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# Consumption Externalities in a Ramsey Model\*

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## Abstract

This chapter studies the effect of consumption externalities on stability properties of a Ramsey model with heterogeneous agents and borrowing constraints. Agents differ in their initial wealth, felicity functions and discount factors. For simplicity, heterogeneity is reduced to two groups. Agents are identical within each group. In order to capture the role of heterogeneity as well as external effects, we introduce intergroup and intragroup consumption externalities. In this setting, we show that the most patient agent holds the entire capital stock at the steady-state whereas the other agent (impatient) consumes his wage-income. Our main result is that, whenever the preferences display keeping up with the Joneses feature with respect to intergroup externalities, the appearance of two-period cycles does not require the relaxation of Income Monotonicity Assumption. Instead, only the external effects in consumption from the other group that plays a crucial role for the appearance of these cycles.

Key words: Consumption externalities; borrowing constraints; heterogeneous agents; Indeterminacy; Bifurcations.

JEL Classification: E22; E23; O40.

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# 1 Introduction

Consumption externalities are considered as a simple way to study non-market interdependence between households in the economy, as stressed by earlier economists, notably, Veblen (1899) and Duesenberry (1949), and recently by empirical studies such as Easterlin (1995), Brekke and Howarth (2002), among others. Moreover, consumption externalities have been extensively studied in dynamic general equilibrium models. This literature is based on the assumption that household's felicity function depends on his own consumption as well as on some reference (external) consumption level which is taken as given. According to Dupor and Liu (2003), the existence of such externalities in household's felicity function gives rise to two distinct effects. Consumption spillovers may either increase or reduce the utility that an individual obtains from his own consumption. Consequently, the agent feels either "admiration" or "jealousy". Moreover, consumption externalities may also increase or reduce agent's marginal utility from private consumption. This effect is respectively referred to as Keeping-Up with the Joneses (KUJ) and Running-Away from the Joneses (RAJ).

The role of consumption externalities has been examined in different contexts, such as, happiness (Tian and Yang 2009), asset pricing (Abel 1990; Galí 1994), optimal taxation (Ljungqvist and Uhlig 2000), equilibrium efficiency (Liu and Turnovsky 2005; Nakamoto 2009), and long-run economic growth (Fisher and Hof 2000). Another strand of literature has focused upon the stability properties in dynamic models and notably, the existence of local indeterminacy and endogenous cycles. It has been shown that, in a representative-agent one-sector growth model with inelastic labor supply, the dynamic behavior of the economy with consumption externalities is equivalent to that of the economy without external effects, that is, the dynamic equilibrium is always unique. However, Alonso-Carrera et al. (2008) show that local indeterminacy appear whenever labor supply is endogenous. Moreover, Chen and Hsu (2007) demonstrate that, if agent's time-preference rate exhibits decreasing marginal impatience in private consumption, then equilibrium indeterminacy can emerge even without elastic labor supply.

A common feature of previous works is that they adopt the representative agent framework and thus external effects are represented by the economy-wide average consumption. As a result, all households have long-run symmetric level of consumptions which in turn implies that the average consumption coincides with the level of individual consumption. Then as recently mentioned by Garcia-Peñalosa and Turnovsky (2008), whenever rational agents realize that everyone's consumption is equal at equilibrium, external effects should disappear. Therefore, the authors argue that heterogeneity across agents is a fundamental component of any analysis of consumption externalities. Subsequently, only few works have investigated the role of consumption externalities in heterogeneous agents framework. In a neoclassical growth model, Garcia-Peñalosa and Turnovsky (2008) assume that households have the same felicity function but different reference groups and initial wealth. They show that, in a simple case where utility

depends on own and aggregate consumption and labor supply is inelastic, KUI accelerates the speed of convergence and thus resulting in less wealth inequality than would prevail in the absence of consumption externalities. However, Mino and Nakamoto (2008 and 2009) assume that, in a neoclassical growth model, there exist two types (groups) of agents who have different felicity function and initial wealth. Labor supply is inelastic. In addition, an agent cares about the consumption level of his group's members as well as the other group members' consumption level. Such a specification is formulated by assuming that the felicity function of an agent depends on his consumption, the average consumption of his group and the average consumption of the other group. In other terms, one agent's utility is affected by intragroup and intergroup externalities. In Mino and Nakamoto (2008), nonlinear income taxation is introduced. The authors show that, if the intragroup externalities dominate the intergroup effects, the symmetric steady state is a saddle. However, if the intergroup external effects have larger impact than the intragroup externalities, then the symmetric steady state is either unstable or locally indeterminate. Mino and Nakamoto (2009) demonstrate that the long-run wealth distribution is highly sensitive to the strength of intergroup and intragroup externalities. Finally, Mino (2008) develops an endogenous growth version of Diamond's (1965) OLG model with consumption externalities. The intragroup and intergroup externalities are respectively replaced with intragenerational and intergenerational external effects. It is shown that balanced growth equilibrium and transitional dynamics depend on the preference structure that characterizes forms of consumption external effects.

In this line, this chapter is interested in studying the impact of consumption externalities on the stability properties of a Ramsey model with heterogeneous agents. Consumers are assumed to have different initial wealth, felicity functions and discount factors. For simplicity, it is assumed that there are two groups of agents. Each group consists of identical households who supply labor inelastically and are subject to a borrowing constraint, that is, they can not borrow against their future income. Ramsey model with this type of heterogeneity but without consumption externalities has been extensively studied by Becker (1980); Becker and Foias (1987; 1994; 2007) and Sorger (1994; 2002).

It is also assumed that an agent's felicity function depends on his private consumption as well as on the average consumption of own group and the other group. This specification distinguishes between two external effects: intragroup and intergroup, that is, an agent's concern with the consumption of his group may be different from his concern with the consumption of the other group. For instance, his preferences can display keeping-up with the consumption behavior of agents' from own group and running-away from the consumption behavior of agents' from the other group. This asymmetry in external effects has been considered in Mino and Nakamoto (2008 and 2009).

This setup provides a rich environment to underline the role of consumption externalities and heterogeneity across agents on the local dynamic of Ramsey model.

Our first result is that only the patient agent owns a positive capital stock in

the long-run while the other (impatient) agent only consumes his wage-income as he does not hold capital. The steady state is thus characterized as in Becker (1980) and consumption externalities have no impact on the steady state.

In this framework but without consumption externalities, Becker and Foias (1994) have shown that the existence of periodic orbit of period two requires a negative response of the patient agent's income to the capital stock. Put it differently, *Income Monotonicity Assumption* is sufficient to rule out these cycles. However, we show that, by introducing consumption externalities in such a framework, these externalities give rise to a new mechanism through which cycles appear. More precisely, two-period cycles can occur even under *Income Monotonicity Assumption*. This result holds if input substitutability is low, intergroup externalities are large and exhibit KUJ feature, and intragroup externalities and intertemporal substitution are small.

The intuition goes as follows: cycles of period two requires that an increase in present capital stock should be followed by a reduction in the next-period capital. The initial increase in capital stock has two opposite effects. On the one side, it raises the impatient agent's income (his wage) and his consumption as well. Low input substitutability and high intergroup external effects imply an increase in the consumption of the patient agent's consumption. On the other side, the patient agent's income goes up under *Income Monotonicity Assumption*. However, as intertemporal substitutability is weak, the sensitivity of his consumption to income change is low. The appearance of two-period cycles requires that the first effect dominates the second effect. This requires large intergroup externalities. Obviously, such a mechanism is more likely to occur whenever intergroup externalities are higher.

To stress on the role of heterogeneity and consumption externalities, we consider particular cases. From one hand, this result still maintains even in the absence of intragroup external effects. That is, cycles appear because of the existence of intergroup externalities. For higher external effects from the other group, the range of saddle-path stability region shrinks and cycles become more likely to emerge. From the other hand, whenever we consider intragroup external effects alone, they do not influence the stability properties of the representative-agent model while, in the heterogeneous agents model, they promote the stability and make the emergence of cycles less likely. Therefore, in this late case, heterogeneity is the only mechanism that gives rise to cycles. These result are in line with Mino and Nakamoto (2008) who have shown that instability and local indeterminacy appear if intergroup externalities dominate intragroup effects.

