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with Endogenous Labor Supply**

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Consumption Externalities in a Ramsey Model with Endogenous Labor Supply*

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Abstract

We consider a Ramsey model with heterogeneous agents and borrowing constraint. Heterogeneity across agents stems from different initial capital endowment, labor supply, felicity function and discount rate. For simplicity, heterogeneity is reduced to only two groups of agents where they are identical in each group. The felicity function of an agent depends on his own consumption as well as on others' consumption. Our objective is to study the effect of consumption externalities on economic stability around the steady state. As in standard models, only the patient agents hold capital at the stationary equilibrium and further, this stationary equilibrium is not affected by the presence of consumption externalities. There are two types of steady states: one with both agents labor supply and the other with only impatient agents supply labor while the patient agents enjoy leisure. Moreover, the interaction between externalities and endogenous labor supply implies the emergence of endogenous fluctuations due to self-fulfilling expectations of agents.

Key words: Consumption externalities; Bifurcations; Heterogeneous agents; Borrowing constraint; Indeterminacy.

JEL Classification: C62; D30; E21; E32.

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1 Introduction

In modern economies, it is observed that individuals care about the social aspect of their consumption in the sense that, it is not the absolute level of consumption that matters, but rather one's consumption relative to other individuals in the society. **The** idea that one's relative consumption represents his social position is not new issue. It has been initially discussed by Veblen (1899) and Duesenberry (1949) who show that individual's satisfaction depends on his position in the society, i.e., his social status.

Such an idea is supported in numerous empirical studies. For instance, Easterlin (1995) deduces, based on American data, that increasing income of all people in a society does not increase their happiness because agents are caring more about their relative income. Moreover, based on British data, Clark and Oswald (1996) find that workers' happiness is inversely related to their comparison wages. Using psychological evidence, Frank (1997) shows that agent's satisfaction depends on his relative position.

The influence of social forces on individual's satisfaction is formulated in theoretical literature by introducing some reference (external) consumption level in the utility function. Such a reference is exogenously given and equal to the average consumption of some reference group which may comprise the entire or a subset of the population. According to Dupor and Liu (2003), when consumption spillovers augment (or decline) agent's marginal rate of substitution between agents' own consumption and leisure, then his preferences exhibit "Keeping-up with the Joneses", hereafter KUJ, (or "Running-away from the Joneses", hereafter RAJ). When KUJ, then agent is willing to imitate the consumption of others while when RAJ, he chooses his consumption in such a way that makes him different from others.

In the real world, such situations can emerge. Consider for example an economy in which the households differ in their wealth. Then, it can be observed that poor agents usually try to imitate the consumption behaviour of rich while the later cares only about the consumption of other rich agents and does not pay any attention to the consumption level of the poor. We thus deduce that, on the one side, the existence of these consumption external effects is plausible and relevant. On the other side, studying these externalities in an economy with heterogeneous agents can provide a richer environment to cover more configurations than that studied in the representative-agent framework.

Along this direction, this paper considers an economy populated with rich and poor and focuses on the most realistic configuration where the consumption behaviour of rich affects the poor's consumption while the inverse does not hold. For this purpose, a simple Ramsey model with heterogeneous agents is considered. Several works have focused on the long-run capital distribution when consumers differ in their initial capital level, discounting and felicity function. In a framework with heterogeneous agents, the borrowing constraint matters. Whenever consumers are allowed to borrow against their income, the steady state does not exist and the impatient agents' consumption asymptotically vanishes (Le Van and Vailakis 2003). Conversely, whenever a borrowing

constraint is imposed, i.e., market is incomplete, then a steady state exists and around it, the capital is concentrated in the hands of the most patient agents (Becker 1980; Becker and Foias 1987; 1994; 2007; and Sorger 1994; 2002). Obviously, it is more likely the later framework gives rise to two types of agents (poor and rich) in the long-run. As a result, this paper adopts the framework initiated by Becker (1980) to analyze the social interaction between poor and rich.

The objective of this paper is to study the role of consumption externalities on the appearance of local indeterminacy and endogenous cycles generated by self-fulfilling expectations in a Ramsey model with heterogeneous agents.

Consumption externalities have been widely studied in various contexts. Among others, Abel (1990) and Galí (1994) introduce externality in consumption in asset pricing model to clarify the divergence between theoretical and empirical findings in the data. Fisher and Hof (2000) and Alvarez-Cuadrado *et al.* (2004) consider consumption externalities to study the economic growth and show that considering consumption externalities increases the speed of convergence. Tian and Yang (2009) give formal explanation to Easterlin's (1995) empirical findings and analyze the impact of consumption externalities on agents' happiness.

Further, the impact of consumption externalities on the appearance of local indeterminacy and endogenous fluctuations has also been treated. In a representative-agent Ramsey model, the dynamic equilibrium is always unique and locally determinate whenever the labor supply is inelastic (Liu and Turnovsky 2005). However, Alonso-Carrera *et al.* (2008) demonstrate that the endogeneity of labor supply is a key requirement for the emergence of local indeterminacy. Chen and Hsu (2007) show that the dynamic equilibrium can be locally indeterminate, even when labor supply is inelastic, if agent's time-preference exhibits decreasing marginal impatience in private consumption.

Another strand of literature has focused on the dynamic effects of consumption externalities in Ramsey model with heterogeneous agents. For example, García-Peñalosa and Turnovsky (2008) show that consumption externalities affect the speed of convergence and reduces wealth inequality. Further, intergroup and intragroup externalities have been introduced by several authors, such as Mino and Nakamoto (2008) who show that whenever the intergroup dominate the intragroup external effects, the symmetric steady state can be locally indeterminate. In addition, Mino and Nakamoto (2009) demonstrate that the long-run wealth distribution is highly sensitive to the strength of intergroup and intragroup externalities.

Unlike these papers that only focus on heterogeneity in wealth, Barbar and Barinci (2010) introduce consumption externalities in the framework of Becker and Foias (1994) and distinguish between intergroup and intragroup external effects. The authors show that it's more likely the intergroup external effects that promote instability and the emergence of deterministic cycles of period two. However, contrary to Becker and Foias (1994), the negative response of the patient agent's income to the capital stock is no longer required for the appearance of these cycles if the patient agent's preferences exhibit KUJ feature.

The result of Barbar and Barinci (2010) is interesting but has two shortcomings. On the one side, the result is obtained in terms of the patient agent's preference only and thus the heterogeneity is not explicitly captured. On the other side, instability requires an unrealistic configuration where the rich is willing to imitate the poor in his consumption behaviour.

In light of the results of Barbar and Barinci (2010), this paper extends their work by including endogenous labor supply. It is assumed that the heterogeneity is reduced to two types of agents who differ in their initial wealth, discounting and the felicity function. However, we only consider intergroup externalities.¹ Under these specifications, we avoid the shortcomings of Barbar and Barinci (2010). The endogeneity of labor supply allows to underline the role of consumption externalities as well as the heterogeneity across agents on the stability properties of the model. In particular, the impact of consumption externalities arise through the interplay between the consumption-labor arbitrage and the mechanism of capital accumulation. Thus, under elastic labor supply, the heterogeneous preferences will have a great influence on the dynamic. In particular, contrary to Barbar and Barinci (2010), the impatient agent's preferences affect the stability properties of the model. Along this direction, in the absence of consumption externalities, Bosi and Seegmuller (2010) show that heterogeneous preferences is the channel that generate instability and flip bifurcation and the impatient agent's preferences play a crucial role on the appearance of endogenous cycles.

Our model provides the following results. We show that there exists a steady state at which only the patient agent holds capital while the impatient agent has to work in order to finance his consumption. This result goes in line with all the variants of Becker (1980). Further, we demonstrate the existence of two economic configurations according to the labor supply: the first is where both agents supply labor whereas in the second, only impatient agent supplies labor while the patient agent enjoys leisure (see Bosi and Seegmuller 2010). For simplicity, in local dynamic analysis, it is focused on the economy where only impatient agent supplies labor.

After the characterization of the steady state, we restrict our attention to a particular configuration, which is the most relevant and the most realistic one, where the poor (impatient) wants to keep-up with the consumption of the rich (patient) while the later does not care about the consumption of the poor. Contrary to Barbar and Barinci (2010), this restriction ensures a realistic result.

Our main result states that there is a room for local indeterminacy and endogenous fluctuations due to self-fulfilling expectations. This result stands to the contrary of all the variants of Becker (1980), mentioned above, in which the steady state is always determinate but changes its stability through the emergence of deterministic cycles of period two.

