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Leisure Externalities and Endogenous Cycles in an OLG Model

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Abstract

This chapter examines the effect of leisure externalities on the appearance of local indeterminacy in a two-period overlapping generations economy with capital accumulation. Households work and consume in both periods of their life. Labor supply is elastic for young while inelastic for adult. It is shown that higher external effects in leisure promotes the occurrence of local indeterminacy, in the sense that the range for the parameter value giving rise to it becomes larger. However, high level of inelastic second-period supply makes the emergence of fluctuations based upon self-fulfilling expectations less likely. In other terms, the second-period inelastic labor supply acts as a stabilizing force while leisure externalities act as a destabilizing force. The analytical findings are complemented by economic interpretations.

Key words: Leisure externalities; Elderly labor supply; Overlapping generations; Determinacy; Endogenous fluctuations.

JEL classification: C62; E32; J2.

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1 Introduction

A growing literature is concerned with the effects of consumption externalities on the occurrence of local indeterminacy and endogenous cycles in a variety of dynamic models.[1] Even though leisure is an important component of an individual's utility, very little research has considered the role of leisure externalities on the dynamic properties of macroeconomic models.

The presence of spillovers associated to leisure activity is plausible since leisure can represent one's social status as much as conspicuous consumption does (Veblen, 1899). Moreover, empirical evidence suggests that many leisure activities are more enjoyable if they are done with others (for example, sports, trips or shopping). Jenkins and Osberg (2005) present evidence of the coordination of working hours by couple using British Household Panel Survey. Hamermesh (2002) finds similar results using United States data. Costa (2000) argues that the compression in the length of the work day distribution over the last century could be partly explained by the increasing coordination of work activities within and across firms and by the increasing arrangement of leisure time activities with those of relatives and friends. Alesina et al. (2005) argue that leisure externalities are an important explanatory factor for the difference in working hours in Europe and the US. However, an increase in the economy-wide leisure level might have disadvantages due to, for example, crowded places such as parks or swimming pools, and so reduce the benefits of own leisure.

In addition to empirical works, there are many theoretical models that study leisure externalities in dynamic general equilibrium models. In a one-sector dynamic general equilibrium model with identical agents, variable capital utilization and production externalities, Weder (2004) shows that the existence of leisure externalities is able to affect significantly the dynamics of the economy, i.e., the emergence of local indeterminacy and expectations-driven business cycles in one-sector general equilibrium model. He demonstrates that the presence of leisure externalities reduces the minimum increasing returns in production needed for the local indeterminacy to emerge and thus makes the conditions required for local indeterminacy more plausible. Therefore, leisure externalities matter for the appearance of endogenous fluctuations. Gomez (2008a, 2008b) analyzes the effects of both consumption and leisure externalities on economic growth and welfare in a two-sector endogenous growth model with human capital accumulation. Gomez (2008b) assume that production is subject to sector-specific externalities associated with physical and human capital. In both papers, Gomez shows that optimal growth paths and competitive equilibrium paths may not coincide. Thus he analyzes the efficiency of the competitive equilibrium and, when it is not efficient, designs a tax policy in order for the decentralized equilibrium path to replicate the optimal growth path. Azariadis et al. (2009) introduce leisure externalities in a two-sector endogenous growth model with physical and human capital are necessary inputs in both sectors. The authors consider a separable felicity function in consumption and leisure

[1] See Mino and Nakamoto (2009), Alonso-Carrera et al. (2007) and Chen and Hsu (2007), among others.

since it permits leisure externalities to influence allocations along a balanced growth path. They quantify the role preferences toward leisure have on economic growth and welfare which in turn can shed some light on the effect of preferences on the observed disparity in hours worked between Americans and Europeans.

While the mentioned literature thoroughly discussed the impact of leisure externalities in models with infinitely-lived agents, the dynamic effects of leisure externalities has not been studied in an overlapping generations OLG model. This chapter fills this gap in literature by introducing external effects in leisure for young agents in a two-period OLG Diamond's (1965) model. Moreover, we consider such a framework because it models explicitly the different periods of life and is usually exploited by literature to study the allocation of resources across different generations. Therefore, using two-period OLG model allows studying explicitly the effect of leisure externalities on agents' consumption and savings.

Further, unlike the mainstream recent contributions, based upon Diamond's framework (e.g., Nourry 2001; Cazzavillan and Pintus 2004; Nourry and Venditti 2006), we assume that households work in both periods of their life. In fact, the introduction of labor supply in the second period is plausible according to statistics. In the European Union, the proportion of persons aged $15-24$ who were employed in 2008 was 37.6%; those aged $25-54$ was 79.6% and those aged $55-64$ was 45.6% (INSEE). Recently, several authors have considered the second-period labor supply in OLG models, for example, Matsuyama (2008), d'Autume (2003) and Michel and Pestieau (1999). They are mainly interested in studying the interaction between the social security system and retirement age choice. For this purpose, they endogenize the second-period labor supply. As we are not interested in this aspect, we assume for simplicity that labor supply is inelastic in the second period. Moreover, such an assumption is supported by the INSEE estimates for employment status in France which show that labor supply is more flexible for young than for old. The fixed-term contracts (CDD) and temporary work are unusual among the oldest groups. The statistics of the INSEE in 2008 shows that 21. 9% of those aged $15-24$ have fixed-term contracts (CDD) and temporary work while only 2.2% of those aged $25-49$ and 0.7% of those aged 50 and over. Further, the percentage of employed persons who have open-ended contracts (CDI) increases with age: it is 49.7% for those aged $15-24$, 7.5% for those aged $25-49$ and 78.4% for those aged 50 and over. Thus old persons have more likely stable jobs relative to young.

This chapter aims at studying the impact of leisure externalities on the appearance of expectations-driven business cycles in a two-period OLG model where agents consume and work in both periods.

Along this line of research, the emergence of fluctuations due to self-fulfilling prophecies has been studied in OLG models. In a standard Diamond's OLG model with inelastic labor supply, local indeterminacy of the steady state cannot appear. On the contrary, the emergence of local indeterminacy is possible when endogenous labor is introduced in the model with separable preferences and Leontief technology and consumption only in the second period (Reichlin

1986). More recently, an OLG model with both-period consumption augmented to include endogenous first-period labor supply has been considered by several papers. Assuming non-separable preferences and gross substitutability of consumption and leisure, Nourry (2001) derives conditions under which the steady state is a sink, saddle-point or a source. Further, Cazzavillan and Pintus (2004) show that, under gross substitutability of consumption and leisure and additively separable preferences assumption, fluctuations due to self-fulfilling prophecies arise under two quite restrictive conditions: weak substitution between capital and labor and high propensity to save. Finally, Nourry and Venditti (2006) study the issue of local indeterminacy under general non-separable preferences and without any restriction on the substitutability properties.

