undertaken by the capital spillover effect $\gamma \geq 0$, it shall hencerforward be denoted as $\Delta(0;\gamma)$. Recalling that in the no capital externality case, $\Delta(0;0)$ locates upon $(A_{\mathscr{D}}C_{\mathscr{D}})$, the equilibrium emerges as being always determinate in such a configuration. As a matter of fact, the equilibrium dynamical system assuming two predetermined variables, namely k_0 and C_{-1}, the steady state currently exhibits a canonical saddle-point structure. Yet, as soon as $\gamma > 0$ and from (14), $\Delta(0;\gamma)$ can be located either on the R.H.S or on the L.H.S. of $(A_{\mathscr{D}}B_{\mathscr{D}})$ according to the value of the elasticity of intertemporal substitution in consumption $1/\eta$. Actually, noticing that $\mathscr{T}(0) \to -\infty$ as $\eta \to 0$, it appears that a flip bifurcation is typically bound to occur, i.e., there exists $1/\sigma^{\mathscr{F}}$ such that one of the characteristic roots goes through -1 as $1/\sigma$ crosses $1/\sigma^{\mathscr{F}}$. In addition, as regards to the saddle-node type bifurcation, that occurs as $\Delta(0;\gamma)$ crosses $(A_{\mathscr{D}}C_{\mathscr{D}})$, it currently does not indicate a persistent exchange of the number of steady-states, that is assumed to be unique, but rather a structurally unstable situation where there are a continuum of steady-states. The latter occurs for a value of the elasticity of substitution between capital and labor, namely $(1-s)/\gamma$, such that the equilibrium marginal product of capital does not depend upon the level of capital.[6]

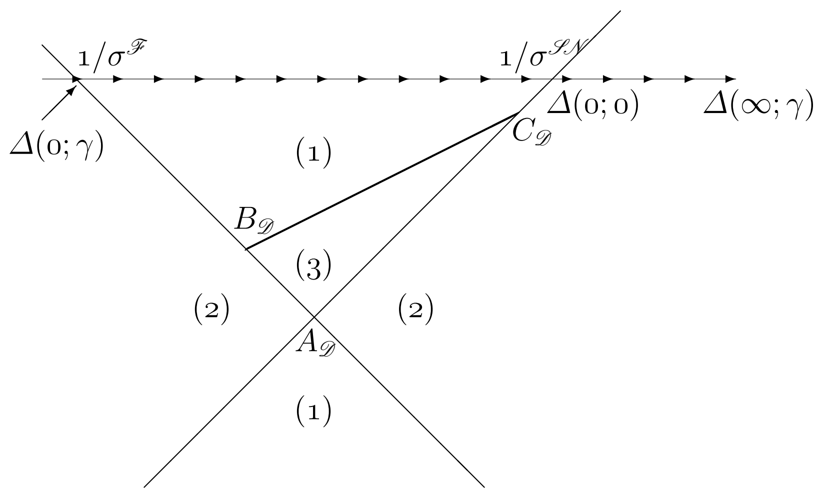

Figure 7: Catching up with the Joneses $[\varsigma > 0, (\gamma + s)\varsigma/\beta s < 1]$

Assume now that $\varsigma > \beta s/(\gamma + s)$. From Figure 8 and as long as $\varsigma < 1$, $\Delta(1/\sigma)$ locates above $C_{\mathscr{D}}$ and the qualitative picture reached in the former case is entirely recovered. Oppositely, when $\varsigma > 1$, $\Delta(1/\sigma)$ being now located below $C_{\mathscr{D}}$, a Poincaré-Hopf bifurcation (quasi-periodic equilibria) is bound to occur even in the absence of capital externality.[7] It is nonetheless worth stressing that the requesite $\varsigma > 1$, which is actually equivalent to $u''_{cc} + u''_{cC} > 0$, means that the external influence of the benchmark level of consumption dominates the direct own consumption effect.

[6]Notice that this is nothing more but the specification allowing for endogenous growth which is always characterised by an unit root.

[7]From the Figures, the flip and saddle-node bifurcations schemes depicted in the first case are both left qualitatively unaltered as $\mathscr{M} > 0$. Thus and for convenience, only the constant returns to scale case is depicted.

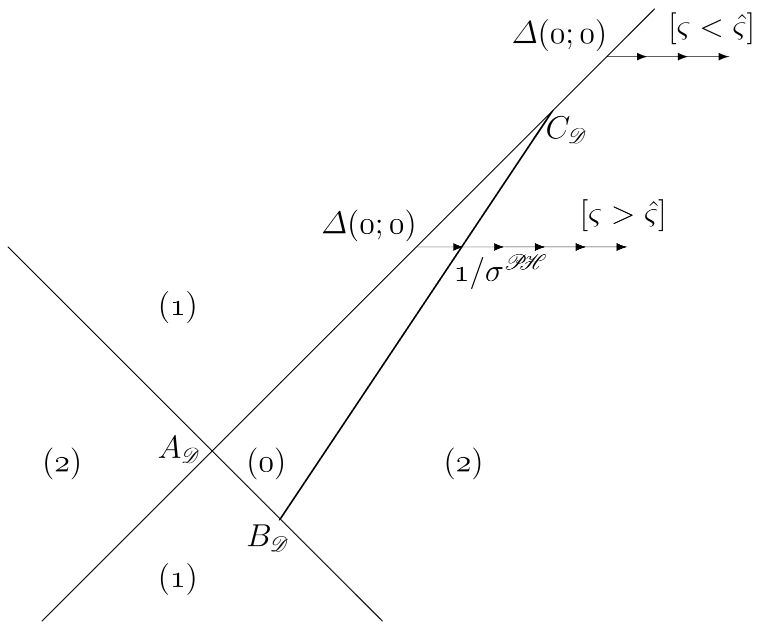

Figure 8: Catching up with the Joneses $[\varsigma > \beta]$

3.2.2 Running-away from the Joneses: $\varsigma < 0$

Contrariwise and for this parameters configuration, $\mathscr{D} < 0$. In addition and from (14), $\mathscr{M} > 2\mathscr{D} + 1$ uniformly holds and the position with respect to the ordinate of $C_{\mathscr{D}}$ is unambiguously established. Assume first that $|\mathscr{D}| < 1$, i.e., $|\varsigma| < \beta s/(\gamma + s)$ on Figure 9. Here again the requesites for the occurrence of a flip bifurcation remain qualitatively unchanged. A more remarkable feature is rather that there exist admissible values for ς, i.e., values within the currently assumed interval, such that $\mathscr{M} < 1-2\mathscr{D}$. Recalling (5), this indicates that there is a non-empty interval for ς ensuring that $\Delta(1/\eta)$ will be located between $B_{\mathscr{D}}$ and $C_{\mathscr{D}}$. Therefore, following the same line of arguments as for the flip bifurcation, one concludes that there are consistent parameterisations such that not only both types of bifurcations will exist but, more remarkably, local indeterminacy (a stable steady state) is bound to emerge as the economy passes through the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ (vide Figure 9).[8]

To sum up, the most stringent results of the above graphical analysis are i) in the Catching-up of the Joneses case, relying upon a strong external effect of the economy average consumption, the model exhibits deterministic quasi-periodic equilibria; ii) in the Running-away from the Joneses formulation, even for moderate influence of the benchmark level of consumption, not only deterministic quasi-periodic paths may occur, but equilibria may also be locally indeterminate.

