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Abstract

While most of the literature concerned with indeterminacy considers a representative agent, some recent works have investigated the role of heterogenous agents on dynamics. This paper adds a contribution to the debate, stressing the effects of heterogeneity in consumers' preferences within an overlapping generations economy with capital accumulation and consumption in both periods. We show that such an heterogeneity can stabilize fluctuations in many cases, by simply reducing the range of parameters compatible with indeterminacy.

JEL classification: C62, E32.

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1 Introduction

In the past two decades, a large number of papers have been devoted to find conditions for indeterminacy and endogenous cycles in macrodynamic models and to shed light on the underlying mechanisms.[1] One of the main criticisms addressed to this literature concerns the assumption of a representative agent to summarize the average behavior either of an infinite-lived population or of a finite-horizon generation: Whether the dynamic properties of the model don't depend on such an assumption has been seldom questioned. The lack of clear-cut results about the role of agents' heterogeneity in the emergence of endogenous cycles, either deterministic or stochastic, encourages us to look into the question.

Recently, Herrendorf, Waldmann and Valentiny (2000) have shown that the introduction of heterogenous agents could stabilize the economy, by ruling out indeterminacy and consequent fluctuations. The effect of heterogeneity on dynamics is unambiguously established, but, unfortunately, their findings rest on

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[1]For a survey, the reader is referred to Benhabib and Farmer (1999).

the assumption of exogenous prices. In order to account for endogenous prices, more recent papers have studied heterogeneity of infinite-lived agents in a general equilibrium framework. Ghiglino and Olszak-Duquenne (2001), Ghiglino (2005) and Ghiglino and Venditti (2005) focus on the role of consumers' diversity in the occurrence of optimal cycles, whereas Ghiglino and Sorger (2002) and Ghiglino and Olszak-Duquenne (2005) are mainly concerned with its impact on indeterminacy.[2] All these papers prove that heterogeneity, often thought as wealth inequality, matters for endogenous fluctuations, but they don't show how. The transmission mechanism from inequalities to fluctuations lies into a black-box.

Similar conclusions hold in the overlapping generations literature. For instance, Ghiglino and Tvede (1995) study an exchange economy with many consumers and commodities. They find an effect of endowments on the cyclical properties of the model, but still provide no clear interpretation about the mechanism.

In line with Ghiglino and Tvede (1995), we develop an overlapping generations model, but we make an effort to know how heterogeneity affects the conditions for indeterminacy and endogenous cycles, to describe the chain of variables and their co-movements. In contrast to them, we don't care about wealth inequalities, in order to focus on the heterogeneity in consumers' preferences, and we account for capital accumulation and elastic labor supply.

We consider a competitive economy, where consumers live two periods, supply labor when young, but consume during the whole life span. Preferences are non-separable in the current and future consumption, but separable in leisure. We introduce heterogeneity, taking into account two consumers' types, who differ by their intertemporal substitution in consumption and their propensity to save.

Our main concern is to understand whether or not differences in tastes play any role in dynamics and, above all, why. In this connection, we study, first, the local dynamics of a constant returns to scale economy. Results are generalized in the second part of the paper, where a labor externality in production makes the social returns increasing.[3]

We find that a degree of heterogeneity rules out indeterminacy, if returns to scale are constant and labor supply highly elastic. It is known that one of the main channels for indeterminacy in overlapping generations economies, goes through the effect of the future real interest rate on labor supply. Keeping in mind this remark, we prove that, in many cases, the introduction of heterogeneous preferences reduces the elasticity of labor supply with respect to the future interest rate: In fact, such an heterogeneity decreases the range of para-

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[2] In Woodford (1986), Becker and Foias (1994) and Sorger (1994), among others, time preference heterogeneity makes impatient agents financially constrained and indeterminacy possible.

[3] Externalities in capital don't promote the occurrence of endogenous fluctuations in overlapping generations economies with consumption in both periods (Cazzavillan and Pintus (2004b)), whereas labor externalities actually matters as stressed by Lloyd-Braga, Nourry and Venditti (2005).

meters compatible with indeterminacy (elasticity of labor supply, capital-labor substitution). We conclude that it stabilizes economic fluctuations.

However, when the returns to scale are constant, the emergence of endogenous fluctuations requires at least two restrictive conditions: A weak substitution between capital and labor[4] and a high propensity to save.[5] Lloyd-Braga, Nourry and Venditti (2005) prove that these requirements are no longer needed in presence of productive labor externalities. The section we devote to the increasing returns, is close to theirs. On the one side is more general, since we account for heterogeneous preferences. On the other side it is less general, since we restrict the analysis, for tractability reasons, to a Cobb-Douglas technology. Nevertheless, the new conclusions reinforces the old ones, obtained in the constant returns to scale economy: Indeterminacy becomes less likely under higher degrees of heterogeneity, because, within a wide parameter range, the elasticity of labor supply with respect to the future real interest rate reduces. Indeed, heterogeneity decreases the range of the elasticity of labor supply with respect to real wage and indeterminacy requires higher increasing returns. We further notice that, as under constant returns, indeterminacy no more occur for a sufficiently elastic labor supply.

Summing up, we show, within an overlapping generation framework, that heterogeneity in consumers' preferences stabilizes expectations-driven fluctuations, by shrinking the parameter range of indeterminacy. Hence, our main conclusion agrees with Herrendorf, Waldmann and Valentiny (2000), but, in contrast to them, arises in a dynamic general equilibrium model.

The paper is organized as follows. In the next section, we present the model and define the intertemporal equilibrium. Section 3 is devoted to the existence of a steady state. In section 4, we analyze local dynamics, considering, first, constant returns and, second, increasing returns. Concluding remarks are provided in section 5, while computational details are gathered in the Appendix.

2 The Model

We present a discrete time overlapping generations model with capital accumulation and consumption in both periods. Markets are supposed to be perfectly competitive. For simplicity, the heterogeneity we take in account, does not concern the production side, but only the consumption one. In other terms, a unique final good is produced by a representative firm by means of a constant returns to scale technology involving capital k_{t-1} and labor l_t. The production function is also affected by aggregate labor externalities $\psi(l)$.[6] The amount of final good y_t, yielded at period t, is given by $y_t = A\psi(l_t) f(a_t) l_t$, where $A > 0$

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[4] Unfortunately, this assumption is not in accordance with some recent empirical studies: see, in particular, Duffy and Papageorgiou (2000).

[5] This condition has been criticized, for example, by Cazzavillan and Pintus (2004a).

[6] Capital externalities are not needed: As shown in Cazzavillan and Pintus (2004b) and Lloyd-Braga, Nourry and Venditti (2005), they don't promote endogenous fluctuations in an overlapping generations model with consumption in both periods.

is a scaling parameter, f and $a_t \equiv k_{t-1}/l_t$ represent, respectively, the intensive production function and the capital intensity. We further assume:

Assumption 1 *The production function $f(a)$ is continuous for $a \geq 0$, positive-valued and differentiable, as many times as needed, for $a > 0$, with $f'(a) > 0 > f''(a)$.*

The externality function $\psi(l)$ is continuous for $l \in \mathbf{R}_+$, positive-valued and differentiable as many times as needed for $l \in \mathbf{R}_+$. Moreover, $\varepsilon_\psi(l) \equiv \psi'(l)\, l/\psi(l) \geq 0$.

Profit maximization determines the real interest rate r and the real wage w as follows:

$$r_t = A\psi(l_t)\rho(a_t) \equiv r(k_{t-1}, l_t)$$
$$w_t = A\psi(l_t)\omega(a_t) \equiv w(k_{t-1}, l_t)$$

with, respectively, $\rho(a) = f'(a)$ and $\omega(a) = f(a) - af'(a)$. Two identities, of interest in the sequel, link ρ and ω with the capital share on total income $s(a) \equiv f'(a)\, a/f(a) \in (0,1)$, and the elasticity of capital-labor substitution $\sigma(a) > 0$:[7] $a\rho'(a)/\rho(a) = -[1 - s(a)]/\sigma(a)$ and $a\omega'(a)/\omega(a) = s(a)/\sigma(a)$.