For simplicity, we follow Cazzavillan and Pintus (2004) by introducing two simplifying assumptions on preferences: we consider additively separable preferences and gross substitutability of consumption and leisure.

In order to examine the effect of leisure externalities on the occurrence of local indeterminacy and local bifurcations, we adopt the geometrical method of Grandmont et al. (1998). Our main result states that the appearance of expectations-driven cycles requires small leisure externalities and high proportion of young labor supply out of total labor supply. Further, similar to Cazzavillan and Pintus (2004), weak substitution between capital and labor and sufficiently high saving levels are also needed for the appearance of local indeterminacy.

Intuitively, indeterminacy requires that the initial expectations of higher next-period interest rate reduces the savings, and thus raises the next-period interest rate, so that expectations-driven equilibrium can be self-fulfilling. Such a mechanism appears as follows: whenever agents anticipate higher future interest rate, they are willing to increase their capital accumulation so as to enjoy higher returns in the next period. This leads each agent to work more and to reduce his consumption of leisure. As all agents' labor supply goes up, the marginal product of labor (i.e., wage) falls, resulting in a decrease in next-period capital. Therefore, the next-period interest rate will increase and thus, initial expectations are self-fulfilling.

Moreover, we show that high level of second-period inelastic labor supply makes the occurrence of expectation-driven fluctuations less likely and therefore, acts as a stabilizing force.

Finally, one can notice that the endogeneity of labor supply is a sufficient condition in order for this mechanism to take place and for local indeterminacy to appear. This is the result of Cazzavillan and Pintus (2004). However, higher external effects in leisure promotes the occurrence of local indeterminacy in the sense that the range for the parameter value giving rise to it becomes larger. Thus leisure externalities act as a destabilizing factor. This later result is consistent with that of Weder (2004) who shows that leisure externalities destabilize in an economy with infinitely-lived agents in the sense that the existence of externalities in the model make the conditions for the appearance of local indeterminacy less demanding.

This paper is organized as follows. In section 2, we present the model and derive the intertemporal equilibrium and the analysis of the steady state. Section 3 studies the occurrence of local indeterminacy. In section 4, we provide the economic intuition of the main result. Finally, section 5 provides an illustrative example and section 6 concludes.

2 The model

We consider a one-sector two-period overlapping generations model with identical households and a representative firm.

2.1 Households

The economy is populated by identical agents who live for two periods, youth and adulthood. In each period, agents are endowed with one unit of time which is allocated between work and leisure. During the first period, an individual supplies elastically an amount $l_t \in [0,1]$ of labor and allocates the wage-income $w_t l_t$ between current consumption c_{1t} and saving s_t. In the second period, he works a fraction of time $h \in [0,1]$. An adult is assumed to be non-altruistic so the adulthood income, consisting of return on savings $r_{t+1}s_t$ and wage-income $w_{t+1}h$, is entirely consumed. Denote the adulthood consumption by c_{2t+1}.

It is also assumed that an agent derives utility not only from consumption and leisure demand, but also from how he compares his level of leisure, when young, with that of others from the same generation. Formally, the disutility from labor supply depends on his own labor supply l_t as well as on the average level in the economy $\bar{l}_t$.

Assuming separable preferences in consumption and labor, a representative agent who is born at time $t \geq 0$ solves the following maximization problem:

$$\max_{c_{1t}, l_t, s_t, c_{2t+1}} \left[u\left(c_{1t}/B\right) + \beta u\left(c_{2t+1}\right) - v\left(l_t, \bar{l}_t\right) \right] \tag{1}$$

subject to the constraints

$$c_{1t} + s_t = w_t l_t \tag{2}$$

$$c_{2t+1} = r_{t+1}s_t + w_{t+1}h \tag{3}$$

$$c_{1t} \geq 0,\ c_{2t+1} \geq 0,\ 0 \leq l_t \leq 1 \text{ and } h \in [0,1], \text{ for all } t \geq 0, \tag{4}$$

where r_{t+1} represents the interest rate at time $t+1$; w_t and w_{t+1} are the real wage at time t and $t+1$, respectively. In addition, $B > 0$ is a scaling parameter which is useful for the normalization of the steady state and $\beta \in (0,1)$ is the discount factor.

The preferences given by (1) are defined in a way so that the following assumption holds:

Assumption 1 *For all* $c \geq 0$*, the function* u *is increasing and concave* $u''(c) < 0 < u'(c)$*. Further, the function* v *is continuous and has continuous derivatives of every required order* $0 \leq l \leq 1$ *and satisfies* $v_1 > 0$ *and* $v_{11} > 0$*, and* $v_{12} \lesseqgtr 0$*.*

Define the following second-order elasticities: Let $\xi \equiv -u'/cu''$ be the elasticity of intertemporal substitution in consumption, let $\varphi_{11} \equiv lv_{11}/v_1$ be the elasticity of disutility of labor [the inverse of the elasticity of labor supply with respect to real wage] and let $\varphi_{12} \equiv lv_{12}/v_1$ be the elasticity of external effects in labor which could be positive or negative.

A positive value of φ_{12} captures the Keeping-Up with the Joneses (KUJ) effect in labor supply; that is, if all agents in the economy enjoy more leisure (i.e., they work less), this affects positively the individual's marginal utility from leisure. Thus leisure becomes more valuable for this individual. As a result, he is willing to work less and consume more leisure. Conversely, whenever φ_{12} is negative, then the preferences display Running-Away from the Joneses (RAJ) effect; that is, if others work more, the agent will work less.

Throughout this chapter, the following restriction is imposed on the strength of external effects:

Assumption 2 $\qquad v_{11} + v_{12} > 0.$

Assumption 2 states that the external effects can not dominate the direct effect of labor supply. In term of elasticities, it is given by $\varphi_{11} + \varphi_{12} \geq 0$.

Under Assumption 1 and 2, the first-order conditions of the representative agent program (1)-(4) imply

$$Bv_1\left(l_t, \bar{l}_t\right) - u'\left(c_{1t}/B\right) w_t = 0 \tag{5}$$

$$v_1\left(l_t, \bar{l}_t\right) - \beta w_t r_{t+1} u'\left(c_{2t+1}\right) = 0 \tag{6}$$

Further, Assumption 1 ensures that the second-order condition is well verified.

2.2 Production

On the production side, a representative firm uses a constant return to scale technology $AF(K_t, L_t)$ to produce one final good. Let $a_t \equiv K_t/L_t$ be the capital-labor ratio. The amount of final good could be written as follows $AF(K_t, L_t) \equiv AL_tF(K_t/L_t, 1) \equiv AL_tf(a_t)$.