[8] As it does not bring any new feature, the last case $|\mathscr{D}| > 1$, i.e., $|\varsigma| > \beta s/(\gamma + s)$ on Figure 9 is not detailed (remark nonetheless that for $\varsigma = -1$, $\mathscr{M} = -1$, i.e., the ordinate of $A_{\mathscr{D}}$).

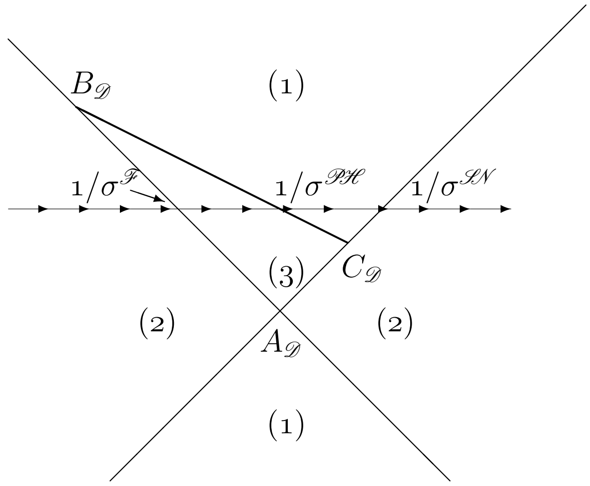

Figure 9: Running Away from the Joneses [$\varsigma < 0, -1 < (\gamma + s)\varsigma/\beta s < 0$]

4 A Heterogeneous Agents Economy

4.1 The setup

As an another illustration, this section will consider a variation of the heterogeneous agents economy introduced by Woodford [16]. The preferences of the capitalists are described by: $\sum_{t=0}^{\infty} \beta^t u^c(c_t^c)$, for $\beta \in]0,1]$ and c_t^c respectively their positive rate of marginal impatience and their consumption at date $t \geq 0$ and for $u^c(\cdot)$ a continuous concave instantaneous utility function which maps $\mathbb{R}_+$ into $\mathbb{R}$, is of class C^k, $k \geq 3$, over $\mathbb{R}_+^*$ and satisfies the Inada conditions at the origin. These agents maximize the above objective over a constraint set defined from their intertemporal budget constraint:

$$(15) \quad c_t^c + K_{t+1}^c + m_{t+1}^c/p_t = \big(r_t + 1 - \delta\big)K_t^c + m_t^c/p_t,$$

for $m_t^c \geq 0$ and $K_t^c \geq 0$ the respective nominal money balances and capital stock held by capitalists at the beginning of period t, p_t the price of the produced good, r_t the real return on capital and $\delta \in]0,1]$ the depreciation rate of the capital stock. Following Woodford [16] and also resting upon the detailed argument available in the Appendix A of Grandmont, Pintus et de Vilder [16], the subsequent argument will focus on equilibria along which $m_t^c = 0$ and $K_t^c > 0$. The latters are thus characterised by the holding of:

$$(16) \quad \begin{aligned} &\big(u^c\big)'\big(c_t^c\big) = \beta R_{t+1}\big(u^c\big)'\big(c_{t+1}^c\big), \\ &c_t^c + K_{t+1}^c = R_t K_t^c. \end{aligned}$$

for $R_t := r_t + 1 - \delta$.

In parallel to this, wage-earners instantaneous preferences are represented by a separable utility function $u^w(c^w) - \alpha v(\ell)$, where $\alpha \in]0,1]$ denotes their common discount rate; c^w and ℓ respectively label their amounts of consumption and worked hours. It is assumed that both $u^w(\cdot)$ and $v(\cdot)$ are increasing maps from $\mathbb{R}_+$ into $\mathbb{R}$ which are C^k, $k \geq 3$, over $\mathbb{R}_+^*$, $u(\cdot)$ and $v(\cdot)$ being respectively strictly concave and convex over that set. Let $m_t^w \geq 0$ and $K_t^w \geq 0$

be, respectively, nominal feature the money balances and capital stock held by workers at the outset of period t. Letting w_t denotes the monetary wage, at time t workers face the following constraint set :

$$(17)\quad c_t^w + K_{t+1}^w + m_{t+1}^w/p_t = w_t\ell_t + \big(r_t + 1 - \delta\big)K_t^w + m_t^w/p_t$$
$$c_t^w + K_{t+1}^w \leq \big(r_t + 1 - \delta\big)K_t^w + m_t^w/p_t.$$

Whilst the first relation depicts an usual budget constraint, the second imposes a finance constraint stipulating that labor incomes are not available to finance good expenditures. The focus shall henceforward be placed on equilibria that satisfy $K_t^w = 0$, $m_t^w > 0$ and a bidding finance constraint. The latter are thus characterized by:

$$(18)\quad w_t\frac{p_t}{p_{t+1}}\big(u^w\big)'\big(c_{t+1}^w\big) = v'\big(\ell_t\big),$$
$$\frac{p_t}{p_{t+1}}c_{t+1}^w = w_t\ell_t.$$

Under the gross substitute axiom that ensures the uniform holding of $(u^w)'(c^w)+c^w(u^w)''(c^w) > 0$, the equations above actually boil down to:

$$(19)\quad c_{t+1}^w = \psi\big(\ell_t\big) := \big[\big(\mathscr{U}^w\big)^{-1}\circ\mathscr{V}\big]\big(\ell_t\big),$$

for $\mathscr{U}^w\big(c^w\big) := c^w u'\big(c^w\big)$ and $\mathscr{V}(\ell) := \ell v'(\ell)$.

The unique good can be consumed or accumulated and is produced by a continuum of firms, normalised to one, that act competitively through a common aggregate production function $F\big(K_t, L_t\big)$, where $F(\cdot,\cdot)$ satisfies to the same range of standard properties listed in Section 3 when it is specialised to constant returns to scale and $\gamma = 0$, the first order conditions of the firm problem being thus $r_t = F_K'\big(K_t, L_t\big)$, $w_t = F_L'\big(K_t, L_t\big)$.

