



DOCUMENT DE RECHERCHE

EPEE

CENTRE D'ETUDES DES POLITIQUES ECONOMIQUES DE L'UNIVERSITE D'EVRY

Transaction Costs in an overlapping Generations Model

Mohanad Ismael

10-07

www.univ-evry.fr/EPEE

Université d'Evry Val d'Essonne, 4 bd. F. Mitterrand, 91025 Evry CEDEX

Transaction costs in an overlapping generations model

Mohanad ISMAEL^{*}

Work in Progress

University D'Evry Val D'Essonne / EPEE

November 8, 2010

Abstract

We study the stability properties of a Diamond (1965) overlapping generations model in which agents have to pay transaction costs related to the capital accumulated. In particular, these costs depend positively on the amount of individual's savings. At first, we show that under standard conditions, the steady state may be dynamically inefficient (efficient) if there is an over-accumulation (under-accumulation) of capital with respect to Golden Rule. Namely, the introduction of transaction costs has a negative impact on capital accumulations. It is also shown that the stationary equilibrium is determinate. Further, transaction costs promote the emergence of cycles of period two and therefore acts as a destabilizing factor. These results are robustly obtained by considering separable and non-separable preferences. The analytical findings are completed by a numerical example.

Key words: Transaction costs, Overlapping generations, Determinacy, Cycles of period two.

JEL classification: E20, E21, E30, E32.

^{*}The author gratefully acknowledges the constructive remarks from Stefano Bosi to achieve this work. I wish to thank also Riham Barbar and Thérèse Rebière for helpful suggestions. The paper benefited from comments received during presentation at the APEA, Hong Kong, 8-9 July 2010. Any errors are solely my own.

[†]Département d'économie, Université d'Evry Val d'Essonne / EPEE. 4, Boulevard François Mitterrand, 91025, Evry Cedex. Tel: + 33 1 69 47 70 96. Fax: + 33 1 69 47 70 50. E-mail address: mohanad.ismael@univ-evry.fr.

1 Introduction:

It is well known that transaction costs in asset and stock markets are considered as important factors in determining the investment portfolio. Without these costs agents can take positions in all existing assets, while the introduction of transaction costs discourages agents from purchasing these assets. For instance, if we consider two different assets without transaction costs, the portfolio choice would be a segment of these two assets. However, in the presence of transaction costs, the investment choice mainly moves to the assets with lower costs. This shows that costs have negative effect on assets' demand.

Transaction costs might consist of communication and time costs, government fees, stamp taxes, information and search costs, administration costs and brokerage commissions. Throughout the literature authors are usually interested in studying how transaction costs can influence the portfolio choice and the stock pricing. They also seek to explain why do not all households take positions in the stock market. They perform portfolio models based on stock market transaction costs in order to match the observed household's participation rate in the data. Among others, Alan (2006) supposes that the costs are paid only one time over the entire life cycle. Once the costs paid, the household is free to re-enter the stock market. He estimates that in the absence of costs, 80 to 90% of households participate in the stock market, while with only 1% costs, the percentage of households declines to around 30%. Constantinides (1986) argues that proportional costs have only a small impact on asset prices. Lo, Mamaysky and Wang (2004) consider a dynamic equilibrium model of trade volume and asset prices when agents face fixed costs. They show that the presence of these costs allows agents to trade infrequently. Further, Vayanos (1998) shows that an increase in transaction costs has two opposite effects on stock prices. On the one hand, agents buy fewer stocks and on the other hand, they hold them for longer periods. More recently, Vissing-Jorgensen (2002) considers several types of costs: fixed entry costs, proportional and per period costs. Using data from 1989-1994, he estimates that a 50 dollar of transaction costs are sufficient to explain the choices of half of the households that do not invest in the stock market. For the same period, a 260 dollar of transaction costs is enough to explain the choices of 75 percent of the nonparticipants.

In the view of above papers, the study of transaction costs seems to be plausible and relevant. However, the effect of transaction costs on economic stability or capital accumulations has not been yet treated in literature. Thus, this paper fills this gap by introducing transaction costs on saving in a standard Diamond OLG model. Contrary to infinite-horizon Ramsey model, the use of OLG framework allows formulating and studying the saving function explicitly. In the standard Diamond model, agents live two periods: youth and adulthood. In the first period, young agents supply labor inelastically and allocate their wage income between consumption and savings. When they are old, they are retired and consume their savings entirely.¹ In this paper, it is assumed that

¹Geanakoplos and Polemarchakis (1986), Reichlin (1986) and Cazzavillan (2001) suppose

young agents have to pay transaction costs related to their level of capital accumulations. Namely, these costs increase with the amount of savings but its marginal increase declines (concave transaction costs).

Our main objective is to analyze the impact of transaction costs on dynamic efficiency, capital accumulations and economic stability. In addition, the robustness of our results is shown by considering two forms of utility function: non-separable and separable in both periods consumption.

Our first result states that, under standard conditions, the steady state may be dynamically inefficient (efficient) if there is an over-accumulation (under-accumulation) of capital with respect to the Golden Rule, i.e. the net returns of capital are lower (higher) than the gross rate of population growth. Further, the presence of costs has a negative impact on capital accumulations.

From a stability point of view, the steady state of the standard Diamond model exhibits unique path stability if savings increase with the rate of return. However, it is possible to have a global indeterminate steady-state whenever the agent's saving is supposed to be a decreasing function of the interest rate, Galor and Ryder (1989).

This paper is mainly interested in the local stability properties of the steady state of Diamond model augmented to include transaction costs, without any restriction on the saving function. It is demonstrated that the steady state is determinate where there is one trajectory that converges to a unique steady state. However, it changes its stability through cycles of period two. It is found that these cycles require a sufficiently high sensitivity of transaction costs with respect to savings, low elasticity of marginal utility with respect to future consumption, high elasticity of marginal utility with respect to current consumption and a high first-period consumption share.

The intuition of these cycles is given as follows: assume that the level of current capital increases from its steady state value. This leads wage income to rise which induces more capital accumulation. However, there are some factors that influence capital accumulation negatively. The presence of high transaction costs associated with savings, the existence of low elasticity of marginal utility of future consumption, high sensitivity of marginal utility of current consumption and high consumption share enforce agents to accumulate low capital. Cycles of period two are obtained whenever the latter effects dominate the former one. Therefore, higher transaction costs make the appearance of cycles of period two more likely. In particular, we show that the range of parameters giving rise to cycles of period two widens with transaction costs.

The remainder of the paper is arranged as follows. In section 2, we present the model with a transaction costs function (the optimization problem of households and firms). The intertemporal equilibrium is presented in section 3. We present the steady state analysis in section 4. We study dynamic efficiency of the intertemporal equilibrium in section 5. In section 6, we present the local dynamics. A numerical example is located in section 7 and we conclude in section 8.

that agents consume only in the second-period.

2 The model

Consider a non-monetary overlapping generations economy with identical agents who live two periods. In each period t , N_t individuals are born and they live for two periods "young and old". In this model, there is a unique good that can be either consumed or invested. In the first period, agents are endowed with one unit of labor which is supplied inelastically to firms. They choose their amounts of consumption and saving along with income. In addition, agents have to pay variable costs related to saving amount "transaction costs". In the second period, they do not work and their income comes from the return of first-period saving.