Each agent lives two periods, supplies labor only in the first period of life, but consumes in both periods. He belongs to a type $i = 1, 2$. Preferences of type i are rationalized by a utility function non-separable in consumption of both periods, but separable in consumption and labor:

$$U_i(c_{i1t}, c_{i2t+1}) - B_i v_i(l_{it}/B_i) \tag{1}$$

where c_{i1t} (c_{i2t+1}) is the consumption during the first (second) period of life and l_{it} is the labor supply. $B_i > 0$ is a scaling parameter. The share of type 1 agents in a generation is constant and denoted by $\lambda \in [0, 1]$. We assume, for simplicity, no population growth. The properties of the utility functions are summarized as follows:[8]

Assumption 2 *The function $U_i(x_1, x_2)$ is continuous for $x_1 \geq 0$ and $x_2 \geq 0$ and has continuous derivatives of every required order for $x_1 > 0$ and $x_2 > 0$. Moreover, $U_i(x_1, x_2)$ is increasing in x_1 and x_2, strictly quasi-concave, homogeneous of degree one and such that the indifference curves don't cross the axes.*

The function $v_i(l_i/B_i)$ is continuous for $0 \leq l_i \leq \bar{\bar{l}}_i$ and has continuous derivatives of every required order for $0 < l_i < \bar{\bar{l}}_i$, where $1 < \bar{l}_i \leq +\infty$ is the labor endowment. Furthermore, we assume that $v_i'(l_i/B_i) > 0$, $v_i''(l_i/B_i) > 0$ and $\lim_{l_i \to \bar{l}_i} v_i'(l_i/B_i) = +\infty$.

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[7] These formulas are easily obtained, by noticing that $1/\sigma(a) = a\omega'(a)/\omega(a) - a\rho'(a)/\rho(a)$ and $\omega'(a) = -a\rho'(a)$.

[8] Similar preferences have been used, among others, by d'Aspremont, Dos Santos Ferreira and Gérard-Varet (1995), Seegmuller (2004), Lloyd-Braga, Nourry and Venditti (2005).

In the youth, the labor income $w_t l_{it}$ is consumed (c_{i1t}) and saved (k_{it}). In the old age, the capital income $r_{t+1} k_{it}$ is consumed (c_{i2t+1}). Thereby, a type i consumer faces two budget constraints

$$c_{i1t} + k_{it} \leq w_t l_{it} \tag{2}$$

$$c_{i2t+1} \leq r_{t+1} k_{it} \tag{3}$$

Maximizing (1) under (2) and (3), the agent finds the optimal consumption and saving:

$$\frac{U_{i1}\left(c_{i1t}, c_{i2t+1}\right)}{U_{i2}\left(c_{i1t}, c_{i2t+1}\right)} = r_{t+1} \tag{4}$$

where $U_{ij} \equiv \partial U_i\left(x_1, x_2\right) / \partial x_j$, or, more explicitly,

$$c_{i1t} = \alpha_i\left(r_{t+1}\right) w_t l_{it} \tag{5}$$

$$c_{i2t+1} = \left(1 - \alpha_i\left(r_{t+1}\right)\right) r_{t+1} w_t l_{it} \tag{6}$$

$$k_{it} = \left(1 - \alpha_i\left(r_{t+1}\right)\right) w_t l_{it} \tag{7}$$

where $\alpha_i\left(r_{t+1}\right) \in (0,1)$ is the propensity to consume when young, while $1 - \alpha_i\left(r_{t+1}\right)$ is the propensity to save (see the Appendix). We define the consumptions ratio as a function H_i of the interest rate:

$$\frac{c_{i1t}}{c_{i2t+1}} = \frac{\alpha_i\left(r_{t+1}\right)}{\left[1 - \alpha_i\left(r_{t+1}\right)\right] r_{t+1}} \equiv H_i\left(r_{t+1}\right)$$

The elasticity of intertemporal substitution η_i is simply the elasticity of H_i in absolute value:

$$\eta_i\left(r_{t+1}\right) = -\frac{H_i'\left(r_{t+1}\right) r_{t+1}}{H_i\left(r_{t+1}\right)} = 1 - \frac{\alpha_i'\left(r_{t+1}\right) r_{t+1}}{\alpha_i\left(r_{t+1}\right)\left(1 - \alpha_i\left(r_{t+1}\right)\right)} \tag{8}$$

We observe that α_i is increasing for $0 < \eta_i < 1$ (intertemporal complementarity), decreasing for $\eta_i > 1$ (intertemporal substitutability) and constant for $\eta_i = 1$.

Replacing (5) and (6) into the function $U_i\left(c_{i1t}, c_{i2t+1}\right)$, we obtain the optimal sub-utility level: $U_i^*\left(r_{t+1}\right) \equiv U_i\left(\alpha_i\left(r_{t+1}\right), \left[1 - \alpha_i\left(r_{t+1}\right)\right] r_{t+1}\right)$. The elasticity of U_i^* is given by $r_{t+1} U_i^{*\prime}\left(r_{t+1}\right) / U_i^*\left(r_{t+1}\right) = 1 - \alpha_i \in (0,1)$ (see the Appendix).

Eventually, agent's arbitrage between consumption and leisure is computed.

$$U_i^*\left(r_{t+1}\right) w_t = v_i'\left(l_{it}/B_i\right) \tag{9}$$

with $i = 1, 2$. The labor supply increases in the real wage with elasticity $1/\varepsilon_{v_i}\left(l_i/B_i\right)$, where $\varepsilon_{v_i}\left(l_i/B_i\right) \equiv \left(l_i/B_i\right) v_i''\left(l_i/B_i\right) / v_i'\left(l_i/B_i\right) > 0$, under Assumption 2.

Propensities $\alpha_i\left(r\right)$'s and elasticities $\eta_i\left(r\right)$'s and $\varepsilon_{v_i}\left(l_i/B_i\right)$'s sum up all the relevant information to preferences and, therefore, to consumers' heterogeneity. Parameters $\alpha_i, \eta_i, \varepsilon_{v_i}$ solely matter in order to study the dynamic effects of heterogeneity.[9] We eliminate r_{t+1} from equations (9) to obtain:

$$\frac{v_2'\left(l_{2t}/B_2\right)}{w_t} = U_2^* \circ U_1^{*-1}\left(\frac{v_1'\left(l_{1t}/B_1\right)}{w_t}\right) \tag{10}$$

[9] The argument is developed in the section devoted to local dynamics.

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Equation (10) enables us[10] to define the labor supply l_{2t} as a function $l_{2t} \equiv h(k_{t-1}, l_{1t})$. The aggregate capital is given by $k_t = \lambda k_{1t} + (1-\lambda) k_{2t}$ and equilibrium dynamics are defined by sequences $\{k_{t-1}, l_{1t}\}_{t=1}^{\infty}$ that meet conditions (7) and (9).

$$
\begin{aligned}
k_t &= (\lambda l_{1t} [1 - \alpha_1 (r(k_t, l_{t+1}))] \\
&\quad + (1-\lambda) h(k_{t-1}, l_{1t}) [1 - \alpha_2 (r(k_t, l_{t+1}))]) w(k_{t-1}, l_t) \quad (11) \\
v_1'(l_{1t}/B_1) &= U_1^*(r(k_t, l_{t+1})) w(k_{t-1}, l_t) \quad (12)
\end{aligned}
$$

where, now, the aggregate supply $l_t = \lambda l_{1t} + (1-\lambda) h(k_{t-1}, l_{1t})$ only depends on k_{t-1} and l_{1t}.

We notice that the capital k_{t-1} is the only predetermined variable of the two-dimensional dynamic system (11)-(12). In order to study the local dynamics, we first establish the existence of a steady state and, then, we linearize the system in a neighborhood.

3 Steady State

Following Cazzavillan, Lloyd-Braga and Pintus (1998), we prove the existence of a normalized steady state $(k, l_1, l_2) = (1, 1, 1)$ by choosing appropriately the scaling parameters $A, B_1, B_2 > 0$.[11] Assume first that $(k, l_1) = (1, 1)$. If

$$
\lim_{x \to 0} v_2'(x) < w(1,1) U_2^* \circ U_1^{*-1} \left(\frac{v_1'(1/B_1)}{w(1,1)} \right) < \lim_{x \to +\infty} v_2'(x) \quad (13)
$$

there is a unique $B_2 > 0$ which ensures $l_2 = h(1,1) = 1$, for every A and B_1. Second, keeping in mind this result, we determine A and B_1 such that $(k, l_1) = (1, 1)$ is a steady state of (11)-(12). In this respect, A has to satisfy

$$
g(A) = 1/(\psi(1)[f(1) - f'(1)]) \quad (14)
$$

where

$$
g(A) \equiv A(\lambda[1 - \alpha_1 (A\psi(1) f'(1))] + (1-\lambda)[1 - \alpha_2 (A\psi(1) f'(1))]) \quad (15)
$$

is an increasing function.[12] Since $\lim_{A \to 0} g(A) = 0$, there is a unique solution A to equation (14) if and only if:

$$
\lim_{A \to +\infty} g(A) > 1/(\psi(1)[f(1) - f'(1)]) \quad (16)
$$

Moreover, B_1 is the unique solution of $v_1'(1/B_1) = U_1^*(r(1,1)) w(1,1)$, if

$$
\lim_{x \to 0} v_1'(x) < U_1^*(r(1,1)) w(1,1) < \lim_{x \to +\infty} v_1'(x) \quad (17)
$$

The result is summed up in the next proposition.