Lastly, noticing that the money market clearing condition writes as $w_t\ell_t = M/p_t = c_t^w$, for M the constant level of money supply, it is readily shown that a dynamic equilibrium summarises to an infinite sequence $\{K_t, L_t, c_t^c\}_{t=0}^{+\infty}$ for which :

$$(20)\quad \big(u^c\big)'\big(c_t^c\big) - \beta\big[F_K'\big(K_{t+1}, L_{t+1}\big) + 1 - \delta\big]\big(u^c\big)'\big(c_{t+1}^c\big) = 0,$$
$$K_{t+1} - \big[F_K'\big(K_t, L_t\big) + 1 - \delta\big]K_t + c_t^c = 0,$$
$$F_L'\big(K_{t+1}, L_{t+1}\big)L_{t+1} - \psi\big(L_t\big) = 0.$$

Linearising again the dynamical system in the neighbourhood of the steady state, maintaining the system of notations introduced in Section 3 for the elasticity of substitution between the productive factors, the share of wages, the share of profits, the elasticity of intertemporal substitution of the capitalist but also introducing the elasticity of the reciprocal offer curve of the worker along $\varepsilon := \psi'(L)L/\psi(L) > 1$, from Appendix 3, the coefficients of the third order associated characteristic polynomial write down as :

$$(21)\quad \mathscr{T} = 1 + \beta^{-1} - \frac{1-s}{\sigma - s}[1 - \eta(1-\beta)]\varrho + \frac{\sigma}{\sigma - s}\varepsilon,$$
$$\mathscr{M} = \big(1 + \beta^{-1}\big)\frac{\sigma\varepsilon}{\sigma - s} + \beta^{-1} - \frac{1-s}{\sigma - s}\varrho - \frac{1-s}{\sigma - s}[1 - \eta(1-\beta)]\varrho\varepsilon,$$
$$\mathscr{D} = \frac{\beta^{-1}\sigma - \varrho(1-s)}{\sigma - s}\varepsilon,$$

for $\varrho := \beta^{-1} - 1 + \delta$. As a matter of comparison, it may be noticed that the specialisation of the capital-holder preferences to a neperian-logarithmic formulation would have entailed a significant formal simplification: it is indeed readily checked that there then exists T and D such that the above coefficients respectively restate as $\mathscr{T} = \beta^{-1} + T$, $\mathscr{M} = D + \beta^{-1}T$ and $\mathscr{D} = \beta^{-1}D$. Otherwise stated, β^{-1} emerges as a trivial eigenvalue and there exists a second-order polynomial $\mathscr{Q}(\cdot)$ with coefficients T and D that satisfies $\mathscr{P}(z) = (1 - z)\mathscr{Q}(z)$. Incidently, further letting — along these authors — $\beta = 1$, the expression of the characteristic polynomial analysed in Grandmont, Pintus and de Vilder [14] is recovered.

4.2 The analysis

It can be established that whenever the capitalists preferences are logarithmic, $\sigma > s$ is sufficient to preclude the emergence of quasi-periodic as well as locally indeterminate equilibria — vide Grandmont, Pintus & de Vilder [14]. A configuration $\sigma > s$ shall henceforth uniformly be retained but the current analysis shall oppositely provide a more comprehensive account based upon a general class of assumptions on capitalists preferences — formally, η is not any longer specialised to 1. The most noticeable feature of (21) for the current purpose then formulates as the independency of the coefficient $\mathscr{D}$ with respect to the elasticity of intertemporal substitution of capitalists, i.e., η. Thus, selecting η as the tuning parameter, it brings about the possibility of a purely geometric argument for appraising the dynamical properties of the model.

It is first noticed that

$$\text{(22)} \quad \mathscr{T}'(\eta) = \frac{1-s}{\sigma - s}(1-\beta)\varrho > 0, \qquad \mathscr{M}'(\eta) = \frac{1-s}{\sigma - s}(1-\beta)\varrho\varepsilon > 0$$

whence a half-line directed towards the north-east as the parameter η is raised:

$$\text{(23)} \quad \Delta' := \frac{\mathscr{M}'(\eta)}{\mathscr{T}'(\eta)} = \varepsilon.$$

A further nice property that holds whatever the considered parameterisation is related to the origin of the half-line $\Delta(\eta)$. As a matter of fact, since $-1 + \mathscr{T}(0) - \mathscr{M}(0) + \mathscr{D} = 0$, $\Delta(0) \in (A_{\mathscr{D}}C_{\mathscr{D}})$ and is fully localised by its ordinate

$$\text{(24)} \quad \mathscr{M}(0) = \frac{(1+\beta^{-1})\sigma\varepsilon - \varrho(1-s)(1+\varepsilon)}{\sigma - s} + \beta^{-1}.$$

Along Section 3, a geometrical approach starts by locating the underlying plane that in turn implies to position $\mathscr{D}$ with respect to $+1$. Firstly remarking that

$$\text{(25)} \quad \mathscr{D} = \beta^{-1}\frac{\sigma - \beta\varrho(1-s)}{\sigma - s}\varepsilon := \beta^{-1}\bar{\mathscr{D}}\varepsilon,$$

for $\beta^{-1} > 1$ and $\varepsilon > 1$, two cases are to be considered.

Case 1 $\beta\varrho(1-s) < s$. In that case, $\mathscr{D} > 1$ whatever the value of ε. As $\Delta' > 1$ and from Figure 10, any scope for a Poincaré-Hopf bifurcation is ruled out. From (23), $\Delta' > 1$ being independent of σ, the discussion shall be organised around the dependency of the location of $\mathscr{M}(0)$ with respect to σ — restricted to $]s, +\infty)$ — and thus based upon a notation $\mathscr{M}(0;\sigma)$. As this is illustrated on Figure 10, the key element will state as the position of $\mathscr{M}(0;\sigma)$ with respect to $\mathscr{M}_{A_\mathscr{D}} = -1$. Facing thus with the sign of $\mathscr{M}(0;\sigma) + 1$, it derives that:

$$(26) \quad \lim_{\sigma\to s}[1 + \mathscr{M}(0;\sigma)] = \pm\infty \Longleftrightarrow \varepsilon \gtreqless \tilde{\varepsilon} := \frac{\beta\varrho(1-s)}{(1+\beta)s - \beta\varrho(1-s)}.$$

Hence, for a given (β, s, δ) such that $\beta\varrho(1-s) < s$, two configurations are to be distinguished

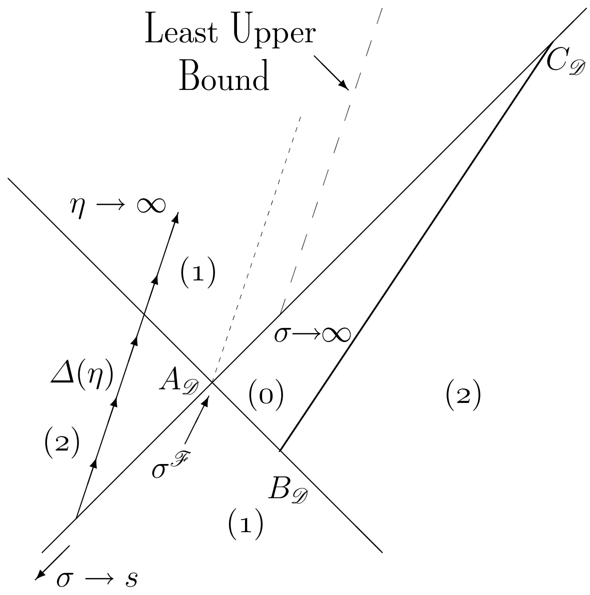

Figure 10: $\beta\varrho(1-s) < s$ $[\varepsilon < \tilde{\varepsilon}]$.