Given the real wage w_t and the real returns R_{t+1} , agents allocate savings and consumptions for both periods to maximize the intertemporal following preferences:

$$u(c_t, d_{t+1}) \tag{1}$$

subject to the constraints

$$c_t + s_t + \xi(s_t) \leq w_t \tag{2}$$

$$d_{t+1} \leq R_{t+1} s_t \tag{3}$$

$$c_t \geq 0, d_{t+1} \geq 0 \text{ for all } t \geq 0$$

where c_t, d_{t+1} is the consumption in first "young" and second "old" period respectively, s_t is the saving, $\xi(s_t)$ is the transaction cost associated with saving².

Assumption (1) $u(c_t, d_{t+1})$ is strictly increasing with respect to each argument $u_1(c_t, d_{t+1}) > 0, u_2(c_t, d_{t+1}) > 0$, concave $u_{11}(c_t, d_{t+1}) < 0, u_{22}(c_t, d_{t+1}) < 0, C^2$ over the interior of the set $R_+^2 = [0, +\infty) \times [0, +\infty)$. Additionally, $\lim_{d_{t+1} \rightarrow +\infty} u_1(c_t, d_{t+1})/u_2(c_t, d_{t+1}) = +\infty$ and $\lim_{d_{t+1} \rightarrow 0^+} u_1(c_t, d_{t+1})/u_2(c_t, d_{t+1}) = 0$ for all $c_t, d_{t+1} > 0$. Furthermore, $\lim_{c_t \rightarrow 0^+} u_1(c_t, d_{t+1})/u_2(c_t, d_{t+1}) = +\infty$ and $\lim_{c_t \rightarrow +\infty} u_1(c_t, d_{t+1})/u_2(c_t, d_{t+1}) = 0$ for all $c_t, d_{t+1} > 0$.³

For future reference, we propose some necessary elasticities: the elasticity of marginal utility with respect to first and second argument are respectively $\varepsilon_{11} \equiv u_{11}(c, d) c / u_1(c, d) < 0, \varepsilon_{22} \equiv u_{22}(c, d) d / u_2(c, d) < 0$. The cross elasticities in consumption are $\varepsilon_{21} \equiv u_{21}(c, d) c / u_2(c, d), \varepsilon_{12} \equiv u_{12}(c, d) d / u_1(c, d)$.

Assumption (2) The cost function is increasing in its argument $\xi'(s_t) > 0$ and concave $\xi''(s_t) < 0$.

²As in standard two-period OLG models, we assume a full depreciation rate that is $s_t = K_{t+1}$.

³Notice that $u_1(c_t, d_{t+1}) \equiv \partial u(c_t, d_{t+1}) / \partial c_t, u_2(c_t, d_{t+1}) \equiv \partial u(c_t, d_{t+1}) / \partial d_{t+1}$.

In order to simplify the notation, we suppose⁴ $\varphi(s_t) = s_t + \xi(s_t)$. The Lagrangian function for household problem is:

$$L = u(c_t, d_{t+1}) + \lambda_t(w_t - c_t - \varphi(s_t)) + \mu_t(R_{t+1}s_t - d_{t+1}) \quad (4)$$

The first-order conditions with respect to c_t , d_{t+1} and s_t are respectively:

$$u_1(c_t, d_{t+1}) = \lambda_t \quad (5)$$

$$u_2(c_t, d_{t+1}) = \mu_t \quad (6)$$

$$\mu_t R_{t+1} = \varphi'(s_t) \lambda_t \quad (7)$$

this gives:

$$\frac{u_1(c_t, d_{t+1})}{u_2(c_t, d_{t+1})} = \frac{R_{t+1}}{\varphi'(s_t)} \quad (8)$$

The LHS is simply the marginal rate of substitution between consumption "today" and consumption "tomorrow". Due to the existence of increasing cost, the associated marginal rate of substitution is smaller than that of the standard Diamond model. In other words, the interest rate factor is higher than that of standard Diamond without costs which implies that agents accumulate lower capital. The model of Diamond (1965) is obtained by setting $\varphi(s_t) = s_t$.

On the production side, a representative firm uses labor and capital to produce final goods using constant returns-to-scale technology $AF(K_t, L_t)$ with $A > 0$ is a productivity scaling factor. Let $a_t = K_t/L_t$ be the capital stock per labor unit, then the production function can be written as $Af(a_t) = AF(a_t, 1)$.

Assumption (3) Let $a \geq 0$, the technology $f(a)$ is continuous and differentiable. It is increasing $f'(a) > 0$ and concave $f''(a) < 0$. Furthermore, $f(0) = 0$, $\lim_{a \rightarrow 0^+} f'(a) = +\infty$ and $\lim_{a \rightarrow +\infty} f'(a) = 0$.

Each representative firm takes real wages w_t and rental prices R_t as given. If we set $\rho(a_t) = f'(a_t)$ and $\omega(a_t) = f(a_t) - a_t f'(a_t)$, then the competitive equilibrium conditions for profit maximization entail that the real interest rate and the real wage satisfy:

$$R_t = A\rho(a_t) \text{ and } w_t = A\omega(a_t) \quad (9)$$

Thus, we can deduce that the elasticity of interest rate $a\rho'(a)/\rho(a) = -(1-\alpha)/\sigma < 0$ and the elasticity of wage $a\omega'(a)/\omega(a) = \alpha/\sigma > 0$, with $\sigma \in (0, +\infty)$ is the elasticity of capital-labor substitution while $\alpha \in (0, 1)$ is the capital share in total income.

⁴Since $\xi(s)$ is increasing, the function $\varphi(s_t)$ has the same properties as mentioned in Assumption (2).

3 Intertemporal equilibrium

The number of households at each generation grows at a constant rate $n > -1$, such that $1 + n = N_{t+1}/N_t$, where N_t is the number of population born at time t . At equilibrium, three markets clear:

1. Capital market clears according to capital-accumulation equation: $K_{t+1} = N_t s_t$.
2. Labor market clears: $L_t = N_t$.
3. By Walras' law, output market also clears: $N_t (c_t + \varphi(s_t)) + N_{t-1} d_t = AF(K_t, L_t)$.

From market clearing conditions, one can demonstrate that:

$$s_t = (1 + n) a_{t+1} \quad (10)$$

Substituting (10) and condition (9) together with the binding budget constraints (2) and (3) into (8) yields the following one-dimensional dynamic system of a .

$$\frac{u_1 [A\omega(a_t) - \varphi[a_{t+1}(1+n)], A\rho(a_{t+1})a_{t+1}(1+n)]}{u_2 [A\omega(a_t) - \varphi[a_{t+1}(1+n)], A\rho(a_{t+1})a_{t+1}(1+n)] A} - \frac{\rho(a_{t+1})}{\varphi'[a_{t+1}(1+n)]} = 0 \quad (11)$$

4 The steady state

At the steady state $a_{t+1} = a_t = a$, so the dynamic system (11) becomes:

$$\frac{u_1 [A\omega(a) - \varphi[a(1+n)], A\rho(a)a(1+n)]}{u_2 [A\omega(a) - \varphi[a(1+n)], A\rho(a)a(1+n)] A} - \frac{\rho(a)}{\varphi'[a(1+n)]} = 0 \quad (12)$$

To simplify the analysis, we follow the method initiated by Cazzavillan et al. (1998) by using a scaling parameter A in order to give conditions for existence of a normalized steady state $a = 1$.

Proposition 1 *Under Assumptions (1) – (3), $a = 1$ is a steady state of the dynamic system (11) if and only if there exists a scaling parameter A such that $A > \hat{A} \equiv \varphi[1+n]/\omega(1)$ and satisfies*

$$\frac{u_1 [A\omega(1) - \varphi[1+n], A\rho(1)(1+n)]}{u_2 [A\omega(1) - \varphi[1+n], A\rho(1)(1+n)] A} = \frac{\rho(1)}{\varphi'[1+n]} \quad (13)$$

The scaling parameter A is a unique solution of (13) if and only if: $(\varepsilon_{11} - \varepsilon_{21})/\gamma + \varepsilon_{12} - \varepsilon_{22} - 1 < 0$ for all A .⁵

⁵We denote $\gamma \equiv c/A\omega(a)$ as the share of first-period consumption over wage income.