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[10] In order to apply the implicit function theorem, we need $\varepsilon_{v_2} \neq (1-\lambda)(s/\sigma - \varepsilon_\psi)[(1-\alpha_2)/(1-\alpha_1) - 1]$.

[11] The steady state value l_2 is also normalized in order to obtain $l = \lambda l_1 + (1-\lambda) l_2 = 1$.

[12] See the Appendix.

Proposition 1 *Under Assumptions 1 and 2, there exists a steady state* $(k, l_1, l_2) = (1, 1, 1)$, *if and only if conditions (13), (16) and (17) hold, and* (A, B_1, B_2) *is the unique solution of the system*

$$
\begin{aligned}
v_1'\left(1/B_1\right) &= U_1^* \left(r\left(1,1\right)\right) w\left(1,1\right) \\
\frac{v_2'\left(1/B_2\right)}{w\left(1,1\right)} &= U_2^* \circ U_1^{*-1} \left(\frac{v_1'\left(1/B_1\right)}{w\left(1,1\right)}\right) \\
g\left(A\right) &= 1/\left(\psi\left(1\right)\left[f\left(1\right) - f'\left(1\right)\right]\right)
\end{aligned}
$$

where $g\left(A\right)$ *is given by (15).*

In the rest of the paper, we assume this proposition to hold and no longer refer to it.

4 Local Dynamics

In order to know how heterogeneity in consumers' preferences affects the occurrence of endogenous fluctuations, we study the local dynamics. The main finding of the section is that the introduction of a distance between two consumers' profiles can stabilize economic fluctuations, by reducing the range of parameters compatible with indeterminacy.

Local analysis consists in differentiating dynamic system (11)-(12) in a neighborhood of the steady state $(k, l_1, l_2) = (1, 1, 1)$. In what follows, we define $s \equiv s\left(1\right)$, $\sigma \equiv \sigma\left(1\right)$, $\varepsilon_\psi \equiv \varepsilon_\psi\left(1\right)$, $\alpha_i \equiv \alpha_i\left(\left(A\psi\left(1\right)f'\left(1\right)\right)\right)$ and $\eta_i \equiv \eta_i\left(A\psi\left(1\right)f'\left(1\right)\right)$, while ε_{v_i} will be the elasticity of $v_i'\left(l_i/B_i\right)$ evaluated at the steady state. The local dynamics are represented by $(dk_t/k, dl_{1t+1}/l_1)^T = J\,(dk_{t-1}/k, dl_{1t}/l_1)^T$, where J denotes the Jacobian matrix evaluated at the steady state. The determinant D and the trace T of the Jacobian matrix are given by[13]:

$$
\begin{aligned}
D &= s\left(\varepsilon_{v_1}\varepsilon_{v_2}A_1/A_0 + 1/\left[\left(1 - s + \sigma\varepsilon_\psi\right)A_1\right]\right) > 0 \quad &(18) \\
T &= A_0^{-1}\Big[\lambda\Big(\left[\left(1 - \alpha_1 - \sigma\right)\varepsilon_\psi + s\right]A_1 + \varepsilon_\psi A_2 + \left(1 - s\right)\left(1 - \alpha_1\right)^2\Big)\varepsilon_{v_2} \\
&\quad + \left(1 - \lambda\right)\Big(\left[\left(1 - \alpha_2 - \sigma\right)\varepsilon_\psi + s\right]A_1 + \varepsilon_\psi A_2 + \left(1 - s\right)\left(1 - \alpha_2\right)^2\Big)\varepsilon_{v_1} \\
&\quad + \varepsilon_{v_1}\varepsilon_{v_2}\left[\sigma A_1 - \left(1 - s\right)A_2\right] - \lambda\left(1 - \lambda\right)\left(\alpha_1 - \alpha_2\right)^2\varepsilon_\psi\Big] \quad &(19)
\end{aligned}
$$

where

$$
\begin{aligned}
A_0 &\equiv \left[\lambda\left(1 - \alpha_1\right)\varepsilon_{v_2} + \left(1 - \lambda\right)\left(1 - \alpha_2\right)\varepsilon_{v_1}\right]\left(1 - s + \sigma\varepsilon_\psi\right)A_1 \\
A_1 &\equiv \lambda\left(1 - \alpha_1\right) + \left(1 - \lambda\right)\left(1 - \alpha_2\right) \\
A_2 &\equiv \lambda\alpha_1\left(1 - \alpha_1\right)\left(1 - \eta_1\right) + \left(1 - \lambda\right)\alpha_2\left(1 - \alpha_2\right)\left(1 - \eta_2\right) \quad (20)
\end{aligned}
$$

In what follows, we evaluate the characteristic polynomial at $-1, 0, 1$ and we study the sign of $P\left(1\right) = 1 - T + D$ and $P\left(-1\right) = 1 + T + D$. The steady state is

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[13]More details are provided in the Appendix.

a sink and, therefore, it is locally indeterminate, when $P(1) > 0$, $P(-1) > 0$ and $D < 1$. A flip bifurcation generically occurs, when $P(-1)$ crosses 0, whereas a Hopf bifurcation generically emerges, when, for $P(1) > 0$ and $P(-1) > 0$, D crosses 1.

Eventually, for simplicity, we focus on the case with no heterogeneity in labor disutility[14]:

Assumption 3 $\varepsilon_{v_1} = \varepsilon_{v_2} \equiv \varepsilon_v$.

Then, the determinant and the trace simplify:

$$D = \frac{s}{1 - s + \sigma \varepsilon_\psi} \frac{1 + \varepsilon_v}{A_1} > 0 \tag{21}$$

$$T = \frac{1}{(1 - s + \sigma \varepsilon_\psi) A_1^2} \Big([A_1 (A_1 - \sigma) + A_2] \varepsilon_\psi - \lambda (1 - \lambda) (\alpha_2 - \alpha_1)^2 \varepsilon_\psi / \varepsilon_v \tag{22}$$
$$+ sA_1 + (1 - s) \Big[\lambda (1 - \alpha_1)^2 + (1 - \lambda) (1 - \alpha_2)^2 \Big] + [\sigma A_1 - (1 - s) A_2] \varepsilon_v \Big)$$

In the next section, we will prove that more heterogeneity in consumers' preferences shrinks the range of parameters for indeterminacy. The cases of constant and increasing returns to scale are progressively studied.

4.1 Constant Returns $(\varepsilon_\psi = 0)$

Under constant returns to scale, the production sector no longer benefits from externalities, or, more formally, $\varepsilon_\psi = 0$. In order to understand how preferences heterogeneity affects the occurrence of indeterminacy and cycles, we compare it with the case with no heterogeneity. Homogeneous preferences require $\alpha_1 = \alpha_2 \equiv \alpha$ and $\eta_1 = \eta_2 \equiv \eta$, or, equivalently, $A_1 = 1 - \alpha$ and $A_2 = \alpha (1 - \alpha) (1 - \eta)$. The determinant and the trace simplify further:

$$D = \frac{s}{1 - s} \frac{1 + \varepsilon_v}{1 - \alpha} \tag{23}$$

$$T = 1 + D - \varepsilon_v \frac{s + \alpha (1 - \eta)(1 - s) - \sigma}{(1 - \alpha)(1 - s)} \tag{24}$$

According to consensual empirical estimates, the capital share in income is supposed to be smaller than one half. This entails, in turn, a moderate propensity to save.

Assumption 4 $s < 1/2$ and $1 - \alpha > s/(1 - s)$.

We are now enabled to characterize the dynamics, using equations (23) and (24).

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[14] Recently, Bosi and Seegmuller (2005) have characterized the role of heterogeneity in labor disutility on the occurrence of local indeterminacy. They have shown that, in an overlapping generations model with consumption only in the second period of life, a representative agent turns out to be an average representation of heterogenous workers.

Proposition 2 *(No heterogeneity and constant returns) Let*

$$\varepsilon_{v_H} \quad \equiv \quad \frac{(1-\alpha)(1-s)-s}{s} \tag{25}$$

$$\varepsilon_{v_F} \quad \equiv \quad 2\frac{(1-\alpha)(1-s)+s}{\alpha(1-\eta)(1-s)-s-\sigma} \tag{26}$$

$$\sigma_H \quad \equiv \quad \alpha(1-\eta)(1-s)-s\left[1+2\frac{(1-\alpha)(1-s)+s}{(1-\alpha)(1-s)-s}\right] \tag{27}$$

Under Assumption 4, the following generically holds.