The intuition goes as follows, the appearance of local indeterminacy requires that an increase in savings raises the marginal product of capital, and thus the

¹Intragroup externalities are not introduced in the economy because, as it is shown by Barbar and Barinci (2010), they fail to promote endogenous cycles and instability in a Ramsey model à la Becker (1980).

interest rate, so that the initial expectations of a higher interest rate can be self-fulfilling. In models with elastic labor supply, the positive relationship between saving and interest rate may arise when an increase in capital accumulation causes an augmentation in next-period labor supply which in turn raises future interest rate. In our model, the next-period labor supply goes up due to two mechanism. The first one is through the elasticity of labor supply with respect to the real wage. A positive elasticity acts as a destabilizing factor. The second mechanism is through the patient agent's capital income. When it increases with the capital accumulation, it acts as a destabilizing factor. As a result, local indeterminacy appears if and only if both effects work in the same direction or, when they are opposite, the destabilizing effect should dominate.

Interestingly, the heterogeneity is captured as the impatient agent's preferences play a crucial role on the emergence of local indeterminacy. This is because the elasticity of labor supply with respect to the real wage is determined by his elasticity of intertemporal substitution in consumption. When the impatient's consumption is substitutable then the elasticity of labor supply is positive and thus endogenous cycles from self-fulfilling expectations become more likely to appear. Further, the relaxation of *Income Monotonicity Assumption* stabilizes. This stands to the contrary of the existing literature à la Becker (1980).

Finally, it is shown that the steady state changes its stability through the emergence of deterministic cycles of period two. However, unlike the existing works, this paper proves that endogenous cycles and local indeterminacy can be ruled out, in certain cases, when the elasticity of input substitution is close to zero.

This paper is organized as follows. The model together with the intertemporal equilibrium and the steady state analysis is presented in section 2 and the local dynamics in section 3. In section 4, we focus on the results and the interpretations and finally in section 5 we conclude. All technical details are gathered in the Appendix.

2 The model

We consider a discrete time growth model with heterogeneous consumers, endogenous labor supply, borrowing constraint and a representative firm.

2.1 Households

The framework we consider assumes that consumers are heterogeneous with respect to their initial endowments, labor supply, discount rates and felicity functions. In order to simplify the presentation but without loss of generality, consumers' heterogeneity is reduced to two types (or groups) of agents, labeled with $i = 1, 2$. Agents are identical in each group. Population size of group i is constant and denoted by $N_i > 0$ and the total population is also constant over time with size $N > 0$. Further, it is assumed that agents of type 2 are more

impatient than agents of type 1, i.e., they discount their future utility more heavily, namely,

$$\textbf{Assumption 1} \quad 0 < \beta_2 < \beta_1 < 1$$

Moreover, it is assumed that the felicity function of the representative agent of group i depends on his consumption level and his labor supply as well as on the consumption level of the representative agent of the other group j . Let the felicity function of agent i be given by $u_i(c_{it}, c_{jt}) - v_i(l_{it})$ for $i, j = 1, 2$ with $i \neq j$, where c_{it} and c_{jt} are the consumption of the representative agent of group i and j respectively while l_{it} is the labor supply of agent i . The felicity function satisfies the following assumption

Assumption 2 For $i, j = 1, 2$ with $i \neq j$, the function $u_i(c_{it}, c_{jt})$ is continuous and twice differentiable and satisfies the following conditions: (i) $u_{i,1}(c_{it}, c_{jt}) > 0 > u_{i,11}(c_{it}, c_{jt})$, (ii) $u_{i,12}(c_{it}, c_{jt}) \geq 0$, (iii) $\lim_{c_{it} \rightarrow 0^+} u_{i,1}(c_{it}, c_{jt}) = +\infty$ and (iv) $\lim_{c_{it} \rightarrow +\infty} u_{i,1}(c_{it}, c_{jt}) = 0$. Moreover, $v_i(l_{it})$ is a continuous function defined on $(0, \varsigma)$ and satisfies $v'_i(l_{it}) > 0$ and $v''_i(l_{it}) < 0$. In addition, the conditions $\lim_{l_{it} \rightarrow 0} v'_i(l_{it}) \geq 0$ and $\lim_{l_{it} \rightarrow \varsigma} v'_i(l_{it}) = +\infty$ are verified.²

Condition (i) states that the utility function of agent i is an increasing function in his own consumption while condition (ii) gives that consumption externality generates either positive or negative effect on the marginal utility from own consumption of agent i . In other terms, if agent i wants to be similar to others (resp., different from others), then his preference displays KUJ feature, i.e., $u_{i,12}(c_{it}, c_{jt}) > 0$ (resp., RAJ feature, i.e., $u_{i,12}(c_{it}, c_{jt}) < 0$).

For further reference, we present some necessary elasticities concerning agent's preference for $i, j = 1, 2$ and $i \neq j$

$$\varepsilon_{i,11} \equiv \frac{u_{i,11}c_i}{u_{i,1}} < 0 \quad (1)$$

$$\varepsilon_{i,12} \equiv \frac{u_{i,12}c_j}{u_{i,1}} \leq 0 \quad (2)$$

where $\varepsilon_{i,11}$ is the elasticity of marginal utility of private consumption for agent i while $\varepsilon_{i,12}$ is the elasticity of private marginal utility with respect to external effects from the group j . Initially, agents are endowed with capital $k_{i0} \geq 0$ such that $K_0 = \sum_{i=1}^2 N_i k_{i0} > 0$, i.e., the condition $k_{i0} > 0$ holds at least for one agent. Given the real interest rate r_t and the real wage w_t , agent i chooses the amount of labor supply l_{it} , consumption c_{it} and capital accumulation k_{it+1} to maximize the following instantaneous separable utility function

$$\sum_{t=0}^{+\infty} \beta_i^t [u_i(c_{it}, c_{jt}) - v_i(l_{it})] \quad (3)$$

²Notice that $u_{i,1}(c_{it}, c_{jt})$ is the partial derivative with respect to the first variable in the utility function, or equivalently, $\partial u_i(c_{it}, c_{jt}) / \partial c_{it}$. Similarly, $u_{i,12}(c_{it}, c_{jt})$ is equivalent to $\partial u_{i,1}(c_{it}, c_{jt}) / \partial c_{jt}$.

subject to

$$c_{it} + k_{it+1} - (1 - \delta) k_{it} \leq r_t k_{it} + w_t l_{it} \quad (4)$$

$$l_{it} \geq 0 \quad (5)$$

$$k_{it+1} \geq 0 \quad (6)$$

where inequality (4) is a sequence of budget constraint, (5) is the positivity of labor supply and (6) is the borrowing constraint which states that agents are not allowed to finance current consumption by borrowing against future income. This borrowing constraint (6) reveals the incompleteness of the market.

First-order conditions imply

$$u_{i,1}(c_{it}, c_{jt}) - \beta_i [r_{t+1} + 1 - \delta] u_{i,1}(c_{it+1}, c_{jt+1}) \geq 0 \quad (7)$$

$$v'_i(l_{it}) - w_t u_{i,1}(c_{it}, c_{jt}) \geq 0 \quad (8)$$

where inequality (7) is the Euler equation and (8) is the consumption-labor arbitrage condition which hold with equality if $k_{it+1} > 0$ and $l_{it} > 0$, respectively. Further, the monotonicity of the utility function gives rise to a binding budget constraint

$$c_{it} + k_{it+1} - (1 - \delta) k_{it} = r_t k_{it} + w_t l_{it} \quad (9)$$

2.2 Production

In contrast to the consumers' side, the production sector is homogeneous. Assume that a representative firm produces the final good using a constant return-to-scale technology $y_t = F(K_t, L_t)$, where K_t and L_t are the aggregate capital and labor. Let $k_t \equiv K_t/L_t$ be the capital-labor ratio, using the homogeneity feature, the production function can be written as $F(K_t, L_t) \equiv f(k_t) L_t$. Representative firm takes prices (real interest rate r_t and real wage w_t) and technology as given and maximizes the profit $\pi_t \equiv F(K_t, L_t) - r_t K_t - w_t L_t$.

Assumption 3 *The production function $f(k)$ is continuous in capital-labor ratio $k > 0$, differentiable with $f'(k) > 0 > f''(k)$, $f(0) = 0$ and satisfies Inada conditions $\lim_{k \rightarrow 0^+} f'(k) = +\infty$ and $\lim_{k \rightarrow +\infty} f'(k) = 0$.*

Given r_t and w_t , then profit maximization implies

$$r_t = f'(k_t) \text{ and } w_t = f(k_t) - k_t f'(k_t) \quad (10)$$

For further reference, we introduce the elasticity of capital-labor substitution $\sigma \equiv [k f'(k) / f(k) - 1] f'(k) / k f''(k) > 0$. The capital share of total income is given by $s \equiv k f'(k) / f(k) \in (0, 1)$. Finally, the elasticities of interest rate with respect to capital and labor are $r_k k / r = -r_l l / r = -(1 - s) / \sigma$, and the elasticities of the real wage with respect to capital and labor are $w_k k / w = -w_l l / w = s / \sigma$.