Proof. The solution $a = 1$ is a steady state if and only if (13) is verified. Moreover, the positivity of first-period consumption requires $A > \hat{A} \equiv \varphi[1+n]/\omega(1)$, so $A \in (\hat{A}, +\infty)$. Let us call the LHS as:

$$G(A) \equiv \frac{u_1 [A\omega(1) - \varphi[1+n], A\rho(1)(1+n)]}{u_2 [A\omega(1) - \varphi[1+n], A\rho(1)(1+n)] A}$$

since it is a continuous function, then based on Assumption (1), it is easy to show that $\lim_{A \rightarrow \hat{A}} G(A) = +\infty$ and $\lim_{A \rightarrow +\infty} G(A) = 0$. Since the RHS is a positive constant, thus there is a steady state for $a = 1$. Concerning the uniqueness of A , it is enough to show that $G(A)$ is monotonic, i.e., $(\varepsilon_{11} - \varepsilon_{21})/\gamma + \varepsilon_{12} - \varepsilon_{22} - 1 < 0$ is satisfied⁶. ■

Assumption (4) The utility function $u(c_t, d_{t+1})$ is homogenous of degree one.

Corollary (1) Under Assumptions (1) – (4), $a = 1$ is a steady state for the dynamic system (11) if and only if there exists a scaling parameter A such that $A > \hat{A} \equiv \varphi[1+n]/\omega(1)$ and satisfies

$$\frac{u_1 \left[\frac{A\omega(1) - \varphi[1+n]}{A\rho(1)(1+n)}, 1 \right]}{u_2 \left[\frac{A\omega(1) - \varphi[1+n]}{A\rho(1)(1+n)}, 1 \right] A} = \frac{\rho(1)}{\varphi'[1+n]} \quad (14)$$

In this case, the scaling parameter A is unique.

Proof. Since the utility function is homogenous of degree one, equality (12) can be written as (14). The solution $a = 1$ is a steady state if and only if equality (14) is satisfied. Notice that the RHS does not change and $A \in (\hat{A}, +\infty)$. If we denote the LHS by Q :

$$Q(A) \equiv \frac{u_1 \left[\frac{\omega(1)}{\rho(1)(1+n)} - \frac{\varphi[1+n]}{A\rho(1)(1+n)}, 1 \right]}{u_2 \left[\frac{\omega(1)}{\rho(1)(1+n)} - \frac{\varphi[1+n]}{A\rho(1)(1+n)}, 1 \right] A} \quad (15)$$

Using Assumption (1), it is easy to show that $\lim_{A \rightarrow \hat{A}} Q(A) = +\infty$ and $\lim_{A \rightarrow +\infty} Q(A) = 0$. Further, from homogeneity property of the utility function, one can prove that $u_{21} > 0$. Consequently, a direct inspection of (15) gives that $Q'(A) = [[\varphi[1+n](u_2 u_{11} - u_1 u_{21})/A\rho(1)(1+n)] - u_1 u_2] / (u_2 A)^2 < 0$ which implies that there is a unique scaling parameter A satisfying (14). ■

Assumption (5) The utility function is separable.

⁶If this inequality is satisfied for all A , then there exists a unique A satisfying (13).

Corollary (2) *Let Assumptions (1) – (3) and (5) be satisfied, then $a = 1$ is a steady state for the dynamic system (11) if there exists a scaling parameter A such that $A > \hat{A} \equiv \varphi[1+n]/\omega(1)$ and satisfies*

$$\frac{v'[A\omega(1) - \varphi[1+n]]}{v'[A\rho(1)(1+n)]} \frac{1}{A} = \frac{\beta\rho(1)}{\varphi'[1+n]} \quad (16)$$

where β is the discount factor. *Further, the scaling parameter A is unique.*

Proof. In a separable case, household's problem is simplified at the steady state to (16). As before, let us call the LHS as:

$$\Pi(A) \equiv \frac{v'[A\omega(1) - \varphi[1+n]]}{v'[A\rho(1)(1+n)]} \frac{1}{A}$$

and A belongs to $(\hat{A}, +\infty)$, then based on Assumption (1) we have $\lim_{A \rightarrow \hat{A}} \Pi(A) = +\infty$ and $\lim_{A \rightarrow +\infty} \Pi(A) = 0$. In order to show the existence of a unique A , it is easy to demonstrate that $\Pi(A)$ is always decreasing i.e., $\varepsilon(1/\gamma - 1) - 1 < 0$, where ε is the elasticity of marginal utility of consumption. ■

Throughout the paper, it is supposed that the above propositions hold for each configuration.

5 Dynamic efficiency

In this section, we analyze the dynamic efficiency of the steady state. Before passing through efficiency analysis, let us define the following useful elasticities: the elasticity of transaction cost with respect to savings $\eta_1 \equiv \varphi'(s)s/\varphi(s) > 0$, the elasticity of marginal transaction cost $\eta_2 \equiv \varphi''(s)s/\varphi'(s) < 0$.

Using above intertemporal equilibrium conditions, we obtain the following stationary resource constraint:

$$c + \frac{d}{1+n} = \Sigma(a) \quad (17)$$

with

$$\Sigma(a) \equiv Af(a) - \varphi[(1+n)a] \quad (18)$$

is the net production and the LHS is simply the stationary aggregate consumption⁷.

Assumption (6): Assume that $-(1-\alpha)/\sigma < \eta_2$.

This assumption is necessary to confirm that the net production $\Sigma(a)$ is concave. Subsequently, this ensures the existence of a unique positive capital-labor ratio that maximizes the net production and so allocates the maximum

⁷Similar to De la Croix and Michel (2002), we define the net production as the production minus investment and its related costs.

amount of consumptions⁸. In order to characterize the Golden Rule capital-labor ratio, we need to make the following assumption.

Assumption (7) Assume that:

$$\begin{aligned} \lim_{a \rightarrow 0^+} Af'(a) &> \lim_{a \rightarrow 0^+} (1+n) \varphi' [(1+n)a] \\ \lim_{a \rightarrow +\infty} Af'(a) &< \lim_{a \rightarrow +\infty} (1+n) \varphi' [(1+n)a] \end{aligned}$$

Following Phelps (1965) and Diamond (1965), we define the Golden Rule level of capital-labor ratio⁹.

Definition (1) (Golden Rule) Under Assumptions (6) and (7), there exists a unique positive capital stock per young agents such that:

$$\frac{Af'(\bar{a})}{\varphi'[(1+n)\bar{a}]} = 1+n \quad (19)$$

with \bar{a} is the Golden Rule capital-labor ratio.

In other words, the Golden Rule (19) determines the level of capital in which the net marginal productivity of capital equals the gross rate of population growth. The Golden Rule capital does not depend on consumption allocations in both periods. At the same time, this level of capital provides the highest level of consumptions.