according to the asymptotic behaviour of $\mathscr{M}(0;\sigma)$. As this is pictured on Figures 10 and 11, a characterisation becomes available:

1. for $\varepsilon < \tilde{\varepsilon}$ $\left[\lim_{\sigma\to s}[1 + \mathscr{M}(0;\sigma)] = -\infty\right]$,

   (a) if $\sigma < \sigma^\mathscr{F} = \beta\varrho(1-s)/(1+\beta) + s/(1+\varepsilon)$, for $\sigma^\mathscr{F}$ the value of σ for which $\Delta(0) \in (A_\mathscr{D}B_\mathscr{D})$ and thus $\mathscr{M}(0;\sigma^\mathscr{F}) = -1$, the system is firstly locally indeterminate with a degree of indeterminacy equal to one, then undergoes a flip bifurcation and is eventually locally determinate ;

   (b) if $\sigma > \sigma^\mathscr{F}$, the system is locally determinate ;

2. for $\varepsilon > \tilde{\varepsilon}$ $\left[\lim_{\sigma\to s}[1 + \mathscr{M}(0;\sigma)] = +\infty\right]$, the system is locally determinate for any $\sigma \in]s, \infty)$.

Case 2 $\beta\varrho(1-s) > s$. In this outstanding configuration, the most significant case turns out to be $\mathscr{D} \in]0, 1[$ — the occurrence of $\mathscr{D} > 1$ mainly boils the argument down to the one developed

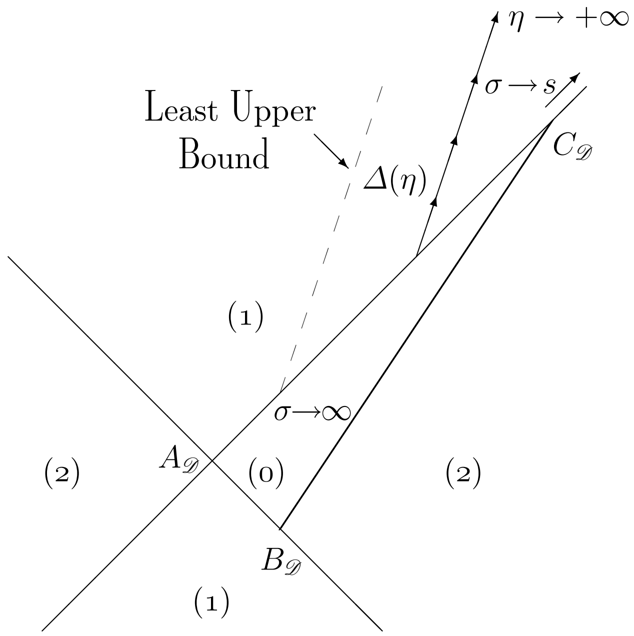

Figure 11: $\beta\varrho(1-s) < s$ $[\varepsilon > \hat{\varepsilon}]$.

in Case 1. This requires the satisfaction of:

$$(27)\quad \beta\varrho(1-s) < \sigma < \frac{\varrho(1-s)\varepsilon - s}{\beta^{-1}\varepsilon - 1}.$$

Letting, for future reference, $\underline{\sigma} := \beta\varrho(1-s)$ and $\bar{\sigma} := [\varrho(1-s)\varepsilon - s]/(\beta^{-1}\varepsilon - 1)$ denote the minimal and maximal admissible values for σ, the main argument will anew be organised around the position of $\Delta(0)$ with respect to σ and ε. As this is illustrated on Figure 12, the key elements are, for $\mathscr{D} \in]0,1[$ and a straight-line whose origin $\Delta(0)$ lies on $(A_{\mathscr{D}}C_{\mathscr{D}})$, the actual location of $\Delta(0)$ with respect to points $A_{\mathscr{D}}$ and $C_{\mathscr{D}}$ and thus of $\mathscr{M}(0)$ with respect to $\mathscr{M}_{A_{\mathscr{D}}} = -1$ and $\mathscr{M}_{C_{\mathscr{D}}} = 1 + 2\mathscr{D}$. Facing first with $\mathscr{M}_{A_{\mathscr{D}}}$ and thus with the sign of $\mathscr{M}(0;\sigma)+1$ at the lower and upper boundaries $\underline{\sigma}$ and $\bar{\sigma}$:

$$(28a)\quad 1 + \mathscr{M}(0;\underline{\sigma}) \begin{cases} < 0 & \text{if } \varepsilon < \underline{\varepsilon}; \\ > 0 & \text{if } \varepsilon > \underline{\varepsilon}. \end{cases}$$

$$(28b)\quad 1 + \mathscr{M}(0,\bar{\sigma}) > 0 \text{ for any } \varepsilon > 1,$$

for $\underline{\varepsilon} = (1+\beta^{-1})s/\beta\varrho(1-s) - 1$, the formal details underlying the derivation of (28b) being available in Appendix 4. Similarly locating $\mathscr{M}(0;\sigma)$ with respect to $\mathscr{M}_{C_{\mathscr{D}}}$ and exploring the involved sign of $\mathscr{M}(0;\sigma) - 1 - 2\mathscr{D}(\sigma)$ at the lower and upper boundaries $\underline{\sigma}$ and $\bar{\sigma}$:

$$(29a)\quad \mathscr{M}(0;\underline{\sigma}) - 1 - 2\mathscr{D}(\underline{\sigma}) \begin{cases} < 0 & \text{if } \varepsilon < \hat{\varepsilon}; \\ > 0 & \text{if } \varepsilon > \hat{\varepsilon}. \end{cases}$$

$$(29b)\quad \mathscr{M}(0;\bar{\sigma}) - 1 - 2\mathscr{D}(\bar{\sigma}) \begin{cases} < 0 & \text{if } \varepsilon < \bar{\varepsilon}; \\ > 0 & \text{if } \varepsilon > \bar{\varepsilon}. \end{cases}$$

for $\hat{\varepsilon} := 1 + (\beta^{-1} - 1)s/\beta\varrho(1-s)$ whilst the existence of $\bar{\varepsilon}$ is proved in Appendix 4. An eventual checking in the latter further ensuring the satisfaction of $\underline{\varepsilon} < \hat{\varepsilon} < \bar{\varepsilon}$, four distinct configurations reveal to deserve an explicit consideration:

- $\varepsilon < \underline{\varepsilon}$.