1. *When $\eta \geq 1 - s/(\alpha(1-s))$ and $\sigma < s + \alpha(1-\eta)(1-s)$, the steady state is locally indeterminate for $\varepsilon_v < \varepsilon_{v_H}$.*

2. *When $\eta < 1 - s/(\alpha(1-s))$,*

    2.1 *if $\sigma < \sigma_H$, the steady state is locally indeterminate for $\varepsilon_v < \varepsilon_{v_F}$;*

    2.2 *if $\sigma_H < \sigma < s + \alpha(1-\eta)(1-s)$, the steady state is locally indeterminate for $\varepsilon_v < \varepsilon_{v_H}$.*

Moreover, when ε_v crosses the critical values ε_{v_F} and ε_{v_H}, the system undergoes, respectively, a flip and a Hopf bifurcation.

Proof. See the Appendix. ∎

Proposition 2 establishes that under Assumption 4, local indeterminacy occurs, if labor supply is sufficiently elastic and capital and labor are weak substitutes. In the particular case $\eta = 1$, savings don't depend on the real interest rate and indeterminacy simply requires $\sigma < s$.[15] When $\eta < 1$ (savings decrease in the real interest rate), endogenous fluctuations can emerge for higher values of the elasticity of capital-labor substitution, whereas when $\eta > 1$ (savings increase in the real interest rate), indeterminacy and cycles require a lower upper bound of this elasticity.

In order to provide an intuition, we notice that, with no heterogeneity in preferences, dynamics are shaped as follows:

$$k_t \quad = \quad [1-\alpha(r_{t+1})]\,w_t l_t \tag{28}$$

$$v'(l_t/B) \quad = \quad U^*(r_{t+1})\,w_t \tag{29}$$

Assume first that $\alpha = 0$ and $\eta = 1$. If consumers expect a higher future real interest rate, they increase the labor supply according to equation (29). When $\sigma < s$, labor income $w_t l_t$ decreases and, as a consequence, savings and capital stock lower as well, according to equation (28). Capital decrease raises the future real interest rate and makes expectations self-fulfilling.

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[15] See Reichlin (1986) and, more recently, Cazzavillan (2001) for similar conditions in overlapping generations model with no consumption in the first period.

We observe that two effects matter for indeterminacy: The impact of the expected real interest rate on labor supply $(dl_t/l_t)/(dr_{t+1}/r_{t+1}) = (1-\alpha)/\varepsilon_v$, and the effect of labor income on capital accumulation. The elasticity of labor supply $(1/\varepsilon_v)$ is required to be sufficiently large, while, when $\alpha > 0$ and $\eta = 1$, the propensity to save $1 - \alpha$ has to be high enough (Assumption 4).

The interpretation of the results is similar, when $\alpha > 0$ and $\eta \neq 1$, but we need to take in account the additional feedback on the propensity to save due to a final increase of the real interest rate. First, we consider the case $\eta < 1$. The final increase of r_{t+1} reduces $1 - \alpha(r_{t+1})$, which reinforces in turn the negative effect of labor income on savings. Hence, capital can decrease and the future real interest rate can increase for elasticities of capital-labor substitution slightly greater than the capital share in income. On the contrary, $\eta > 1$ means that savings increase with respect to future real interest rate. In this case, the occurrence of expectations-driven fluctuations requires a more restrictive condition on factors substitutability.

We now introduce heterogeneity in consumers' preferences, by setting $\alpha_1 \neq \alpha_2$ and $\eta_1 \neq \eta_2$. Determinant and trace become

$$
D = \frac{s}{1-s}\frac{1+\varepsilon_v}{A_1} > 0
$$

$$
T = \frac{s}{1-s}\frac{1}{A_1} + \frac{\lambda(1-\alpha_1)^2 + (1-\lambda)(1-\alpha_2)^2}{A_1^2} + \frac{\varepsilon_v}{A_1}\left(\frac{\sigma}{1-s} - \frac{A_2}{A_1}\right)
$$

Restrictions close to those in Assumption 4 are required, in order to study the emergence of endogenous fluctuations and to compare the new findings with the previous ones:

Assumption 5 $s < 1/2$, $A_1 > s/(1-s)$.

Results can be organized within a proposition.

Proposition 3 *(Heterogeneity and constant returns) Let*

$$
\varepsilon_{v_H} \equiv A_1(1-s)/s - 1 \tag{30}
$$

$$
\varepsilon_{v_T} \equiv \frac{1-s}{A_1}\frac{\lambda(1-\lambda)(\alpha_2-\alpha_1)^2}{\sigma_T - \sigma} \tag{31}
$$

$$
\varepsilon_{v_F} \equiv \frac{1-s}{A_1}\frac{(1-\alpha_1)^2\lambda + (1-\alpha_2)^2(1-\lambda) + A_1(A_1 + 2s/(1-s))}{\sigma_F - \sigma} \tag{32}
$$

$$
\sigma_{H_1} \equiv \sigma_F - \frac{s(1-s)}{A_1}\frac{\lambda(1-\alpha_1)^2 + (1-\lambda)(1-\alpha_2)^2 + A_1(A_1 + 2s/(1-s))}{A_1(1-s) - s} \tag{33}
$$

$$
\sigma_{H_2} = \sigma_T - \frac{s(1-s)}{A_1}\frac{\lambda(1-\lambda)(\alpha_2-\alpha_1)^2}{A_1(1-s) - s} \tag{34}
$$

with

$$
\sigma_T \equiv (1-s)A_2/A_1 + s \tag{35}
$$

$$
\sigma_F \equiv (1-s)A_2/A_1 - s \tag{36}
$$

Under Assumption 5, the steady state is locally indeterminate, if and only if

1. $\sigma < \sigma_{H_1}$ and $\varepsilon_{v_T} < \varepsilon_v < \varepsilon_{v_F}$, or

2. $\sigma_{H_1} < \sigma < \sigma_{H_2}$ and $\varepsilon_{v_T} < \varepsilon_v < \varepsilon_{v_H}$.

Moreover, when ε_v crosses the value ε_{v_F}, a flip bifurcation emerges, whereas when ε_v crosses the value ε_{v_H}, a Hopf bifurcation emerges.

Proof. See the Appendix. ∎

This proposition allows us to understand how heterogeneity affects local dynamics. First, we notice that indeterminacy no longer occurs for $\lambda \neq 0, 1$ and $\alpha_1 \neq \alpha_2$, when ε_v is sufficiently close to 0, *i.e.* for a labor supply sufficiently or even infinitely elastic.

Second, to focus on the dynamic effects of heterogeneous propensities to consume, we assume, for simplicity $\eta_1 = \eta_2 = 1$, *i.e.* $A_2 = 0$. In this case, σ_{H_1} becomes strictly negative and only configuration 2 of Proposition 3 applies. Moreover we assume, without loss of generality, $\alpha_2 > \alpha_1$. Let, now, $\alpha \equiv (\alpha_1 + \alpha_2)/2$ be the midpoint and $\varepsilon \equiv \alpha_2 - \alpha_1$ be the heterogeneity measure. Then, we have:

$$\varepsilon_{v_T} = \frac{1-s}{s-\sigma} \frac{\lambda(1-\lambda)\varepsilon^2}{A_1}$$

$$\sigma_{H_2} = s - \frac{(1-s)s}{A_1(1-s)-s} \frac{\lambda(1-\lambda)\varepsilon^2}{A_1}$$

while ε_{v_H} is still given by (30), with now $A_1 = 1 - \alpha + \varepsilon(\lambda - 1/2)$.

Evidently, heterogeneity in consumers' preferences appears in the model as soon as ε increases from 0. Moreover, we say that heterogeneity stabilizes fluctuations, if it shrinks the range of parameters compatible with indeterminacy. In our case, this happens, if $\lambda \leq 1/2$, because ε_{v_T} increases. Moreover, when $\lambda < 1/2$, ε_{v_H} and σ_{H_2} decrease as long as ε becomes strictly positive. Heterogeneity stabilizes fluctuations due to self-fulfilling expectations for a large range of parameters ε and λ.

To provide more intuition for indeterminacy, we follow the interpretative lines of Proposition 2.[16] As seen above, one of the main channel goes through the effect of the expected real interest rate on labor supply. For each kind of agent, the elasticity of labor supply in this rate is equal to $(dl_{it}/l_{it})/(dr_{t+1}/r_{t+1}) = (1-\alpha_i)/\varepsilon_v$. Since $l_t = \lambda l_{1t} + (1-\lambda)l_{2t}$, we have $(dl_t/l_t)/(dr_{t+1}/r_{t+1}) = A_1/\varepsilon_v$ at the aggregate level. Therefore, expectations-driven fluctuations appear less likely under heterogeneity, whenever A_1 is smaller than $1-\alpha$, which is actually the meaning of inequality $\lambda < 1/2$.