Proposition 2 Under Assumptions (6)–(7), there is a unique optimal stationary path, the Golden Rule, which is characterized by $a = \bar{a}$ and by \bar{c}, \bar{d} satisfying the following conditions:

$$\bar{c} + \frac{\bar{d}}{1+n} = \Sigma(\bar{a}) \quad (20)$$

$$u_1(\bar{c}, \bar{d}) = (1+n) u_2(\bar{c}, \bar{d}) \quad (21)$$

Proof. The maximum of $\Sigma(\bar{a})$ ¹⁰ is satisfied using the Golden Rule (19) and the optimal allocation of first-period and second-period consumptions (\bar{c}, \bar{d}) that maximize household's preferences (1) under the constraint (20) is illustrated by the first-order necessary condition (21). ■

Definition (2) (Feasible path of capital) A sequence of capital stock per young agents $a_t \geq 0$ is a feasible path if the corresponding production net of investment i.e., $\Sigma(a_t, a_{t+1}) \equiv Af(a_t) - \varphi[(1+n)a_{t+1}] \geq 0$ is non-negative for all $t > 0$.

⁸If this Assumption is violated, we can not determine the stationary capital-labor ratio that maximizes the net production.

⁹The term "Golden Rule" was introduced by Phelps (1961).

¹⁰ $\Sigma(\bar{a})$ is defined in (18).

Definition (3) (Efficiency) A feasible sequence of capital per young agents $\{a_t\}_{t \geq 0}$ is efficient if it is impossible to raise agent's consumption at one date without reducing it at another date, i.e., if there does not exist another feasible path $\{\bar{a}_t\}_{t \geq 0}$ with $\bar{a}_0 = a_0$ such that:

- (i) $\Sigma(\bar{a}_t, \bar{a}_{t+1}) \geq \Sigma(a_t, a_{t+1})$, for all $t \geq 0$;
- (ii) $\Sigma(\bar{a}_t, \bar{a}_{t+1}) > \Sigma(a_t, a_{t+1})$, for some $t \geq 0$.

Now, let us consider a feasible path of capital-labor ratio a_t where this path converges to the normalized steady state value $a^* = 1$. Then, we deduce the following result:

Proposition 3 Under Assumptions (6) and (7), then:

- (i) The steady state is characterized by over-accumulation of capital for $Af'(a^*)/\varphi'[(1+n)a^*] < 1+n$ and the competitive equilibrium is dynamically inefficient.
- (ii) The steady state is characterized by under-accumulation of capital for $Af'(a^*)/\varphi'[(1+n)a^*] > 1+n$ and the competitive equilibrium is dynamically efficient.

Proof. See Appendix (B). ■

Dynamic efficiency in terms of aggregate consumptions states that it is not possible to raise total consumption at one date without reducing it in another date. Proposition (3) states that there is an under-accumulation (or an over-accumulation) of capital comparing to the Golden Rule level if the *net* capital rate of return is higher (or lower) than the gross rate of population growth.

Corollary (3) Let Assumptions (1) - (3), (6) and (7) be satisfied, then comparing to standard Diamond model the steady state is characterized by an under-accumulation of capital.

Proof. Let us define the Golden-Rule level of capital in Diamond (1965) as a^D where a^D satisfies $Af'(a^D) = 1+n$. However, the Golden-Rule level of capital with the presence of costs is \bar{a} such as $Af'(\bar{a}) = (1+n)\varphi'[(1+n)\bar{a}]$. Given that $\varphi'[(1+n)\bar{a}] > 1$, this implies that $Af'(\bar{a}) = (1+n)\varphi'[(1+n)\bar{a}] > (1+n) = Af'(a^D)$. Therefore, capital accumulations with costs \bar{a} is lower than that in standard Diamond model without costs a^D . ■

6 Local dynamics

In this section, we study the economic stability locally around the normalized steady state $a = 1$. It is demonstrated that the introduction of transaction costs in a standard OLG à la Diamond affects the appearance of cycles of period two.

Linearizing the dynamic equation (11) around the steady state $a = 1$ yields the following eigenvalue $J \equiv da_{t+1}/da_t$:

$$J = \frac{\alpha \left(\frac{\varepsilon_{12}}{\eta_1} \frac{1}{1-\gamma} - \frac{\varepsilon_{11}}{\gamma} \right)}{\sigma \left(2\varepsilon_{12} - \varepsilon_{22} - \eta_1 \frac{1-\gamma}{\gamma} \varepsilon_{11} + \eta_2 \right) + (1-\alpha)(1-\varepsilon_{12} + \varepsilon_{22})} \quad (22)$$

Notice that a_t is a predetermined variable, therefore the steady state of system (11) is determinate. Further, the steady state is stable whenever the unique eigenvalue belongs to the interior of the unit circle, i.e. belongs to the interval $(-1, 1)$. The second-order conditions associated with household problem imply that¹¹

$$2\varepsilon_{12} - \varepsilon_{22} - \eta_1 \frac{1-\gamma}{\gamma} \varepsilon_{11} + \eta_2 > 0$$

A sufficient condition for the emergence of cycles of period two is generically $J(\sigma) = -1$. This holds at $\sigma = \sigma^F$, where

$$\sigma^F \equiv - \frac{\alpha \left(\frac{\varepsilon_{12}}{\eta_1} \frac{1}{1-\gamma} - \frac{\varepsilon_{11}}{\gamma} \right) + (1-\alpha)(1-\varepsilon_{12} + \varepsilon_{22})}{2\varepsilon_{12} - \varepsilon_{22} - \eta_1 \frac{1-\gamma}{\gamma} \varepsilon_{11} + \eta_2} \quad (23)$$

Before going on, we present some critical values for η_1 , γ , ε_{22} and ε_{11} .

$$\begin{bmatrix} \eta_1^* \\ \gamma^* \\ \varepsilon_{11}^* \\ \varepsilon_{22}^* \end{bmatrix} \equiv \begin{bmatrix} \varepsilon_{12} / \left[(1-\gamma) \left(\varepsilon_{11} / \gamma - \frac{1-\alpha}{\alpha} (1-\varepsilon_{12} + \varepsilon_{22}) \right) \right] \\ \alpha \varepsilon_{11} / \left[(1-\alpha) (1-\varepsilon_{12} + \varepsilon_{22}) \right] \\ (1-\alpha) (1-\varepsilon_{12} + \varepsilon_{22}) / \alpha \\ \varepsilon_{12} - 1 \end{bmatrix} \quad (24)$$

In the next proposition, we present the sufficient conditions for the appearance of cycles of period two.

Proposition 4 *In view of Assumptions (1) - (3) together with (24), flip bifurcation exists when σ is close to σ^F if one of the following conditions holds:*

1. For $\varepsilon_{11} > \varepsilon_{11}^*$, $\varepsilon_{22} < \varepsilon_{22}^*$ and $\gamma > \gamma^*$ with either: (i) $\varepsilon_{12} > 0$, $\eta_1 > \eta_1^*$; or (ii) $\varepsilon_{12} < 0$, $\eta_1 > 0$.
2. For $\varepsilon_{12} < 0$, $\varepsilon_{22} < \varepsilon_{22}^*$ and $\eta_1 < \eta_1^*$ with either: (i) $\varepsilon_{11} < \varepsilon_{11}^*$, for all $\gamma > 0$; or (ii) $\varepsilon_{11} > \varepsilon_{11}^*$, $\gamma < \gamma^*$.
3. For $\varepsilon_{12} < 0$, $\varepsilon_{22} > \varepsilon_{22}^*$ and $\eta_1 < \eta_1^*$ for all $\gamma > 0$ and $\varepsilon_{11} < 0$.

Proof. See Appendix (C). ■

This proposition shows that flip cycles arise for different configurations concerning the preferences, the cost function and the first-period consumption share. In order to study the mechanism behind the emergence of these cycles, let us initially focus on the Benchmark model without costs.

¹¹See Appendix (A).

7 Discussion of the results

In order to understand the role of transaction costs on economic stability, let us investigate the cases without costs in separable and non-separable preferences respectively.