2.3 Intertemporal equilibrium

We start by providing a standard definition of equilibrium for the economy described above:

Definition 1 An equilibrium of the economy $E = \left(F, (k_{i0}, \beta_i, u_i, v_i, N_i)_{i=1}^2 \right)$ is an intertemporal sequence $\left(r_t, w_t, K_t, L_t, (c_{it}, k_{it}, l_{it})_{i=1}^2 \right)_{t=0}^{+\infty}$ which satisfies the following conditions:

- (D1) $(r_t, w_t)_{t=0}^{+\infty}$ is a sequence of strictly positive prices;
- (D2) given $(r_t, w_t)_{t=0}^{+\infty}$, (K_t, L_t) solves the firm's program for $t = 0, 1, \dots, +\infty$;
- (D3) given $(r_t, w_t)_{t=0}^{+\infty}$, $(c_{it}, k_{i,t+1}, l_{it})_{t=0}^{+\infty}$ solves the i th consumer's program for $i = 1, 2$;
- (D4) the capital market clears $K_t = N_1 k_{1t} + N_2 k_{2t}$, for $t = 0, 1, \dots, +\infty$;
- (D5) the labor market clears $L_t = N_1 l_{1t} + N_2 l_{2t}$, for $t = 0, 1, \dots, +\infty$;
- (D6) the product market clears $\sum_{i=1}^2 N_i [c_{it} + k_{i,t+1} - (1 - \delta) k_{it}] = F(K_t, L_t)$.

In the following Lemma, we present a set of equilibrium conditions for our model.

Lemma 1 Let the economy $E = \left(F, (k_{i0}, \beta_i, u_i, v_i, N_i)_{i=1}^2 \right)$ satisfying Assumptions 1 – 3. Consider the following conditions for $t = 0, 1, \dots, +\infty$ and $i, j = 1, 2$ with $i \neq j$:

- (L1) $c_{it} > 0$, $k_{it} > 0$, $0 < l_{it} < \varsigma$, $K_t > 0$, $L_t > 0$;
- (L2) $r_t = f'(k_t)$ and $w_t = f(k_t) - k_t f'(k_t)$ with $k_t \equiv K_t/L_t$;
- (L3) $v'_i(l_{it}) \geq u_{i,1}(c_{it}, c_{jt}) w_t$ with equality when $l_{it} > 0$;
- (L4) $u_{i,1}(c_{it}, c_{jt}) \geq \beta_i (r_{t+1} + 1 - \delta) u_{i,1}(c_{i,t+1}, c_{j,t+1})$ with equality if $k_{i,t+1} > 0$;
- (L5) $c_{it} + k_{i,t+1} - (1 - \delta) k_{it} = r_t k_{it} + w_t l_{it}$;
- (L6) $K_t = N_1 k_{1t} + N_2 k_{2t}$;
- (L7) $L_t = N_1 l_{1t} + N_2 l_{2t}$.

Then, if the sequence $\left(r_t, w_t, K_t, L_t, (c_{it}, k_{it}, l_{it})_{i=1}^2 \right)_{t=0}^{+\infty}$ is a competitive equilibrium, the conditions (L1)–(L7) hold. Further, if the sequence $\left(r_t, w_t, K_t, L_t, (c_{it}, k_{it}, l_{it})_{i=1}^2 \right)_{t=0}^{+\infty}$ satisfies the conditions (L1) – (L7) and the transversality condition

$$\lim_{t \rightarrow +\infty} \beta_i^t u_{i,1}(c_{it}, c_{jt}) k_{i,t+1} = 0 \quad (11)$$

for $i, j = 1, 2$ with $i \neq j$, it is an equilibrium for the economy E .

Proof. See the Appendix. ■

In a Ramsey model with incomplete markets and without consumption externalities, the heterogeneity of discounting promotes the concentration of capital stock in the hands of the most patient agent, as demonstrated by Becker (1980) and Becker and Foias (1987; 1994). Such a result maintains either by the inclusion of endogenous labor supply (Bosi and Seegmuller 2010a) or by the introduction of intergroup and intragroup external effects in consumption (Barbar and Barinci 2010).

As the following Proposition shows, whenever the benchmark model of Becker and Foias (1994) is augmented to include both consumption externalities and endogenous labor supply, the benchmark result still holds. Namely, at a neighborhood of the steady state, the patient agent 1 owns the entire capital in the economy while agent 2 does not hold capital, provided that he is sufficiently impatient.

Proposition 1 *There exists a steady state defined as follows:*

- (S1) r and w are constant;
- (S2) $r + 1 - \delta = 1/\beta_1 < 1/\beta_2$;
- (S3) $k_1 > 0$ and $k_2 = 0$;
- (S4) $u_{1,1}(c_1, c_2)w \leq v'_1(l_1)$ and $u_{2,1}(c_2, c_1)w = v'_2(l_2)$;
- (S5) $c_1 = wl_1 + (r - \delta)k_1$ and $c_2 = wl_2$;
- (S6) $K = N_1k_1 > 0$;
- (S7) $L = N_1l_1 + N_2l_2$.

Proof. See the Appendix. ■

According to Proposition 1, the real gross interest rate should be equal to $1/\beta_1$ around the steady state because firms prefer renting capital with the lowest costs, i.e., from patient agent and therefore, the impatient agent ends up holding no capital. Further, it is important to mention that the presence of endogenous labor supply with different agents' preferences implies that agents could have different levels of labor supply. In particular, Proposition 1 shows that impatient agent is obliged to supply labor to finance his consumption while the patient agent has two choices either to supply labor or to enjoy leisure. In other words, Proposition 1 allows for the existence of *two* economic configurations: in the first one, both agents supply labor and, in the second one, only the impatient agent works while the patient agent enjoys leisure.

In the sequel, we consider a realistic configuration in which the poor agent (impatient) is willing to imitate the consumption of the rich (patient), i.e., the impatient agent's preferences display a Keeping-Up with the Joneses feature. However, the rich does not care about the poor's consumption. Namely, consider the following Assumption:

Assumption 4 *Let $u_{1,12} = 0$ and $u_{2,12} > 0$.*

In the following, we characterize the condition under which the patient agent decides to supply labor at a neighborhood of the steady state. But before proceeding, we need to ensure the existence of the function $l_2 \equiv l_2(l_1)$ for all $l_1 \in (0, \varsigma)$, as demonstrated by the following Lemma.

Lemma 2 *Given Assumptions 1 – 5 and let $\Psi(l_1, l_2) = 1$ where*

$$\Psi(l_1, l_2) \equiv v'_2(l_2) / u_{2,1}(wl_2, c_1(l_1, l_2)) w$$

then there exists an implicit function $l_2 \equiv l_2(l_1)$ for all $l_1 \in (0, \varsigma)$ if

$$\frac{1}{w} v''_2 - u_{2,11} w - \frac{1 - \beta_1}{\beta_1} \frac{N_2 k}{N_1} u_{2,12} \neq 0 \quad (12)$$

Further, for $l_1 = 0$, there exists a unique l_2^ that solves $\Psi(0, l_2^*) = 1$ if and only if*

$$\frac{1}{w} v''_2 - u_{2,11} w - \frac{1 - \beta_1}{\beta_1} \frac{N_2 k}{N_1} u_{2,12} > 0 \quad (13)$$

Proof. See the Appendix. ■

Now, taking account the function $l_2(l_1)$ as characterized by Lemma 2, the next proposition presents explicitly the condition under which the patient agent supplies labor in a neighborhood of the steady state.

Proposition 2 *Given Assumptions 1–5 and let the condition (13) of Lemma 2 hold, the patient agent supplies labor at the steady state if and only if*

$$\lim_{l_1 \rightarrow 0^+} v'_1(l_1) < \lim_{l_1 \rightarrow 0^+} u'_1 \left(\left[w + \frac{1 - \beta_1}{\beta_1} k \right] l_1 + \frac{1 - \beta_1}{\beta_1} \frac{N_2 k}{N_1} l_2(l_1) \right) w \quad (14)$$

where $k = r^{-1} (1/\beta_1 - (1 - \delta))$.

Proof. See the Appendix.