Assumption 3 *For all* $a \geq 0$*, the technology* $f(a)$ *is* a continuous function*, positive-valued and differentiable. Furthermore,* $f''(a) < 0 < f'(a)$*, for* $a > 0$*, and* $f(0) = 0$*,* $\lim_{a \to 0} f'(a) = +\infty$ *and* $\lim_{a \to +\infty} f'(a) = 0$*.*

A representative firm takes the technology and input prices as given then maximizes its profit. If we set $\rho(a_t) \equiv f'(a_t)$ and $\omega(a_t) \equiv f(a_t) - a_tf'(a_t)$, we then obtain

$$r_t = A\rho\left(a_t\right) \equiv r\left(a_t\right) \quad \text{and} \quad w_t = A\omega\left(a_t\right) \equiv w\left(a_t\right) \tag{7}$$

Let $\alpha \equiv af'(a)/f(a) \in (0,1)$ be the share of capital in total income and $\sigma > 0$ be the elasticity of capital-labor substitution, we can now define the following elasticities: $a\rho'(a)/\rho(a) = -(1-\alpha)/\sigma$ and $a\omega'(a)/\omega(a) = \alpha/\sigma$.

2.3 Intertemporal equilibrium

Assume a constant rate of population growth, so that $(1+n) = N_{t+1}/N_t$, where N_t is the number of individual born at time t. Equilibrium in inputs market implies $K_{t+1} = N_t s_t$ and $L_t = N_t l_t + N_{t-1} h$, and therefore the saving is given by $s_t = a_{t+1}\left[(1+n) l_{t+1} + h\right]$, for every $t \geq 0$. Output market clears by Walras law.

Since agents in the same generation are identical, individual and average labor supply coincide at equilibrium, i.e., $l_t = \bar{l}_t$. Substituting (2), (3) in (5)-(6) and taking account of the equilibrium prices (7), one gets the intertemporal competitive equilibria with perfect foresight which can be described by a positive sequence $\{a_t, l_t\}_{t=0}^{\infty}$ satisfying for every $t \geq 0$ the following equations

$$A\omega\left(a_t\right) u'\left(\left[A\omega\left(a_t\right) l_t - s_t\right]/B\right)/B - v_1\left(l_t, l_t\right) = 0 \tag{8}$$

$$\beta A\omega\left(a_t\right) A\rho\left(a_{t+1}\right) u'\left(A\rho\left(a_{t+1}\right) s_t + A\omega\left(a_{t+1}\right) h\right) - v_1\left(l_t, l_t\right) = 0 \tag{9}$$

subject to the initial endowment of capital $a_0 > 0$ and $s_t = a_{t+1}\left[(1+n) l_{t+1} + h\right]$.

2.4 Steady state

A steady state for the two-dimensional dynamic system (8)-(9) is a solution (a, l) of the system:

$$A\omega(a) u'\left(\left[A\omega(a) l - s\right]/B\right)/B = v_1(l, l) \tag{10}$$

$$\beta A\omega(a) A\rho(a) u'\left(A\rho(a) s + A\omega(a) h\right) = v_1(l, l) \tag{11}$$

where $s = a\left[(1+n) l + h\right]$.

In the rest of the chapter, we restrict our attention to realistic values for the share of capital in total income $\alpha \approx 1/3$. Moreover, we focus on the values of elasticity of intertemporal substitution in consumption which ensure an increasing saving function with respect to interest rate, provided that this configuration is the most plausible empirically. Whenever $\xi > 1$, the substitution effect dominates the income effect as the interest rate goes up. This implies that the households consume less today to enjoy the rise in the return of savings. As a result, savings increase as a response to the rise in interest rate. Conversely, whenever $\xi < 1$, the income effect dominates and thus, savings decrease when interest rate goes up. Let us thus consider the following assumption:[2]

Assumption 4 $\alpha \leq 1/2$ and $\xi > 1$.

In order to study the local properties of this economy, we focus on proving the existence of at least one steady state which may not be necessarily unique.[3] We follow the procedure provided by Cazzavillan, at al. (1998) and show the existence of a Normalized Steady State (NSS in the sequel), $(a, l) = (1, 1)$, by selecting appropriately the scaling parameters $A > 0$ and $B > 0$. Such a NSS is characterized by the following proposition:

Proposition 1 *Let Assumptions* $1-4$ *be satisfied, there exists a steady state of the dynamic system (8)-(9) such that* $a = 1$ and $l = 1$ *if and only if*

$$\beta\rho(1)\, s^2 u'\left(s\left[s\rho(1) + \omega(1)\, h\right]/\omega(1)\right)/\omega(1) < v_1(1, 1) \tag{12}$$

and the scaling parameters A and B are set at the levels $A^* > s/\omega(1)$ *and* $B^* > 0$ *that are the unique solutions of*

$$A^*\omega(1)\, u'\left(\left[A^*\omega(1) - s\right]/B^*\right)/B^* = v_1(1, 1), \tag{13}$$

$$\beta A^*\omega(1)\, A^*\rho(1)\, u'\left(A^*\left[\rho(1)\, s + \omega(1)\, h\right]\right) = v_1(1, 1), \tag{14}$$

where $s = (1 + n + h)$.

Proof. See the Appendix. ■

In the following we assume that the conditions of Propositions 1 are satisfied in order to guarantee the existence of the NSS. Before studying the appearance of local indeterminacy and endogenous fluctuations, we evaluate the following at $(a, l) = (1, 1)$: The share of saving out of first-period wage-income is $\theta \equiv s/A^*\omega(1)$, the share of second-period consumption paid with previous savings is $\eta \equiv A^*\rho(1)\, s/c_2$ and the young agents' labor supply relative to the total labor supply is $\lambda \equiv (1 + n)/(1 + n + h)$, with $c_2 \equiv A^*\left[\rho(1)\, s + \omega(1)\, h\right]$ is the second-period consumption and $s = (1 + n + h)$ is the saving.

3 Local indeterminacy

The objective of this section is to study the role of leisure externalities and the second-period labor supply on the occurrence of endogenous fluctuations due to self-fulfilling prophecies. The two-dimensional dynamic system (8)-(9) is linearized around the NSS $(a, l) = (1, 1)$, we obtain $(da_{t+1}/a, dl_{t+1}/l)^T =$

[2] Such an assumption has also been made by Cazzavillan and Pintus (2006), Lloyd-Braga et al (2007) and Bosi and Seegmuller (2010).

[3] Cazzavillan et al. (1998) studied the multiplicity of steady states in an OLG framework.