7.1 Benchmark model

7.1.1 Non-separable utility

We recover the basic model studied by Diamond (1965) where agents are not imposed to pay transaction costs by setting $\eta_1 = 1$ and $\eta_2 = 0$. Then, $J(\sigma) = -1$ holds at $\sigma = \sigma_{BM}^F$ where

$$\sigma_{BM}^F = -\frac{\alpha \left(\varepsilon_{12} \frac{1}{1-\gamma} - \frac{\varepsilon_{11}}{\gamma} \right) + (1-\alpha)(1-\varepsilon_{12} + \varepsilon_{22})}{2\varepsilon_{12} - \varepsilon_{22} - \frac{1-\gamma}{\gamma} \varepsilon_{11}} \quad (25)$$

where $2\varepsilon_{12} - \varepsilon_{22} - \frac{1-\gamma}{\gamma} \varepsilon_{11} > 0$ by SOCs. Given the critical values $\varepsilon_{22}^{bb} \equiv \varepsilon_{12} - 1$, $\varepsilon_{12}^b \equiv -(1-\alpha)/(2\alpha-1)$, $\varepsilon_{22}^b \equiv -\varepsilon_{12}(2\alpha-1)/(1-\alpha) - 1$, $\varepsilon_{11}^b \equiv \varepsilon_{12}\gamma/(1-\gamma) + (1-\alpha)(1-\varepsilon_{12} + \varepsilon_{22})\gamma/\alpha$ and $\gamma^b \equiv (\alpha\varepsilon_{12}/(1-\alpha)(1-\varepsilon_{12} + \varepsilon_{22})) + 1$, then, the positivity of σ_{BM}^F in (25) requires one of the following conditions:¹²

1. $\varepsilon_{11} > \varepsilon_{11}^b$ and $\gamma < \gamma^b$ for *either* (i) $\varepsilon_{12} > \varepsilon_{12}^b$, $\varepsilon_{22} < \varepsilon_{22}^{bb}$, *or* (ii) $0 < \varepsilon_{12} < \varepsilon_{12}^b$ and $\varepsilon_{22} < \varepsilon_{22}^b$.
2. $\varepsilon_{11} > \varepsilon_{11}^b$ and $\varepsilon_{12} < 0$ for *either* (i) $\varepsilon_{22} < \varepsilon_{22}^{bb}$ for all $\gamma > 0$, *or* (ii) $\varepsilon_{22} > \varepsilon_{22}^b$ and $\gamma > \gamma^b$.

Cycles of period two appear if capital increases in the current period and then it decreases in the following period. One can easily observe from (25) that the appearance of cycles of period two depends on agents' preferences and on first-period consumption share. Notice that, a low (high) ε_{22} means that as second-period consumption increases, its marginal utility declines significantly (slightly). In addition, a high (low) ε_{11} implies that as first-period consumption augments, its marginal utility declines slightly (significantly).

The intuition for the existence of cycles of period two is the following: Focus on case (1) and suppose that K_t increases from its steady state value, then w_t augments which induces a higher capital accumulation K_{t+1} . The presence of a sufficiently high ε_{11} and a small ε_{22} encourages agents to raise current consumption and to reduce future one, and thus, to accumulate low of capital. However, the presence of a small γ has a positive effect on capital accumulation, it enforces agents to consume less today and to accumulate more capital. As a result, flip cycles require that the first effect (ε_{11} and ε_{22}) dominates the effect of γ . In case (2), the economic intuition is mainly the same.

¹²Computations are available from the author upon request.

7.1.2 Separable utility

The separability of preferences can be obtained by setting $\varepsilon_{12} = 0$. For simplicity, it is supposed that agents have the same utility in both periods, then flip cycles emerge at $\sigma = \sigma_{BM,S}^F$, where

$$\sigma_{BM,S}^F \equiv \frac{\alpha\varepsilon/\gamma - (1-\alpha)(1+\varepsilon)}{-\varepsilon\left(1 + \frac{1-\gamma}{\gamma}\right)} \quad (26)$$

Where ε is the elasticity of marginal utility in consumption and so the elasticity of intertemporal substitution in consumption is given by $-1/\varepsilon$. Notice that if we assume high substitutability between consumptions in both periods, that is, $1+\varepsilon > 0$, then the result of Diamond (1965) is obtained with $\sigma_{BM,S}^F < 0$ which implies a unique stable steady state. However, since it is not the case here, then $\sigma_{BM,S}^F > 0$ for $\varepsilon < \min(-1, \varepsilon^{sb})$ with $\varepsilon^{sb} \equiv (1-\alpha)/(2\alpha-1)$ and for $\gamma > \gamma^{sb}$ with $\gamma^{sb} \equiv \alpha\varepsilon/((1-\alpha)(1+\varepsilon))$. Hence, income effect dominates substitution effect and so agents are not interested in future consumptions. The presence of high consumption share together with low elasticity of substitution makes agents more incentive to accumulate low capital. This means that a rise K_t in current period is followed by a decline in K_{t+1} in next period.

Furthermore, the eigenvalue (22) is simplified to¹³

$$J = \frac{\alpha}{\sigma} \frac{-\varepsilon}{\frac{\gamma(1-\alpha)(1+\varepsilon)}{\sigma} - \varepsilon} > 0 \quad (27)$$

Equation (27) argues that the unique equilibrium path of standard Diamond (1965) is recovered as $\sigma > \alpha$.¹⁴ However, Nourry (2001) recovers Diamond with a range of elasticity of input substitution such that $\sigma \geq 1$. In order to clarify more, let us take logarithmic formulations for the utility functions, i.e., $\varepsilon = -1$ with a Cobb-Douglas technology, $\sigma = 1$. Therefore, the eigenvalue (27) is simplified to $J = \alpha \in (0, 1)$, so a unique-path steady state.

7.2 Our model

7.2.1 Separable preferences

As before, it is supposed that agents have the same utility in both periods, i.e., $\varepsilon_{11} = \varepsilon_{22} = \varepsilon$. The eigenvalue (22) is simplified to:

$$J = \frac{-\alpha\frac{\varepsilon}{\gamma}}{\sigma\left(\eta_2 - \varepsilon - \eta_1\frac{1-\gamma}{\gamma}\varepsilon\right) + (1-\alpha)(1+\varepsilon)} \quad (28)$$

¹³Diamond finds that the steady state exhibits a saddle-path stability if and only if the elasticity of saving with respect to interest is not negative, which means a high elasticity of intertemporal substitution in consumption. Simply, in this model, high substitutability implies $1+\varepsilon > 0$.

¹⁴Cazzavillan and Pintus (2004) find that endogenous fluctuations require $\sigma < \alpha$.

The necessary condition for the existence of cycles of period two is $1 + \varepsilon < 0$. Along with (28), the numerator is positive and the SOCs state that $\eta_2 - \varepsilon - \eta_1 \frac{1-\gamma}{\gamma} \varepsilon > 0$. Flip cycles arise at $\sigma = \sigma_S^F$ where

$$\sigma_S^F \equiv \frac{\alpha\varepsilon/\gamma - (1-\alpha)(1+\varepsilon)}{\eta_2 - \varepsilon - \eta_1 \frac{1-\gamma}{\gamma} \varepsilon} \quad (29)$$

Remark that cycles appear for the same conditions as in the Benchmark model without costs, i.e., for $\varepsilon < \min(-1, \varepsilon^{sb})$ and $\gamma > \gamma^{sb}$. This provides that transaction costs do not affect the appearance of these cycles and the existence of cycles requires $\alpha\varepsilon/\gamma > (1-\alpha)(1+\varepsilon)$. In this case, the SOCs dominate the effect of transaction costs.