■

Inequality (14) states that patient agent supplies labor if and only if the marginal dis-utility of labor, whenever labor supply is close to zero, is lower than

the marginal utility of consumption. However, if (14) is not verified, then agent prefers enjoying leisure and pays their consumptions through capital income. For the sake of simplicity, we only focus on the steady state where patient agent does not supply labor, i.e., inequality (14) is *not* verified. **Namely, consider the following Assumption:**

Assumption 5 Let $u_{1,12} = 0$ and $u_{2,12} > 0$.

Further, it is worth to notice that consumption externalities affect the steady-state through the consumption-labor arbitrage condition of the impatient agent, namely,

$$u_{2,1}(wl_2, c_1)w - v_2'(l_2) = 0 \quad (15)$$

This implies that the labor supply of impatient agent is determined by the consumption level of patient agent c_1 and the real wage w . Namely, agent 2's labor supply can be defined implicitly as a function of c_1 and w , i.e., we have

$$l_2 \equiv l_2(c_1, w) \quad (16)$$

The existence of the function $l_2 \equiv l_2(c_1, w)$ is ensured by Lemma 2 and gives rise to the following elasticities:

$$\xi_{lc} \equiv \frac{\partial l_2 / \partial c_1}{c_1 / l_2} = \frac{\varepsilon_{2,12}}{\varphi_2 - \varepsilon_{2,11}} \quad (17)$$

$$\xi_{lw} \equiv \frac{\partial l_2 / \partial w}{w / l_2} = \frac{1 + \varepsilon_{2,11}}{\varphi_2 - \varepsilon_{2,11}} \quad (18)$$

where $\varphi_2 \equiv v_2''(l_2)l_2/v_2'(l_2)$ is the elasticity of marginal dis-utility of impatient labor supply while $\varepsilon_{2,11}$ and $\varepsilon_{2,12}$ are summarized respectively in (3) and (4). Further, using labor market equilibrium condition (S7), one can easily verify that

$$\frac{L_c c_1}{L} = \frac{\xi_{lc}}{1 + \frac{s}{\sigma} \xi_{lw}} \quad (19)$$

$$\frac{L_{k_1} k_1}{L} = \frac{\xi_{lw}}{\frac{\sigma}{s} + \xi_{lw}} \quad (20)$$

Therefore, given the steady state characterized by Proposition 1, a Ramsey equilibrium is a sequence of $\{c_{1t}, k_{1t}\}_{t=0}^{+\infty}$ that is described by patient agent's Euler equation and budget constraint which solves the following two-dimensional dynamic system

$$\frac{u_{1,1}(c_{1t}, w(N_1 k_{1t} / L(c_{1t}, k_{1t}))) l_2(c_{1t}, k_{1t})}{u_{1,1}(c_{1t+1}, w(N_1 k_{1t+1} / L(c_{1t+1}, k_{1t+1}))) l_2(c_{1t+1}, k_{1t+1})} = \beta_1 [1 - \delta + r(N_1 k_{1t+1} / L(c_{1t+1}, k_{1t+1}))] \quad (21)$$

$$c_{1t} + k_{1t+1} - (1 - \delta) k_{1t} = g(c_{1t}, k_{1t}) \quad (22)$$

subject to the initial endowment $k_{10} > 0$ and the transversality condition (11), where $g(c_{1t}, k_{1t}) \equiv r(N_1 k_{1t}/L(c_{1t}, k_{1t})) k_{1t}$ is the capital-income of patient agent and the effect of capital on capital-income is given by

$$g_k = \frac{1}{\beta} [1 - \beta(1 - \delta)] \left(1 - \frac{1 - s}{\sigma + s\xi_{lw}} \right) \quad (23)$$

3 Local dynamics

In this section, we analyze the stability properties of the economy described above and study the occurrence of indeterminacy and local bifurcations. We start by linearizing the system (21)-(22) around the steady state where only the patient agent holds capital and only the impatient agent supplies labor. We get the linear system $(dk_{1t+1}/k_1, dc_{1t+1}/c_1)^T = J (dk_{1t}/k_1, dc_{1t}/c_1)^T$, where J is the associated Jacobian matrix which is given by

$$J = \begin{bmatrix} -\rho & \rho\xi_{lc} + (\sigma + s\xi_{lw})\varepsilon_{1,11}/s \\ \beta_1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & (\sigma + s\xi_{lw})\varepsilon_{1,11}/s \\ 1 - \frac{s\rho}{\sigma + s\xi_{lw}} & \frac{s\rho\xi_{lc}}{\sigma + s\xi_{lw}} - (1 - \beta_1) \end{bmatrix} \quad (24)$$

where

$$\rho \equiv \frac{1}{s} (1 - s) [1 - \beta_1 (1 - \delta)] > 0 \quad (25)$$

The trace T and the determinant D of the Jacobian matrix (24) are given by:

$$T = 1 + D + \frac{s\rho(1 - \beta_1)}{\beta_1} \frac{\xi_{lc} - 1}{(\sigma + s\xi_{lw})\varepsilon_{1,11} + s\rho\xi_{lc}} \quad (26)$$

$$T = -1 - D + \frac{1}{\beta_1} \frac{(1 + \beta_1) [2(\sigma + s\xi_{lw})\varepsilon_{1,11} + s\rho\xi_{lc}] - s\rho(2\varepsilon_{1,11} + 1 - \beta_1)}{(\sigma + s\xi_{lw})\varepsilon_{1,11} + s\rho\xi_{lc}} \quad (27)$$

$$D = \frac{1}{\beta_1} \frac{(\sigma + s\xi_{lw} - s\rho)\varepsilon_{1,11}}{(\sigma + s\xi_{lw})\varepsilon_{1,11} + s\rho\xi_{lc}} \quad (28)$$

where ξ_{lc} and ξ_{lw} are respectively given by (17) and (18). The model of Bosi and Seegmuller (2010a) without consumption externalities is recovered by setting $\xi_{lc} = 0$ while the model of Barbar and Barinci (2010) is obtained by considering $\xi_{lc} = 0$ and $\xi_{lw} = 0$.

Using the fact that the trace T and the determinant D are the sum and the product of the eigenvalues of the Jacobian matrix J , our analysis consists of two parts: the occurrence of local indeterminacy and the saddle-path stability.

Since k_1 is the only predetermined variable, the stationary equilibrium is locally indeterminate if and only if both eigenvalues of the Jacobian matrix J lie inside the unit circle. Namely, the conditions $D < 1$, $T < 1 + D$ and $T > -1 - D$ should be verified simultaneously. Before providing our main results, let us define the following critical values:³

Let $\sigma = \sigma_{AB}$ be the critical value at which $T = -1 - D$ holds and given by

$$\sigma_{AB} \equiv -\frac{2\varepsilon_{1,11} [(1 + \beta_1) \xi_{lw} - \rho] - \rho(1 - \beta_1) + \rho(1 + \beta_1) \xi_{cl}}{2^{\frac{1}{s}} (1 + \beta_1) \varepsilon_{1,11}} \quad (29)$$

Let $\sigma = \sigma_D$ be the solution of $D = 1$ which is given by

$$\sigma_D \equiv \frac{[\rho - (1 - \beta_1) \xi_{lw}] \varepsilon_{1,11} + \rho \beta_1 \xi_{cl}}{\frac{1}{s} (1 - \beta_1) \varepsilon_{1,11}} \quad (30)$$

The conditions in which local indeterminacy appears around the steady state are summarized in the next proposition.⁴

Proposition 3 *Let Assumptions 1 – 5 hold and given the critical values (29)-(30) and the other critical values provided in the Appendix, then:*

Case 1 *Whenever $\xi_{lw} > 0$, then the steady state is locally indeterminate under the following conditions:*

- (1.1) *For $\sigma < \sigma_D$ with $1 < \xi_{cl} < \xi_{cl}^{AB}$ and $\varepsilon_{1,11} < -1$ and $\rho < \xi_{lw} < \ddot{\xi}_{lw}$.*
- (1.2) *For $\sigma_{AB} < \sigma < \sigma_D$ with $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^{ABD}$ and $\varepsilon_{1,11} < -1$ and $\rho < \xi_{lw} < \ddot{\xi}_{lw}$.*
- (1.3) *For $\sigma_D < \sigma < \sigma_{AB}$ with $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and either $-1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\hat{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ or $\varepsilon_{1,11} > -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} < \ddot{\xi}_{lw}$.*
- (1.4) *For $\sigma < \sigma_{AB}$ with $\xi_{cl}^{AB} < \xi_{cl} < 1$ and $\varepsilon_{1,11} > -1$ and $\rho < \xi_{lw} < \xi_{lw}^Z$.*

Case 2 *Whenever $\xi_{lw} < 0$, then the steady state is locally indeterminate under the following conditions:*

- (2.1) *For $\varepsilon_{111} < -1$ and all $\xi_{lw} < 0$ and $1 < \xi_{cl} < \xi_{cl}^{ABD}$ and $\sigma_{AB} < \sigma < \sigma_D$.*
- (2.2) *For $\xi_{cl}^{ABD} < \xi_{cl} < 1$ and $-1 < \varepsilon_{1,11} < -\beta_1$ and all $\xi_{lw} < 0$ and $\sigma_D < \sigma < \sigma_{AB}$;*
- (2.3) *For $\xi_{cl}^{ABD} < \xi_{cl} < 1$ and $\varepsilon_{1,11} > -\beta_1$ and $\xi_{lw} < \tilde{\xi}_{lw}$ and $\sigma_D < \sigma < \sigma_{AB}$;*

³More critical values are provided in the Appendix.