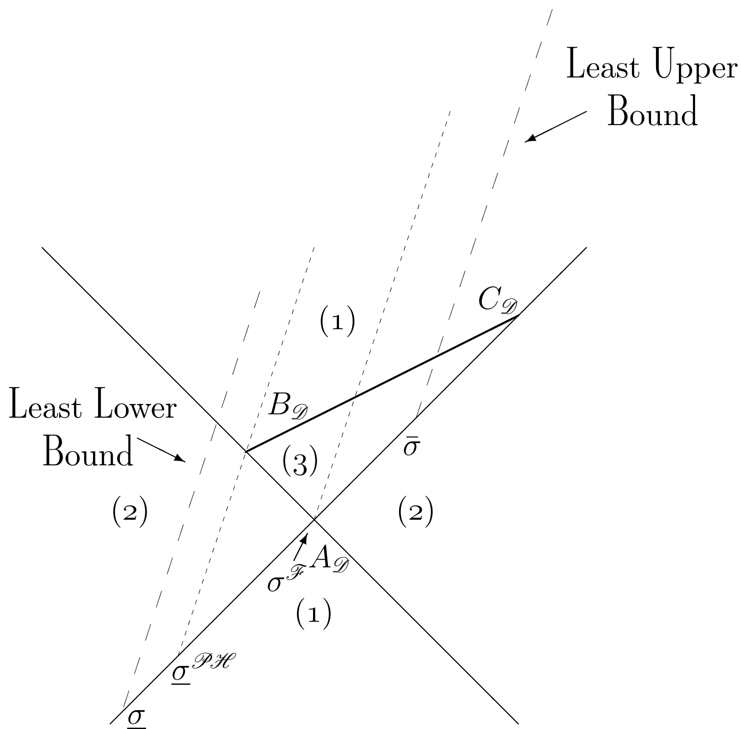

Figure 12: $\beta\varrho(1-s) > s$, $[\varepsilon < \underline{\varepsilon}]$

This is unambiguously the most complex occurrence, noticeably because it raises an extra formal issue that is illustrated by Figure 12: for $\underline{\sigma} < \sigma^{\mathscr{F}}$, does there exist a critical value for σ, henceforward $\underline{\sigma}^{\mathscr{PH}}$, that is associated with the emergence of a Poincaré-Hopf bifurcation ? This boundary value $\underline{\sigma}^{\mathscr{PH}}$ is defined from the requirement that the half-line $\Delta(\eta)$ with slope ε goes through $B_{\mathscr{D}}$. From a given ε, one then seeks σ such that $[\mathscr{M}(0;\sigma) - \mathscr{M}_{B_{\mathscr{D}}}]/[\mathscr{T}(0;\sigma) - \mathscr{T}_{B_{\mathscr{D}}}] = \Delta' = \varepsilon$, that gives:

$$\underline{\sigma}^{\mathscr{PH}} = \frac{(\beta^{-1}-1)(1-\varepsilon)-4\varepsilon}{(\beta^{-1}-1)(\varepsilon^2+2\varepsilon+1)}s + \frac{\beta\varrho(1-s)}{1-\beta}.$$

At that stage and although this proceeds from a boundary argument, it is worth noticing that this parameters configuration — the occurrence of which is illustrated through Figure 12 — is non-empty in a somewhat trivial way in that it embeds $\varepsilon = 1$. A typology hence becomes available:

— for $\sigma \in \left]\underline{\sigma}, \underline{\sigma}^{\mathscr{PH}}\right[$ and as η is increased, the economy will be locally indeterminate with a degree of indeterminacy equal to one, then undergo a flip bifurcation and finally be locally determinate ;

— for $\sigma \in \left]\underline{\sigma}^{\mathscr{PH}}, \sigma^{\mathscr{F}}\right[$, an even more complex scenario emerges as an increase in η implies that the system starts from an area with a unique degree of indeterminacy, then undergoes a flip bifurcation and falls in an area — the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ — with a degree of indeterminacy of two, then undergoes a Poincaré-Hopf bifurcation and eventually recovers local determinacy ;

— for $\sigma \in \,]\underline{\sigma}^{\mathscr{F}}, \bar{\sigma}[$, the system starts into an area with two degrees of indeterminacy, then undergoes a Poincaré-Hopf bifurcation and finally exhibits local determinacy.

It should be emphasised that the illustration of this configuration through Figure 12 is not the sole one that fits $\varepsilon < \underline{\varepsilon}$. The occurrence of $\underline{\sigma}^{\mathscr{PH}} < \underline{\sigma} < \sigma^{\mathscr{F}}$ would equally be admissible. The transposition of the preceding typology being however straightforward, it will not be detailed further.

- $\underline{\varepsilon} < \varepsilon < \hat{\varepsilon}$.

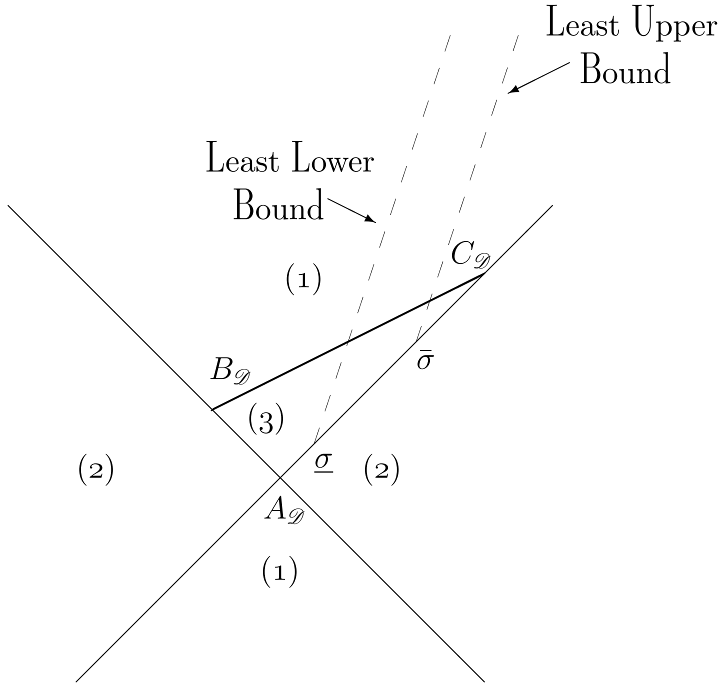

Figure 13: $\beta\varrho(1-s) > s$, $[\underline{\varepsilon} < \varepsilon < \hat{\varepsilon}]$

This second occurrence — depicted in Figure 13 — is oppositely quite simple since the whole interval of admissible values for the elasticity of substitution between the inputs, namely $]\underline{\sigma}, \bar{\sigma}[$, will entail a local indeterminacy with a degree of two — the economy starts in the triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ as η is increased —, the occurrence of a Poincaré-Hopf bifurcation and eventually recover local uniqueness, the uniformity of this configuration being noticeably strengthened by the disappearance of any area for a flip bifurcation.

