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## Sunspot Bubbles

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# Sunspot Bubbles*

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**Abstract**

Bubbles and money holding can reabsorb capital overaccumulation and restore the MGR in exogenous growth OLG economies. In contrast, under productive externalities and endogenous growth, bubbles and monetary saving can worsen an already inefficient underaccumulation. Under a simple credit market imperfection, we prove that bubbles can fluctuate around a bubbly balanced growth path as sunspot equilibria with short-run effects on the saving rate.

*JEL classification*: D9, E4, G1.

*Keywords*: overlapping generations model, bubbles, cash-in-advance constraint, sunspot equilibria.

## 1 Introduction

Tirole (1985) was the first to focus on the existence of rational bubbles in a general equilibrium model à la Diamond (1965) with overlapping generations and capital accumulation. Within the Tirole model, the seminal Diamond (1965) can be reinterpreted as a bubbleless equilibrium, while the introduction of a bubbly asset can resorb the possible oversaving, typically arising in a Diamond economy, and, then, restore the modified golden rule.

Grossman and Yanagawa (1993) criticize the virtues of bubbles, by developing an endogenous growth version of Tirole (1985), where positive productive externalities play a rôle of growth engine: now, bubbles worsens a market regime of underaccumulation, where the beneficial external effects of production are not internalized.

Our model is close to Grossman and Yanagawa (1993), but takes in account a credit market imperfection as in Crettez, Michel and Wigniolle (1999), Michel and Wigniolle (2003), Polemarchakis and Rochon (2005), where generations are financially constrained by a cash-in-advance constraint.

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Michel and Wigniolle (2003) study the market movements from a regime, where money is a dominated asset, to a regime, where the cash-in-advance is no longer binding and the economy experiences a temporary bubble.

Another paper close to ours is Guillard (1998), an overlapping generation model of endogenous growth, which focuses on the effects of a particular financial constraint: agents are required to invest in money an exogenous share of saving. This hypothesis underlines a rationale for money demand slightly different from our cash-in-advance assumption, since the nominal interest rate, that is the opportunity cost of holding money, amplifies the effects of saving on the second-period consumption and entails different dynamics.

On the one side we can have rational bubbles *à la* Tirole, essentially due to agents' finite life-span, while, on the other side, we can obtain multiple equilibria (local indeterminacy), because of the cash-in-advance constraints. The added value of the paper mainly consists in relating conditions for rational bubbles with conditions for sunspot equilibria and understanding the joint effect of them on optimality.

What is really new in our paper is that, now, since the growth rate of the bubble is non-predetermined, the size of the bubble can be self-fulfilling, when the bubbly steady state is locally stable. In other words the growth factor of the bubble converges in the long run as in Grossman and Yanagawa (1993), but in the short run the size of the bubble can be shorter or larger than the balanced value with a real effect on the saving rates.

In a different perspective, we provide also an alternative way of conceiving stochastic rational bubbles with respect to the seminal paper by Weil (1987), where bubbles burst according to an exogenous probability, and to more sophisticated approaches such as Bertocchi (1991): our bubbles can be self-fulfilling and their endogenous fluctuations follow a stochastic sunspot process.

The remainder of the paper is organized as follows. In Section 2 we present the model and derive the intertemporal demands, while in section 3 we compute the general equilibrium. Section 4 is devoted to the study of the stationary solutions and their local stability. Section 5 concludes the paper.

# 2 The model

We study a decentralized market economy populated by two generations of price-taker agents: the young and the old. As in the Reichlin (1986) model, individuals born at time $t$, supply inelastically a unit of labor when young and consume $c_{t+1}$ when old. In order to ensure the consumption during the retirement age, people save and buy a diversified portfolio of money balances $M_{t+1}$, public bonds $B_{t+1}$ and physical capital $K_{t+1}$. Money provides liquidity services, bonds give an interest rate, capital is used by the firms to produce the consumption good. The relevant prices are the real wage $w_t$, the rental factors $i_{t+1}$ and $r_{t+1}$ on bonds and capital and the price $p_t$ of consumption good.