⁴For further detailed conditions, see the Appendix.

(2.4) For $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma_{AB}$ and either $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw} > \tilde{\xi}_{lw}$; or $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$.

Proof. See the Appendix. ■

In Proposition 3, two cases are distinguished according to the elasticity of labor supply with respect to the real wage ξ_{lw} . Note that $\xi_{lw} > 0$ if and only if the impatient agent's intertemporal substitution in consumption is sufficiently high, i.e., $\varepsilon_{2,11} > -1$. In either of these cases, local indeterminacy occurs if consumption externalities are small, i.e., $\xi_{lc} < 1$ (resp. high, i.e., $\xi_{cl} > 1$) and patient agent's substitutability is high, i.e., $\varepsilon_{1,11} > -1$ (resp. small, i.e., $\varepsilon_{1,11} < -1$). Further, one can notice that the elasticity of input substitution can take small or moderate values.

In view of Proposition 3, a Ramsey model with heterogeneous agents and endogenous labor supply under the presence of consumption externalities displays local indeterminacy. Such a result is of interest for two reasons: First, it provides new dynamic configurations comparing to the existing literature à la Becker (1980) in which the model only exhibits deterministic cycles of period two. Second, it stresses the role of heterogeneity, i.e., even if the dynamic system (21)-(22) is described by patient agent's Euler equation and budget constraint, Proposition 3 shows that impatient agent's preferences also play a crucial role in the appearance of endogenous fluctuations. On the one side, the elasticity of labor supply with respect to the real wage is determined by the impatient agent's elasticity of intertemporal substitution in consumption. On the other side, the elasticity of consumption externalities also affect the stability properties of the model.

Further, Proposition 3 shows that the steady state changes its stability through the occurrence of flip bifurcation at $\sigma = \sigma_{AB}$ and so there is a room for deterministic cycles of period two. On the one side, such a result goes in line with previous literature, namely, Becker and Foias (1987, 1994), Bosi and Seegmuller (2010) and Barbar and Barinci (2010). On the other side, while these papers show that a sufficiently high capital-labor substitution can rule out the occurrence of fluctuations, the next Proposition proves that endogenous cycles and local indeterminacy can be also ruled out under sufficiently *weak* elasticity of input substitution, whenever the intertemporal substitution in the patient agent's consumption is low while both of the external effects and the impatient agent's consumption substitutability are high.

Proposition 4 *Let Assumptions 1–5 hold. Whenever $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^{ABD}$ and $\varepsilon_{1,11} < -1$ and $\rho < \xi_{lw} < \tilde{\xi}_{lw}$, then there exists a critical value of the elasticity of capital-labor substitution $\sigma = \sigma_{AB}$ such that $\sigma < \sigma_{AB}$ implies saddle-path stability.*

Proof. See the Appendix. ■

Finally, in order to complete the local analysis, consider the derivative of (29) and (30) with respect to ξ_{cl} , we get $\partial\sigma_D/\partial\xi_{cl} < 0$ and $\partial\sigma_{AB}/\partial\xi_{cl} > 0$. This gives rise to two cases. From one side, whenever the patient agent's intertemporal substitutability is low then the indeterminacy region is given by $\sigma_{AB} < \sigma < \sigma_D$ or $0 < \sigma < \sigma_D$ and thus, consumption externalities make the emergence of endogenous fluctuations due to self-fulfilling expectations less likely. From the other side, whenever the patient agent's intertemporal substitutability is high then the local indeterminacy region $\sigma_D < \sigma < \sigma_{AB}$ or $0 < \sigma < \sigma_{AB}$ widens with consumption external effects and so these externalities promote the appearance of endogenous fluctuations.

4 Discussion

The model we consider in this paper is an extension of Barbar and Barinci (2010) augmented to include the endogenous labor supply. It is shown that such a framework allows to stress on the role of consumption externalities and the heterogeneity across agents. To understand the added value of this paper, consider three variants of Ramsey model when the labor supply is inelastic: i) with identical agents; ii) with heterogeneous agents; and with heterogeneous agents and consumption externalities. Finally, we discuss the results obtained in the previous section and compare them to the existing literature.

i) Ramsey model with identical agents

In a standard Ramsey model with identical agents and inelastic labor supply, the steady state is always a saddle point.

ii) Ramsey model with heterogeneous agents

Once the heterogeneity across agents is introduced in the standard Ramsey model where both agents supply work inelastically, Becker and Foias (1994) show that the heterogeneity gives rise to a new dynamic configuration. In particular, the model exhibits deterministic cycles of period two. The authors prove that a necessary condition for the appearance of these cycles is the relaxation of *Income Monotonicity Assumption*, i.e., the patient agent's income should be decreasing in capital stock. Further, cycles are ruled out whenever input substitutability is sufficiently high.

iii) Ramsey model with heterogeneous agents and consumption externalities

Subsequently, Barbar and Barinci (2010) extend the framework of Becker and Foias (1994) by introducing consumption externalities. In particular, they distinguish between the intergroup and intragroup consumption externalities. The purpose of such a specification is to capture the role of heterogeneity as well as the external effects. Barbar and Barinci (2010) show that the steady

state is still determinate but changes its stability through the emergence of flip bifurcation. However, contrary to Becker and Foias (1994), they demonstrate that such a result is obtained without the need to relax the *Income Monotonicity Assumption*. Instead, it is more likely the consumption externalities are the mechanism that generate cycles. More precisely, if the patient agent is willing keep-up with the impatient, then cycles of period two occur even under *Income Monotonicity Assumption*, i.e., input substitutability is required to be high.

The added value of Barbar and Barinci (2010) is simple, clear and interesting but has two weak points. From the one side, even when the intergroup externalities that generate cycles rather than the intragroup externalities, the role of heterogeneity across agents is not explicitly captured. In other terms, the impatient agent's preferences do not play any role in the emergence of cycles. From the other side, such a result is not realistic. That is, it is based on the configuration in which rich agent imitates the poor while the contrary happens in the real world.

iv) Ramsey model with heterogeneous agents, consumption externalities and endogenous labor supply

This paper avoids the weakness in Barbar and Barinci (2010) by considering two main assumptions. First, we assume an elastic labor supply. From the one side, the endogeneity of labor supply gives rise to the positive relation between saving and labor supply that makes the initial expectations about higher future interest rate self-fulfilling, and thus local indeterminacy appears. From the other side, it allows to capture the heterogeneity across agent, that is, the role of the impatient agent's preferences. In particular, when impatient agent's elasticity of intertemporal substitution in consumption is higher than one, the elasticity of labor supply to the real wage becomes positive, making the emergence of endogenous fluctuations due to self-fulfilling expectations more likely, and vice versa. Along this direction, Bosi and Seegmuller (2010) also stress on the role of heterogeneous preferences on the stability properties of the steady state, in the absence of consumption externalities. They show that when the impatient agents' elasticity of intertemporal substitution is greater than one, the occurrence of cycles is less likely, and vice versa.

Second, we consider a realistic configuration in which the poor (impatient) agent wants to keep-up with the rich (patient) while the rich does not pay any attention to the poor's consumption. Such an assumption ensures a result that is based upon plausible conditions.

Under these specifications, we show that there is a room for local indeterminacy. More precisely, we show that the introduction of elastic labor supply is a key requirement for the appearance of local indeterminacy in a Ramsey model with heterogeneous agents and consumption externalities. The emergence of indeterminacy requires that an increase in savings raises the marginal product of capital, and thus the interest rate, so that the initial expectations of higher interest rate can be self-fulfilling. If labor supply is inelastic, an increase in savings has a negative effect on the marginal product of capital, and thus there

is no room for self-fulfilling expectations. Then the presence of consumption externalities only gives rise to cycles of period two (Barbar and Barinci 2010). Conversely, if labor supply is elastic, the positive relationship between saving and interest rate may arise when an increase in the amount of saving causes an increase in the next-period labor supply which in turn affects the future interest rate, resulting in self-fulfilling initial expectations. Finally, this result goes in line with Alonso-Carrera *et al.* (2008) who show that consumption externalities, in a representative-agent Ramsey model, are a source of local indeterminacy if labor supply is endogenous.