- $\hat{\varepsilon} < \varepsilon < \bar{\varepsilon}$.

This third configuration — illustrated through Figure 14 — reintroduces a certain degree of complexity in the analysis. Firstly characterising the limit value for σ such that the ordinate of $\Delta(0)$ locates on $C_{\mathscr{D}}$, $\bar{\sigma}^{\mathscr{PH}}$ is found by solving $\mathscr{M}(0;\sigma) - 1 - 2\mathscr{D}(\sigma) = 0$. Its expression derives as:

$$\bar{\sigma}^{\mathscr{PH}} = \frac{\beta\varrho(1-s)}{1-\beta} - \frac{s}{\varepsilon - 1},$$

It is finally obtained that:

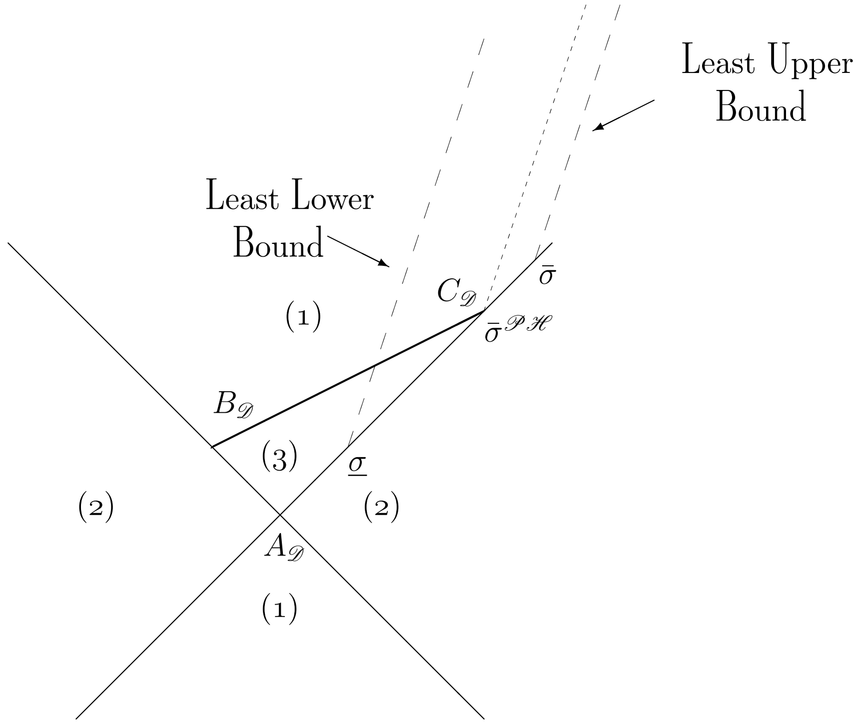

Figure 14: $\beta\varrho(1-s) > s$, $[\hat{\varepsilon} < \varepsilon < \bar{\varepsilon}]$

— for $\sigma \in]\underline{\sigma}, \bar{\sigma}^{\mathscr{P}\mathscr{H}}[$ and as η is increased, the system replicates the kind of behaviours that uniformly prevailed in the former occurrence, namely, an inception with a high degree of local indeterminacy that is followed by a Poincaré-Hopf bifurcation and a final recovering of local determinacy;

— for $\sigma \in]\bar{\sigma}^{\mathscr{P}\mathscr{H}}, \bar{\sigma}[$, local determinacy will be available throughout as η spans its interval.

- $\varepsilon > \bar{\varepsilon}$.

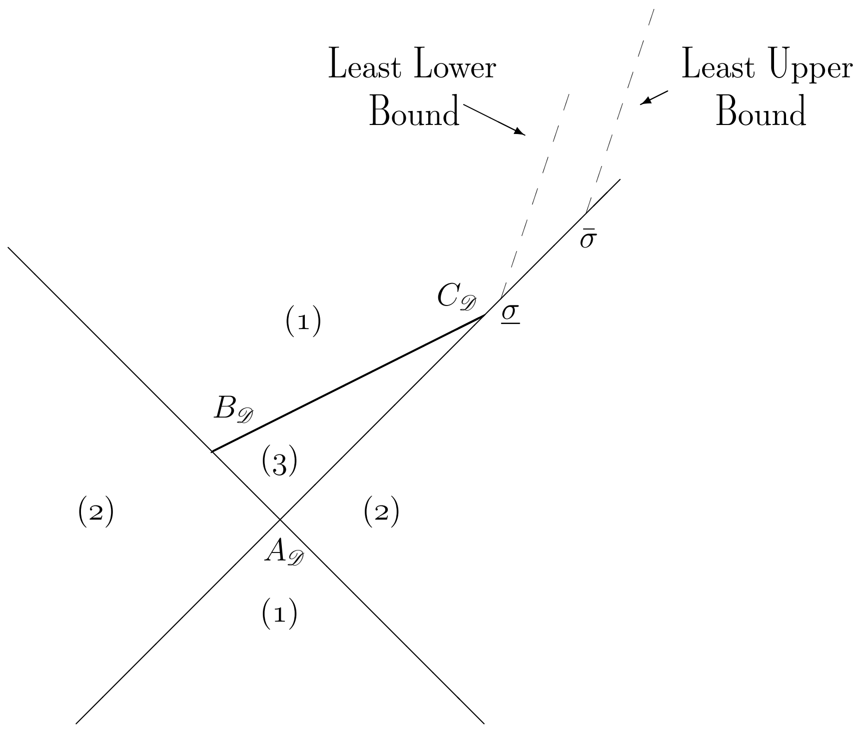

Figure 15: $\beta\varrho(1-s) > s$, $[\varepsilon > \bar{\varepsilon}]$

Finally, this ultimate occurrence portrayed in Figure 15 is unequivocally the simplest to grasp in being characterised by local uniqueness for the whole interval $]\underline{\sigma}, \bar{\sigma}[$ and for any $\eta \in]0, +\infty)$.

In conclusive terms and to sum up, the central insights of this appraisal on an enriched version of the Woodford [16] setup list as the appearance of a class of indeterminacy results that do not necessarily involve low orders for factors substitutability and are, e.g., available for baseline Cobb-Douglas technologies. They overall appear to be linked to low orders for the elasticity of the offer curve and thus to high values for the elasticity of labour supply as well as to low orders for the intertemporal elasticity of substitution in consumption for the capital-holder.

5 References

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6 Proofs

6.1 The Implicit Function Theorem Argument

The following lemma characterizes the modulus behaviour near the critical loci.