$J\left(da_t/a, dl_t/l\right)^T$, where J is the Jacobian matrix evaluated at $(a,l)=(1,1)$ and is given, in terms of the elasticity of external effects and the elderly labor-supply, as follows:

$$J=\left[\begin{array}{cc} 1 & \lambda \\ \frac{\alpha+(\sigma-1)\eta+(1-\alpha)\xi}{\sigma} & \lambda\eta \end{array}\right]^{-1}\left[\begin{array}{cc} \frac{\alpha[1-(1-\theta)\xi]}{\theta\sigma} & \frac{1+(1-\theta)(\varphi_{11}+\varphi_{12})\xi}{\theta} \\ \frac{\alpha}{\sigma}\xi & -\left(\varphi_{11}+\varphi_{12}\right)\xi \end{array}\right] \quad (15)$$

The characteristic polynomial of the Jacobian matrix J in (15) is $P(\rho)=\rho^2-T\rho+D$, where the trace $T=\rho_1+\rho_2$ and the determinant $D=\rho_1\rho_2$ are respectively given by

$$T=1+\frac{\sigma\eta+\alpha\xi+H_1+[\alpha+H_2]\left(\varphi_{11}+\varphi_{12}\right)\xi}{\theta\lambda\left[(1-\alpha)\xi+\alpha-\eta\right]} \quad (16)$$

$$D=\frac{\left(1+\varphi_{11}+\varphi_{12}\right)\alpha\xi}{\theta\lambda\left[(1-\alpha)\xi+\alpha-\eta\right]} \quad (17)$$

where

$$H_1\equiv(1-\theta\lambda)\left[\left[(1-\alpha)\xi+\alpha-\eta\right]-\alpha\xi\right]-\alpha\lambda\eta\left[1-(1-\theta)\xi\right] \quad (18)$$

$$H_2\equiv\sigma\left[\theta+(1-\theta)\eta\right]+(1-\theta)\left[(1-\alpha)\xi+\alpha-\eta\right]-\alpha \quad (19)$$

Since k_t is a predetermined variable, fixing l_t is equivalent to fixing a_t according to $a_t\equiv k_t/l_t$. Thus the only one independently non-predetermined variable is l_t. As there is only one non-predetermined variable, the equilibrium is locally indeterminate if and only if the steady state is a sink, i.e., both eigenvalues of the Jacobian matrix J lie inside the unit circle. Using the fact that the trace T and the determinant D are the sum and the product of the eigenvalues, local indeterminacy requires that the conditions $D<1$, $D>T-1$ and $D>-1-T$ to be verified simultaneously, where

$$T=1+D+\frac{\sigma\eta+H_1+H_2\left(\varphi_{11}+\varphi_{12}\right)\xi}{\theta\lambda\left[(1-\alpha)\xi+\alpha-\eta\right]} \quad (20)$$

$$\begin{aligned} T &= -1-D \\ &+\frac{H_1+\sigma\eta+2\alpha\xi+2\theta\lambda\left[(1-\alpha)\xi+\alpha-\eta\right]+[2\alpha+H_2]\left(\varphi_{11}+\varphi_{12}\right)\xi}{\theta\lambda\left[(1-\alpha)\xi+\alpha-\eta\right]} \end{aligned} \quad (21)$$

The standard Diamond model with endogenous labor supply in the first period could be obtained by setting $\varphi_{12}=0$ and $\lambda=\eta=1$. We then get the linear system of Cazzavillan and Pintus (2004).

In the following we chose to discuss our results in terms of the parameters that capture the main features of the economy, namely σ, φ_{12}, θ and λ. As previously defined, σ summarizes the properties of the technology, φ_{12} captures the external effects in labor, θ measures the propensity to save of young generation and λ is the proportion of young labor supply. Our objective is to study the influence of leisure externalities and the second-period labor supply on the occurrence of endogenous fluctuations due to self-fulfilling prophecies.

In the spirit of Grandmont et al. (1998), we apply the geometrical method and we characterize the locus $\Sigma \equiv \{(T(\sigma), D(\sigma)) : \sigma \geq 0\}$ obtained by varying the elasticity of external effects σ from 0 to $+\infty$ in the (T, D) −plane.

Our main result follows as an implication of the following lemmas:

Lemma 1 *The locus Σ describes a horizontal line as a function of $\sigma \geq 0$ with origin (T_0, D_0) such that*

$$\begin{aligned} T_0 &= 1 + D_0 + \frac{H_1 + [(1-\theta)[(1-\alpha)\xi + \alpha - \eta] - \alpha](\varphi_{11} + \varphi_{12})\xi}{\theta\lambda[(1-\alpha)\xi + \alpha - \eta]} \\ D_0 &= \frac{(1 + \varphi_{11} + \varphi_{12})\alpha\xi}{\theta\lambda[(1-\alpha)\xi + \alpha - \eta]} \end{aligned} \tag{22}$$

satisfying $T_0 > -1 - D_0$ and endpoint

$$(T_\infty, D_\infty) = \left(+\infty, \frac{(1 + \varphi_{11} + \varphi_{12})\alpha\xi}{\theta\lambda[(1-\alpha)\xi + \alpha - \eta]}\right) \tag{23}$$

Moreover, Σ moves to the right with $\sigma \geq 0$, i.e., $T'(\sigma) > 0$.

Proof. See the Appendix. ■

Our main objective is to give conditions for local indeterminacy of equilibria. For this purpose, we should study the intersection of the half-line Σ with the sink-stability triangle ABC. Both $D_0 > -1 - T_0$ and $T'(\sigma) > 0$ imply that the condition $D > -1 - T$ is always verified for all $\sigma \geq 0$. It thus remains to determine the range of parameters for which the inequalities $D < 1$ and $D > T - 1$ are met simultaneously. For this purpose, we have to analyze how the starting point (T_0, D_0) of Σ, given in (22), changes with the elasticity of external effects in leisure φ_{12}, when φ_{12} is made to increase from $-\varphi_{11}$ to $+\infty$ (By Assumption 2: $\varphi_{11} + \varphi_{12} \geq 0$). Proceeding in a similar way as for the Σ-half-line, we define a linear relationship linking the initial points T_0 and D_0 for different values of $\varphi_{12} \in (-\varphi_{11}, +\infty)$ by $\Sigma_1 \equiv \{(T_0(\varphi_{12}), D_0(\varphi_{12})) : \varphi_{12} \geq -\varphi_{11}\}$. The following Lemma characterizes the location of Σ_1 in the (T, D) −plane.

Lemma 2 *The half-line Σ_1 is linear in φ_{12} with origin*

$$\begin{aligned} T_0(-\varphi_{11}) &= 1 + D_0(-\varphi_{11}) + \frac{H_1}{\theta\lambda[(1-\alpha)\xi + \alpha - \eta]} \\ D_0(-\varphi_{11}) &= \frac{\alpha\xi}{\theta\lambda[(1-\alpha)\xi + \alpha - \eta]} \end{aligned} \tag{24}$$

endpoint $(T_\infty, D_\infty) = (+\infty, +\infty)$ *and slope*

$$S_{\Sigma_1} = \frac{\alpha}{(1-\theta)\left[(1-\alpha)\xi + \alpha - \eta\right]} > 0 \tag{25}$$

Σ_1 *moves upward in the* (T, D) *−plane when* φ_{12} *increases from* $-\varphi_{11}$ *to* $+\infty$. *Moreover, it makes counter-clockwise rotation when* θ *goes up and its origin, given by (24) moves with* θ *such that* $\partial T_0(-\varphi_{11})/\partial\theta < 0$ *and* $\partial D_0(-\varphi_{11})/\partial\theta < 0$.