However, the occurrence of endogenous fluctuations under constant returns to scale requires at least two demanding conditions. On the one side, we need a

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[16] For simplicity, we only focus on $\lambda < 1/2$.

sufficiently weak substitution between capital and labor, which is not in accordance with empirical results (see Duffy and Papageorgiou, 2000). As shown in Lloyd-Braga, Nourry and Venditti (2005) and discussed in the next section, this condition is no longer required as soon as labor externalities are introduced in the production sector. On the other side, a too high level of propensity to save is criticizable.[17] However, as stressed also by Lloyd-Braga, Nourry and Venditti (2005), under productive labor externalities, this assumption is no more needed. Thus, now, we have two good reasons to study the case, where returns to scale are increasing through the existence of labor externalities.

4.2   Increasing Returns $(\varepsilon_\psi > 0)$

Henceforth, we assume $\varepsilon_\psi > 0$, in order to make returns increasing. We aim at shedding light on the role of heterogeneous preferences on local dynamics and, as above, we start with the homogeneous case to better appreciate, subsequently, the impact of such an heterogeneity.

Here, homogeneity means the same preferences for everybody: Formally, $\alpha_1 = \alpha_2 \equiv \alpha$ and $\eta_1 = \eta_2 \equiv \eta$, which entail $A_1 = 1-\alpha$ and $A_2 = \alpha (1-\alpha)(1-\eta)$. The following assumption allows us to simplify computations.

Assumption 6 $s < 1/2$, $\eta = \sigma = 1$, $\max\{1/2, (1-2s)/(1-s)\} < \alpha < (1+2s)/(1+3s)$.

First, according to the empirical literature, the capital share in income is kept smaller than one half. Second, we focus on the case with unit elasticities of capital-labor and intertemporal substitution, *i.e.* on Cobb-Douglas technology and preferences. Finally, we assume the propensity to save not too high and, in any case, smaller than one half, but not too close to 0.[18] Hence, the model presented here, can be viewed as an extension of Lloyd-Braga, Nourry and Venditti (2005) who, more generally, discuss the emergence of local indeterminacy through bifurcations for whatever level of capital-labor substitution, but, less generally, don't deal with heterogeneity, as we do. Expressions (21) and (22) simplify:

$$D = \frac{s}{1-s+\varepsilon_\psi}\frac{1+\varepsilon_v}{1-\alpha} > 0 \qquad (37)$$

$$T = \frac{1-\alpha(1-s+\varepsilon_\psi)+\varepsilon_v}{(1-\alpha)(1-s+\varepsilon_\psi)} \qquad (38)$$

to obtain:

---

[17]See, among others, Cazzavillan and Pintus (2004a).

[18]If we set, as usually, $s = 1/3$, the double inequality of Assumption 6 becomes $1/2 < \alpha < 5/6$. **We observe that** $\alpha > (1-2s)/(1-s)$ **is equivalent to** $1-\alpha < s/(1-s)$**, an opposite condition to that met in Assumption 4.**

Proposition 4 *(No heterogeneity and increasing returns)* Let

$$\varepsilon_{\psi_H} \equiv s(2-\alpha)/(1-\alpha) - 1 \tag{39}$$

$$\varepsilon_{\psi_F} \equiv s - 1 - (1+s)/(1-2\alpha) \tag{40}$$

$$\varepsilon_{v_H} \equiv (1-\alpha)(1-s+\varepsilon_\psi)/s - 1 \tag{41}$$

Under Assumption 6 and inequalities $\varepsilon_{\psi_H} < \varepsilon_\psi < \varepsilon_{\psi_F}$, the steady state is locally indeterminate, provided that $\varepsilon_v < \varepsilon_{v_H}$. When ε_v crosses the value ε_{v_H}, a Hopf bifurcation occurs.

Proof. See the Appendix. ∎

Proposition 4 shows that local indeterminacy and cycles can occur under small increasing returns, when propensity to save is weak and production factors are substitutable.[19] In this respect, endogenous fluctuations actually arise under mild conditions in our overlapping generations economy.

In order to provide an intuition about the possibility of oscillating trajectories, we follow Lloyd-Braga, Nourry and Venditti (2005). With no heterogeneity, dynamics are defined by equations (28) and (29). Assume, now, that the economy is at the steady state and we allow a deviation through an increase of k_{t-1}. According to equation (28), the shock raises the labor income $w_t l_t$ and consequently the savings k_t, which, in turn, lower the future real interest rate r_{t+1}. This decrease of the interest rate has a negative feedback effect on the current labor supply l_t. According to equation (29), we have, indeed, in elasticity terms: $(dl_t/l_t)/(dr_{t+1}/r_{t+1}) = (1-\alpha)/\varepsilon_v > 0$. Eventually, a lowering labor supply entails a similar negative effect on the labor income, according to the relevant elasticity $[d(w_t l_t)/(w_t l_t)]/(dl_t/l_t) = 1-s+\varepsilon_\psi > 0$, and on the capital stock k_t. An increase in k_{t-1} is thus followed by a decrease in k_t: Oscillatory dynamics arise, whenever the negative feedback dominates.

The joint product of these elasticities reveals that feedback dominance and oscillations happen under a weak propensity to save (small $1-\alpha$) and elastic labor supply (small ε_v), coupled with increasing returns ($\varepsilon_\psi > 0$).

We now introduce preferences heterogeneity in the framework with externalities. In order to compare the new results with the findings obtained under constant returns, we assume:

Assumption 7 $1/3 \leq s < 1/2$, $\sigma = 1$, $\eta_1 = \eta_2 = 1$, $s/(1+3s) < A_1 < 1/2$.

These conditions are quite similar to Assumption 6. In particular the elasticities of intertemporal substitution of both the agents' types are equal to one.[20] However, for simplicity, we set a lower bound for the capital share (one third). The last double inequality in Assumption 7 is identical to the corresponding one

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[19] If $\epsilon_\psi > \epsilon_{\psi_F}$, there is room for local indeterminacy. However, since increasing returns are required to be high, in contrast with the empirical studies, we have omitted this case in Proposition 4.

[20] According to (20), we have $A_2 = 0$.

in Assumption 6, where, now, A_1 replaces $1 - \alpha$ and, since $s \geq 1/3$, we have $s/(1-s) \geq 1/2$.

Under Assumptions 3 and 7, the heterogeneity in consumers' preferences is captured by the difference between α_1 and α_2. Determinant and trace become

$$D = \frac{s}{1 - s + \varepsilon_\psi} \frac{1 + \varepsilon_v}{A_1} > 0 \tag{42}$$

$$T = \frac{1}{A_1^2 (1 - s + \varepsilon_\psi)} \Big(A_1 [s + \varepsilon_v + (A_1 - 1)\varepsilon_\psi] - \lambda (1 - \lambda) (\alpha_2 - \alpha_1)^2 \varepsilon_\psi / \varepsilon_v$$
$$+ (1 - s) \left[\lambda (1 - \alpha_1)^2 + (1 - \lambda) (1 - \alpha_2)^2 \right] \Big) \tag{43}$$

Conditions for indeterminacy and endogenous cycles are now provided.

Proposition 5 *(Heterogeneity and increasing returns) Define*

$$\varepsilon_{\psi_H} \equiv s(1 + A_1)/A_1 - 1 \tag{44}$$

$$\varepsilon_{\psi_F} \equiv 2[s + (1 - s)A_1]/(1 - 2A_1) \tag{45}$$

$$\varepsilon_{v_H} \equiv A_1 (1 - s + \varepsilon_\psi)/s - 1 \tag{46}$$

Let Assumption 7 hold and $\varepsilon_{\psi_H} < \varepsilon_\psi < \varepsilon_{\psi_F}$. If ε_ψ is sufficiently close to ε_{ψ_F} and A_1 to $1/2$, then the steady state is locally indeterminate, provided that $\varepsilon_{v_F} < \varepsilon_v < \varepsilon_{v_H}$, where ε_{v_F} is given by expression (58), detailed in the Appendix. Moreover, when ε_v crosses the critical values ε_{v_H} and ε_{v_F}, the system undergoes, respectively, a Hopf and a flip bifurcation.

Proof. See the Appendix. ∎

Proposition 5 gives conditions for indeterminacy and cycles under assumptions very similar to those met in the homogeneous case. Assumptions 6 and 7 are close enough and the two bounds ε_{ψ_H} and ε_{ψ_F} are, now, the same than in Proposition 4, provided that A_1 replaces $1 - \alpha$.