This chapter is organized as follows. The next section presents the model. Section 3 defines the intertemporal equilibrium and provides the analysis of the steady state. Section 4 focuses on the stability properties and the occurrence of bifurcations. The results on local dynamics are discussed in section 5. An economic intuition for the main result is provided in section 6. Finally, section 7 concludes. All technical details are gathered in the Appendix.

## 2 The model

We consider a discrete-time Ramsey model with heterogeneous consumers and a representative firm.

### 2.1 Households

Consumers are assumed to be heterogeneous with respect to their discount rate, felicity function and initial wealth. In order to keep things as simple as possible, but without loss of generality, we reduce consumers' heterogeneity to two groups of agents, labeled with  $i = 1, 2$ . Individuals are identical in each group. Total population is constant over time, with size  $N > 0$ . The population size of each group is also constant, denoted by  $N_i$ , for type  $i$ .

Consider a representative agent for each group  $i$  ( $i = 1, 2$ ). In the following, it is assumed that agents of type 2 are more impatient than agents of type 1, that is, they discount the future utility more heavily:

**Assumption 1**  $0 < \beta_2 < \beta_1 < 1$ .

It is also assumed that the felicity function of the representative agent  $i$  ( $i = 1, 2$ ) depends on his private consumption and on consumption spillovers from his own group and from the other group. Let the preferences of agent  $i$  be represented by the instantaneous utility function  $u_i(c_{it}, \bar{c}_{it}, \bar{c}_{jt})$ , for  $i = 1, 2$ , where  $c_{i,t}$  is agent  $i$ 's consumption, and  $\bar{c}_{it}$  and  $\bar{c}_{jt}$  respectively represent the average-consumption in group  $i$  and  $j$ , for  $i \neq j$ . By considering this specification, we distinguish intragroup externalities from intergroup externalities. That is, an agent's concern with the consumption levels of members in his own group may be different from the concern with consumption of agents in the other group. A crucial feature of this setup is that the representative agent of group  $i$  takes the intragroup and intergroup average-consumption levels, i.e., respectively,  $\bar{c}_{it}$  and  $\bar{c}_{jt}$ , for  $i \neq j$ , as given. That is, each individual is assumed to be small enough to neglect his own contribution to the average consumption level of his group.

Moreover, as each group consists of identical agents, it is assumed that, for each group, the individual consumption and the average level of consumption coincide at equilibrium, namely,  $c_i = \bar{c}_{it}$ .

The felicity function satisfies the following assumption:

**Assumption 2** For  $i \neq j$  and  $i, j = 1, 2$ , the instantaneous utility function  $u_i(c_{it}, c_{it}, c_{jt})$  is twice continuously differentiable and satisfies the following conditions: (i)  $u_{i1}(c_{it}, c_{it}, c_{jt}) > 0 > u_{i11}(c_{it}, c_{it}, c_{jt})$ ; (ii)  $u_{i12}(c_{it}, c_{it}, c_{jt}) \leq 0$  and (iii)  $u_{i13}(c_{it}, c_{it}, c_{jt}) \leq 0$ .

In condition (i) of Assumption 2, the instantaneous utility function  $u_i(\cdot)$  is assumed to be monotonically increasing and strictly concave in private consumption,  $c_i$ . Condition (ii) and (iii) state that the external effect of consumption

could generate positive or negative effect on the marginal utility from own consumption of agent  $i$ . Namely, if agent  $i$  wants to be similar to others (resp., different from others), then agent  $i$ 's preferences display keeping up with the Joneses (KUJ) feature, i.e.,  $u_{i1l} > 0$ , for  $l = 2, 3$  (resp. running away from the Joneses, RAJ) feature, i.e.,  $u_{i1l} < 0$ , for  $l = 2, 3$ ). Further, Assumption 2 allows for the possibility that, for instance, agent  $i$  is willing to imitate the consumption behavior of his own group's members, i.e.,  $u_{i12}(c_{it}, c_{it}, c_{jt}) > 0$ , but different from the consumption behavior of the other group's members, i.e.,  $u_{i13}(c_{it}, c_{it}, c_{jt}) < 0$ .

We should impose the following restrictions on external effects:

**Assumption 3** (i)  $u_{i11} + u_{i1l} < 0$ , for  $l = 2, 3$ , and (ii)  $u_{i11} + u_{i12} + u_{i13} < 0$ .

Assumption 3 imposes restrictions on the strength of intergroup and intra-group external effects; they can not dominate the direct effect of private consumption: either the externalities augment the direct effect of own consumption, or, if they are offsetting, they are dominated by the own effect.

For  $i = 1, 2$ , we introduce the following elasticities:

$$\varepsilon_{i11} \equiv \frac{u_{i11}c_i}{u_{i1}} < 0 \quad (1)$$

$$\varepsilon_{i12} \equiv \frac{u_{i12}c_i}{u_{i1}} \leq 0 \quad (2)$$

$$\varepsilon_{i13} \equiv \frac{u_{i13}c_j}{u_{i1}} \leq 0 \quad (3)$$

where let  $\varepsilon_{i11}$  is the marginal elasticity of private consumption;  $\varepsilon_{i12}$  is the elasticity of external effects from own group  $i$  and  $\varepsilon_{i13}$  is the elasticity of external effects from the other group  $j$ , with  $i \neq j$ . In view of Assumption 3, the following conditions hold:

$$\varepsilon_{i11} + \varepsilon_{i12} < 0 \quad (4)$$

$$\varepsilon_{i11} + \varepsilon_{i13} < 0 \quad (5)$$

$$\varepsilon_{i11} + \varepsilon_{i12} + \varepsilon_{i13} < 0 \quad (6)$$

Assume that the representative agent of type  $i$  is initially endowed by  $k_{i0} \geq 0$ , such that the initial aggregate capital stock is strictly positive,  $K_0 = \sum_{i=1}^2 N_i k_{i0} > 0$ , i.e.,  $k_{i0} > 0$  holds at least for one agent. Denote the depreciation rate of capital by  $\delta \in (0, 1)$ . Given a sequence of real interest rates on capital  $\{r_t\}$  and wages rate  $\{w_t\}$ , the representative agent  $i$  chooses a pattern of consumption and capital holdings at each time that maximizes his lifetime utility subject to a sequence of budget constraints and a sequence of borrowing constraints. Formally, consumer  $i$ 's maximization program is written as

$$\max_{c_{it}, k_{it+1}} \sum_{t=0}^{+\infty} \beta_i^t u_i(c_{it}, \bar{c}_{it}, \bar{c}_{jt}) \quad (7)$$

subject to

$$c_{it} + k_{it+1} - (1 - \delta) k_{it} \leq r_t k_{it} + w_t \quad (8)$$

$$k_{it+1} \geq 0 \quad (9)$$

According to the non-negativity constraint (9), agents are not allowed to finance present consumption by borrowing against future income. This borrowing constraint reflects the incomplete market structure of the Ramsey model.

One can easily show that the necessary first-order conditions imply Euler equation

$$u_{i1}(c_{it}, \bar{c}_{it}, \bar{c}_{jt}) \geq \beta_i (r_{t+1} + 1 - \delta) u_{i1}(c_{it+1}, \bar{c}_{it+1}, \bar{c}_{jt+1}) \quad (10)$$

which holds with equality if  $k_{it+1} > 0$ . Moreover, the monotonicity of the utility function gives rise to a binding budget constraint

$$c_{it} + k_{it+1} - (1 - \delta) k_{it} = r_t k_{it} + w_t \quad (11)$$

## 2.2 Production

In contrast to the consumers' side, the production sector is homogeneous. Assume that a representative firm produces the final good using a constant return to scale technology  $y_t = F(K_t, L_t)$ , where  $K_t$  and  $L_t$  are the aggregate capital and labor. Let  $k_t \equiv K_t/L_t$  be the capital-labor ratio, using homogeneity of degree one, the production function can be written as  $F(K_t, L_t) \equiv f(k_t) L_t$ . Suppose that the representative firm maximizes the profit  $\pi_t \equiv y_t - r_t K_t - w_t L_t$  taking factor prices (the real interest rate  $r_t$  and the real wage  $w_t$ ) and the technology as given.