Proof. See the Appendix. ■

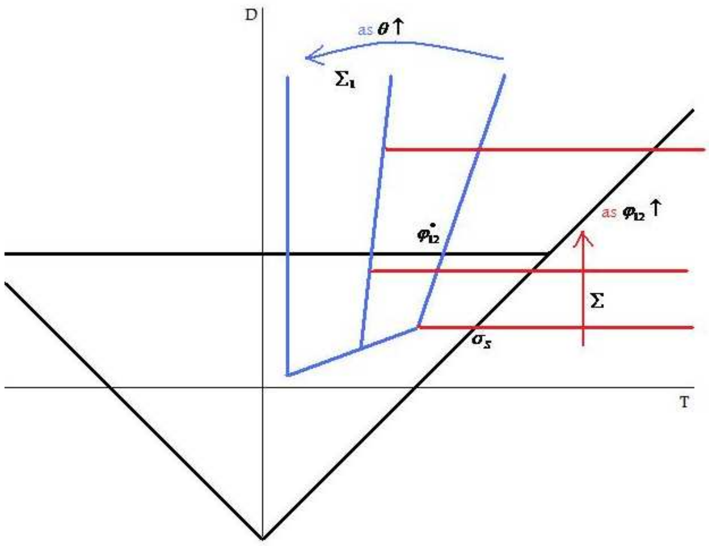

Fig. 1

Once the location of both half-lines Σ and Σ_1 are determined, a critical issue in order for local indeterminacy to occur is to ensure that Σ starts inside the triangle ABC, i.e., the initial point of Σ_1, given by (24), should lie initially inside the triangle ABC. In other terms, a *sufficient condition* for the appearance of local indeterminacy is that $(T_0(-\varphi_{11}), D_0(-\varphi_{11}))$ should satisfy the inequalities $D_0(-\varphi_{11}) < 1$ and $D_0(-\varphi_{11}) > T_0(-\varphi_{11}) - 1$.

Lemma 3 *Let Assumptions* 1−4 *be satisfied, the origin* $(T_0(-\varphi_{11}), D_0(-\varphi_{11}))$ *lies inside the triangle* ABC, *i.e., the conditions* $D_0(-\varphi_{11}) < 1$ *and* $D_0(-\varphi_{11}) > T_0(-\varphi_{11}) - 1$ *hold, if* $\max\{\theta_1, \theta_2\} < \theta \leq 1$, $\max\{\lambda_1, \lambda_2\} < \lambda \leq 1$ and *either* (*i*) $0 \leq \eta < 1-\alpha$; *or* (*ii*) $1-\alpha < \eta \leq 1$ and $\xi > \xi^*$, *where* $\xi^* \equiv (\eta - \alpha)/(1-2\alpha)$ *and*

$$\left[\begin{array}{cc} \theta_1 & \theta_2 \\ \lambda_1 & \lambda_2 \end{array}\right] \equiv \left[\begin{array}{cc} \frac{\alpha\xi}{\lambda[(1-\alpha)\xi+\alpha-\eta]} & \frac{([(1-\alpha)\xi+\alpha-\eta]-\alpha\xi)+\lambda\alpha\eta(\xi-1)}{\lambda[(((1-\alpha)\xi+\alpha-\eta)-\alpha\xi)+\alpha\eta\xi]} \\ \frac{\alpha\xi}{(1-\alpha)\xi+\alpha-\eta} & \frac{[(1-\alpha)\xi+\alpha-\eta]-\alpha\xi}{((1-\alpha)\xi+\alpha-\eta)-\alpha\xi+\alpha\eta} \end{array}\right] \tag{26}$$

Proof. See the Appendix. ■

Lemma 3 shows that the existence of local indeterminacy requires fundamentally sufficiently high levels of capital accumulation which are ensured if the share of saving over the youth wage-income θ is large enough. This result stands in line with economies without leisure externalities nor second-period labor supply such as Nourry (2001), Cazzavillan and Pintus (2004) and Nourry and Venditti (2006), among others. Moreover, large young labor supply proportion of total labor supply λ is also required. High λ means that the endogenous part of labor supply more important than the inelastic one. In other terms, high level of second-period inelastic labor supply relative to total labor supply makes the emergence of fluctuations based upon self-fulfilling expectations less likely.

Before proceeding, let us define the following critical values. Let $\sigma = \sigma_S$ be the critical value that solves $T = 1 + D$ and is given by

$$\sigma_S \equiv \frac{[\alpha - (1-\theta)\left[(1-\alpha)\xi + \alpha - \eta\right]](\varphi_{11} + \varphi_{12})\xi - H_1}{\eta + [\theta + (1-\theta)\eta](\varphi_{11} + \varphi_{12})\xi} \tag{27}$$

The half-line Σ_1 crosses the segment BC at $\varphi_{12} = \varphi_{12}^*$ which solves $D_0 = 1$ and is given by

$$\varphi_{12}^* \equiv -\varphi_{11} + \frac{\theta\lambda\left[(1-\alpha)\xi + \alpha - \eta\right] - \alpha\xi}{\alpha\xi} \tag{28}$$

Given the results provided in Lemmas $1-3$, we can now characterize the local dynamic of the NSS $(1,1)$:

Proposition 2 *Under Assumptions* $1-4$ *and let Lemmas* $1-3$ *hold, there exist* σ_S *and* φ_{12}^**, given by (27) and (28), such that the following results generically hold:*

(i) Whenever $-\varphi_{11} \leq \varphi_{12} < \varphi_{12}^*$*, the NSS* $(1,1)$ *is locally indeterminate for* $0 \leq \sigma < \sigma_S$*, undergoes a saddle-node bifurcation at* $\sigma = \sigma_S$ *and becomes a saddle-point for* $\sigma > \sigma_S$*.*

(ii) Whenever $\varphi_{12} > \varphi_{12}^*$*, the NSS* $(1,1)$ *is a source for* $0 \leq \sigma < \sigma_S$*, undergoes a saddle-node bifurcation at* $\sigma = \sigma_S$ *and becomes a saddle-point for* $\sigma > \sigma_S$*.*

Proof. See the Appendix. ■

In view of Lemma 3 and Proposition 2, endogenous fluctuations due to self-fulfilling expectations appear if the elasticity of input substitution σ is small and the elasticity of external effects φ_{12} is small. In addition, local indeterminacy requires sufficiently high saving levels, namely, the share of saving over the youth wage-income θ and the proportion of young labor supply λ are large enough. The intuition of this result is provided in the next section.