Under the heterogeneity hypothesis, as in the constant returns economy, indeterminacy no longer emerges with an infinitely elastic labor supply, whereas it could occur in the homogeneous case. Indeed, ε_{v_F} is strictly positive for $\lambda \neq 0, 1$ as long as α_1 and α_2 are different, whereas ε_{v_F} goes to 0, when the difference between α_1 and α_2 vanishes.

To understand better the role of heterogeneity on local indeterminacy, we assume without loss of generality, as done in section 4.1, that $\alpha_2 > \alpha_1$. As above, let $\alpha \equiv (\alpha_1 + \alpha_2)/2$ be the midpoint and $\varepsilon \equiv \alpha_2 - \alpha_1$ be the heterogeneity measure. We still have $A_1 = 1 - \alpha + \varepsilon (\lambda - 1/2)$. Heterogeneity in consumers' preferences appears in the model as soon as ε increases from 0.

Using the expressions for ε_{ψ_H}, ε_{ψ_F}, ε_{v_H}, one can show that, if $\lambda < 1/2$, ε_{ψ_H} increases, while ε_{ψ_F} and ε_{v_H} decrease. Hence, heterogeneity actually reduces the range of the elasticity of labor supply and the level of externalities compatible with indeterminacy. Moreover, the introduction of a degree of heterogeneity, makes the lower bound for increasing returns higher in order to get endogenous fluctuations.

We are allowed to conclude that, when consumers have heterogenous preferences and $\lambda \leq 1/2$, indeterminacy requires more restrictive conditions and, therefore, heterogeneity stabilizes expectations-driven fluctuations, as it does under constant returns to scale.[21]

Such a stabilizing power of heterogeneity can be intuitively enlightened. In line with the interpretation of Proposition 4, we stress that the emergence of oscillatory dynamics requires a sufficiently large feedback of the future real interest rate on the current labor supply. We know that the relevant elasticity $(dl_t/l_t)/(dr_{t+1}/r_{t+1})$ is equal to $(1-\alpha)/\varepsilon_v$, when agents are identical, whereas to A_1/ε_v, when they are heterogeneous (see the end of Section 4.1). Then endogenous fluctuations appear less likely under heterogeneity, provided that $A_1 < 1 - \alpha$. But this inequality is just equivalent to $\lambda < 1/2$ and justifies our ultimate result.

5 Conclusion

Herrendorf, Waldmann and Valentiny (2000) have shown that the introduction of heterogenous agents in a model, where prices are exogenous, rules out indeterminacy and, in this respect, stabilizes the economy. We have yielded a general equilibrium model in order to check the robustness of their conclusion. Constant and increasing returns to scale economies have been characterized within an overlapping generations framework with consumption in both periods and elastic labor supply.

Introducing heterogeneity under constant returns, indeterminacy no longer occurs when the labor supply is sufficiently or even infinitely elastic. Moreover, endogenous fluctuations appear under smaller ranges of factors substitution and elasticity of labor supply. Therefore, expectations-driven fluctuations emerge less likely.

As in the constant returns economy, under heterogeneity and increasing returns, indeterminacy no longer emerges with an infinitely elastic labor supply, whereas it could occur in the homogeneous case. Heterogeneity reduces the range of the elasticity of labor supply and the level of externalities compatible with indeterminacy. Moreover, the introduction of a degree of heterogeneity, makes the lower bound for increasing returns higher in order to get endogenous fluctuations. We are allowed to conclude that, when consumers have heterogenous preferences and the share of first type's agents in a generation is not too high, indeterminacy requires more restrictive conditions and, therefore, heterogeneity stabilizes expectations-driven fluctuations, as it does under constant returns to scale.

In short conclusion, we show the robustness of Herrendorf, Waldmann and Valentiny (2000) in a dynamic general equilibrium framework, by proving that heterogeneity stabilizes fluctuations due to animal spirits.

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[21] When $\lambda = 1/2$, heterogeneity stabilizes the expectations-driven fluctuations, ruling them out for highly elastic labor supply.

6 Appendix

The existence of $\alpha_i(r_{t+1})$

Equation (4) writes equivalently:

$$z_i(c_{i1t}/c_{i2t+1}) \equiv \frac{U_{i1}(c_{i1t}/c_{i2t+1}, 1)}{U_{i2}(1, c_{i2t+1}/c_{i1t})} = r_{t+1}$$

where z_i is a strictly decreasing function. Then z_i is invertible and $c_{i1t} = z_i^{-1}(r_{t+1})c_{i2t+1}$. Consequently, the propensity to consume when young $\alpha_i(r_{t+1})$ becomes: $\alpha_i(r_{t+1}) = r_{t+1}z_i^{-1}(r_{t+1}) / [1 + r_{t+1}z_i^{-1}(r_{t+1})]$.

Elasticity of U_i^*

Under Assumption 2, the Euler identity applies and, jointly with the first order condition (4), gives

$$U_i = \alpha_i U_{i1} + (1 - \alpha_i) U_{i2} r_{t+1} = U_{i1} \tag{47}$$

Using (47) and still (4), we have:

$$
\begin{aligned}
U_i^{*\prime}(r_{t+1}) r_{t+1} &= [(U_{i1} - U_{i2}r_{t+1})\alpha_i' + U_{i2}(1 - \alpha_i)]r_{t+1} \\
&= (1 - \alpha_i) U_{i2} r_{t+1} = (1 - \alpha_i) U_{i1} = (1 - \alpha_i) U_i^*
\end{aligned}
$$

$g(A)$ is an increasing function

Using (8), we obtain the elasticity of g:

$$
\begin{aligned}
\frac{g'(A)A}{g(A)} &= 1 - \frac{\lambda\alpha_1(1 - \alpha_1)(1 - \eta_1) + (1 - \lambda)\alpha_2(1 - \alpha_2)(1 - \eta_2)}{\lambda(1 - \alpha_1) + (1 - \lambda)(1 - \alpha_2)} \\
&> 1 - \frac{\lambda\alpha_1(1 - \alpha_1) + (1 - \lambda)\alpha_2(1 - \alpha_2)}{\lambda(1 - \alpha_1) + (1 - \lambda)(1 - \alpha_2)} > 0
\end{aligned}
$$

since $\eta_i > 0$, for $i = 1, 2$.

Determinant and trace

First, we compute the elasticities ε_{hk} and ε_{hl_1} of $h(k, l_1)$ with respect to k and l_1. Totally differentiating (10), we obtain:

$$\varepsilon_{hk} = \frac{s(\alpha_2 - \alpha_1)}{\sigma(1 - \alpha_1)\varepsilon_{v_2} + (1 - \lambda)(\alpha_2 - \alpha_1)(s - \sigma\varepsilon_\psi)} \tag{48}$$

$$\varepsilon_{hl_1} = \frac{\sigma(1 - \alpha_2)\varepsilon_{v_1} - \lambda(\alpha_2 - \alpha_1)(s - \sigma\varepsilon_\psi)}{\sigma(1 - \alpha_1)\varepsilon_{v_2} + (1 - \lambda)(\alpha_2 - \alpha_1)(s - \sigma\varepsilon_\psi)} \tag{49}$$

Second, the Jacobian matrix, evaluated at the steady state, is given by $J = V^{-1}W$, where

$$V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \text{ and } W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \tag{50}$$

with

$$
\begin{aligned}
v_{11} &= \sigma A_1 + [(1 - s + \sigma \varepsilon_\psi)(1 - \lambda)\varepsilon_{hk} + s - 1] A_2 \\
v_{12} &= (1 - s + \sigma \varepsilon_\psi)[\lambda + (1 - \lambda)\varepsilon_{hl_1}] A_2 \\
v_{21} &= (1 - \alpha_1)[(1 - s + \sigma \varepsilon_\psi)(1 - \lambda)\varepsilon_{hk} + s - 1] \\
v_{22} &= (1 - \alpha_1)(1 - s + \sigma \varepsilon_\psi)[\lambda + (1 - \lambda)\varepsilon_{hl_1}] \quad\quad (51) \\
w_{11} &= sA_1 + [A_1(\sigma \varepsilon_\psi - s) + (1 - \alpha_2)\sigma](1 - \lambda)\varepsilon_{hk} \\
w_{12} &= [\lambda + (1 - \lambda)\varepsilon_{hl_1}](\sigma \varepsilon_\psi - s)A_1 + \sigma[\lambda(1 - \alpha_1) + (1 - \lambda)(1 - \alpha_2)\varepsilon_{hl_1}] \\
w_{21} &= (s - \sigma \varepsilon_\psi)(1 - \lambda)\varepsilon_{hk} - s \\
w_{22} &= \sigma \varepsilon_{v_1} + (s - \sigma \varepsilon_\psi)[\lambda + (1 - \lambda)\varepsilon_{hl_1}] \quad\quad (52)
\end{aligned}
$$