## 2.1 Households

The representative household born at time $t$ derives the consumption and assets demand, by maximizing $c_{t+1}$ under two budget constraints and a long-run cash-in-advance.[1]

$$\begin{aligned} \frac{M_{t+1}}{N_t} + \frac{B_{t+1}}{N_t} + p_t \frac{K_{t+1}}{N_t} &\leq p_t w_t + \tau_t \\ p_{t+1} c_{t+1} &\leq \frac{M_{t+1}}{N_t} + i_{t+1} \frac{B_{t+1}}{N_t} + r_{t+1} p_{t+1} \frac{K_{t+1}}{N_t} \\ q p_{t+1} c_{t+1} &\leq \frac{M_{t+1}}{N_t} \end{aligned}$$

This program deserves some few comments and definitions The CIA constraint is partial ($q \in (0,1)$) because the second constraint involves a positive amount of capital. $r$ are $i$ are factors (1 plus the real or nominal interest rate). $M_{t+1}/N_t$, $B_{t+1}/N_t$ and $K_{t+1}/N_t$ represent, respectively, the individual demand for money balances, public bonds and capital at time $t$, while $N_t$ is the size of the generation born at time $t$. A capital letter just denotes the aggregate level.

Setting and maximizing the Lagrangian, after canceling out the multipliers we obtain a no-arbitrage condition:

$$i_{t+1} = r_{t+1} \pi^e_{t+1} \tag{1}$$

where $\pi^e_{t+1} \equiv p_{t+1}/p_t$ is the expected inflation factor. The monotonicity of the utility function and the positivity of the nominal interest rate ensure, respectively, the budget constraint and the cash-in-advance to be binding (see the fourth section and the steady state analysis for more details). We write down the equilibrium budget constraints and the cash-in-advance in real terms:

$$\left[ \left( m_{t+1} + b_{t+1} \right) \pi_{t+1} + k_{t+1} \right] n = w_t + \tau_t / p_t \tag{2}$$

$$c_{t+1} = \left( m_{t+1} + i_{t+1} b_{t+1} + r_{t+1} k_{t+1} \right) n \tag{3}$$

$$q c_{t+1} = m_{t+1} n \tag{4}$$

From now on, $m_t \equiv M_t / \left( p_t N_t \right)$, $b_t \equiv B_t / \left( p_t N_t \right)$ and $k_t \equiv K_t / N_t$ will denote the real assets per capita, while $n \equiv N_{t+1}/N_t$ will stand for the demographic growth factor.

## 2.2 Firms

Following Grossman and Yanagawa (1993), we consider a representative firm endowed with constant private returns to scale technology and affected by aggregate externalities. As the length of the period is equal to the half-life of a generation, we plausibly assume a full capital depreciation.

[1] We notice that whatever strictly increasing utility function $u(c_{t+1})$ can be composed with the strictly increasing transformation $u^{-1}$ to give the identity $c_{t+1}$, as a new utility function, without altering the demand functions.

**Assumption 1** $Y_t = A_t N_t f(k_t)$, *with* $k_t \equiv K_t/N_t$, $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$ *for every* $k > 0$, $f'(0) = +\infty$, $f'(+\infty) = 0$. *The external effects depend on the capital intensity:* $A_t = A(k_t)$, *with* $\varepsilon \equiv kA'(k)/A(k) = 1 - kf'(k)/f(k)$, *a constant.*

In other words, $\varepsilon$ can be interpreted as an externality measure.

Private profit is the firm's objective, but firms don't take into account the impact of factors demand on $A_t$: $\max_{K_t,N_t} A_t N_t f(k_t) - r_t K_t - w_t N_t$, where the wage is per unit of labor services. Firm's equilibrium requires, as usual: $r_t = A_t f'(k_t)$, $w_t = A_t [f(k_t) - k_t f'(k_t)]$.