4 Discussion

In order to complete the characterization of local indeterminacy, we provide the economic interpretation of local indeterminacy. Then we study in the following the effect of leisure externalities φ_{12} and the propensity to save θ on the indeterminacy range $\sigma \in (0, \sigma_S)$.

4.1 Intuition for local indeterminacy

The objective of this section is to explain why prophecies can be self-fulfilling in OLG model based on Cazzavillan and Pintus (2004) framework with leisure externalities and inelastic second-period labor-supply. Assume that agents anticipate an increase in the future interest rate r_{t+1}. Then this expectation is self-fulfilling if it results in a reduction in capital accumulation because, in this case, the future interest rate will increase. To check it, we compute the effect of initial expectation of higher r_{t+1} on K_{t+1}.

When the an agent anticipates higher future interest rate, he is willing to raise his savings because, by doing so, he gets higher income in the second period and so higher consumption when old. This leads him to work more in the first period and to reduce his consumption of leisure. Formally we can show this positive effect of the expected higher future interest rate on the individual labor supply l_t, using the budget constraint (3) and equation (6)

$$v_1(l_t, l_t) - \beta w_t r_{t+1} u'(r_{t+1} s_t + w_{t+1} h) = 0 \tag{29}$$

from which we obtain

$$\frac{dl_t}{l_t} = \frac{1 - \eta/\xi}{\varphi_{11} + \varphi_{12}} \frac{dr_{t+1}}{r_{t+1}} \tag{30}$$

Under gross substitutability, i.e. $\xi > 1$, equation (30) shows that individual labor supply increases as a response to the initial expectation of high future interest rate.

From the budget constraint (2) and capital market clearing condition, we get

$$K_{t+1} = N_t \theta_t l_t w(K_t, L_t) \tag{31}$$

where θ is the propensity to save. From (31) we observe that an increase of the individual labor supply l_t has two effects on capital accumulation: The first one is through the individual supplies

$$dK_{t+1}/K_{t+1} = dl_t/l_t \tag{32}$$

which in turn implies the effect of initial anticipation on capital accumulation:

$$\frac{dK_{t+1}}{K_{t+1}} = \frac{1-\eta/\xi}{\varphi_{11}+\varphi_{12}}\frac{dr_{t+1}}{r_{t+1}} \tag{33}$$

The other is through the aggregate labor supply and its impact on the wage: The labor market clearing condition $L_t = N_t l_t + N_{t-1}h$ implies that the increase in individual labor supply has positive effect on the aggregate labor supply as follows: $dL_t/L_t = \lambda dl_t/l_t$. Then the increase in L_t reduces the wage:

$$\frac{dK_{t+1}}{K_{t+1}} = \frac{L_t w_L}{w_t}\frac{dL_t}{L_t} = -\frac{\alpha}{\sigma}\frac{dL_t}{L_t} \tag{34}$$

and thus, using (30) and (34), the effect of initial expectation on capital accumulation is given as follows:

$$\frac{dK_{t+1}}{K_{t+1}} = -\frac{\alpha}{\sigma}\lambda\frac{1-\eta/\xi}{\varphi_{11}+\varphi_{12}}\frac{dr_{t+1}}{r_{t+1}} \tag{35}$$

In sum, the variation of capital accumulation as a consequence of an increase in the expected interest rate depends on two opposite effects:

1. $\frac{dK_{t+1}}{K_{t+1}} = \frac{1-\eta/\xi}{\varphi_{11}+\varphi_{12}}\frac{dr_{t+1}}{r_{t+1}} > 0$
2. $\frac{dK_{t+1}}{K_{t+1}} = -\frac{\alpha}{\sigma}\lambda\frac{1-\eta/\xi}{\varphi_{11}+\varphi_{12}}\frac{dr_{t+1}}{r_{t+1}} < 0$

Expectations are self-fulfilling if capital accumulation reduces because this in turn will increase the future interest rate. Therefore, the second (negative) effect should dominate the first one. Without leisure externalities nor second-period labor supply ($\eta = \lambda = 1$ and $\varphi_{12} = 0$), this requires $\sigma < \sigma_S < \alpha$. However, the introduction of the second-period labor supply ($0 < \lambda < 1$ and $0 < \eta < 1$) reinforces the first effect which promotes determinacy. This means that the second-period inelastic labor supply has a stabilizing power. Moreover, one can notice that the above mechanism can appear if young labor supply is endogenous and old inelastic labor supply is small enough. In other terms, the existence of leisure externalities does not have the crucial role for the appearance of local indeterminacy. Instead, as we will show below, leisure externalities can significantly influence the values of parameters that give rise to local indeterminacy.

4.2 Leisure externalities and indeterminacy range

Consider the saddle-node bifurcation value of $\sigma = \sigma_S$, given by (27), and differentiate it with respect to φ_{12}, we obtain

$$\frac{\partial \sigma_S}{\partial \varphi_{12}} = \frac{\left(1 - \lambda\eta - \theta\lambda\left(1-\eta\right)\right)\left[\theta\left[\left(\left(1-\alpha\right)\xi + \alpha - \eta\right) - \left(1-\eta\right)\alpha\xi\right] - \alpha\eta\left(\xi - 1\right)\right]\xi}{\left[\eta + \left(\theta + \left(1-\theta\right)\eta\right)\xi\left(\varphi_{11} + \varphi_{12}\right)\right]^2}$$

which is strictly positive since $\left[\theta\left[\left(\left(1-\alpha\right)\xi + \alpha - \eta\right) - \left(1-\eta\right)\alpha\xi\right] - \alpha\eta\left(\xi - 1\right)\right] > 0$ for all $\theta > \max\{\theta_1, \theta_2\}$ and $\left[1 - \lambda\eta - \theta\lambda\left(1-\eta\right)\right]$ is always positive.

The indeterminacy range $(0, \sigma_S)$ *widens* with the elasticity of leisure externalities in the interval $-\varphi_{11} \leq \varphi_{12} < \varphi_{12}^*$ (see also Fig. 1). This means that leisure externalities *destabilizes*: the higher the leisure externalities, the more likely the occurrence of self-fulfilling expectations. This result goes in line with Weder (2004) who also shows that leisure externalities act as a destabilizing factor in a neoclassical model with infinitely lived agents. More precisely, he demonstrates that whenever preferences display KUJ feature, the degree of increasing return to scale needed for the emergence of local indeterminacy is reduced.