where A_1 and A_2 are given by (20). We compute $\det V$ and $\det W$, to obtain the determinant of the Jacobian: $\det J = \det W / \det V$. Using (48), (49), (51) and (52), we have

$$
\det V = \frac{\sigma^2 (1 - \alpha_1)[\lambda(1 - \alpha_1)\varepsilon_{v_2} + (1 - \lambda)(1 - \alpha_2)\varepsilon_{v_1}] A_1 (1 - s + \sigma \varepsilon_\psi)}{(1 - \alpha_1)\sigma \varepsilon_{v_2} + (1 - \lambda)(\alpha_2 - \alpha_1)(s - \sigma \varepsilon_\psi)} \quad (53)
$$

$$
\det W = \frac{\sigma^2 (1 - \alpha_1)[\lambda(1 - \alpha_1)\varepsilon_{v_2} + (1 - \lambda)(1 - \alpha_2)\varepsilon_{v_1} + A_1 \varepsilon_{v_1}\varepsilon_{v_2}] s}{(1 - \alpha_1)\sigma \varepsilon_{v_2} + (1 - \lambda)(\alpha_2 - \alpha_1)(s - \sigma \varepsilon_\psi)}
$$

and, eventually, (18). The trace of the Jacobian is obtained from (50): $T = (v_{11}w_{22} - v_{12}w_{21} - v_{21}w_{12} + v_{22}w_{11})/\det V$. Using (48), (49), (51), (52) and (53), after some computations, we obtain (19).

Proof of Proposition 2

Under Assumption 4, $D < 1$, if and only if $\varepsilon_v < \varepsilon_{v_H}$, where ε_{v_H} is given by (25). Using equations (23) and (24), we find also:

$$
P(1) = \varepsilon_v \frac{s + \alpha(1 - \eta)(1 - s) - \sigma}{(1 - \alpha)(1 - s)} \quad\quad (54)
$$

$$
P(-1) = 2 + \frac{2s + \varepsilon_v[s - \alpha(1 - \eta)(1 - s) + \sigma]}{(1 - \alpha)(1 - s)} \quad\quad (55)
$$

We observe that $P(1) > 0$, if and only if $\sigma < s + \alpha(1 - \eta)(1 - s)$. If

$$
\eta \geq 1 - s/[\alpha(1 - s)] \quad\quad (56)
$$

then $P(-1) > 0$, whatever σ and ε_v. When (56) is not satisfied, $P(-1)$ can become negative. All these pieces of information prove that, under condition (56), the occurrence of local indeterminacy requires $\sigma < s + \alpha(1 - \eta)(1 - s)$ and $\varepsilon_v < \varepsilon_{v_H}$.

Assume now that (56) no longer holds. If $\sigma < \alpha(1 - \eta)(1 - s) - s$, then $P(1) > 0$. Furthermore, $P(-1) > 0$, if $\varepsilon_v < \varepsilon_{v_F}$, where ε_{v_F} is given by (26). Local indeterminacy requires now $\varepsilon_v < \min\{\varepsilon_{v_H}, \varepsilon_{v_F}\}$. We notice that $\varepsilon_{v_F} < \varepsilon_{v_H}$, if and only if $\sigma < \sigma_H$, where σ_H is given by (27). Then indeterminacy arises, if $\varepsilon_v < \varepsilon_{v_F}$ and $0 < \sigma < \sigma_H$; or if $\varepsilon_v < \varepsilon_{v_H}$ and $\sigma_H < \sigma < \alpha(1 - \eta)(1 - s) - s$. If

$\alpha\left(1-\eta\right)\left(1-s\right)-s<\sigma<\alpha\left(1-\eta\right)\left(1-s\right)+s$, then $P\left(1\right)>0$ and $P\left(-1\right)>0$ for all ε_v: Hence, the steady is locally indeterminate if $\varepsilon_v<\varepsilon_{v_H}$. Eventually, if $\sigma>\alpha\left(1-\eta\right)\left(1-s\right)+s$, $P\left(1\right)<0$ and $P\left(-1\right)>0$: Thus, the steady state ends up to be a saddle and becomes locally determinate. ∎

Proof of Proposition 3

Under Assumption 5, $D<1$, if and only if $\varepsilon_v<\varepsilon_{v_H}$, where ε_{v_H} is given by (30). Moreover, we have:

$$P\left(1\right)=A_1^{-2}\left(\varepsilon_v\left[A_1\left(s-\sigma\right)/\left(1-s\right)+A_2\right]-\lambda\left(1-\lambda\right)\left(\alpha_2-\alpha_1\right)^2\right)$$

$$P\left(-1\right)=A_1^{-2}\left(\varepsilon_v\left[A_1\left(s+\sigma\right)/\left(1-s\right)-A_2\right]+\lambda\left(1-\alpha_1\right)^2+\left(1-\lambda\right)\left(1-\alpha_2\right)^2\right)$$
$$+1+2s/\left[\left(1-s\right)A_1\right]$$

By direct inspection of expressions (31), (32), (35), (36), we remark that: $P\left(1\right)>0$, if and only if $\sigma<\sigma_T$ and $\varepsilon_v>\varepsilon_{v_T}$; and $P\left(-1\right)>0$, if and only if $\sigma<\sigma_F$ and $\varepsilon_v<\varepsilon_{v_F}$. Furthermore, we observe that $\sigma<\sigma_F$ entails $\sigma_T>\sigma_F$ and $\varepsilon_{v_T}<\varepsilon_{v_F}$. Then $P\left(1\right)$ and $P\left(-1\right)$ are both strictly positive in two cases:

1. $\sigma<\sigma_F$ and $\varepsilon_{v_T}<\varepsilon_v<\varepsilon_{v_F}$,

2. $\sigma_F\leq\sigma<\sigma_T$ and $\varepsilon_v>\varepsilon_{v_T}$.

Finally, we notice that $\sigma_{H_1}<\sigma_F$ and $\sigma_{H_1}<\sigma_{H_2}<\sigma_T$, where σ_{H_1} and σ_{H_2} are respectively given by (33) and (34). Therefore, $\varepsilon_{v_H}>\varepsilon_{v_F}$, if and only if $\sigma<\sigma_{H_1}$, and $\varepsilon_{v_H}>\varepsilon_{v_T}$, if and only if $\sigma<\sigma_{H_2}$. These last comments jointly with inequalities in 1 and 2 conclude the proof. ∎

Proof of Proposition 4

Using (37) and (38), we get:

$$P\left(1\right)=\frac{\varepsilon_\psi-\left(1-s\right)\varepsilon_v}{\left(\varepsilon_\psi+1-s\right)\left(1-\alpha\right)}$$

$$P(-1)=\frac{\varepsilon_\psi+\left(1+s\right)\varepsilon_v+2}{\left(\varepsilon_\psi+1-s\right)\left(1-\alpha\right)}-\frac{2\alpha}{1-\alpha}$$

We observe that $\varepsilon_{v_T}\equiv\varepsilon_\psi/\left(1-s\right)>-\left(\varepsilon_\psi+2\left[1-\alpha\left(1-s+\varepsilon_\psi\right)\right]\right)/\left(1+s\right)\equiv\varepsilon_{v_F}$. Then $P\left(-1\right)>0$, if $\varepsilon_v>\max\{0,\varepsilon_{v_F}\}$, and $P\left(1\right)>0$, if $\varepsilon_v<\varepsilon_{v_T}$. Since $\alpha>1/2$, $\varepsilon_{v_F}>0$, if and only if $\varepsilon_\psi>\varepsilon_{\psi_F}$, where ε_{ψ_F} is given by (40). Using equation (37), we have $D<1$, if $\varepsilon_v<\varepsilon_{v_H}$, which requires $\varepsilon_\psi>\varepsilon_{\psi_H}$ (ε_{ψ_H} and ε_{v_H} are, respectively, given by (39) and (41)). From Assumption 6, we deduce that $\varepsilon_{\psi_H}<\varepsilon_{\psi_F}$ and $\varepsilon_{v_H}<\varepsilon_{v_T}$. The proposition follows these last remarks. ∎