To keep things as simple as possible, we focus on the Cobb-Douglas case: $f(k_t) = k_t^{\alpha}$, and we set: $A_t = Ak_t^{\varepsilon}$. As in Romer (1986), $\varepsilon = 1 - \alpha$ ensures an endogenous growth. The reduced form is close to Rebelo's $Ak$ model: $y_t \equiv Y_t/N_t = Ak_t$, and firm's equilibrium becomes:

$$r_t = \alpha A \equiv r \quad (5)$$
$$w_t = (1-\alpha) A k_t \quad (6)$$

## 2.3 Government

For simplicity, we assume that money is "helicoptered" to the young:

$$\tau_t = (M_{t+1} - M_t)/N_t \quad (7)$$

while the fiscal authority is assumed to roll over the debt.[2]

$$B_{t+1} = i_t B_t \quad (8)$$

# 3 Equilibrium

In order to obtain the general equilibrium, markets for money, bonds and good are required to clear.

## 3.1 Money market

Money demand is for transaction purpose and comes from the CIA constraint (4), now binding:

$$\frac{c_{t+1}}{c_t} = \frac{m_{t+1}}{m_t} = \frac{M_{t+1}/M_t}{n\pi_{t+1}} \quad (9)$$

On the supply side, monetary transfers (7) become in real terms:

$$\tau_t/p_t = m_{t+1}\pi_{t+1}n - m_t \quad (10)$$

Monetary growth $M_{t+1}/M_t$ is determined by the rule the central bank chooses. In the following, we will consider a control of the monetary mass in response to

[2] Bonds can be viewed as pure bubbles, with no fundamental value.

inflation.[3]

$$M_{t+1}/M_t = \sigma\left(\pi_t, \pi^e_{t+1}\right) \tag{11}$$

The elasticities of the rule $\varepsilon_1 \equiv \pi\sigma_1/\sigma \leq 0$ and $\varepsilon_2 \equiv \pi\sigma_2/\sigma \leq 0$, where $\sigma_i$ is the $i$th partial derivative, capture the reactivity of the monetary authority to the observed or expected inflation, respectively.

The following assumption exclude a very eccentric behavior of the central bank with respect to the expected inflation.

**Assumption 2** $\varepsilon_2 < 1$.

From now, we don't need other particular restriction on $\varepsilon_1$ and $\varepsilon_2$, even if they can be required to be not too positive, in order to rule out a hyper-inflationary regime: $\varepsilon_1 + \varepsilon_2 < 1$ (for more details, see equation (32) below). Clearly, the usual policies are characterized by a non-positive response to inflation: $\varepsilon_1, \varepsilon_2 \leq 0$. A constant money growth can be viewed as a particular case, when $\sigma(\pi_t, \pi_{t+1}) = \sigma$, a constant and, thereby, $\varepsilon_1 = \varepsilon_2 = 0$.

## 3.2 Bonds market

According to (8), we assume no tax and no public spending: the new national debt pays the interest on the current one. In real terms the government budget becomes:

$$i_t b_t = b_{t+1}\pi_{t+1} n \tag{12}$$

Two policies are allowed: the bubbleless regime ($b_t = 0$, for every $t$) and the bubbly regime ($b_t$ for some $t$).

## 3.3 Goods market

Substituting (6) and (10) in the constraint (2) of the young born in $t$ and replacing (5) and (12) in the constraint (3) of the old born in $t-1$, we obtain, respectively,

$$\left(b_{t+1}\pi_{t+1} + k_{t+1}\right) n = (1-\alpha) A k_t - m_t \tag{13}$$

$$c_t/n = m_t + b_{t+1}\pi_{t+1} n + \alpha A k_t \tag{14}$$

In order to clear the good market, we have to aggregate side by side (13) and (14), and to simplify:

$$c_t = n\left(A k_t - n k_{t+1}\right) \tag{15}$$

which is the resource constraint of period $t$.