It worth noticing that such a result is obtained without **considering social concavity restriction** i.e., $\varepsilon_{i,11} + \varepsilon_{i,12} < 0$. In fact, this condition states that either the externality augments the direct effect of own consumption, or, if it is offsetting, it is dominated by the own consumption effect. Thus it imposes an upper bound on the KUJ effect. Such a restriction is essentially a stability condition and relaxing it means that we consider a KUJ effect which is too intense that it induces instability. The existing literature, such as Alonso-Carrera *et al.* (2008) and Barbar and Barinci (2010), considers this restriction. Conversely, this paper does not impose it in order for the calculations to be less complicated. In addition, it does not change the main result.

In the next section, we provide the detailed intuition for our main result in Proposition 3.

5 Intuition for local indeterminacy

The appearance of local indeterminacy requires that an increase in savings raises the marginal product of capital, and thus the interest rate, so that the initial expectations of higher interest rate can be self-fulfilling. In models with elastic labor supply, the positive relationship between saving and interest rate may arise when an increase in the amount of saving causes an augmentation in next-period labor supply which in turn raises future interest rate.

The initial expectation of a higher future interest rate induces patient agent to increase his savings and reduces his current consumption. In this framework, this has two effects on next-period labor supply:

1. The first effect is through the future wage w_{t+1} : If patient agent expects a higher future interest rate r_{t+1} , he accumulates more capital k_{1t+1} which implies an increase in wage w_{t+1} . The effect of the rise in wage on the next-period labor supply depends on the sign of elasticity of labor supply ξ_{lw} .
2. The increase in savings influences next-period patient agent's capital-income g . This effect also depends on the sign of the elasticity of labor supply ξ_{lw} and on the elasticity of input substitution σ as follows

$$g_k = \frac{1}{\beta} [1 - \beta(1 - \delta)] \left(1 - \frac{1 - s}{\sigma + s\xi_{lw}} \right) \quad (31)$$

The consumption of patient agent at $t+1$ reacts to the change in his income according to the elasticity of intertemporal substitution $-1/\varepsilon_{1,11}$. Further, the presence of consumption externalities implies a positive influence of c_{1t+1} on next-period labor supply, namely,

$$\xi_{lc} \equiv \frac{\partial l_2 / \partial c_1}{c_1 / l_2} = \frac{\varepsilon_{2,12}}{\varphi_2 - \varepsilon_{2,11}} > 0 \quad (32)$$

According to these effects, three cases should be distinguished in Proposition 3:

Whenever $\xi_{lw} > 0$ and $g_k < 0$, then from the one side, following the augmentation in the wage, the next-period labor supply L_{t+1} increases and does the future interest rate r_{t+1} . Therefore, the positivity of ξ_{lw} acts as a destabilizing factor. Such an effect is sufficiently large because the elasticity of input substitution is low.

From the other side, the increase in next-period capital k_{1t+1} , due to the expectation of a higher interest rate r_{t+1} , is followed by a decrease in future capital income g , then according to the budget constraint, patient agent's future consumption falls but slightly because the intertemporal substitutability is very low. As the impatient agent wants to keep up ($\varepsilon_{2,12} > 0$) and so to reduce his consumption, he will thus work less, i.e., l_{2t+1} decreases. This results in a decrease in future interest rate r_{t+1} . Therefore, the negative response of capital-income to k_1 plays a stabilizing role and makes the appearance of fluctuations due to self-fulfilling expectations less likely.

In this case, in order for the initial expectations to be self-fulfilling, the first effect should dominate the second one. Such a requirement is ensured because of the small elasticity of input substitutability that generates large wage effect and the low sensitivity of the patient agent's consumption to income that results in a small second effect.

Whenever $\xi_{lw} > 0$ and $g_k > 0$, then from the one side, following the augmentation in the wage, the next-period labor supply L_{t+1} increases and does the future interest rate r_{t+1} . Therefore, the positivity of ξ_{lw} acts as a destabilizing factor. As the elasticity of input substitution is high, such an effect is not large.

From the other side, as the elasticity of input substitutability is high, then $g_k > 0$ and thus future capital income g goes up in response to the increase in next-period capital k_{1t+1} . Then patient agent's future consumption augments. Since $\varepsilon_{2,12} > 0$, then impatient agent is willing to imitate patient agent and raise his consumption. This induces him to increase labor supply l_{2t+1} which yields to a rise in r_{t+1} . Therefore, such an effect works in the same direction as the first effect (due to the positive ξ_{lw}). In other terms, the positive response of capital-income to k_1 destabilizes and promotes the appearance of local indeterminacy.

In this case, both effects work in the same direction, i.e., both act as destabilizing effects and thus give rise to local indeterminacy.

Whenever $\xi_{lw} < 0$ and $g_k > 0$, then from the one side, a rise in k_{1t+1} raises w_{t+1} which in turn affects l_{2t+1} negatively (since $\xi_{lw} < 0$). Then L_{t+1} decreases and thus the future interest rate r_{t+1} falls as well. In other terms, negative elasticity of labor supply ξ_{lw} acts as a stabilizing effect and makes the emergence of local indeterminacy less likely.

From the other side, the increase in k_{1t+1} will raise g_{t+1} , because $g_k > 0$. Then patient agent augments his future consumption c_{1t+1} as the intertemporal substitutability of the patient agent's consumption is high. Since $\varepsilon_{2,12} > 0$, then impatient agent is willing to raise his future consumption in response to the increase in c_{1t+1} and so he works more, i.e., l_{2t+1} goes up. This implies that L_{t+1} increases, resulting in a higher future interest rate r_{t+1} . So the positive effect of capital on patient agent's income destabilizes and promotes the occurrence of endogenous cycles.

In this case, endogenous fluctuations driven from self-fulfilling expectation appear if and only if the second effect dominates the first one. Such a requirement is ensured because of the high input substitutability σ which provides a small effect of K_{t+1} on w_{t+1} , and both high intertemporal substitutability of patient agent's consumption and high consumption externalities result in large second effect.

Notice that the counter-intuitive configuration for the appearance of local indeterminacy, that is, $\xi_{lw} < 0$, $g_k < 0$ can also emerge in proposition 3.

6 Conclusion

This paper considers a Ramsey heterogeneous agents model of Becker (1980), Becker and Foias (1987, 1994) with borrowing constraint and consumption externalities augmented to include endogenous labor supply. For simplicity, it is supposed the existence of two types of identical agents. Agents are different in their initial endowment of capital, labor supply, time preference and felicity function. Agents utility function depends on its own consumption as well as on consumptions of the other group. It is shown the existence of two steady states: the first one consists of both agents supply labor and the second one with only impatient labor supply while the patient agents enjoy leisure. As in the literature, the entire capital market is held by the most patient agents while the impatient agents have to supply labor always to finance their consumptions. Along this paper, it is focused on the steady state where patient agents enjoy leisure. In the stationary equilibrium, it is assumed that patient agents are not affected by impatient agents consumptions but impatient agents have KUU feeling toward the patient agents. The main contribution of this paper is the existence of local indeterminacy due to self-fulfilling expectations. These expectations emerge since the presence of endogenous labor supply together with

consumption spillover give rise to an intratemporal effect which can generate endogenous fluctuations.

7 Appendix

7.1 Proof of Lemma 1

Necessity. Conditions (L1), (L6) and (L7) directly come from the equilibrium definition. In addition, (L2) corresponds to the first-order necessary condition of profit maximization of the firm. (L3) and (L4) are the first-order necessary conditions of the utility maximization of household i and (L5) is the corresponding budget constraint.