**Assumption 4** *The technology  $f(k)$  is a continuous function of the capital-labor ratio  $k \geq 0$ , positive-valued and differentiable. Furthermore,  $f''(k) < 0 < f'(k)$ , for  $k > 0$ , and  $f(0) = 0$ ,  $\lim_{k \rightarrow 0} f'(k) = +\infty$  and  $\lim_{k \rightarrow +\infty} f'(k) = 0$ .*

Under Assumption 4, profit maximization implies

$$r_t = f'(k_t) \quad \text{and} \quad w_t = f(k_t) - k_t f'(k_t) \quad (12)$$

We define the following elasticities. The elasticity of capital-labor substitution is given by  $\sigma \equiv [k f'(k) / f - 1] f'(k) / k f''(k)$ . The capital share of the total income is given by  $s \equiv f'(k) k / f(k) \in (0, 1]$ . Finally, the elasticities of the interest rate with respect to capital and labor are  $kr_k/r = -lr_l/r = -(1 - s) / \sigma$ , and the elasticities of the real wage with respect to capital and labor are  $kw_k/w = -lw_l/w = s / \sigma$ .



### 3 Intertemporal equilibrium

We provide a standard definition of equilibrium for the economy described above:

**Definition 1** *An equilibrium of the economy  $E = \left( F, (k_{i0}, \beta_i, u_i, N_i)_{i=1}^2 \right)$  is an intertemporal sequence  $\left( r_t, w_t, K_t, L_t, (c_{it}, k_{it})_{i=1}^2 \right)_{t=0}^{+\infty}$  which satisfies the following conditions:*

- (D1)  $(r_t, w_t)_{t=0}^{+\infty}$  is a sequence of strictly positive prices;
- (D2) given  $(r_t, w_t)_{t=0}^{+\infty}$ ,  $(K_t, L_t)$  solves the firm's program for  $t = 0, 1, \dots, \infty$ ;
- (D3) given  $(r_t, w_t)_{t=0}^{+\infty}$ ,  $(c_{it}, k_{i,t})_{t=0}^{+\infty}$  solves the  $i$ th consumer's program for  $i = 1, 2$ ;
- (D4) the capital market clears  $K_t = N_1 k_{1t} + N_2 k_{2t}$ , for  $t = 0, 1, \dots, \infty$ ;
- (D5) the labor market clears  $L_t = N_1 + N_2$ , for  $t = 0, 1, \dots, \infty$ ;
- (D6) the product market also clears  $\sum_{i=1}^2 N_i [c_{it} + k_{it+1} - (1 - \delta) k_{it}] = F(K_t, L_t)$ .

In the following Lemma, we state a set of equilibrium conditions for our model.

**Lemma 1** *Let  $E = \left( F, (k_{i0}, \beta_i, u_i, N_i)_{i=1}^2 \right)$  be an economy satisfying Assumptions 1 – 4. Consider the following conditions for  $t = 0, 1, \dots, \infty$  and  $i = 1, 2$ :*

- (a)  $c_{it} > 0$ ,  $k_{it} > 0$ ,  $K_t > 0$ ,  $L_t > 0$ ;
- (b)  $r_t = f'(k_t)$  and  $w_t = f(k_t) - k_t f'(k_t)$ , with  $k_t \equiv K_t/L_t$ ;
- (c)  $u_{i1}(c_{it}, \bar{c}_{it}, \bar{c}_{jt}) \geq \beta_i [r_{t+1} + 1 - \delta] u_{i1}(c_{it+1}, \bar{c}_{it+1}, \bar{c}_{jt+1})$  with equality when  $k_{it+1} > 0$ ;
- (d)  $c_{it} + k_{it+1} - (1 - \delta) k_{it} = r_t k_{it} + w_t$ ;
- (e)  $K_t = N_1 k_{1t} + N_2 k_{2t}$ ;
- (f)  $L_t = N_1 + N_2$ .

Then, if the sequence  $\left( r_t, w_t, K_t, L_t, (c_{it}, k_{it})_{i=1}^2 \right)_{t=0}^{+\infty}$  is a competitive equilibrium, the conditions (a)–(f) hold. Further, if the sequence  $\left( r_t, w_t, K_t, L_t, (c_{it}, k_{it})_{i=1}^2 \right)_{t=0}^{+\infty}$  satisfies (a) – (f) and the transversality condition

$$\lim_{t \rightarrow +\infty} \beta_i^t u_{i1}(c_{it}, \bar{c}_{it}, \bar{c}_{jt}) k_{it+1} = 0 \quad (13)$$

for  $i = 1, 2$ , it is an equilibrium for the economy  $E$ .

**Proof.** See the Appendix. ■

The theory of heterogeneous Ramsey equilibrium has been developed by Becker (1980) and Becker and Foias (1987; 1994). These papers have proved that, in a framework where agents have different time-preference rates, only the most patient agent owns a positive capital stock in the long-run. We show below that this result maintains and the steady state is the same as in the Ramsey model without consumption externalities. That is, at a neighborhood of the steady state, the patient agent 1 owns all the capital while agent 2 chooses to hold no capital provided that he is sufficiently impatient.

**Proposition 1** *There exists a unique steady state defined as follows:*

- (S1)  $r$  and  $w$  are constant;
- (S2)  $r + 1 - \delta = 1/\beta_1 < 1/\beta_2$ ;
- (S3)  $k_1 > 0$  and  $k_2 = 0$ ;
- (S4)  $c_1 = [(r + 1 - \delta) - 1]k_1 + w$  and  $c_2 = w$ ;
- (S5)  $K = N_1 k_1 > 0$ .

**Proof.** See the Appendix. ■

According to Proposition 1, the real gross return on capital should be  $1/\beta_1$  near the steady state since a firm prefers to have the capital with the lowest cost.

Let  $n_i \equiv N_i/(N_1 + N_2)$  be the mass of agents of type  $i$  ( $i = 1, 2$ ) as well as the population share of that type. Nearby the steady state characterized by Proposition 1, a Ramsey equilibrium is a sequence of  $\{c_{1t}, k_{1t}\}_{t=0}^{+\infty}$  that solves the following two-dimensional dynamic system

$$\frac{u_{11}(c_{1t}, c_{1t}, f(n_1 k_{1t}) - n_1 k_{1t} f'(n_1 k_{1t}))}{u_{11}(c_{1t+1}, c_{1t+1}, f(n_1 k_{1t+1}) - n_1 k_{1t+1} f'(n_1 k_{1t+1}))} = \beta_1 [1 - \delta + f'(n_1 k_{1t+1})] \quad (14)$$

$$c_{1t} + k_{1t+1} - (1 - \delta) k_{1t} = f(n_1 k_{1t}) + (1 - n_1) k_{1t} f'(n_1 k_{1t}) \quad (15)$$

subject to the initial aggregate endowment  $k_{10} > 0$  and the transversality condition (13).

## 4 Local dynamics

This section deals with the local dynamics of system (14)-(15) around the interior steady state. In order to save notations, it is convenient to drop the index of the patient agent; hence  $\beta_1 \equiv \beta$ ,  $n_1 \equiv n$ ,  $c_{1t} \equiv c_t$ , and  $k_{1t} \equiv k_t$ . We linearize the system (14)-(15) around the steady state, we get

$$(dk_{t+1}/k, dc_{t+1}/c)^T = J (dk_t/k, dc_t/c)^T$$

Let  $g(k) \equiv f(nk) + (1-n)kf'(nk)$  be the income of patient agent 1. We compute

$$g' = \frac{1}{\beta} [1 - \beta(1 - \delta)] \left( 1 - \frac{(1-s)(1-n)}{\sigma} \right) \quad (16)$$

$$\frac{c}{k} = \frac{1}{\beta} [1 - \beta(1 - \delta)] \left( 1 + \frac{1-s}{s}n \right) - \delta \quad (17)$$

Let

$$B_0 \equiv [1 - \beta(1 - \delta)](1 - s) \quad (18)$$

The trace and the determinant of the Jacobian matrix:

$$J = \begin{bmatrix} \frac{s\varepsilon_{13} - B_0}{\sigma} & \varepsilon_{11} + \varepsilon_{12} \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \frac{s\varepsilon_{13}}{\sigma} & \varepsilon_{11} + \varepsilon_{12} \\ 1 - \delta + g' & -c/k \end{bmatrix} \quad (19)$$

are respectively given by

$$T = 1 + \frac{1}{\beta} - \left( \frac{(1-n)B_0}{\beta} + \frac{c}{k} \frac{B_0 - s\varepsilon_{13}}{\varepsilon_{11} + \varepsilon_{12}} \right) \frac{1}{\sigma} \quad (20)$$

$$D = \frac{1}{\beta} - \left( \frac{B_0(1-n)}{\beta} - \frac{c}{k} \frac{s\varepsilon_{13}}{\varepsilon_{11} + \varepsilon_{12}} \right) \frac{1}{\sigma} \quad (21)$$