4.3 Propensity to save and indeterminacy range

Differentiating (27) with respect to θ, we get

$$\frac{\partial \sigma_S}{\partial \theta} = \frac{\begin{pmatrix} \left[\left(\left(1-\alpha\right)\xi + \alpha - \eta\right) - \left(1-\eta\right)\alpha\xi\right]\left(1 + \left(\varphi_{11} + \varphi_{12}\right)\xi\right) \\ +\alpha\left(1-\eta\right)\left(\xi - 1\right)\left(\varphi_{11} + \varphi_{12}\right)\xi \end{pmatrix}\left(\lambda\eta + \left(\varphi_{11} + \varphi_{12}\right)\xi\right)}{\left[\eta + \left(\theta + \left(1-\theta\right)\eta\right)\left(\varphi_{11} + \varphi_{12}\right)\xi\right]^2}$$

which is strictly *positive*. Therefore, in line with Cazzavillan and Pintus (2004), higher propensity to save makes the occurrence of local indeterminacy more likely.

Considering the case without external effects in preferences nor second-period labor supply, i.e., $\varphi_{12} = 0$ and $h = 0$, we obtain the framework of Cazzavillan and Pintus (2004). Setting $\lambda = \eta = 1$ and $\varphi_{12} = 0$ in (27), the bifurcation value becomes $\sigma_{S,CP} \equiv \alpha - \left(1-\theta\right)\left(1-\alpha\right)\left(\xi - 1\right)$. We notice that the positivity of $\sigma_{S,CP}$ requires sufficiently high propensity to save, $\theta > 1 - \alpha / \left(1-\alpha\right)\left(\xi - 1\right)$. In addition, $\sigma_{S,CP} < \alpha$ and $\partial \sigma_S / \partial \theta > 0$. That is, local indeterminacy requires complementary inputs and is more likely to appear for higher propensity to save. This confirms the result of Cazzavillan and Pintus (2004); local indeterminacy occurs if the elasticity of inputs substitution and the share of first-period consumption over young wage-income are sufficiently large.

5 Conclusion

In this chapter we have extended a one-sector OLG model with capital accumulation and endogenous labor supply by introducing leisure externalities and second-period inelastic labor supply. The occurrence of local indeterminacy and endogenous fluctuations has been studied. We have shown that higher external effects in leisure makes the emergence of fluctuations based upon self-fulfilling expectations more likely, in the sense that the range for the parameter value giving rise to it becomes larger. That is, leisure externalities act as a destabilizing force. However, high second-period inelastic labor supply promotes stability.

6 Appendix

6.1 Proof of Proposition 1

Consider (10) and (11) at a NSS $(1,1)$, we have

$$A\omega(1)\, u'\left(\left[A\omega(1) - s\right]/B\right)/B = v_1(1,1) \tag{36}$$

$$A\omega(1)\, A\rho(1)\, u'\left(A\rho(1)\, s + A\omega(1)\, h\right) = v_1(1,1) \tag{37}$$

We start with (37): RHS is a constant and LHS is increasing in A since $\xi > 1$ (gross substitutability assumption). Thus, whenever it exists, there is a unique A^* which solves equation (37). In view of equations (2) and (7), the following condition should be imposed on A^* in order to insure the positivity of first-period consumption:

$$A^* > s/\omega(1) \equiv \underline{A} \tag{38}$$

Since $\lim_{A\to+\infty}(LHS) = +\infty$, there exists a unique A^* solution of (24) if and only if $\lim_{A\to\underline{A}}(LHS) < v_1(1,1)$ which is equivalent to the following boundary condition:

$$\frac{\rho(1)\, s^2}{\omega(1)} u'\left(\frac{s\left[\rho(1)\, s + \omega(1)\, h\right]}{\omega(1)}\right) < v_1(1,1) \tag{39}$$

Thus, there exists a unique $A^* > 0$ which solves (A.1.2).

Given $A = A^*$, RHS is a constant while LHS is decreasing in $B > 0$ since $\xi > 1$ (gross substitutability assumption). In addition, $\lim_{B\to 0}(LHS) = +\infty$ and $\lim_{B\to+\infty}(LHS) = 0$. Thus, there exists a unique $B^* > 0$ which solves (36).

Consequently, we have shown that there are unique constants $A^* > 0$ and $B^* > 0$ such that $(a,l) = (1,1)$ is a steady state of the system (8)-(9).

6.2 Proof of Lemma 1

The coordinates of the origin of Σ are computed at $\sigma = 0$. Using (16) and (17), we obtain

$$\begin{aligned} T_0 &\equiv T(0) \\ &= 1 + D_0 + \frac{H_1 + \xi\left(\varphi_{11} + \varphi_{12}\right)\left[(1-\theta)\left((1-\alpha)\,\xi + \alpha - \eta\right) - \alpha\right]}{\theta\lambda\left((1-\alpha)\,\xi + \alpha - \eta\right)} \end{aligned}$$

$$D_0 \equiv D(0) = \frac{(1+\varphi_{11}+\varphi_{12})\,\alpha\xi}{\theta\lambda\,((1-\alpha)\,\xi+\alpha-\eta)}$$

where H_1 is given by (18).

$$\begin{aligned} T_0 &= -1 - D_0 \\ &+ \frac{(1+\theta\lambda)+(1-\theta)\,(\varphi_{11}+\varphi_{12})\,\xi}{\theta\lambda} \\ &+ \frac{\theta\lambda\,(1-\eta)\,\alpha\xi+\alpha\lambda\eta\,(\xi-1)+(1+\varphi_{11}+\varphi_{12})\,\alpha\xi}{\theta\lambda\,((1-\alpha)\,\xi+\alpha-\eta)} \end{aligned}$$

which implies that $T_0 > -1 - D_0$. The endpoint of Σ is obtained by evaluating (16) and (17) as σ tends to $+\infty$: $T_\infty \equiv T(+\infty) \to +\infty$ and $D_\infty \equiv D(+\infty) = D_0$. The slope of Σ is obtained by computing $D'(\sigma)$, $T'(\sigma)$ and the ratio $S_\Sigma = D'(\sigma)/T'(\sigma)$. Since $D'(\sigma) = 0$, the half-line Σ is horizontal, i.e., $S_\Sigma = 0$.