[3] Loosely speaking, monetary rules can be divided in two main categories, according to the policy of tuning the "quantity" (the monetary mass) or the "price" (the interest rate) in response to some observed or expected variables. The arguments of the rules can be various as, for instance, in the Taylor rule. In our specific case a more general rule could take into account the output gap: $M_{t+1}/M_t = \sigma\left(\pi_t, \pi^e_{t+1}, y_t, y^e_{t+1}\right)$, where $y$ denote the output with $\sigma_3, \sigma_4 < 0$.

# 4 Dynamics

The main mechanism of the model is now provided.

**Proposition 1** *The growth factor*

$$\gamma_{t+1} \equiv k_{t+1}/k_t \tag{16}$$

*moves according to a one-dimensional law:*

$$\gamma_{t+2} = \frac{A}{n}\left(1 - \frac{\alpha}{1-q}\left[q + (1-\alpha-q)\frac{A}{n}\frac{1}{\gamma_{t+1}}\right]\right) \tag{17}$$

*while the inflation path is subsequently determined by the implicit equation*

$$\sigma\left(\pi_t, \pi_{t+1}\right) = n\pi_{t+1}\gamma_{t+1}\frac{A - n\gamma_{t+2}}{A - n\gamma_{t+1}} \tag{18}$$

**Proof.** See the Appendix. ■

The first equation determines separately and independently the path $\left\{\gamma_{t+1}\right\}_{t=0}^{+\infty}$, while the path $\{\pi_t\}_{t=0}^{+\infty}$ is jointly determined by both the equations. We observe that money supply does not affect the real variables ($\sigma$ does not appear in (17)) and only matters for inflation, because of equation (18). In contrast, the money demand matters ($q$ plays a role in (17)).

## 4.1 Steady state

Equation (17) has two steady states:

$$\gamma_0 = \frac{1-\alpha-q}{1-q}\frac{A}{n} \tag{19}$$

$$\gamma_1 = \alpha\frac{A}{n} \tag{20}$$

The positivity of capital, requires

$$q < 1-\alpha \tag{21}$$

In other words, the cash-in-advance in an OLG framework has to be partial: otherwise agents holds only money instead of capital and there is no longer space for productive equilibria.

The stationary inflation is computed, defining the function $\varphi(\pi) \equiv \sigma(\pi,\pi)/\pi$ and solving the equation $\varphi(\pi) = n\gamma$ obtained from (18):[4]

$$\pi_0 = \varphi^{-1}\left(\frac{1-\alpha-q}{1-q}A\right)$$

$$\pi_1 = \varphi^{-1}(\alpha A)$$

[4] Under a rule of constant monetary growth, we explicitly find

$$\pi_0 = \frac{\sigma}{A}\frac{1-q}{1-\alpha-q},\ \pi_1 = \frac{\sigma}{\alpha A}$$

We observe that the CIA constraint is required to be binding around both the steady states. Consider, for simplicity, the case of a constant money growth. The positivity of the nominal interest rate ($i = r\pi > 1$) is needed, in order to make the return on money dominated by that on capital: $1/\pi < r$, or, from (18), evaluated at the steady state ($\sigma = n\pi\gamma$):

$$\sigma > n\gamma/r \tag{22}$$

Noticing that $r = \alpha A$ and taking in account expressions (19) and (20), inequality (22) becomes, respectively, in the two steady states:

$$\gamma_0 \quad : \quad \sigma > \frac{1}{\alpha}\frac{1-\alpha-q}{1-q} \equiv \sigma^* \tag{23}$$

$$\gamma_1 \quad : \quad \sigma > 1 \tag{24}$$

Then, in the bubbly steady state, a positive monetary growth, given by (24), is sufficient to make the CIA binding, while condition (23) deserves more comments. We observe that $\sigma^* < 1$, if and only if

$$q > \frac{1-2\alpha}{1-\alpha} \tag{25}$$

If (25) is satisfied, then (24) implies (23) and a positive monetary growth ensures a binding CIA in both the steady states. If (25) is violated, then condition (23) becomes more demanding, since now a monetary growth rate greater than $\sigma^* - 1 > 0$ is required, in order to have both the steady states with a binding CIA constraint.