Sufficiency. Condition (D1) is guaranteed by condition (L2) and Assumption 4. In order to show that (D2) is verified, notice that for every alternative pair $(\tilde{K}_t, \tilde{L}_t)$, we have:

$$\begin{aligned}
& F(K_t, L_t) - w_t L_t - r_t K_t - \left[F(\tilde{K}_t, \tilde{L}_t) - w_t \tilde{L}_t - r_t \tilde{K}_t \right] \\
& f(K_t/L_t) L_t - w_t L_t - r_t K_t - \left[f(\tilde{K}_t/\tilde{L}_t) \tilde{L}_t - w_t \tilde{L}_t - r_t \tilde{K}_t \right] \\
= & f(K_t/L_t) L_t - f(\tilde{K}_t/\tilde{L}_t) \tilde{L}_t - r_t (K_t - \tilde{K}_t) - w_t (L_t - \tilde{L}_t) \\
\geq & f'(K_t/L_t) (K_t - \tilde{K}_t) + [f(K_t/L_t) - f'(\tilde{K}_t/\tilde{L}_t) (\tilde{L}_t/L_t)] (L_t - \tilde{L}_t) \\
& - r_t (K_t - \tilde{K}_t) - w_t (L_t - \tilde{L}_t) \\
= & 0
\end{aligned}$$

Let us now consider a sequence $(\tilde{k}_{it}, \tilde{l}_{it}, \tilde{c}_{it})$ satisfying the constraints of agent's i 's maximization problem and the initial condition. Then, we have

$$\begin{aligned}
& \sum_{t=0}^{\infty} \beta_i^t \left[u_i(c_{it}, c_{jt}) - v_i(l_{it}) \right] - \left[u_i(\tilde{c}_{it}, c_{jt}) - v_i(\tilde{l}_{it}) \right] \\
= & \sum_{t=0}^{\infty} \beta_i^t \left[u_i(c_{it}, c_{jt}) - u_i(\tilde{c}_{it}, c_{jt}) + v_i(\tilde{l}_{it}) - v_i(l_{it}) \right] \\
\geq & \sum_{t=0}^{\infty} \beta_i^t \left[u'_i(c_{it}, c_{jt}) (c_{it} - \tilde{c}_{it}) - v'_i(l_{it}) (l_{it} - \tilde{l}_{it}) \right] \\
= & \sum_{t=0}^{\infty} \beta_i^t \left[u'_i(c_{it}, c_{jt}) \left[(r_t + \delta - 1) (k_{it} - \tilde{k}_{it}) + w_t (l_{it} - \tilde{l}_{it}) - (k_{it+1} - \tilde{k}_{it+1}) \right] - v'_i(l_{it}) (l_{it} - \tilde{l}_{it}) \right] \\
= & \sum_{t=0}^{\infty} \beta_i^t u'_i(c_{it}, c_{jt}) \left[(r_t + \delta - 1) (k_{it} - \tilde{k}_{it}) - (k_{it+1} - \tilde{k}_{it+1}) \right] + \sum_{t=0}^{\infty} \beta_i^t [u'_i(c_{it}, c_{jt}) w_t - v'_i(l_{it})] (l_{it} - \tilde{l}_{it})
\end{aligned}$$

$$\begin{aligned}
&= \lim_{T \rightarrow \infty} \left[\begin{aligned} &\sum_{t=0}^T \beta_i^t u'_i(c_{it}, c_{jt}) (r_t + \delta - 1) (k_{it} - \tilde{k}_{it}) - \sum_{t=0}^T \beta_i^t u'_i(c_{it}, c_{jt}) (k_{it+1} - \tilde{k}_{it+1}) \\ &+ \sum_{t=0}^T \beta_i^t [u'_i(c_{it}, c_{jt}) w_t - v'_i(l_{it})] (l_{it} - \tilde{l}_{it}) \end{aligned} \right] \\
&= \lim_{T \rightarrow \infty} \left[\begin{aligned} &u'_i(c_{i0}, c_{j0}) (r_0 + \delta - 1) (k_{i0} - \tilde{k}_{i0}) + \sum_{t=1}^T \beta_i^t u'_i(c_{it}, c_{jt}) (r_t + \delta - 1) (k_{it} - \tilde{k}_{it}) \\ &- \sum_{t=0}^{T-1} \beta_i^t u'_i(c_{it}, c_{jt}) (k_{it+1} - \tilde{k}_{it+1}) - \beta_i^T u'_i(c_{iT}, c_{jT}) (k_{iT+1} - \tilde{k}_{iT+1}) \\ &+ \sum_{t=0}^T \beta_i^t [u'_i(c_{it}, c_{jt}) w_t - v'_i(l_{it})] (l_{it} - \tilde{l}_{it}) \end{aligned} \right] \\
&= \lim_{T \rightarrow \infty} \left[\begin{aligned} &\sum_{t=0}^{T-1} \beta_i^t [\beta_i u'_i(c_{it+1}, c_{jt+1}) (r_{t+1} + \delta - 1) - u'_i(c_{it}, c_{jt})] (k_{it+1} - \tilde{k}_{it+1}) \\ &- \beta_i^T u'_i(c_{iT}, c_{jT}) (k_{iT+1} - \tilde{k}_{iT+1}) \\ &+ \sum_{t=0}^T \beta_i^t [u'_i(c_{it}, c_{jt}) w_t - v'_i(l_{it})] (l_{it} - \tilde{l}_{it}) \end{aligned} \right] \\
&\geq \lim_{T \rightarrow \infty} \left[\begin{aligned} &\sum_{t=0}^{T-1} \beta_i^t [\beta_i u'_i(c_{it+1}, c_{jt+1}) (r_{t+1} + \delta - 1) - u'_i(c_{it}, c_{jt})] k_{it+1} \\ &- \beta_i^T u'_i(c_{iT}, c_{jT}) k_{iT+1} \\ &+ \sum_{t=0}^T \beta_i^t [u'_i(c_{it}, c_{jt}) w_t - v'_i(l_{it})] l_{it} \end{aligned} \right] \\
&= 0
\end{aligned}$$

This proves that condition (D3) holds. Finally, conditions (D4), (D5) and (D6) are easily obtained using (L2), (L5), (L6) and (L7).

7.2 Proof of Proposition 1

This proof consists of four steps:

1. For $i = 1$, conditions (S2) – (S5) satisfy the optimality conditions in Lemma 1. Further, the transversality condition $\lim_{t \rightarrow +\infty} \beta_1^t u_{11}(c_1, c_2) k_1 = 0$ holds since $\beta_1^t \in (0, 1)$ and c_1 and c_2 are constant.
2. For $i = 2$, we show that it is optimal that the impatient agent holds no capital $k_2 = 0$ and consume the entire of his wage-income, i.e., $c_2 = wl_2$. For this purpose, let us consider a feasible sequence $(\tilde{k}_2, \tilde{l}_2, \tilde{c}_2)$ starting from $\tilde{k}_{20} = 0$ and compare this path to the steady state solution (l_2, c_2) with $l_2 \in (0, \varsigma)$ and $c_2 = wl_2$. Given the external effect $c_1 = wl_1 + (r - \delta)k_1$, then in the following proof, we show that the stationary solution is optimal.

We have:

$$\begin{aligned}
& \sum_{t=0}^{\infty} \beta_2^t \left[u_2(c_2, c_1) - v_2(l_2) - \left(u_2(\tilde{c}_2, c_1) - v_2(\tilde{l}_2) \right) \right] \\
&= \sum_{t=0}^{\infty} \beta_2^t \left[u_2(c_2, c_1) - u_2(\tilde{c}_2, c_1) - \left(v_2(l_2) - v_2(\tilde{l}_2) \right) \right] \\
&\geq \sum_{t=0}^{\infty} \beta_2^t \left[u_{2,1}(c_2, c_1) [c_2 - \tilde{c}_2] - v'_2(l_{2t}) (l_2 - \tilde{l}_2) \right] \\
&= \sum_{t=0}^{\infty} \beta_2^t \left[u_{2,1}(w_2 l_2, c_1) [w_2 l_2 - \tilde{c}_2] - v'_2(l_{2t}) (l_2 - \tilde{l}_2) \right] \\
&= u_{2,1}(w_2 l_2, c_1) \sum_{t=0}^{\infty} \beta_2^t [w_2 l_2 - \tilde{c}_2] = u_{2,1}(w_2 l_2, c_1) \sum_{t=0}^{\infty} \beta_2^t [\tilde{k}_{2t+1} - (1/\beta_1) \tilde{k}_{2t}] \\
&= u_{2,1}(w_2 l_2, c_1) \lim_{T \rightarrow \infty} \left[\sum_{t=0}^T \beta_2^t \tilde{k}_{2t+1} - \sum_{t=0}^T \beta_2^t (1/\beta_1) \tilde{k}_{2t} \right] \\
&= u_{2,1}(w_2 l_2, c_1) \lim_{T \rightarrow \infty} \left[\beta_2^T \tilde{k}_{2T+1} + \sum_{t=0}^{T-1} \beta_2^t \tilde{k}_{2t+1} - \sum_{t=1}^T \beta_2^t (1/\beta_1) \tilde{k}_{2t} - (1/\beta_1) \tilde{k}_{20} \right] \\
&= u_{2,1}(w_2 l_2, c_1) \lim_{T \rightarrow \infty} \left[\beta_2^T \tilde{k}_{2T+1} + \sum_{t=1}^T \frac{\beta_2^t \tilde{k}_{2t}}{\beta_2} - \sum_{t=1}^T \beta_2^t (1/\beta_1) \tilde{k}_{2t} - (1/\beta_1) \tilde{k}_{20} \right] \\
&= u_{2,1}(w_2 l_2, c_1) \lim_{T \rightarrow \infty} \left[\beta_2^T \tilde{k}_{2T+1} - (1/\beta_1) \tilde{k}_{20} + \left(\frac{1}{\beta_2} - \frac{1}{\beta_1} \right) \sum_{t=1}^T \beta_2^t \tilde{k}_{2t} \right] \\
&\geq -u_{2,1}(w_2 l_2, c_1) (1/\beta_1) \tilde{k}_{20} \\
&= 0
\end{aligned}$$

Because $1/\beta_2 > 1/\beta_1$ and $\tilde{k}_{2t} \geq 0$. Notice that $l_2 = 0$ is not optimal and since $c_2 = w_2 l_2$ and $\lim_{c_2 \rightarrow 0} u'(c_2) = +\infty$, thus a small increase of labor supply l_2 will generate a very huge increase in the welfare.