7.2.2 Non-separable preferences

In the non-separable case, Proposition (4) summarizes the conditions under which flip cycles appear. We only focus on case (1) of Proposition 4, since other cases have similar intuition.

Suppose that, at period t , capital stock K_t increases from its value of the steady state which augments the wage w_t which induces more capital accumulation. The presence of high ε_{11} and γ and small ε_{22} induces agents to consume more today and to lower their capital accumulation and future consumption. Then, this gives rise to two subcases:¹⁵

From one hand, whenever $\varepsilon_{12} > 0$, this effect is offset because a reduction in future consumption decreases the marginal utility from first-period consumption and thus leads agents to reduce present consumption. Hence, in order to ensure a reduction of capital accumulation and therefore the emergence of cycles of period two, a sufficiently high sensitivity of costs is required, $\eta_1 > \eta_1^*$.

From the other hand, whenever $\varepsilon_{12} < 0$, then a reduction in future consumption increases the marginal utility from the first-period consumption and thus induces agents to augment present consumption and to reduce their capital accumulation, resulting in the emergence of cycles of period two. It is important to notice that in this subcase, cycles appear without any restriction on transaction costs.

In the numerical example, we clarify the effect of transaction costs on stability range in both separable and non-separable cases with isoelastic cost formulation.

8 Numerical example

In this section, we confirm numerically our theoretical results presented in the previous section. Let us consider a CES production function $f(a) =$

¹⁵If the utility function exhibits homogenous of degree one, then $\varepsilon_{12} > 0$, as a result, the conditions under which two-period cycles appear are summarized in case (1) of Proposition (4).

$A[\alpha\alpha^{-\varkappa} + (1 - \alpha)]^{-1/\varkappa}$ with $\alpha \in (0, 1)$, $A > 0$, $\varkappa > -1$ and $\varkappa \neq 0$ and an isoelastic cost function: $\varphi(s) = s^\eta$ with $\eta \in (0, 1)$ ¹⁶. The annual ratio of personal consumption expenditures over GDP have an average of 0.65 over the period (1959 - 2008) for US economy.¹⁷ Our objective is to determine the critical values under which flip cycles appear, i.e. $\sigma^F > 0$.

8.1 Non-separable preferences

As Venditti (2003), we consider the following utility function

$$u(c, d) = \frac{1}{\theta} [\zeta c^{-\rho} + (1 - \zeta) d^{-\rho}]^{-\frac{\theta}{\rho}} \quad (30)$$

with $\theta \leq 1$, $\zeta \in (0, 1)$ and $\rho > -1$, the discount factor is $(1 - \zeta)/\zeta$ and the elasticity of intertemporal substitution is $1/(1 + \rho)$ and $u_{12} < 0$ if and only if $\rho + \theta < 0$. Given the above technology and the cost function, we get: $c = A(1 - \alpha) - (1 + n)^\eta$ and $d = A\alpha(1 + n)$. Using the above consumption values, then the steady state value of A can be implicitly obtained using condition (13):

$$\frac{\zeta}{1 - \zeta} (A(1 - \alpha) - (1 + n)^\eta)^{-\rho-1} \eta - (1 + n)^{-\rho-\eta} A^{-\rho} \alpha^{-\rho} = 0 \quad (31)$$

The positivity of first-period consumption requires that

$$A > \frac{(1 + n)^\eta}{1 - \alpha} \equiv \hat{A} \quad (32)$$

The steady state value of γ can be endogenously obtained using:

$$\gamma = 1 - \frac{(1 + n)^\eta}{A(1 - \alpha)} \quad (33)$$

One can directly show that the elasticities of preferences as:

$$\begin{aligned} \varepsilon_{11} &= (\theta + \rho) \frac{\zeta c^{-\rho}}{\zeta c^{-\rho} + (1 - \zeta) d^{-\rho}} - (\rho + 1) \\ \varepsilon_{22} &= (\theta + \rho) \frac{(1 - \zeta) d^{-\rho}}{\zeta c^{-\rho} + (1 - \zeta) d^{-\rho}} - (\rho + 1) \\ \varepsilon_{12} &= (\theta + \rho) \frac{(1 - \zeta) d^{-\rho}}{\zeta c^{-\rho} + (1 - \zeta) d^{-\rho}} \end{aligned} \quad (34)$$

Let us set $\alpha = 0.33$, $\zeta = 0.5$, $n = 0.5175$, $\rho = 7$, $\theta = 0.5$, then using (31), we obtain $A = 4.1816 > 1.6915 = \hat{A}$. Given A , we obtain $\gamma = 0.59550$. Given these values, then we get: $\varepsilon_{12} = 1.2697$, $\varepsilon_{11} = -1.7697$, $\varepsilon_{22} = -6.7303$. The

¹⁶Consistent with previous notations together with an isoelastic cost function $\varphi(s) = s^\eta$, we obtain $\eta_1 = \eta$ and $\eta_2 = \eta - 1$.

¹⁷For more information, see Economic Report of the President, 2008.

SOCs is verified as well $2\varepsilon_{12} - \varepsilon_{22} - \eta^{\frac{1-\gamma}{\gamma}}\varepsilon_{11} + \eta - 1 = 8.9303 > 0$ and finally $\sigma^F = 0.02872 > 0$.

η	0.3	0.5	0.7	0.9
σ^F	0.02872	4.9344×10^{-2}	5.8739×10^{-2}	6.3552×10^{-2}

From the table, we observe that $\partial\sigma^F/\partial\eta > 0$ and further $\varepsilon_{12} > 0$ and so $\partial J/\partial\sigma < 0$. Notice that the basic model without costs is recovered by setting $\eta = 1$. Hence, the more η is far from 1, the higher sensitivity of costs is. As a result, transaction costs act as a destabilizing factor in the sense that it widens the range of parameters giving rise to cycles of period two, that is, $(\sigma^F, +\infty)$.

8.2 Separable preferences

Suppose that agents have the same utility in both periods with a CIES preferences:

$$v(x) = \frac{x^{1-\delta}}{1-\delta}, \delta > 0 \text{ and } x = c, d \quad (35)$$

One can easily show that the elasticity of marginal utility in consumption $\varepsilon = -\delta$. As before, the steady state value of A can be obtained using (16) together with above cost and production functions. Then, we get:

$$(A(1-\alpha) - (1+n)^\eta)^{-\delta} \eta - \beta\alpha^{1-\delta} (1+n)^{1-\eta-\delta} A^{1-\delta} = 0 \quad (36)$$

where A is restricted to positivity condition of first-period consumption (32). Additionally, the share of consumption is obtained by (33).

Let us set $\alpha = 0.33$, $\delta = 4.44$, $n = 0.5175$ and $\beta = 0.3$. Therefore, $\varepsilon = -4.44 < -1.9706 \equiv \varepsilon^{sb}$. As a result, we get the following table:

η	0.3	0.5	0.7	0.9
σ_S^F	2.6044×10^{-6}	3.5750×10^{-2}	0.05233	6.1279×10^{-2}

Transaction costs influence the stability region through its effects on σ_S^F . Consequently, one can easily obtain that $\partial\sigma_S^F/\partial\eta > 0$ and from (28) $\partial J/\partial\sigma < 0$. Similar to previous explanation, transaction costs act as a destabilizing factor. For all values of η , the steady state value of A (32) is verified as well as the SOC.