## 4.2 Local stability

We linearize the system (17-18) around a generic steady state $(\gamma, \pi)$:

$$\begin{aligned} \frac{d\gamma_{t+2}}{\gamma} &= \alpha\frac{1-\alpha-q}{1-q}\left(\frac{A}{n\gamma}\right)^2 \frac{d\gamma_{t+1}}{\gamma} \\ \frac{n}{A-n\gamma}\frac{d\gamma_{t+2}}{\gamma} + (\varepsilon_2 - 1)\frac{d\pi_{t+1}}{\pi} &= \left(1 + \frac{n\gamma}{A-n\gamma}\right)\frac{d\gamma_{t+1}}{\gamma} - \varepsilon_1\frac{d\pi_t}{\pi} \end{aligned}$$

We obtain a triangular Jacobian matrix because (17) describes a separate dynamics from (18):

$$J = \begin{bmatrix} \alpha\frac{1-\alpha-q}{1-q}\left(\frac{A}{n\gamma}\right)^2 & 0 \\ \frac{1}{1-\varepsilon_2}\frac{A}{A-n\gamma}\left(\alpha\frac{1-\alpha-q}{1-q}\frac{A}{n\gamma^2} - 1\right) & \frac{\varepsilon_1}{1-\varepsilon_2} \end{bmatrix}$$

We notice that the eigenvalues appear explicitly on the main diagonal and are both real. The key eigenvalue is

$$\alpha\frac{1-\alpha-q}{1-q}\left(\frac{A}{n\gamma}\right)^2$$

which can take two different values in the two different steady states (19) and (20):

$$\begin{aligned} \lambda_0 &= \alpha \frac{1-q}{1-\alpha-q} > 0 \\ \lambda_1 &= \frac{1}{\lambda_0} > 0 \end{aligned} \tag{26}$$

Both the inequalities hold under restriction (21): dynamics are monotonic.

Since $\gamma_{t+1}$ is a non-predetermined variable[5], local indeterminacy requires $\lambda < 1$. Clearly the first steady state is stable, if and only if the second one is unstable. The determinacy of a steady state entails the indeterminacy of the other: this is the basic feature of the transcritical bifurcation, which is characterized by the existence of at least two steady states and an exchange of their stability properties, when the relevant eigenvalue goes through 1.

## 4.3 Bubbly regime

The steady state $\gamma_1$ is associated to the existence of a stationary growth factor for the bubble $b > 0$. Using (1) and (12), we obtain

$$\frac{b_{t+1}}{b_t} = \alpha \frac{A}{n} \frac{\pi_t}{\pi_{t+1}}$$

and, at the steady state, the bubble follows the balanced growth: $b_{t+1}/b_t = \alpha A/n$. The bubbly steady state is stable, if and only if $\lambda_1 < 1$, that is, if and only if

$$q > \frac{1-2\alpha}{1-\alpha} \tag{27}$$

We notice that $(1-2\alpha)/(1-\alpha) < 1-\alpha$, always and, therefore, there are feasible CIA constraints characterized by a parameter

$$q \in \left( \frac{1-2\alpha}{1-\alpha}, 1-\alpha \right) \tag{28}$$

[5] Why is $\gamma_{t+1}$ non-predetermined? When the young choose at time $t$ how much to invest in capital, that is $k_{t+1}$, they fix the consumption as well as the real balances of the old, according to their CIA constraint: $m_t = q(Ak_t - nk_{t+1})$. Since $M_t$ is given, the price level $p_t$ adjusts, in order to ensure the equilibrium. Thereby, the young choose indirectly $\gamma_{t+1} = k_{t+1}/k_t$, where $k_t$ is predetermined. Consider, now, the second period equation:

$$\frac{k_{t+1}}{k_t} = \frac{\alpha A}{n} \frac{(1-\alpha)A - m_t/k_t}{(1-q)(A - nk_{t+2}/k_{t+1})}$$