Lemma 1 Consider the eigenvalue $z(\mathscr{T},\mathscr{M},\mathscr{D})$. By construction, $z(\mathscr{T},\mathscr{M},\mathscr{D}) = -1$ when $\mathscr{P}(-1) = 0$, $z(\mathscr{T},\mathscr{M},\mathscr{D}) = 1$ when $\mathscr{P}(+1) = 0$ and $|z(\mathscr{T},\mathscr{M},\mathscr{D})| = 1$, $\mathscr{P}(\mathscr{D}) = 0$ for $|\mathscr{T}-\mathscr{D}| < 2$. Then:

1. for $z'_{\mathscr{M}} : = dz(\mathscr{T},\mathscr{M},\mathscr{D})/d\mathscr{M}|_{z(\mathscr{T},\mathscr{M},\mathscr{D})=-1}$,
   (a) $z'_{\mathscr{M}} < 0$ if $\mathscr{T}-\mathscr{D} < -2$,
   (b) $z'_{\mathscr{M}} > 0$ if $\mathscr{T}-\mathscr{D} > -2$;
2. for $z'_{\mathscr{M}} := dz(\mathscr{T},\mathscr{M},\mathscr{D})/dx|_{z(\mathscr{T},\mathscr{M},\mathscr{D})=1}$,
   (a) $z'_{\mathscr{M}} < 0$ if $\mathscr{T}-\mathscr{D} < 2$,
   (b) $z'_{\mathscr{M}} > 0$ if $\mathscr{T}-\mathscr{D} > 2$;
3. $d_{\mathscr{M}}|z_c| := d|z_c(\mathscr{T},\mathscr{M},\mathscr{D})|/d\mathscr{M}|_{|z_c(\mathscr{T},\mathscr{M},\mathscr{D})|=1} > 0$.

Proof :
1. The argument builds from a straightforward application of the Implicit Function Theorem and it is assumed that $z(\mathscr{T},\mathscr{M},\mathscr{D}) = -1$ is a regular zero of $\mathscr{P}(\cdot)$. Fixing $\mathscr{T}$ and $\mathscr{D}$ and

taking the differential of $-[z(\mathscr{T},\mathscr{M},\mathscr{D})]^3+\mathscr{T}[z(\mathscr{T},\mathscr{M},\mathscr{D})]^2-\mathscr{M}z(\mathscr{T},\mathscr{M},\mathscr{D})+\mathscr{D}=0$, this yields :

$$\frac{dz(\mathscr{T},\mathscr{M},\mathscr{D})}{d\mathscr{M}}=\frac{z(\mathscr{T},\mathscr{M},\mathscr{D})}{-[z(\mathscr{T},\mathscr{M},\mathscr{D})]^3+\mathscr{T}[z(\mathscr{T},\mathscr{M},\mathscr{D})]^2-\mathscr{M}z(\mathscr{T},\mathscr{M},\mathscr{D})}.$$

Evaluating then the former equation at $z(\mathscr{T},\mathscr{M},\mathscr{D})=-1$ and recalling that $1+\mathscr{T}+\mathscr{M}+\mathscr{D}=0$:

$$\frac{dz(\mathscr{T},\mathscr{M},\mathscr{D})}{d\mathscr{M}}=\frac{1}{2+\mathscr{T}-\mathscr{D}},$$

whence the statement.

2. This follows from a simple adaptation of the arguments used in 1.

3. The argument of the proof builds from a twice application of the Implicit Function Theorem. Actually, the determinant being equal to the product of the eigenvalues, in the present configuration, it may be restated as

$$\mathscr{D}=z_r(\mathscr{T},\mathscr{M},\mathscr{D})\left|z_c(\mathscr{T},\mathscr{M},\mathscr{D})\right|^2,$$

where $z_r(\mathscr{T},\mathscr{M},\mathscr{D})$, $z_c(\mathscr{T},\mathscr{M},\mathscr{D})$ and $|z_c(\mathscr{T},\mathscr{M},\mathscr{D})|$ denote the real eigenvalue, the nonreal eigenvalue and its norm, respectively. Taking the total differential of the above equality for fixed values of $\mathscr{T}$ and $\mathscr{D}$, it comes, with non confusing notations :

$$\begin{aligned} 0 &= z_r'|z_c|^2 d\mathscr{M}+2z_r|z_c|d|z_c|,\\ d_{\mathscr{M}}|z_c| &= \frac{-z_r'|z_c|^2}{2z_r|z_c|}. \end{aligned}$$

By now, applying the Implicit Function Theorem to the relation $-[z(\mathscr{T},\mathscr{M},\mathscr{D})]^3+\mathscr{T}[z(\mathscr{T},\mathscr{M},\mathscr{D})]^2-\mathscr{M}z(\mathscr{T},\mathscr{M},\mathscr{D})$, it is readily obtained that

$$z_r'(\mathscr{T},\mathscr{M},\mathscr{D})=\frac{z_r(\mathscr{T},\mathscr{M},\mathscr{D})}{-[z(\mathscr{T},\mathscr{M},\mathscr{D})]^3+\mathscr{T}[z(\mathscr{T},\mathscr{M},\mathscr{D})]^2-\mathscr{M}z(\mathscr{T},\mathscr{M},\mathscr{D})}.$$

Whenever $|z_c(\mathscr{T},\mathscr{M},\mathscr{D})|=1$, $z_r(\mathscr{T},\mathscr{M},\mathscr{D})=\mathscr{D}$. Reminding that in order for the modulus to be equal to one the relation $\mathscr{M}=1+(\mathscr{T}-\mathscr{D})\mathscr{D}$ must hold, it is obtained that :

$$d_{\mathscr{M}}|z_c(\mathscr{T},\mathscr{M},\mathscr{D})|=\frac{1}{2}\frac{1}{2\mathscr{D}^2-\mathscr{T}\mathscr{D}+1}.$$

As for the sign of the denominator, consider a small perturbation $\mathscr{T}=2+\mathscr{D}+\varepsilon$, for ε small. Plugging this expression into the second-order polynomial $2\mathscr{D}^2-\mathscr{T}\mathscr{D}+1=0$, it restates as :

$$\mathscr{D}^2-(2+\varepsilon)\mathscr{D}+1=0.$$

Its discriminant is given by $\Delta=(2+\varepsilon)^2-4$ and it will assume complex roots if and only if $\varepsilon<0$, i.e., for $\mathscr{T}-\mathscr{D}<2$. Completing a mirror approach for $\mathscr{T}=-2+\mathscr{D}$, it finally derives that $2\mathscr{D}^2-\mathscr{T}\mathscr{D}+1>0$ as long as the condition $|\mathscr{T}-\mathscr{D}|<2$ will be satisfied. The statement 3 follows. △