9 Conclusion

This paper analyzes the economic stability in an overlapping generations model with exogenous labor supply. We extend the standard one-dimension OLG by introducing transaction costs related to the amount of investment. Young agents consume and save according to the wage income, while in the next period, old agents who are retired consume all their saving returns. We mainly consider

two different aspects with respect to household preferences. Initially, we focus on a general non-separable formulation of preferences then it is supposed to be separable. It is shown that the presence of transaction costs with respect to saving promote cycles and it is proved that these costs act as a destabilizing factor. It is also demonstrated that under specific conditions, the steady state may be dynamically inefficient (or efficient) if there is an over-accumulation (or under-accumulation) of capital with respect to the Golden Rule, i.e. the net return of capital is higher (or lower) than the population growth. Comparing to the model of Diamond (1965) where he proposes a high substitutability between current and future consumptions, the main contribution of this paper is the emergence of cycles of period two. This paper generalizes the stability condition of a steady state equilibrium obtained by Diamond.

10 Appendix

(A) Sufficient conditions for utility maximization

Using the Lagrangian function (4), we calculate the associated Hessian matrix with respect to $(\lambda_t, \mu_t, c_t, d_{t+1}, s_t)$ ¹⁸:

$$H \equiv \begin{bmatrix} 0 & 0 & -1 & 0 & -\varphi' \\ 0 & 0 & 0 & -1 & r \\ -1 & 0 & u_{11} & u_{12} & 0 \\ 0 & -1 & u_{12} & u_{22} & 0 \\ -\varphi' & r & 0 & 0 & -\lambda_1 \varphi'' \end{bmatrix}$$

Household problem is considered as a maximization problem if and only if the determinant of the leading principal minors of above Hessian matrix changes its sign. If the determinant of H has the same sign as $(-1)^n$ and the last $n - m$ diagonal principal minors have alternative signs. Here, the number of variables $n = 3$ and the number of constraints $m = 2$. Thus, the optimum is a local maximum only if $\det H < 0$. We need to find the conditions under which the matrix H is negative definite (negative semi-definite) over the set of values satisfying the first-order conditions and the constraints. Therefore,

$$\text{Det } H = r^2 u_{22} - 2r\varphi' u_{12} - \varphi'' \lambda_1 + (\varphi')^2 u_{11} < 0$$

Using (8) and the FOCs (5), (6) and (7), we obtain a lower bound for the elasticity of transaction cost with respect to savings, that is,

$$\eta_1 \frac{1 - \gamma}{\gamma} \varepsilon_{11} - 2\varepsilon_{12} + \varepsilon_{22} < \eta_2 \quad (37)$$

(B) Proof of Proposition (3)

¹⁸For simplicity, we omit the arguments and the time subscripts related with the functions.

Proof. According to definitions (1) – (3) and Assumptions (6) and (7), over-accumulation of capital gives $Af'(a^*)/\varphi'[(1+n)a^*] < 1+n$. This demonstration is based on previous work of De la Croix and Michel (2002) and Drugeon et al. (2010). We have to prove that we can decrease capital stock and raise consumption at one date without reducing consumption at another date. In a neighborhood $(a^* - 2\kappa, a^* + 2\kappa)$ of a^* , we have $Af'(a)/\varphi'[(1+n)a] < 1+n$. After some date t_0 , we have $a_t \in (a^* - \kappa, a^* + \kappa)$ with $Af'(a_t)/\varphi'[(1+n)a_{t+1}] < 1+n$ and $Af'(a_t - \kappa)/\varphi'[(1+n)(a_{t+1} - \kappa)] < 1+n$. The concavity of $f(\cdot)$ and $\varphi[\cdot]$ implies respectively:

$$Af(a - \kappa) - Af(a) \geq -Af'(a - \kappa)\kappa$$

and

$$\varphi[(1+n)(a - \kappa)] - \varphi[(1+n)a] \geq -\varphi'[(1+n)(a - \kappa)](1+n)\kappa$$

Let us decline capital stock by κ after date t_0 and forever. Investment a_{t_0+1} is reduced by κ and consumption $\Sigma(a_{t_0}, a_{t_0+1})$ is increased by $\varphi'[(1+n)a_{t_0+1}](1+n)\kappa$. At date $t > t_0$, the new consumption level is:

$$\begin{aligned} & \Sigma(a_t - \kappa, a_{t+1} - \kappa) \\ = & Af(a_t - \kappa) - \varphi[(1+n)(a_{t+1} - \kappa)] \\ \geq & Af(a_t) - Af'(a_t - \kappa)\kappa - (\varphi[(1+n)a_{t+1}] - \varphi'[(1+n)(a_{t+1} - \kappa)](1+n)\kappa) \\ \geq & Af(a_t) - \varphi[(1+n)a_{t+1}] + [\varphi'[(1+n)(a_{t+1} - \kappa)](1+n) - Af'(a_t - \kappa)]\kappa \\ > & Af(a_t) - \varphi[(1+n)a_{t+1}] = \Sigma(a_t, a_{t+1}) \end{aligned}$$

So, consumption can be increased for all future periods and the path is dynamically inefficient.

Now, we go forward to show that whenever $Af'(a^*)/\varphi'[(1+n)a^*] > 1+n$ then there exists an under-accumulation of capital. To prove this, it is enough to show the impossibility of raising one period t_1 consumption without reducing other period's consumption. Moreover, $Af'(a^*)/\varphi'[(1+n)a^*] > 1+n$ gives that $Af'(a^*)/\varphi'[(1+n)a^*] > b(1+n)$ with some $b > 1$. Along an equilibrium path and for $t \geq t_0$, we have $Af'(a_t)/\varphi'[(1+n)a_t] > b(1+n)$. At any date t , the difference from another feasible path \tilde{a}_t satisfies:

$$\begin{aligned} \Delta C_t &= Af(\tilde{a}_t) - Af(a_t) - (\varphi[(1+n)\tilde{a}_{t+1}] - \varphi[(1+n)a_{t+1}]) \\ &\leq Af'(a_t)(\tilde{a}_t - a_t) - \varphi'[(1+n)a_{t+1}](1+n)(\tilde{a}_{t+1} - a_{t+1}) \end{aligned}$$

Where ΔC_t is the difference of total consumption. This implies:

$$\varphi'[(1+n)a_{t+1}](1+n)(\tilde{a}_{t+1} - a_{t+1}) \leq Af'(a_t)(\tilde{a}_t - a_t) - \Delta C_t \quad (38)$$

Assume that consumption never decreases which means that capital never increases. Indeed, by induction if $\tilde{a}_t - a_t \leq 0$, which is true at $t = 0$, and if $\Delta C_t \geq 0$, then (38) implies $\tilde{a}_{t+1} - a_{t+1} \leq 0$. Moreover, suppose that consumption

increases at time t_1 : $\Delta C_{t_1} > 0$, then the previous argument gives that $\tilde{a}_t - a_t < 0$, for all $t > t_1$. This implies that for $t > t_2 = \max\{t_0, t_1\}$:

$$\begin{aligned} (1+n)(\tilde{a}_{t+1} - a_{t+1}) &\leq \frac{Af'(a_t)}{\varphi'[(1+n)a_{t+1}]}(\tilde{a}_t - a_t) \\ &< b(1+n)(\tilde{a}_t - a_t) \end{aligned}$$

since $\tilde{a}_t - a_t < 0$ and $Af'(a_t)/\varphi'[(1+n)a_{t+1}] > b(1+n)$. Hence,

$$\tilde{a}_{t+1} - a_{t+1} < b(\tilde{a}_t - a_t)$$

and

$$\tilde{a}_{t+1} - a_{t+1} < b^{t-t_2}(\tilde{a}_{t_2} - a_{t_2}) < 0$$

As $b > 1$ and a_{t+1} converges to the steady state, we have $\tilde{a}_t - a_t$ converges to $-\infty$ and \tilde{a}_{t+1} becomes negative, which is impossible. ■

(C) Proof of Proposition (4)

Proof. In this proof, we show the existence of flip bifurcation according to different configurations. As shown before the flip cycles appear at $\sigma = \sigma^F$ given by (23) and since its denominator is positive, then the existence of flip bifurcation require a negative numerator, that is:

$$\eta_1 \left(\frac{\varepsilon_{11}}{\gamma} - \frac{1-\alpha}{\alpha} (1 - \varepsilon_{12} + \varepsilon_{22}) \right) > \frac{\varepsilon_{12}}{(1-\gamma)} \quad (39)$$

In order to simplify, let us take two different cases concerning the sign of ε_{12} .