Further, consider

$$T = 1 + D - \frac{c}{k} \frac{B_0}{\sigma} \frac{1}{\varepsilon_{11} + \varepsilon_{12}} \quad (22)$$

$$\begin{aligned} T &= -1 - D \\ &+ 2 \left( 1 + \frac{1}{\beta} \right) - \left( \frac{2(1-n)B_0}{\beta} - \frac{c}{k} \frac{2s\varepsilon_{13} - B_0}{\varepsilon_{11} + \varepsilon_{12}} \right) \frac{1}{\sigma} \end{aligned} \quad (23)$$

In the following, we use the fact that the trace  $T$  and the determinant  $D$  are respectively the sum and the product of the eigenvalues.<sup>1</sup> The stability

<sup>1</sup>The characteristic polynomial of the Jacobian matrix  $J$  is  $P(\lambda) = \lambda^2 - T\lambda + D$ , where the trace  $T = \lambda_1 + \lambda_2$  and the determinant  $D = \lambda_1\lambda_2$ .

properties of the system, i.e., the location of the eigenvalues with respect to the unit circle, is equivalently and conveniently characterized in the  $(T, D)$ -plane.

In the spirit of Grandmont, Pintus and de Vilder (1998), we apply the geometrical method and we characterize the locus  $\Sigma \equiv \{(T(\sigma), D(\sigma)) : \sigma \geq 0\}$  obtained by varying the elasticity of inputs substitution  $\sigma$  in the  $(T, D)$ -plane. Then we examine the impact of  $\varepsilon_{13}$  on the location of  $\Sigma$ , by considering  $\min\{-\varepsilon_{11}, -(\varepsilon_{11} + \varepsilon_{12})\}$  as the upper bound of the intergroup external effects  $\varepsilon_{13}$ , see Assumption 3 and (5)-(6).

The following lemma provides some technical results involved in the main propositions.

**Lemma 2** *The half-line  $\Sigma$  is linear in  $\sigma \geq 0$  with endpoint  $(T_f, D_f) = (1/\beta, 1 + 1/\beta)$  and slope*

$$S = \frac{(1-n)B_0 - \beta \frac{c}{k} \frac{s\varepsilon_{13}}{\varepsilon_{11} + \varepsilon_{12}}}{(1-n)B_0 + \beta \frac{c}{k} \frac{B_0 - s\varepsilon_{13}}{\varepsilon_{11} + \varepsilon_{12}}} \quad (24)$$

and make clockwise rotation with  $\varepsilon_{13}$ , i.e.,  $S'(\varepsilon_{13}) < 0$ , and  $S(-\infty) = 1$ . Further,

1.  $D'(\sigma) < 0$  whenever  $\varepsilon_{13} < \bar{\varepsilon}_{13}$ , i.e., the locus  $(T(\sigma), D(\sigma))$  moves downwards in the  $(T, D)$ -plane when  $\sigma$  goes up; however,  $D'(\sigma) > 0$  whenever  $\varepsilon_{13} > \bar{\varepsilon}_{13}$ , where  $\bar{\varepsilon}_{13}$  is the solution of  $S = 0$ :

$$\bar{\varepsilon}_{13} = \frac{(1-n)B_0}{s\beta c/k} (\varepsilon_{11} + \varepsilon_{12}) \quad (25)$$

2. Let

$$B_1 \equiv \frac{s}{B_0} + \frac{1-n}{\beta c/k} \quad (26)$$

There exists a critical value  $\varepsilon_{13} = \tilde{\varepsilon}_{13}$  at which the slope of  $\Sigma$  is equal to  $-1$ , i.e.,  $S = -1$ , and which is given by

$$\tilde{\varepsilon}_{13} = \frac{B_0}{s} \left( \frac{1}{2} + \frac{(1-n)[\varepsilon_{11} + \varepsilon_{12}]}{\beta c/k} \right) \quad (27)$$

whenever either (i)  $-\infty < \varepsilon_{12} < \hat{\varepsilon}_{12}$  and  $\varepsilon_{11} < -1/2B_1$ ; or (ii)  $\varepsilon_{12} < \hat{\varepsilon}_{12} < 0$  and  $\varepsilon_{11} > -1/2B_1$ , where

$$\tilde{\varepsilon}_{12} \equiv -\frac{1}{2B_1} (1 + 2B_1\varepsilon_{11}) \quad (28)$$

$$\hat{\varepsilon}_{12} \equiv -\frac{1}{2} \frac{\beta c/k}{1-n} (1 + 2B_1\varepsilon_{11}) \quad (29)$$

**Proof.** See the Appendix. ■

Let  $\sigma = \sigma_F$  be the critical value that solves  $T = -1 - D$ :

$$\sigma_F \equiv \frac{1}{1 + \beta} \left[ (1 - n) B_0 - \beta \frac{c}{k} \frac{s\varepsilon_{13} - B_0/2}{\varepsilon_{11} + \varepsilon_{12}} \right] \quad (30)$$

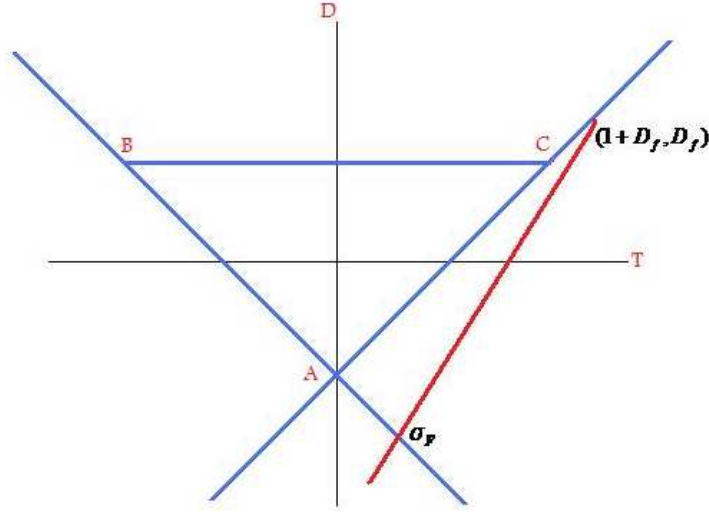
The characterization of the local dynamics is given by the following proposition:

**Proposition 2** *Let Assumptions 1 – 4 hold. Then the following generically occurs:*

(i) *Whenever  $-\infty < \varepsilon_{13} < \tilde{\varepsilon}_{13}$ , the steady state is a saddle point for all  $\sigma \geq 0$ .*

(ii) *Whenever  $\tilde{\varepsilon}_{13} < \varepsilon_{13} < \min\{-\varepsilon_{11}, -(\varepsilon_{11} + \varepsilon_{12})\}$ , the steady state is a source for all  $\sigma < \sigma_F$  and a saddle for all  $\sigma > \sigma_F$ . When  $\sigma$  crosses  $\sigma_F$ , the system undergoes a flip bifurcation.*

**Proof.** See the Appendix. ■



**Fig. 1**

In the following Proposition, we show that it is possible to find a range for the main parameters of the model, that belongs to intervals in Proposition 2, such that the appearance of two-period cycles does no longer require a negative response of the patient agent's income to the capital stock. In other terms, the following Proposition shows that two-period cycles can occur even under the *Income Monotonicity Assumption*, contrary to Becker and Foias (1994), and that only the external effects in consumption that give rise to flip bifurcation.