On the RHS, $m_t/k_t$ is now given and, since $k_{t+1}$ has been chosen in the first period, market clearing fixes $k_{t+2}$. But $\gamma_{t+2} = k_{t+2}/k_{t+1}$ and, therefore, $\gamma_{t+2}$ is indirectly fixed by the first period choice of $k_{t+1}$. Summing up, we obtain that the young choose $k_{t+1}$ at period $t$, or equivalently $\gamma_{t+1}$, while the price level $p_t$ adjusts in consequence, but the choice of $\gamma_{t+1}$, pegs $\gamma_{t+2}$, through the market mechanism, summarized by equation (17).

such that the bubbly steady state is indeterminate and restriction (21) is satisfied, in order to have positive variables.[6]

Clearly, in the bubbly steady state, growth is balanced

$$\frac{m_{t+1}}{m_t} = \frac{b_{t+1}}{b_t} = \frac{k_{t+1}}{k_t} = \frac{c_{t+1}}{c_t} = \alpha \frac{A}{n}$$

What is really new in our paper is that, now, the size of the bubble can be self-fulfilling, since the growth rate of the bubble is non-predetermined. In other words the growth factor of the bubble converges in the long run to $\alpha A/n$ as in Grossman and Yanagawa (1993), but in the short run the size of the bubble can be shorter or larger than the balanced value and, possibly, confined within a bounded support.

$$b_{t+1} \gtreqless b_t \alpha A/n$$

Agents can coordinate themselves on a size or another according to an extrinsic stochastic process. In this sense we interpret the self-fulfilling fluctuations of the bubble in the short run as a sunspot bubble.

As seen in the introduction, however, the bubble no longer restores the Modified Golden Rule *à la* Phelps, since now productive externalities matter and the bubble can only worsen capital underaccumualtion, which typically arises in a decentralized economy.

## 4.4 Bubbleless regime (possibly in the long-run)

In order to provide a more suitable interpretation, let's focus on the alternative steady state $\gamma_0$, which is characterized by the eigenvalue $\lambda_0$ in the separated dynamics (17), and make a step backward. Assume $b_t = b_{t+1} = 0$. Substituting in (14) $c_t = m_t n/q$ and $m_t = (1-\alpha) Ak_t - nk_{t+1}$, obtained, respectively, from (4) and (13), we find the endogenous growth factor:

$$\gamma_{t+1} \equiv \frac{k_{t+1}}{k_t} = \frac{1-\alpha-q}{1-q}\frac{A}{n} \tag{29}$$

This is the endogenous growth version of the bubbleless Diamond regime (1965). A the steady state, growth is balanced and the other variables grow as the capital.

$$\frac{m_{t+1}}{m_t} = \frac{k_{t+1}}{k_t} = \frac{c_{t+1}}{c_t} = \frac{1-\alpha-q}{1-q}\frac{A}{n} \tag{30}$$

A new feature emerging in our paper is that, now, sufficiently small bubbles can also arise on the right-side of the bubbleless equilibrium, provided that the latter is stable. In other words, we require the inverse of inequality (27) to hold, that is

$$q < \frac{1-2\alpha}{1-\alpha} \tag{31}$$

[6] If, for instance, $\alpha = 1/3$, local indeterminacy of the bubbly steady state, jointly with capital positivity, requires $1/2 < q < 2/3$.

We observe that (31) satisfies the positivity requirement (21) and that, surprisingly, the bubbleless regime can be bubbly at least in the short run. In other words short-run positive bubbles can arise arbitrarily close to zero, because of the indeterminacy, but in the long run the regime tends to be bubbleless. When the bubbleless regime is bubbly in the short run, the other steady state (bubbly in the long-run) becomes determinate and there is non longer room for sunspot bubbles around (see equation (26)).