Proof of Proposition 5

$D<1$ needs $\varepsilon_\psi>\varepsilon_{\psi_H}$ and $\varepsilon_v<\varepsilon_{v_H}$, where ε_{ψ_H} and ε_{v_H} are, respectively, given by (44) and (46). Furthermore, using (42) and (43), we have $P\left(1\right)=Q_1\left(\varepsilon_v\right)/\left[\varepsilon_v A_1^2\left(1-s+\varepsilon_\psi\right)\right]$, with

$$Q_1\left(\varepsilon_v\right)\equiv\left[\varepsilon_\psi-\left(1-s\right)\varepsilon_v\right]\left[\varepsilon_v A_1+\lambda\left(1-\lambda\right)\left(\alpha_2-\alpha_1\right)^2\right]$$

We observe that $Q_1\left(\varepsilon_{v_T}\right) = 0$ is satisfied by $\varepsilon_{v_T} \equiv \varepsilon_\psi / \left(1 - s\right) > \varepsilon_{v_H}$. In other words, $P\left(1\right) > 0$, if and only if $\varepsilon_v < \varepsilon_{v_T}$. Using (42) and (43), we also have $P\left(-1\right) = Q_{-1}\left(\varepsilon_v\right) / \left[\varepsilon_v A_1^2 \left(1 - s + \varepsilon_\psi\right)\right]$ with $Q_{-1}\left(\varepsilon_v\right) \equiv a\varepsilon_v^2 + b\varepsilon_v + c$ and

$$
\begin{aligned}
a &\equiv \left(1 + s\right) A_1 > 0 \\
b &\equiv \left(2\left[s + A_1\left(1 - s + \varepsilon_\psi\right)\right] - \varepsilon_\psi\right) A_1 + \lambda\left(1 - \lambda\right)\left(1 - s\right)\left(\alpha_2 - \alpha_1\right)^2 \\
c &\equiv -\lambda\left(1 - \lambda\right)\left(\alpha_2 - \alpha_1\right)^2 \varepsilon_\psi < 0
\end{aligned}
\tag{57}
$$

We observe that $P\left(-1\right)$ and $Q_{-1}\left(\varepsilon_v\right)$ have the same sign, and that $b > 0$, if $\varepsilon_\psi < \varepsilon_{\psi_F}$, where ε_{ψ_F} is given by (45). Under Assumption 7, $\varepsilon_{\psi_H} < \varepsilon_{\psi_F}$: Henceforth, we reduce the analysis to the case $\varepsilon_{\psi_H} < \varepsilon_\psi < \varepsilon_{\psi_F}$. Straightforward computations show that $Q_{-1}\left(0\right) < 0$, $Q_{-1}\left(+\infty\right) = +\infty$ and $Q'_{-1}\left(\varepsilon_v\right) > 0$, entailing the existence of a unique critical value ε_{v_F}, such that $Q_{-1}\left(\varepsilon_v\right) > 0$ if and only if $\varepsilon_v > \varepsilon_{v_F}$, with:

$$
\varepsilon_{v_F} \equiv \frac{\sqrt{b^2 - 4ac} - b}{2a}
\tag{58}
$$

In order to establish the conditions for indeterminacy, we have now to see when $\varepsilon_{v_F} < \varepsilon_{v_H}$. First, we notice that for $\varepsilon_\psi = \varepsilon_{\psi_H}$, then $Q_{-1}\left(\varepsilon_{v_H}\right) = c\left(\varepsilon_{\psi_H}\right) < 0$, where c is given by (57), and, therefore, $\varepsilon_{v_H} < \varepsilon_{v_F}$. By continuity, this implies that indeterminacy is ruled out, when ε_ψ is sufficiently close to ε_{ψ_H}.

Secondly, we observe that

$$
\varepsilon_{v_F} < \sqrt{-c/a} = \sqrt{\lambda\left(1 - \lambda\right)\left(\alpha_2 - \alpha_1\right)^2 \varepsilon_\psi / \left(\left(1 + s\right) A_1\right)}
$$

When ε_ψ tends to ε_{ψ_F}, ε_{v_H} becomes strictly greater than $\sqrt{-c/a}$, if

$$
\begin{aligned}
&\left(1 + s\right) A_1 \left(A_1^2 \varepsilon_{\psi_F}^2 + \left[s - \left(1 - s\right) A_1\right]\left[s - \left(1 - s + 2\varepsilon_{\psi_F}\right) A_1\right]\right) \\
&> \lambda\left(1 - \lambda\right) s^2 \left(\alpha_2 - \alpha_1\right)^2 \varepsilon_{\psi_F}
\end{aligned}
$$

Since $\lim_{A_1 \to \left(1/2\right)^-} \varepsilon_{\psi_F} = +\infty$, the inequality is satisfied, when A_1 is sufficiently close to $1/2$. This remark concludes the proof. ∎

References

[1] D'Aspremont, C., R. Dos Santos Ferreira and L.-A. Gérard-Varet (1995), "Market Power, Coordination Failures and Endogenous Fluctuations," in *The New Macroeconomics: Imperfect Markets and Policy Effectiveness*, ed. by H. D. Dixon, and N. Rankin, pp. 94-138, Cambridge. Cambridge University Press.

[2] Becker, R. A. and C. Foias (1994), "The Local Bifurcation of Ramsey Equilibrium," *Economic Theory*, 4, 719-744.

[3] Benhabib, J. and R. Farmer, (1999), "Indeterminacy and Sunspots in Macroeconomics", in *Handbook of Macroeconomics,* (J. B. Taylor and M. Woodford, Eds.), North-Holland, Amsterdam, ch. 6, 387-448.

[4] Bosi, S. and T. Seegmuller (2005), "Animal Spirits in Woodford and Reichlin Economies: The Representative Agent Does Matter," EPEE Working Paper 05-01, Evry, France.

[5] Cazzavillan, G. (2001), "Indeterminacy and Endogenous Fluctuations with Arbitrarily Small Externalities," *Journal of Economic Theory*, 101, 133-157.

[6] Cazzavillan, G. and P. Pintus (2004a), "Robustness of Multiple Equilibria in OLG Economies," *Review of Economic Dynamics*, 7, 456-475.

[7] Cazzavillan, G. and P. Pintus (2004b), "Capital Externalities in OLG Economies," GREQAM Working Paper 2004-40, Aix-Marseille, France.

[8] Cazzavillan, G., T. Lloyd-Braga and P. Pintus (1998), "Multiple Steady States and Endogenous Fluctuations with Increasing Returns to Scale in Production," *Journal of Economic Theory*, 80, 60-107.

[9] Duffy, J. and C. Papageorgiou (2000), "A Cross-Country Empirical Investigation of the Aggregate Production Function Specification," *Journal of Economic Growth*, 5, 87-120.

[10] Ghiglino, C. (2005), "Wealth Inequality and Dynamic Stability," *Journal of Economic Theory* (forthcoming).

[11] Ghiglino, C. and M. Olszak-Duquenne (2001), "Inequalities and Fluctuations in a Dynamic General Equilibrium Model," *Economic Theory*, 17, 1-24.

[12] Ghiglino, C. and M. Olszak-Duquenne (2005), "On the Impact of Heterogeneity on Indeterminacy," *International Economic Review* (forthcoming).

[13] Ghiglino, C. and M. Tvede (1995), "Endowments, Stability and Fluctuations in OG Models," *Journal of Economic Dynamics and Control*, 19, 621-653.

[14] Ghiglino, C. and G. Sorger (2002), "Poverty Traps, Indeterminacy, and the Wealth Distribution," *Journal of Economic Theory*, 105, 102-139.

[15] Ghiglino, C. and A. Venditti (2005), "Wealth Inequality and Macroeconomic Volatility in Two-sectors Economies," GREQAM Working Paper 2005-05, Aix-Marseille, France.

[16] Herrendorf, B., R. Waldmann and A. Valentiny, (2000), "Ruling Out Multiplicity and Indeterminacy: The Role of Heterogeneity," *Review of Economic Studies*, 67, 295-307.

[17] Lloyd-Braga, T., C. Nourry and A. Venditti (2005), "Indeterminacy in Dynamic Models: When Diamond Meets Ramsey," mimeo.

[18] Reichlin, P. (1986), "Equilibrium Cycles in an Overlapping Generations Economy with Production," *Journal of Economic Theory*, 40, 89-103.

[19] Seegmuller, T. (2004), "Capital-labor Substitution and Endogenous Fluctuations: A Monopolistic Competition Approach with Variable Mark-up," mimeo.

[20] Sorger, G. (1994), "On the Structure of Ramsey Equilibrium: Cycles, Indeterminacy, and Sunspots," *Economic Theory*, 4, 745-764.

[21] Woodford, M., (1986), "Stationary Sunspot Equilibria in a Finance Constrained Economy," *Journal of Economic Theory*, 40, 128-137.

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