**Proposition 3** *Let Assumptions 1 – 4 hold. Given that  $\varepsilon_{11} < 1/2B_2$ ,  $\bar{\varepsilon}_{12} < \varepsilon_{12} < \varepsilon_{12}^*$  and  $n > n^*$ , where*

$$\varepsilon_{12}^* \equiv \frac{1}{2B_2} - \varepsilon_{11} > 0 \quad (31)$$

$$\bar{\varepsilon}_{12} \equiv \beta B_0 B_2 \frac{c/k}{1-n} \frac{\varepsilon_{11} - 1/2B_2}{B_0 - (1-s)[1+\beta]} < 0 \quad (32)$$

with

$$B_2 \equiv \frac{1-n}{\beta c/k} \left[ \frac{1}{B_0} (1-s)(1+\beta) - 1 \right] - \frac{s}{B_0} \quad (33)$$

and

$$n^* \equiv \frac{\beta - s + \beta(1-\delta)(1-s)}{(1-s)(1+\beta)} \in (0, 1) \quad (34)$$

Then cycles of period two appear under *Income Monotonicity Assumption* for all  $\varepsilon_{13} > \varepsilon_{13}^* > 0$  and  $\sigma^* < \sigma < \sigma_F$ , where

$$\sigma^* \equiv (1-s)(1-n) \quad (35)$$

$$\varepsilon_{13}^* \equiv \frac{B_0}{2s} + \frac{(1-n)[B_0 - (1-s)(1+\beta)]}{s\beta c/k} (\varepsilon_{11} + \varepsilon_{12}) \quad (36)$$

**Proof.** See the Appendix. ■

## 5 Discussion of the results

To understand the impact of consumption externalities and heterogeneity on stability, we start with two particular cases: heterogeneity without externalities (framework of Backer and Foias 1994) then intragroup externalities with and without heterogeneity. After that, we consider the general case with heterogeneous agents intergroup and intragroup consumption externalities.

### 5.1 Heterogeneous agents without consumption externalities

In the absence of external effects, the emergence of two-period cycles requires the relaxation of *Income Monotonicity Assumption*, i.e., the income of the household accumulating the capital (the patient) must be decreasing in capital.

To show that, we set  $\varepsilon_{12} = \varepsilon_{13} = 0$  in (22) and (23). At first, as  $\varepsilon_{11} < 0$ , one deduce from (22) that  $1 - T + D < 0$ . Then in order to determine whether the steady state is a source or a saddle point, consider (23) which becomes

$$1 + T + D = 2(1 + g') + 2(1 - \delta) - B_0 \frac{c/k}{\varepsilon_{11}} \frac{1}{\sigma} \quad (37)$$

where  $g(k)$  is the income of the patient agent and  $g'(k)$  is given by (16). From (37), there exist cycles of period two, it must be the case that  $1 + T + D = 0$ . This requires that the term in square brackets should be negative, i.e.,  $g'(k)$  should be negative. This confirms the result of Becker and Foias (1994).

In fact, the heterogeneity is the mechanism that gives rise to this result. Assume that agents are identical, i.e.,  $n = 1$ , then  $g(k)$  satisfies the *Income Monotonicity Assumption*, that is,  $g'(k) = 1/\beta > 0$ , and so the steady state is a saddle point. Further, we notice that the bifurcation value  $\sigma_F$  (given by (30)) becomes  $\sigma_{F,BF} = B_0 [1 - n + \beta(c/k)/2\varepsilon_{11}] / (1 + \beta)$  which is strictly positive if and only if the patient agent has a weak elasticity of intertemporal substitution in consumption, i.e.,  $(-1/\varepsilon_{11}) < 2(1 - n)/\beta(c/k)$ . In addition,  $d\sigma_{F,BF}/d\varepsilon_{11} < 0$  which means that weaker intertemporal substitutability is, more likely two-period cycles to appear. Finally, one can verify that  $g'(k) < 0$  for all  $\sigma < \sigma_{F,BF}$ . That is, cycles are ruled out whenever the *Income Monotonicity Assumption* holds.

## 5.2 Intragroup consumption externalities

Consider now the case with consumption externalities from own group only, that is,  $\varepsilon_{12} \neq 0$  and  $\varepsilon_{13} = 0$ .

From the one hand, assume that agents are *identical*  $n = 1$ , then the income of the representative agent is increasing in capital, i.e.,  $g'(k) = [1 - \beta(1 - \delta)]/\beta > 0$ . Further, by setting  $n = 1$  and  $\varepsilon_{13} = 0$  in (22) and (23) and under condition (4), one can directly deduce that the steady state is always a *saddle point* and that there is no room for cycles. Therefore, the dynamic behavior of the model with consumption externalities is exactly the same as that of the Ramsey model with inelastic labor supply and without external effects.

From the other hand, assume that agents are *heterogeneous*  $n < 1$ , then cycles of period two arise and require the relaxation of *Income Monotonicity Assumption*. Setting  $\varepsilon_{13} = 0$  in (30), one obtains the critical value at which flip bifurcation occurs,  $\sigma_F$ . A higher elasticity of intragroup externalities  $\varepsilon_{12}$  decreases  $\sigma_F$ , so the range of saddle-path stability  $(\sigma_F, +\infty)$  widens, making the emergence of cycles less likely.

As a result, the presence of intragroup external effects has no impact on the stability of the representative-agent model while it promotes the stability and makes the appearance of cycles less likely in the heterogeneous agents case.

## 5.3 Intergroup and intragroup consumption externalities

Let us turn back to the heterogeneous agents framework  $n < 1$  with intergroup and intragroup consumption externalities, i.e.,  $\varepsilon_{13} \neq 0$  and  $\varepsilon_{12} \neq 0$ .

According to Proposition 2, the emergence of two-period cycles requires weak inputs substitutability, high intergroup consumption spillovers.

We also find that, whenever we consider *intragroup* externalities alone, they promote stability and make the emergence of cycles less likely, see subsection (2.5.2). On the contrary, whenever we consider *intergroup* externalities alone, we find that  $\sigma_F$  in (30) increases with  $\varepsilon_{13}$  which means that, for a higher external effects from the other group, the range of saddle-path stability ( $\sigma_F, +\infty$ ) shrinks and instability and cycles become more likely. Deriving the critical bifurcation value  $\sigma_F$  in (30) with respect to two elasticities of consumption externalities, one gets positive effect, i.e.,  $\partial^2 \sigma_F / \partial \varepsilon_{12} \partial \varepsilon_{13} > 0$ . This means that intergroup consumption externalities are crucial for the appearance of cycles rather than intragroup externalities.

Moreover, in Proposition 3, we show that there is a range of the main parameters that belongs to intervals in Proposition 2 for which cycles of period two appear under the *Income Monotonicity Assumption*. This is obtained whenever intertemporal substitutability is weak, intergroup spillovers are positive and large, and external effects from the consumption of the same group are small in absolute value (i.e., either small  $\varepsilon_{12} > 0$  or large  $\varepsilon_{12} < 0$ ). In addition, such a result holds even in the absence of intragroup externalities ( $\varepsilon_{12} = 0$ ). Therefore, contrary to Becker and Foias (1994), the relaxation of *Income Monotonicity Assumption* is no longer needed for the appearance of cycles. Instead, it is the intergroup external effects in consumption that play a crucial role for the emergence of cycles.

## 6 Interpretation

The objective of this section is to explain why cycles of period two appear in a Ramsey model with heterogeneous agents and consumption externalities. We first start by the economy without external effects and then we consider our model.

### 6.1 Benchmark framework

We recover the framework of Becker and Foias (1994), by setting  $\varepsilon_{12} = \varepsilon_{13} = 0$ . As we have shown in subsection (5.1), the existence of two-period cycles requires the relaxation of *Income Monotonicity Assumption*. That is, the income of the household accumulating the capital (the patient) must be decreasing in capital. Further, the elasticity of intertemporal substitution in consumption of the patient agent should be sufficiently weak. The intuition is given as follows:

Suppose that  $k_t$  increases. Then the income of the patient agent 1 decreases, since  $g'(k) < 0$ . This implies to a reduction in his current consumption  $c_t$ . However, the patient agent 1 is willing to smooth his consumption  $c_t$  since his elasticity of intertemporal substitution is sufficiently low. He thus decreases his consumption level  $c_t$  slightly. As a result, the reduction in income will be absorbed by a drop of  $k_{t+1}$ . Two-period cycles appear.



## 6.2 Our model

Now we focus on the result stated by Proposition 3: cycles of period two appear under *Income Monotonicity Assumption* whenever input substitutability is small, intergroup spillovers are large, intertemporal substitution is small. The intuition is given as follows:

Suppose that  $k_t$  goes up. This generates two opposite effects:

On the one side, the income of the patient agent 1 increases by *Income Monotonicity Assumption*. However, as the elasticity of intertemporal substitution of agent 1 is sufficiently weak, the sensitivity of his consumption to the increase in income is low.