### 4.5 Hyperinflation regime

When the path $\{\gamma_{t+1}\}_{t=0}^{+\infty}$ is determined, convergence of the path $\{\pi_t\}_{t=0}^{+\infty}$ requires the second eigenvalue to belong to the unit circle:

$$\left|\frac{\varepsilon_1}{1-\varepsilon_2}\right| < 1 \tag{32}$$

Under the Assumption 2, inequality (32) is equivalent to

$$\varepsilon_1 + \varepsilon_2 < 1 \tag{33}$$

and

$$\varepsilon_2 < 1 + \varepsilon_1 \tag{34}$$

Inequalities (33) and (34) are obviously satisfied under the usual restriction $\varepsilon_1, \varepsilon_2 \leq 0$ and $\varepsilon_1, \varepsilon_2$ not too negative, and ensure the stability of $\pi$, when $\gamma$ is yet stable. We observe that (33) and (34) hold under a constant money growth: $\varepsilon_1, \varepsilon_2 = 0$.

However (32) is a less restrictive condition than $\varepsilon_1, \varepsilon_2 \leq 0$ in order to ensure inflation stability. Positive responses to inflation ($\varepsilon_1, \varepsilon_2 > 0$) are also compatible with a convergent inflationary path, provided that not only (34), but especially $\varepsilon_1 + \varepsilon_2 < 1$ hold. Clearly, if (33) is violated, monetary policy becomes dangerous and the economy experiences a pathological regime: the hyperinflation.

### 4.6 Main results

All these findings are summarized in the following proposition.

**Proposition 2** *Let*

$$q < 1 - \alpha$$

*Then, there are three steady states:*

$$\begin{aligned}
\textit{trivial} &: \quad (b_t, m_t, k_{t+1}) = (0,0,0) \\
\textit{bubbleless} &: \quad \left(b_{t+1}, \frac{m_{t+1}}{m_t}, \frac{k_{t+1}}{k_t}\right) = \left(0, \frac{1-q-\alpha}{1-q}\frac{A}{n}, \frac{1-q-\alpha}{1-q}\frac{A}{n}\right) \\
\textit{bubbly} &: \quad \left(\frac{b_{t+1}}{b_t}, \frac{m_{t+1}}{m_t}, \frac{k_{t+1}}{k_t}\right) = \left(\alpha\frac{A}{n}, \alpha\frac{A}{n}, \alpha\frac{A}{n}\right)
\end{aligned}$$

*The bubbly steady state is indeterminate, if and only if the bubbleless steady state is determinate. The corresponding eigenvalues are*

$$\lambda_1 = \frac{1}{\alpha}\frac{1-\alpha-q}{1-q} = \frac{1}{\lambda_0}$$

*Therefore the bubbly steady state is indeterminate, if and only if*

$$\frac{1-2\alpha}{1-\alpha} < q < 1-\alpha$$

*The bubbleless steady state is bubbly in the short run, if and only if*

$$q < \frac{1-2\alpha}{1-\alpha}$$

*The critical value*

$$q^* \equiv \frac{1-2\alpha}{1-\alpha}$$

*defines a transcritical bifurcation.*

*Under Assumption 2, the hyperinflationary regime is locally ruled out, under very mild restrictions:* $\varepsilon_1 + \varepsilon_2 < 1$ *and* $\varepsilon_2 < 1 + \varepsilon_1$*, where* $\varepsilon_1$ *and* $\varepsilon_2$ *are the elasticities of money supply with respect to the observed and expected inflation factor, respectively.*

# 5 Conclusion

We originally provide a condition for the existence of locally indeterminate equilibria of endogenous growth around a bubbly steady state. The bubble growth rate can fluctuate around the balanced growth rate and the real value of the bubble around its balanced growth path. This fluctuations are typically driven by shocks on the believes and constitute a simple example of stochastic rational bubbles with endogenous probabilities, in contrast with Weil (1987), where the probabilities (of bursting) were given as exogenous.

Other extensions of interest of the paper are, on the one side, considering other regimes, where the CIA constraint turns out to be no longer binding, in line with Michel and Wigniolle (2003), or, on the other side, focussing on indeterminate bubbly regimes, where the local multiplicity of equilibria is no longer due to a credit market imperfection or incompleteness, but to other kinds of imperfections such as market power, externalities and, more generally, asymmetric informations.