6.2 Catching Up with the Joneses: Derivation of the Coefficients of the Characteristic Polynomial

Linearising the characteristic polynomial in the neighbourhood of the steady state leads to:

$$\begin{bmatrix} \dfrac{dc_{t+1}}{c^\star} \\[2ex] \dfrac{dK_{t+1}}{K^\star} \\[2ex] \dfrac{dX_{t+1}}{X^\star} \end{bmatrix} = \begin{bmatrix} 1-\dfrac{\eta^{\mathscr{E}}}{\eta}-\dfrac{(1-s)}{\sigma}\dfrac{c^\star}{K^\star} & \dfrac{1}{\eta}\dfrac{(1-s)}{\sigma}\left(\dfrac{c^\star}{K^\star}+1\right) & \dfrac{\eta^{\mathscr{E}}}{\eta} \\[2ex] -\dfrac{c^\star}{K^\star} & \left(\dfrac{c^\star}{K^\star}+1\right)s & 0 \\[2ex] 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{dc_t}{c^\star} \\[2ex] \dfrac{dK_t}{K^\star} \\[2ex] \dfrac{dX_t}{X^\star} \end{bmatrix}$$

Further noticing that $c^\star/K^\star + 1 = 1/\beta s$ and letting, e.g. J_{11} denote the first element of the first row in the above Jacobian Matrix, the expressions of $\mathscr{T}$, $\mathscr{M}$ and $\mathscr{D}$ in the main text are straighforwardly derived by noticing that they respectively correspond to $J_{11}+J_{22}$, $J_{11}J_{22}-J_{12}J_{21}-J_{13}$ and $-J_{13}J_{22}$. △

6.3 Derivation of the Coefficients of the Characteristic Polynomial

Omitting arguments, a linear approximation of the dynamical system at the steady state delivers:

$$\begin{aligned} (u^c)''dc_t-(u^c)''dc_{t+1}-\beta(F''_{KK})h'dK_{t+1}-\beta(F''_{KL})h'dL_{t+1} &= 0, \\ dK_{t+1}-\left[(F''_{KK}+1-\delta)\right]dK_t-(F''_{KL})KdL_t+dc_t &= 0, \\ (F''_{LK}L)dK_{t+1}+(F''_{LL}L+F'_L)dL_{t+1}-\psi'dL_t &= 0. \end{aligned}$$

Letting

$$\eta := -\frac{(u^c)'}{c^c(u^c)''}, \quad \sigma := \frac{F'_K F'_L}{FF''_{KL}}, \quad s := \frac{KF'_K}{F}, \quad 1-s := \frac{LF'_L}{F}, \quad \varepsilon := \frac{L\psi'}{\psi}$$

and noticing that $-KF''_{KK}/F'_K = (1-s)/\sigma$, $-LF''_{LL}/F'_L = s/\sigma$ the earlier system restates, for $\varrho := \beta^{-1}-1+\delta$, as:

$$\begin{bmatrix} -\beta\rho\left(-\dfrac{1-s}{\sigma}\right) & -\beta\rho\left(\dfrac{1-s}{\sigma}\right)\dfrac{K}{L} & \dfrac{1}{\eta(\beta^{-1}-1)} \\[2ex] 1 & 0 & 0 \\[2ex] \left(\dfrac{s}{\sigma}\right)\dfrac{L}{K} & \left(1-\dfrac{s}{\sigma}\right) & 0 \end{bmatrix} \begin{bmatrix} dK_{t+1} \\[2ex] dL_{t+1} \\[2ex] dc_{t+1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \dfrac{1}{\eta(\beta^{-1}-1)} \\[2ex] \rho\left[-\dfrac{1-s}{\sigma}+(\beta\rho)^{-1}\right] & \rho\left(\dfrac{1-s}{\sigma}\right)\dfrac{K}{L} & -1 \\[2ex] 0 & \varepsilon & 0 \end{bmatrix} \begin{bmatrix} dK_t \\[2ex] dL_t \\[2ex] dc_t \end{bmatrix}$$

It is finally straightforward to recover the expression (21) of the main text. △

6.4 The case $\beta\varrho(1-s) > s$.

a/ Facing with the upper boundary $\bar{\sigma}$:

$$1+\mathscr{M}\left(0,\bar{\sigma}\right) = \frac{\varrho(1-s)(1+\varepsilon)^2 - \left(1+\beta^{-1}\right)^2 s\varepsilon}{\varrho(1-s)\varepsilon - \beta^{-1}s\varepsilon}.$$

The numerator being a polynomial in ε that reformulates along

$$\mathscr{P}(\varepsilon) = \varepsilon^2 + \left[2 - \frac{(1+\beta^{-1})^2 s}{\varrho(1-s)}\right]\varepsilon + 1,$$

its roots $\varepsilon_1, \varepsilon_2$ satisfy $\varepsilon_1 \cdot \varepsilon_2 = 1$ and $\varepsilon_1 + \varepsilon_2 < 2$ that readily implies the satisfaction of $\mathscr{P}(\varepsilon) > 0$ for every $\varepsilon > 1$. Whence the statement (29) in the main text.

b/ $\mathscr{M}\left(0,\bar{\sigma}\right) - 1 - 2\mathscr{D}(\bar{\sigma})$ currently depicts a polynomial of degree two in ε. The numerator of the latter is available as

$$\mathscr{P}(\varepsilon) = \varrho(1-s)(\varepsilon-1)^2 - \left(1-\beta^{-1}\right)^2 s\varepsilon,$$

that in turn describes a U-shaped convex polynomial. Further noticing that $\mathscr{P}(\varepsilon)|_{\varepsilon=1} = -\left(1-\beta^{-1}\right)^2 s\varepsilon < 0$ and that $\mathscr{P}'(\varepsilon) = 0$ for $\varepsilon = \left[\left(1-\beta^{-1}\right)^2\right]/2\varrho(1-s)+1 > 1$, this ensures the existence of a zero for $\mathscr{P}(\varepsilon)$, say $\bar{\varepsilon}$, such that $\bar{\varepsilon} > 1$. In addition, and though its expression remains implicit, the statement (31) becomes available.

c/ The eventual task lies in the ranking of the various values for ε, namely $\underline{\varepsilon}, \hat{\varepsilon}, \bar{\varepsilon}$ that have appeared from the analysis of the values undertaken by $\mathscr{M}(0,\sigma)+1$ and $\mathscr{M}\left(0;\sigma\right)-1-2\mathscr{D}(\sigma)$ at $\underline{\sigma}$ and $\bar{\sigma}$. First recall that, by definition, $\mathscr{P}\left(\bar{\varepsilon}\right) = 0$. Since it is readily checked that

$$\mathscr{P}\left(\hat{\varepsilon}\right) = \left(\beta^{-1}-1\right)^2 s\left[\frac{s}{\beta\varrho(1-s)} - 1\right] < 0,$$

this in turn implies $\hat{\varepsilon} < \bar{\varepsilon}$. In the same vein, as

$$\hat{\varepsilon} - \underline{\varepsilon} = 2\left[1 - \frac{s}{\beta\varrho(1-s)}\right] > 0,$$

the rank of the main text becomes available. △

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