On the other side, the income of impatient agent 2 (wage) also moves up and so does his consumption ( $c_2 = w$ ). The increase in agent 2's consumption gives agent 1 the incentive to raise his own consumption, since the intergroup external effects are large and KUJ,  $\varepsilon_{13} > 0$ . Thus this leads agent 1 to increase his consumption.

Cycles of period two will emerge if the consumption of agent 1 increases sufficiently and implies a reduction in the level of next period capital stock. Such a scenario will be obtained if the elasticity of inputs substitution is weak and the intergroup externalities  $\varepsilon_{13} > 0$  are sufficiently high. The former generates a strong effect of initial increase in capital on wage and so on the impatient's consumption. The second results in a strong response of the patient agent's consumption to the impatient's consumption. Both effects imply that the initial increase in capital stock will produce an increase in the consumption of agent 1, even under weak intertemporal substitutability. Then according to the patient agent's budget constraint, next-period capital  $k_{t+1}$  decreases. So two-period cycles appear.

## 7 Conclusion

In this chapter, we have introduced consumption externalities in a Ramsey model with heterogeneous agents and borrowing constraints. We have assumed that there are two groups of infinitely-lived agents. Each group consists of a continuum identical agents. The representative agents in each group differ in their initial wealth, felicity functions and discount rates. We have also assumed that an agent's felicity function is affected by consumption spillovers from his own group and from the other group. In this setting, we have shown that consumption externalities do not influence the steady state. As in Becker (1980), only the patient agent holds positive capital stock in the long-run whereas the other (impatient) agent holds no capital and consumes his wage-income only. Moreover, we have shown that, under KUJ intergroup external effects, the relaxation of *Income Monotonicity Assumption* is no longer required for the emergence of two-period cycles. Instead, the existence of intergroup consumption externalities provides a new mechanism through which flip bifurcation appears.

## 8 Appendix

### 8.1 Proof of Lemma 1

*Necessary condition:*

Conditions (a), (e) and (f) results from the definition of equilibrium. Condition (b) is the first order condition of profit maximization problem of the representative firm. Condition (c) corresponds to the first order condition of the maximization problem of agent  $i$ . Finally, condition (d) is the budget constraint of agent  $i$ .

*Sufficient condition:*

Condition (D1) is satisfied because of (b) and Assumption 4. In order to show that condition (D2), consider any  $\tilde{k}_t \equiv \tilde{K}_t/\tilde{L}_t \geq 0$  different from  $k_t$ , we have

$$\begin{aligned}
& [F(K_t, L_t) - r_t K_t - w_t L_t] - [F(\tilde{K}_t, \tilde{L}_t) - r_t \tilde{K}_t - w_t \tilde{L}_t] \\
= & F(K_t, L_t) - F(\tilde{K}_t, \tilde{L}_t) - r_t (K_t - \tilde{K}_t) - w_t (L_t - \tilde{L}_t) \\
\geq & F_K(K_t, L_t) (K_t - \tilde{K}_t) + F_L(K_t, L_t) (L_t - \tilde{L}_t) - r_t (K_t - \tilde{K}_t) - w_t (L_t - \tilde{L}_t) \\
= & f'(K_t/L_t) (K_t - \tilde{K}_t) + [f(K_t/L_t) - f'(K_t/L_t) K_t/L_t] (L_t - \tilde{L}_t) - r_t (K_t - \tilde{K}_t) - w_t (L_t - \tilde{L}_t) \\
= & r_t (K_t - \tilde{K}_t) + w_t (L_t - \tilde{L}_t) - r_t (K_t - \tilde{K}_t) - w_t (L_t - \tilde{L}_t) \\
= & 0
\end{aligned}$$

The feasibility of the sequence  $(c_{it}, k_{i,t})$  for agent  $i$ 's maximization problem is ensured by (a). Now consider an alternative sequence  $(\tilde{c}_{it}, \tilde{k}_{i,t})$  satisfying the constraints in agent  $i$ 's program and the initial condition, we have

$$\begin{aligned}
& \sum_{t=0}^{+\infty} \beta_i^t [u_i(c_{it}, \bar{c}_{it}, \bar{c}_{jt}) - u_i(\tilde{c}_{it}, \bar{c}_{it}, \bar{c}_{jt})] \\
= & \lim_{T \rightarrow +\infty} \sum_{t=0}^T \beta_i^t [u_i(c_{it}, \bar{c}_{it}, \bar{c}_{jt}) - u_i(\tilde{c}_{it}, \bar{c}_{it}, \bar{c}_{jt})] \\
\geq & \lim_{T \rightarrow +\infty} \sum_{t=0}^T \beta_i^t [u_{i1}(c_{it}, \bar{c}_{it}, \bar{c}_{jt}) (c_{it} - \tilde{c}_{it})] \\
= & \lim_{T \rightarrow +\infty} \sum_{t=0}^T \beta_i^t u_{i1}(c_{it}, \bar{c}_{it}, \bar{c}_{jt}) \left[ (r_t + 1 - \delta) (k_{it} - \tilde{k}_{it}) - (k_{it+1} - \tilde{k}_{it+1}) \right] \\
= & \lim_{T \rightarrow +\infty} \left\{ \sum_{t=0}^T \beta_i^t u_{i1}(c_{it}, \bar{c}_{it}, \bar{c}_{jt}) (r_t + 1 - \delta) (k_{it} - \tilde{k}_{it}) - \sum_{t=0}^T \beta_i^t u_{i1}(c_{it}, \bar{c}_{it}, \bar{c}_{jt}) (k_{it+1} - \tilde{k}_{it+1}) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{T \rightarrow +\infty} \left\{ \begin{aligned} &u_{i1}(c_{i0}, \bar{c}_{i0}, \bar{c}_{j0})(r_0 + 1 - \delta)(k_{i0} - \tilde{k}_{i0}) \\ &+ \sum_{t=0}^{T-1} \beta_i^{t+1} u_{i1}(c_{it+1}, \bar{c}_{it+1}, \bar{c}_{jt+1})(r_{t+1} + 1 - \delta)(k_{it+1} - \tilde{k}_{it+1}) \\ &- \sum_{t=0}^{T-1} \beta_i^t u_{i1}(c_{it}, \bar{c}_{it}, \bar{c}_{jt})(k_{it+1} - \tilde{k}_{it+1}) - \beta_i^T u_{i1}(c_{iT}, \bar{c}_{iT}, \bar{c}_{jT})(k_{iT+1} - \tilde{k}_{iT+1}) \end{aligned} \right\} \\
&= \lim_{T \rightarrow +\infty} \left\{ \begin{aligned} &\sum_{t=0}^{T-1} \beta_i^t [\beta_i(r_{t+1} + 1 - \delta) u_{i1}(c_{it+1}, \bar{c}_{it+1}, \bar{c}_{jt+1}) - u_{i1}(c_{it}, \bar{c}_{it}, \bar{c}_{jt})] (k_{it+1} - \tilde{k}_{it+1}) \\ &- \beta_i^T u_{i1}(c_{iT}, \bar{c}_{iT}, \bar{c}_{jT})(k_{iT+1} - \tilde{k}_{iT+1}) \end{aligned} \right\} \\
&\geq \lim_{T \rightarrow +\infty} \left\{ \begin{aligned} &\sum_{t=0}^{T-1} \beta_i^t [\beta_i(r_{t+1} + 1 - \delta) u_{i1}(c_{it+1}, \bar{c}_{it+1}, \bar{c}_{jt+1}) - u_{i1}(c_{it}, \bar{c}_{it}, \bar{c}_{jt})] k_{it+1} \\ &- \beta_i^T u_{i1}(c_{iT}, \bar{c}_{iT}, \bar{c}_{jT}) k_{iT+1} \end{aligned} \right\} \\
&= 0
\end{aligned}$$

So we have just proved that (D3) is satisfied. Condition (D4) and (D5) are respectively identical to (e) and (f). Finally, condition (D6) results from (d) and (e).