# 6 Appendix

**Proof of Proposition 1** From the good market equilibrium (15), we obtain

$$\frac{c_{t+1}}{c_t} = \frac{Ak_{t+1} - nk_{t+2}}{Ak_t - nk_{t+1}} \tag{35}$$

Coupling (35) with the binding CIA (3), we get

$$\frac{c_{t+1}}{c_t} = \frac{m_{t+1} + i_{t+1} b_{t+1} + \alpha A k_{t+1}}{m_t + i_t b_t + \alpha A k_t} \tag{36}$$

while, coupling (35) with the binding CIA (9), we get (18). Combining (1), (5), (12) and (16), we find

$$\pi_{t+1} \frac{b_{t+1}}{k_{t+1}} = \frac{\alpha A}{n\gamma_{t+1}} \frac{b_t}{k_t} \pi_t \tag{37}$$

Taking (3) and (4) at time $t$ and normalize them by $k_{t+1}$, we obtain:

$$\frac{c_t}{k_t} = \left( \frac{m_t}{k_t} + i_t \frac{b_t}{k_t} + r_t \right) n \tag{38}$$

$$q \frac{c_t}{k_t} = \frac{m_t}{k_t} n \tag{39}$$

Comparing (38) and (39), after canceling out $m_t/k_t$, we find

$$\frac{c_t}{k_t} = \frac{n}{1-q} \left( i_t \frac{b_t}{k_t} + r_t \right) \tag{40}$$

Dividing (13) by $k_t$ and replacing $m_t$ with the expression given by (4) taken at time $t$, we have:

$$\left( 1 + \pi_{t+1} \frac{b_{t+1}}{k_{t+1}} \right) n\gamma_{t+1} = (1-\alpha) A - \frac{q}{n} \frac{c_t}{k_t} \tag{41}$$

Putting the expression (37) in the LHS of (41) and expression (40) in the RHS, we find

$$n\gamma_{t+1} + \alpha A \pi_t \frac{b_t}{k_t} = (1-\alpha) A - \frac{q}{1-q} \alpha A \left( 1 + \pi_t \frac{b_t}{k_t} \right)$$

and solving for $\pi_t b_t / k_t$, we get

$$\pi_t \frac{b_t}{k_t} = (1-q) \frac{A - n\gamma_{t+1}}{\alpha A} - 1 \tag{42}$$

Substituting now (42) in (37), we obtain the separated dynamics (17) for $\gamma_{t+1}$. ■

# 7 References


Bertocchi G. 1991. Bubbles and inefficiencies. *Economics Letters* **35**, 117-122.

Crettez B., P. Michel and B. Wigniolle. 1999. Cash-in-advances constraints in the Diamond overlapping generations model: neutrality and optimality of monetary policy. *Oxford Economic Papers* **51**, 431–452.

Diamond P. 1965. National debt in a neoclassical growth model. *American Economic Review* **55**, 1127-1155.

Grossman G. and N. Yanagawa. 1993. Asset bubbles and endogenous growth. *Journal of Monetary Economics* **31**, 3-19.

Guillard M. 1998. Croissance, inflation et bulles. Document de recherche 98-01, EPEE, Université d'Evry.

Michel P. and B. Wigniolle. 2003. Temporary Bubbles. *Journal of Economic Theory* **112**, 173–183.

Polemarchakis and C. Rochon. 2005. Debt, liquidity and dynamics. *Economic Theory*, forthcoming.

Reichlin, P. 1986. Equilibrium Cycles in an Overlapping Generations Economy with Production. *Journal of Economic Theory* **40**, 89-103.

Romer P.M. 1986. Increasing Returns and Long-Run Growth. *Journal of Political Economy* **94**, 1002-37.

Tirole J. 1985. Asset bubbles and overlapping generations. *Econometrica* **53**, 1071-1100.

Weil P. 1987. Confidence and the real value of money in overlapping generations models. *Quarterly Journal of Economics* **102**, l-22.

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