1. $\varepsilon_{12} > 0$.

In (39), the sign of $\varepsilon_{11}/\gamma - (1-\alpha)(1 - \varepsilon_{12} + \varepsilon_{22})/\alpha$ is unknown. However, condition (39) can not hold whenever $\varepsilon_{11}/\gamma - (1-\alpha)(1 - \varepsilon_{12} + \varepsilon_{22})/\alpha < 0$. Thus, $\varepsilon_{11}/\gamma - (1-\alpha)(1 - \varepsilon_{12} + \varepsilon_{22})/\alpha > 0$ is a necessary condition in order for condition (39) to verify and thus $\sigma^F > 0$ is satisfied for $\eta_1 > \eta_1^*$.

Moreover, $\varepsilon_{11}/\gamma - (1-\alpha)(1 - \varepsilon_{12} + \varepsilon_{22})/\alpha > 0$ requires $\varepsilon_{22} < \varepsilon_{22}^*$ and can be written as $\gamma > \gamma^*$. Since γ represents the consumption share out of wage, that is, $\gamma \in (0, 1)$. Therefore, $\gamma^* < 1$ if and only if $\varepsilon_{11} > \varepsilon_{11}^*$.

As a result flip bifurcation at $\sigma = \sigma^F$ whenever $\varepsilon_{11} > \varepsilon_{11}^*$, $\gamma > \gamma^*$, $\varepsilon_{22} < \varepsilon_{22}^*$ and $\varepsilon_{12} > 0$ (condition (1.i)).

2. $\varepsilon_{12} < 0$.

We consider the following configurations:

A. $\varepsilon_{11}/\gamma - (1-\alpha)(1 - \varepsilon_{12} + \varepsilon_{22})/\alpha > 0$.

As before, $\sigma^F > 0$ for all $\eta_1 > 0$. Condition A requires $\varepsilon_{22} < \varepsilon_{22}^*$ and is equivalent to $\gamma > \gamma^*$. However, $\gamma^* < 1$ for $\varepsilon_{11} > \varepsilon_{11}^*$. This implies condition (1.ii).

B. $\varepsilon_{11}/\gamma - (1 - \alpha)(1 - \varepsilon_{12} + \varepsilon_{22})/\alpha < 0$.

In this case, $\sigma^F > 0$ if and only if $\eta_1 < \eta_1^*$. For $\varepsilon_{22} > \varepsilon_{22}^*$, then condition (B) is verified for all $\gamma \in (0, 1)$. However, for $\varepsilon_{22} < \varepsilon_{22}^*$, the condition (B) is equivalent to $\gamma < \gamma^*$. In this case, $\gamma^* < 1$ if and only if $\varepsilon_{11} > \varepsilon_{11}^*$. (conditions (2) and (3)). ■

References

- [1] Alan, S. (2006), "Entry Costs and Stock Market Participation over the Life Cycle", *Review of Economic Dynamics* 9, 588-611.
- [2] Bosi, S., and Magris, F. (2002), "Pervasiveness of Sunspot Equilibria", *Document de Recherche EPEE 02-08*.
- [3] Bosi, S., Magris, F., and Cazzavillan, G. (2005), "Plausibility of Indeterminacy and Complex Dynamics", *Annales D'Economie et de Statistique* 78.
- [4] Cazzavillan, G. Llyod-Braga, T., and Pintus, P. (1998), "Multiple Steady States and Endogenous Fluctuations with Increasing Returns to Scale in Production", *Journal of Economic Theory* 80, 60-107.
- [5] Cazzavillan, G. (2001), "Indeterminacy and Endogenous Fluctuations with Arbitrarily Small Externalities", *Journal of Economic Theory* 101, 133-157.
- [6] Cazzavillan, G., and Pintus, P. (2004), "Robustness of Multiple Equilibria in OLG Economies", *Review of Economic Dynamics* 7, 456-475.
- [7] Constantinides, G. M. (1986), "Capital Market Equilibrium with Transaction Costs", *Journal of Political Economy* 94, 842-862.
- [8] De la Croix, D., and Michel, P. (2002), "A Theory of Economic Growth Dynamics and Policy in Overlapping Generations", Cambridge University Press.
- [9] Diamond, P. (1965), "National Debt in a Neoclassical Model", *American Economic Review* 55, 1126-1150.
- [10] Drugeon, J-P, Nourry, C., Venditti, A. (2010), "On Efficiency and Local Uniqueness in Two-sector OLG Economies", *Mathematical Social Sciences* 59, 120-144.
- [11] Galor, O., and Ryder, H. E. (1989), "Existence, Uniqueness, and Stability of Equilibrium in an Overlapping Generations Model with Productive Capital", *Journal of Economic Theory* 49, 360-375.
- [12] Geanakoplos, J., Polemarchakis, H. (1986), "Walrasian Indeterminacy and Keynesian Macroeconomics", *Review of Economic Studies* 103, 755-779.

- [13] Grandmont, J.-M. (2008), "Nonlinear Difference Equations, Bifurcations and Chaos: An Introduction", *Research in Economics* 62, 122-177.
- [14] Lloyd-Braga, T., Nourry, C., and Venditti, A. (2007), "Indeterminacy in Dynamic Models: When Diamond Meets Ramsey", *Journal of Economic Theory* 134, 513-536.
- [15] Lo, A. W., Mamaysky, H., and Wang, J. (2004), "Asset Prices and Trading Volume under Fixed Transaction Costs", *Journal of Political Economy* 112, no. 5.
- [16] Michel, P. and Venditti, A. (1997), "Optimal Growth and Cycles in Overlapping Generations Model", *Economic Theory* 9, 511-528.
- [17] Nourry, C. (2001), "Stability of Equilibria in the Overlapping Generations with Endogenous Labor Supply", *Journal of Economic Dynamics and Control* 25, 1647-1663.
- [18] Phelps, E. (1961), "The Golden Rule of Accumulation: A Fable for Growthmen", *The American Economic Review* 51, 638-643.
- [19] Phelps, E. (1965), "Second Essay on the Golden Rule of Accumulation", *American Economic Review* 55, 793-814.
- [20] Reichlin, P. (1986), "Equilibrium Cycles in an Overlapping Generations Economy with Production", *Journal of Economic Theory* 40, 89-102.
- [21] Vayanos, D. (1998), "Transaction Costs and Asset Prices: A Dynamic Equilibrium Model", *The Review of Financial Studies* 11, 1-58.
- [22] Venditti, A. (2003), "Altruism and Determinacy of Equilibria in Overlapping Generations Models with Externalities", *The Japanese Economic Review* 54.
- [23] Vissing-Jorgensen, A. (2002), "Towards an Explanation of Household Portfolio Choice Heterogeneity: Nonfinancial Income and Participation Cost Structures", NBER Working Paper 8884.