## 8.2 Proof of Proposition 1

Here we show that consumption externalities have no impact on the steady state. The proof consists of four steps:

1. For  $i = 1$ , conditions (S<sub>1</sub>) – (S<sub>4</sub>) satisfy the optimality conditions in Lemma 1. Further, the transversality condition holds since  $0 < \beta_1 < 1$  and consumption levels, i.e.,  $c_1$  and  $c_2$ , are constant, with  $\bar{c}_1$  and  $\bar{c}_2$  are taken as "exogenously given". Hence  $\lim_{t \rightarrow +\infty} \beta_1^t u_{11}(c_1, \bar{c}_1, \bar{c}_2) k_1 = 0$  is satisfied.
2. For  $i = 2$ , we show that it is optimal to hold no capital, i.e.,  $k_2 = 0$  and to consume his wage-income, i.e.,  $c_2 = w$ . For this purpose, consider a feasible sequence  $(\tilde{c}_{2t}, \tilde{k}_{2t})$  starting from  $\tilde{k}_{20} = 0$  and compare this path with the stationary solution  $(c_2, k_2)$  such that  $c_2 = w$  and  $k_2 = 0$ . We show that this stationary solution is optimal. Using the assumption that at equilibrium,  $\bar{c}_i = c_i$ , for  $i = 1, 2$ , and  $\bar{c}_1$  is exogenously taken by agent 2, we then have the following:

$$\begin{aligned}
&\sum_{t=0}^{+\infty} \beta_2^t [u_2(w, w, c_1) - u_2(\tilde{c}_{2t}, \tilde{c}_{2t}, c_1)] \\
&\geq \sum_{t=0}^{+\infty} \beta_2^t [[w - \tilde{c}_{2t}] u_{21}(w, w, c_1) + [w - \tilde{c}_{2t}] u_{22}(w, w, c_1)]
\end{aligned}$$

$$\begin{aligned}
&= [u_{21}(w, w, c_1) + u_{22}(w, w, c_1)] \sum_{t=0}^{+\infty} \beta_2^t [w - \tilde{c}_{2t}] \\
&= [u_{21}(w, w, c_1) + u_{22}(w, w, c_1)] \sum_{t=0}^{+\infty} \beta_2^t \left[ \tilde{k}_{2t+1} - \frac{1}{\beta_1} \tilde{k}_{2t} \right] \\
&= [u_{21}(w, w, c_1) + u_{22}(w, w, c_1)] \lim_{T \rightarrow \infty} \left\{ \sum_{t=0}^T \beta_2^t \tilde{k}_{2t+1} - \frac{1}{\beta_1} \sum_{t=0}^T \beta_2^t \tilde{k}_{2t} \right\} \\
&= [u_{21}(w, w, c_1) + u_{22}(w, w, c_1)] \lim_{T \rightarrow \infty} \left\{ \beta_2^T \tilde{k}_{2T+1} + \frac{1}{\beta_2} \sum_{t=1}^T \beta_2^t \tilde{k}_{2t} - \frac{1}{\beta_1} \sum_{t=1}^T \beta_2^t \tilde{k}_{2t} - \frac{1}{\beta_1} \tilde{k}_{20} \right\} \\
&= [u_{21}(w, w, c_1) + u_{22}(w, w, c_1)] \lim_{T \rightarrow \infty} \left\{ \beta_2^T \tilde{k}_{2T+1} + \left( \frac{1}{\beta_2} - \frac{1}{\beta_1} \right) \sum_{t=1}^T \beta_2^t \tilde{k}_{2t} - \frac{1}{\beta_1} \tilde{k}_{20} \right\}
\end{aligned}$$

3. Under Assumption 4, there is a unique finite and strictly positive value of  $k$  such that  $r = f'(k)$ .
4. It remains to show now that  $r+1-\delta = 1/\beta_1$  is the only stationary solution, with  $k = k_1 > 0$  and  $k_2 = 0$ . From one hand, if  $r+1-\delta > 1/\beta_1$  then it is optimal for agent 1 to accumulate more capital. However, this cannot be a stationary solution because of the decreasing returns. From the other side, if  $r+1-\delta < 1/\beta_1 < 1/\beta_2$  then it is optimal for both agents to decumulate to zero in a finite time, i.e.,  $k \rightarrow 0$  and  $\lim_{k \rightarrow 0} f'(k) = +\infty$ , which contradicts the stationarity.

### 8.3 Proof of Lemma 2

First, the endpoint is obtained by taking the limit of (20) and (21) as  $\sigma$  approaches  $+\infty$ , it is given by  $(T_f, D_f) \equiv (T(+\infty), D(+\infty)) = (1 + D_f, 1/\beta)$ . The end point lies on the line  $(AC)$ , above the point  $C$ . The slope is obtained by computing  $T'(\sigma)$  and  $D'(\sigma)$  and is given by the ratio  $S = D'(\sigma)/T'(\sigma)$ . Thus one gets the expression (24) in Lemma 2.

Then deriving the slope  $S(\varepsilon_{13})$  in (24) with respect to  $\varepsilon_{13}$  gives  $S'(\varepsilon_{13}) < 0$ , i.e., the half-line  $\Sigma$  makes a clockwise rotation with  $\varepsilon_{13}$ , for all  $-\infty < \varepsilon_{13} < \min\{-\varepsilon_{11}, -(\varepsilon_{11} + \varepsilon_{12})\}$ . One can also compute the slope whenever  $\varepsilon_{13}$  tends  $-\infty$ , one gets  $S(-\infty) = 1$ . Moreover, deriving (20) to obtain

$$D'(\sigma) = \left( \frac{(1-n)B_0}{\beta} - \frac{c}{k} \frac{s\varepsilon_{13}}{\varepsilon_{11} + \varepsilon_{12}} \right) \frac{1}{\sigma^2}$$

$D'(\sigma)$  determines the direction of the movement of the half-line  $\Sigma$  with  $\sigma \geq 0$  and depends on the value of  $\varepsilon_{13}$ . Obviously, there is a critical value  $\varepsilon_{13} = \bar{\varepsilon}_{13}$  which solves  $D'(\sigma) = 0$  and at which  $\Sigma$  is horizontal. That is, whenever  $\varepsilon_{13} < \bar{\varepsilon}_{13}$ , then  $D'(\sigma) < 0$  and so  $\Sigma$  makes a downward movement with  $\sigma \geq 0$ ; and

whenever  $\varepsilon_{13} > \bar{\varepsilon}_{13}$  then  $\Sigma$  makes an upward movement with  $\sigma \geq 0$ , where  $\bar{\varepsilon}_{13}$  is given by (25).

Finally, solving  $S = -1$  for  $\varepsilon_{13}$ , one get  $\varepsilon_{13} = \tilde{\varepsilon}_{13}$  which is given by (27). Then we should ensure that  $\tilde{\varepsilon}_{13}$  is lower than  $\min\{-\varepsilon_{11}, -(\varepsilon_{11} + \varepsilon_{12})\}$ , this depends on the sign of  $\varepsilon_{12}$ :

- (i) Whenever agent's preferences display KUJ feature with respect to intragroup consumption externalities, i.e.,  $\varepsilon_{12} > 0$ , then  $\tilde{\varepsilon}_{13} < -(\varepsilon_{11} + \varepsilon_{12})$  holds if and only if  $\varepsilon_{12} < \hat{\varepsilon}_{12}$ . Note that  $\hat{\varepsilon}_{12} < -\varepsilon_{11}$  since  $B_1 > 0$ . One can verify that  $\tilde{\varepsilon}_{12} > 0$  for  $\varepsilon_{11} < -1/2B_1$ .
- (ii) Whenever agent's preferences display RAJ feature with respect to intragroup consumption externalities, i.e.,  $\varepsilon_{12} < 0$ , then  $\tilde{\varepsilon}_{13} < -\varepsilon_{11}$  holds if and only if  $\varepsilon_{12} < \hat{\varepsilon}_{12}$ , with  $\hat{\varepsilon}_{12} < 0$  if  $\varepsilon_{11} > -1/2B_1$ . As a result,  $\tilde{\varepsilon}_{13} < -\varepsilon_{11}$  is satisfied in either of the following cases: (i)  $\varepsilon_{12} < \hat{\varepsilon}_{12} < 0$  and  $\varepsilon_{11} > -1/2B_1$ ; or (ii)  $\varepsilon_{11} < -1/2B_1$  and all  $\varepsilon_{12} < 0$ .