6.3 Proof of Lemma 2

From (22), we observe that the locus (T_0, D_0) depends on the elasticity of external effects φ_{12} and, as a function of φ_{12}, describes a half-line Σ_1 whose starting point is $(T_0, D_0)_{\varphi_{12}=-\varphi_{11}}$ and slope S_{Σ_1} is given by $(\partial D_0/\partial\varphi_{12})/(\partial T_0/\partial\varphi_{12})$. By increasing φ_{12} from $-\varphi_{11}$ to $+\infty$, (T_0, D_0) moves continuously along Σ_1. The origin of Σ_1 is

$$T_{0,\varphi_{12}=-\varphi_{11}} = 1+\frac{(1-\theta\lambda)\,[((1-\alpha)\,\xi+\alpha-\eta)-\alpha\xi]+\alpha\lambda\eta\,(1-\theta)\,\xi+\alpha\,(\xi-\lambda\eta)}{\theta\lambda\,((1-\alpha)\,\xi+\alpha-\eta)}$$

$$D_{0,\varphi_{12}=-\varphi_{11}} = \frac{\alpha\xi}{\theta\lambda\,((1-\alpha)\,\xi+\alpha-\eta)}$$

In addition, the slope of Σ_1 is

$$S_{\Sigma_1} \equiv \frac{D_0'(\varphi_{12})}{T_0'(\varphi_{12})} = \frac{\alpha}{(1-\theta)\,((1-\alpha)\,\xi+\alpha-\eta)}$$

Note that S_{Σ_1} is positive and depends on the propensity to save $\theta \in (0,1)$. In addition, $\partial S_{\Sigma_1}/\partial\theta > 0$, that is, Σ_1 rotates counter-clockwise as θ increases. Finally, deriving (24) with respect to θ, we obtain $\partial T_0(-\varphi_{11})/\partial\theta < 0$ and $\partial D_0(-\varphi_{11})/\partial\theta < 0$.

6.4 Proof of Lemma 3

Consider $D_0(-\varphi_{11})$, given by (24). The condition $D_0(-\varphi_{11}) < 1$ is satisfied if and only if $\theta > \theta_1$, where

$$\theta_1 \equiv \alpha\xi/\lambda\,[(1-\alpha)\,\xi+\alpha-\eta] > 0 \tag{40}$$

Note that if $\theta_1 > 1$ then $D_0(-\varphi_{11}) > 1$ for all $\theta \in (0,1)$. Therefore, in order for the condition $D_0(-\varphi_{11}) < 1$ to hold, the critical value θ_1 should be lower than one. Then $D_0(-\varphi_{11}) < 1$ for all $\theta > \theta_1$ and $D_0(-\varphi_{11}) > 1$ for all $\theta < \theta_1$. We focus on the range $\theta > \theta_1$ for which $D_0(-\varphi_{11}) < 1$, with $\theta_1 < 1$ if and only if $\lambda > \lambda_1$, where

$$\lambda_1 \equiv \alpha\xi / [(1-\alpha)\xi + \alpha - \eta] > 0 \tag{41}$$

Obviously, λ_1 should be lower than one which holds if and only if $\xi > \xi_1$, where

$$\xi_1 \equiv (\eta - \alpha) / (1 - 2\alpha) \tag{42}$$

with $\xi_1 > 1$ if and only if $1 - \alpha < \eta \leq 1$. In sum, $D_0(-\varphi_{11}) < 1$ holds if $\theta > \theta_1$, $\lambda > \lambda_1$ and either of the following conditions is satisfied: (1) $0 \leq \eta < 1 - \alpha$ and all $\xi \geq 1$; or (2) $1 - \alpha < \eta \leq 1$ and $\xi > \xi_1$.

Consider now $T_0(-\varphi_{11})$, given by (24), and assume that the condition $D_0(-\varphi_{11}) < 1$ is verified. The second condition $T_0(-\varphi_{11}) < 1 + D_0(-\varphi_{11})$ is met if and only if $\theta > \theta_2$, where

$$\theta_2 \equiv \frac{(((1-\alpha)\xi + \alpha - \eta) - \alpha\xi) + \lambda\alpha\eta(\xi - 1)}{\lambda[(((1-\alpha)\xi + \alpha - \eta) - \alpha\xi) + \alpha\eta\xi]} > 0 \tag{43}$$

with $\theta_2 < 1$ is insured if and only $\lambda > \lambda_2$, where

$$\lambda_2 \equiv \frac{((1-\alpha)\xi + \alpha - \eta) - \alpha\xi}{((1-\alpha)\xi + \alpha - \eta) - \alpha\xi + \alpha\eta} \tag{44}$$

Obviously, $\lambda_2 \in (0,1)$. Therefore, $T_0(-\varphi_{11}) < 1 + D_0(-\varphi_{11})$ holds if $\theta > \theta_2$ and $\lambda > \lambda_2$.

As a result, given that $\max\{\theta_1, \theta_2\} < \theta \leq 1$ and $\max\{\lambda_1, \lambda_2\} < \lambda \leq 1$, the sufficient conditions for the appearance of local indeterminacy, i.e., $D_0(-\varphi_{11}) < 1$ and $T_0(-\varphi_{11}) < 1 + D_0(-\varphi_{11})$, are met if (i) $0 \leq \eta < 1 - \alpha$ and all $\xi \geq 1$; or (ii) $1 - \alpha < \eta \leq 1$ and $\xi > \xi^*$.

6.5 Proof of Proposition 2

Given the sufficient conditions of local indeterminacy as provided by Lemma 3, the resulted configuration depends on the slope of Σ_1, given by (25). Obviously, $S_{\Sigma_1} > 1$ if and only if $\theta > \theta_3$, where

$$\theta_3 \equiv \frac{((1-\alpha)\xi + \alpha - \eta) - \alpha}{(1-\alpha)\xi + \alpha - \eta} \tag{45}$$

with $\theta_3 \in (0,1)$ since $[((1-\alpha)\xi + \alpha - \eta) - \alpha] > 0$. However, one can verify that $\theta_2 > \theta_3$. This excludes the case in which $S_{\Sigma_1} < 1$ because, given that $\theta_2 > \theta_3$, the intersection of $\theta < \theta_3$ with $\theta > \max\{\theta_1, \theta_2\}$ is empty. Therefore, $S_{\Sigma_1} > 1$ is a direct implication of the conditions $D_0(-\varphi_{11}) < 1$ and $T_0(-\varphi_{11}) < 1 + D_0(-\varphi_{11})$.

As a result, the initial point of Σ_1 lies inside the unit circle, that is, $D_0(-\varphi_{11}) < 1$ and $T_0(-\varphi_{11}) < 1 + D_0(-\varphi_{11})$. Further, $S_{\Sigma_1} > 1$ for all $\theta > \max\{\theta_1, \theta_2\}$ and $\partial S_{\Sigma_1}/\partial\theta > 0$, that is, Σ_1 makes counter-clockwise rotation when θ increases from $\max\{\theta_1, \theta_2\}$ $[S_{\Sigma_1}(\theta = \max\{\theta_1, \theta_2\}) > 1]$ to $+1$ $[S_{\Sigma_1}(1) \to +\infty]$.

For each $\theta > \max\{\theta_1, \theta_2\}$, Σ_1 makes upward movement with φ_{12} and therefore, the half-line Σ shifts upward along Σ_1 as shown in Fig. 1. Consequently, we obtain the configuration described by Proposition 2 and Fig. 1.