## 8.4 Proof of Proposition 2

From (22), we have we have  $T > 1 + D$  as  $(\varepsilon_{11} + \varepsilon_{12}) < 0$  (by Assumption 2). Thus the eigenvalues are real and (at least) one is unstable, i.e. has a norm greater than one. In other terms, the steady state is either a saddle point or (locally) unstable. Yet, the existence of sustained cycles is not precluded. Indeed, whenever it happens that one of the eigenvalues equals  $-1$ , the Flip Bifurcation Theorem (see Ruelle 1989, pp. 67-39) teaches us that, generically, there exists a periodic orbit of period two. In order for an eigenvalue to be equal to  $-1$ , it must be the case that  $P(-1) = 1 + T + D = 0$ .

Consider now (23): In order to examine the effect of  $\varepsilon_{13}$  on the location of  $\Sigma$ , consider the initial slope, evaluated as  $\varepsilon_{13}$  tends to  $-\infty$ , we get  $S(-\infty) \rightarrow +1$ . Then by increasing  $\varepsilon_{13}$ , the half-line  $\Sigma$  rotates clockwise.

The movement of  $\Sigma$  with  $\sigma \geq 0$  (upward or downward) is crucial: As previously shown that  $D'(\sigma) < 0$  for all  $\varepsilon_{13} < \bar{\varepsilon}_{13}$  and  $D'(\sigma) > 0$  for all  $\varepsilon_{13} > \bar{\varepsilon}_{13}$ , where  $\bar{\varepsilon}_{13}$  is given by (25).

1. Whenever  $-\infty < \varepsilon_{13} < \bar{\varepsilon}_{13}$ , the half-line  $\Sigma$  makes a *downward* movement with  $\sigma \geq 0$ , that is,  $D'(\sigma) < 0$ . The origin is obtained by taking the limit of (20) and (21) as  $\sigma$  approaches 0 from above  $(T_0, D_0) \rightarrow (+\infty, +\infty)$ . In this case, the steady state is a saddle point for all  $\sigma \geq 0$ .
2. Whenever  $\varepsilon_{13} > \bar{\varepsilon}_{13}$ , the half-line  $\Sigma$  makes an *upward* movement with  $\sigma \geq 0$ , that is,  $D'(\sigma) > 0$  and  $D_0 \rightarrow -\infty$ . However, the sign of  $T_0$  depends on  $\varepsilon_{13}$ . Let  $\varepsilon_{13} = \tilde{\varepsilon}_{13}$  be the solution of  $S = -1$  and given by (27).
  - (a) For all  $\bar{\varepsilon}_{13} < \varepsilon_{13} < \tilde{\varepsilon}_{13}$ , we have  $-1 < S < 0$  and  $T_0 \rightarrow +\infty$ . Thus the steady state is a saddle point for all  $\sigma \geq 0$ .

- (b) However, for all  $\tilde{\varepsilon}_{13} < \varepsilon_{13} < \min\{-\varepsilon_{11}, -(\varepsilon_{11} + \varepsilon_{12})\}$ , then  $S > -1$ . In this case, the half-line  $\Sigma$  crosses the line  $(AB)$  and flip bifurcation arises whenever  $\sigma$  is close to  $\sigma_F$ , where  $\sigma_F$  is the solution of  $T(\sigma_F) = -1 - D(\sigma_F)$  and given by (30), with  $\sigma_F > 0$  for all  $\tilde{\varepsilon}_{13} < \varepsilon_{13} < \min\{-\varepsilon_{11}, -(\varepsilon_{11} + \varepsilon_{12})\}$ . Therefore, the steady state is a source for all  $\sigma < \sigma_F$  and a saddle for all  $\sigma > \sigma_F$ . When  $\sigma$  crosses  $\sigma_F$ , the system undergoes a flip bifurcation.

### 8.5 Proof of Proposition 3

Assume that *Income Monotonicity Assumption*  $g'(k) > 0$  holds, this requires  $\sigma > (1-s)(1-n) \equiv \sigma^*$  (see (16)). However, according to Proposition 2, two-period cycles appear for all  $\sigma < \sigma_F$ . This implies that cycles appear under *Income Monotonicity Assumption* if and only if  $\sigma^* < \sigma_F$ . This latter inequality holds for all  $\varepsilon_{13} > \varepsilon_{13}^*$ , where

$$\begin{aligned}\varepsilon_{13}^* &\equiv \frac{B_0}{s} \left[ \frac{1}{2} + \frac{1-n}{B_0\beta c/k} [B_0 - (1-s)(1+\beta)] (\varepsilon_{11} + \varepsilon_{12}) \right] \\ &= \tilde{\varepsilon}_{13} - \frac{(1-s)(1-n)(1+\beta)}{s\beta c/k} (\varepsilon_{11} + \varepsilon_{12})\end{aligned}\quad (38)$$

As  $(\varepsilon_{11} + \varepsilon_{12}) < 0$  and  $[B_0 - (1-s)(1+\beta)] < 0$ , then  $\varepsilon_{13}^* > 0$  and  $\varepsilon_{13}^* > \tilde{\varepsilon}_{13}$ .

We should make sure that  $\varepsilon_{13}^*$  is lower than  $\min\{-\varepsilon_{11}, -(\varepsilon_{11} + \varepsilon_{12})\}$ . If so, then flip bifurcation appears for all  $\varepsilon_{13} > \varepsilon_{13}^* > 0$  and  $\sigma^* < \sigma < \sigma_F$ ; otherwise, there is no room for cycles under *Income Monotonicity Assumption*.

Let

$$B_2 \equiv \frac{1-n}{\beta c/k} \left[ \frac{1}{B_0} (1-s)(1+\beta) - 1 \right] - \frac{s}{B_0}\quad (39)$$

1. If agent's preferences display KUJ feature with respect to intragroup consumption externalities, i.e.,  $\varepsilon_{12} > 0$ , then we should verify that  $\varepsilon_{13}^* < -(\varepsilon_{11} + \varepsilon_{12})$ . This latter inequality holds if and only if  $\varepsilon_{12} < \varepsilon_{12}^*$ :

$$\varepsilon_{12}^* \equiv \frac{1}{2B_2} - \varepsilon_{11}\quad (40)$$

with  $\varepsilon_{12}^*$  should be positive which requires that  $\varepsilon_{11} < 1/2B_2$  and  $B_2 < 0$ . This ensures that  $\varepsilon_{12}^* < -\varepsilon_{11}$ . Further, one can easily verify that  $\varepsilon_{12}^* < \tilde{\varepsilon}_{12}$  since  $1/2B_2 < -1/2B_1$ . As a result, flip bifurcation appears, under *Income Monotonicity Assumption*, for  $\sigma^* < \sigma < \sigma_F$ ,  $\varepsilon_{13} > \varepsilon_{13}^* > 0$ ,  $\varepsilon_{11} < 1/2B_2$  and  $0 < \varepsilon_{12} < \varepsilon_{12}^*$ .

2. If agent's preferences display RAJ feature with respect to intragroup consumption externalities, i.e.,  $\varepsilon_{12} < 0$ , then  $\varepsilon_{13}^* < -\varepsilon_{11}$  holds for all  $\varepsilon_{12} > \tilde{\varepsilon}_{12}$ , where

$$\tilde{\varepsilon}_{12} \equiv \beta B_0 B_2 \frac{c/k}{1-n} \frac{\varepsilon_{11} - 1/2B_2}{B_0 - (1-s)(1+\beta)}\quad (41)$$

with  $[B_0 - (1 - s)(1 + \beta)] < 0$ . In order for  $\varepsilon_{13}^* < -\varepsilon_{11}$  to hold, we should have  $\bar{\varepsilon}_{12} < 0$  which in turn requires that  $\varepsilon_{11} < 1/2B_2$  and  $B_2 < 0$ . Therefore, two-period cycles appear, under *Income Monotonicity Assumption*, for  $\sigma^* < \sigma < \sigma_F$ ,  $\varepsilon_{13} > \varepsilon_{13}^* > 0$ ,  $\varepsilon_{11} < 1/2B_2$  and  $\bar{\varepsilon}_{12} < \varepsilon_{12} < 0$ .

It remains to mention that  $B_2 < 0$  if and only if  $n > n^*$ , where

$$\begin{aligned} n^* &\equiv \frac{(1-s)[1+\beta] - B_0 - s(1-\beta)}{(1-s)[1+\beta]} \\ &= \frac{\beta - s + \beta(1-\delta)(1-s)}{(1-s)[1+\beta]} \in (0, 1) \end{aligned}$$

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