3. Under Assumption 3, there is a unique finite and strictly positive value of k such that $r = f'(k) = 1/\beta_1 - (1 - \delta)$.
4. In the last point, we show that $r + 1 - \delta = 1/\beta_1$ is the only and unique steady state solution with $K = N_1 k_1 > 0$ and $k_2 = 0$. From one hand, if $r + 1 - \delta > 1/\beta_1$, then it is optimal for agent 1 to accumulate more capital. However, this can not be a stationary solution because of the decreasing returns. From the other hand, if $r + 1 - \delta < 1/\beta_1 < 1/\beta_2$, then it is optimal for both agents to deaccumulate capital to zero in a finite time, i.e., $k \rightarrow 0$ and $\lim_{k \rightarrow 0} f'(k) = +\infty$ which contradicts the stationarity.

7.3 Proof of Lemma 2

At the steady state, the prices r and w are constant and taken as given by households. Further, at the steady state, we have $r = 1/\beta_1 - (1 - \delta)$ which determines the capital-labor ratio $k = r^{-1} (1/\beta_1 - (1 - \delta))$. Thus k is obtained independently of the labor choice of agents.

Moreover, the budget constraints for patient and impatient agent are respectively given by $c_1 = wl_1 + (r - \delta)k_1$ and $c_2 = wl_2$, with $r = 1/\beta_1 - (1 - \delta)$ and using $K = kL$ with $K = N_1k_1$ and $L = N_1l_1 + N_2l_2$, we get $k_1 = [l_1 + N_2l_2/N_1]k$. Therefore, patient agent's consumption can be written in terms of both agents' labor supply as follows

$$c_1 = \left(w + \frac{1 - \beta_1}{\beta_1} k \right) l_1 + \frac{1 - \beta_1}{\beta_1} \frac{N_2 k}{N_1} l_2 \equiv c_1(l_1, l_2) \quad (33)$$

Then using consumption-leisure arbitrage conditions

$$u'_1(c_1(l_1, l_2))w - v'_1(l_1) = 0 \quad (34)$$

$$u_{2,1}(wl_2, c_1(l_1, l_2))w - v'_2(l_2) = 0 \quad (35)$$

we obtain l_1 and l_2 as implicit functions.

- (i) In order to demonstrate the existence of the function $l_2(l_1)$, we use impatient agent's consumption-leisure arbitrage condition (35) which can be written as follows

$$\Psi(l_1, l_2) = 1 \quad (36)$$

where $\Psi(l_1, l_2) \equiv v'_2(l_2)/u_{2,1}(wl_2, c_1)w$ is a continuous function. Moreover, consider

$$\frac{\partial \Psi(l_1, l_2)}{\partial l_2} = \frac{1}{u_{2,1}} \left[\frac{1}{w} v''_2 - u_{2,11}w - \frac{1 - \beta_1}{\beta_1} \frac{N_2 k}{N_1} u_{2,12} \right]$$

with $\partial \Psi(l_1, l_2)/\partial l_2 \neq 0$ if and only if

$$\frac{1}{w} v''_2 - u_{2,11}w - \frac{1 - \beta_1}{\beta_1} \frac{N_2 k}{N_1} u_{2,12} \neq 0$$

Given that condition (12) is verified for all $l_1, l_2 \in (0, \varsigma)$. In the sequel, for simplicity, we consider the following assumption

$$\frac{1}{w} v''_2 - u_{2,11}w - \frac{1 - \beta_1}{\beta_1} \frac{N_2 k}{N_1} u_{2,12} > 0 \quad (37)$$

Then, by implicit function theorem, l_2 can be expressed as a function of l_1 , i.e., there exists a function $l_2 \equiv l_2(l_1)$ defined over the domain $l_1 \in (0, \varsigma)$.

- (ii) Whenever $l_1 = 0$ then impatient agent's consumption-leisure arbitrage condition implicitly becomes $\Psi(l_2) = 1$ with $\Psi(l_2) \equiv v'_2(l_2)/wu_{2,1}(wl_2, c_1)$. Using Assumption 2, we get that $\lim_{l_2 \rightarrow \varsigma} \Psi(l_2) = +\infty$ and $\lim_{l_2 \rightarrow 0^+} \Psi(l_2) = 0$. Further,

$$\frac{\partial \Psi(l_2)}{\partial l_2} = \frac{1}{u_{2,1}} \left[\frac{1}{w} v''_2 - wu_{2,11} - \frac{1 - \beta_1}{\beta_1} \frac{N_2 k}{N_1} u_{2,12} \right]$$

is positive under the condition (13). As a result, given any w , there exists a unique value of l_2 which solves $\Psi(l_2) = 1$.

7.4 Proof of Proposition 2

Given the function $l_2 \equiv l_2(l_1)$, for all $l_1 \in (0, \varsigma)$, the consumption-leisure arbitrage equation (34) can be written as $\Phi(l_1) = 1$, with $\Phi(l_1) \equiv v'_1(l_1)/u'_1(c_1(l_1, l_2(l_1)))w$. Under Assumption 2, we have $\lim_{l_1 \rightarrow \varsigma} \Phi(l_1) = +\infty$. Further, the condition (13) implies that $\partial l_2 / \partial l_1 > 0$ and thus $\Phi'(l_1) > 0$. Then there exists a unique l_1 that solves $\Phi(l_1) = 1$ (and so equation (34)) if and only if $\lim_{l_1 \rightarrow 0^+} \Phi(l_1) < 1$, namely, the following inequality should hold

$$\lim_{l_1 \rightarrow 0^+} v'_1(l_1) < \lim_{l_1 \rightarrow 0^+} u'_1 \left(\left[w + \frac{1 - \beta_1}{\beta_1} k \right] l_1 + \frac{1 - \beta_1}{\beta_1} \frac{N_2 k}{N_1} l_2(l_1) \right) w$$

provided that $l_2(0)$ is well-defined in Lemma 2.

7.5 Proof of Proposition 3

Conditions of local indeterminacy are $D < 1$, $T < 1 + D$ and $T > -1 - D$. We first study the sign of the denominator:

$$DEN \equiv (\sigma + s\xi_{lw}) \varepsilon_{1,11} + s\rho\xi_{lc} \quad (38)$$

then $DEN > 0$ iff $\sigma < \sigma^*$. We observe that if $\xi_{lw} < 0$ then $\sigma^* > 0$. However, if $\xi_{lw} > 0$ then $\sigma^* > 0$ if and only if $\xi_{cl} > \xi_{cl}^*$, where

$$\sigma^* \equiv -s \left(\xi_{lw} + \frac{1}{\varepsilon_{1,11}} \rho \xi_{cl} \right) \quad (39)$$

$$\xi_{cl}^* \equiv -\frac{1}{\rho} \varepsilon_{1,11} \xi_{lw} \quad (40)$$

This implies that $DEN > 0$ if either $\xi_{lw} < 0$ and $\sigma < \sigma^*$ or $\xi_{lw} > 0$ and $\xi_{cl} > \xi_{cl}^*$ and $\sigma < \sigma^*$. However, $DEN < 0$ if either $\xi_{lw} < 0$ and $\sigma > \sigma^*$ or $\xi_{lw} > 0$ and either $\xi_{cl} < \xi_{cl}^*$ and all $\sigma > 0$ or $\xi_{cl} > \xi_{cl}^*$ and $\sigma > \sigma^*$.

Then the proof is divided into two parts: The first one assumes that $DEN > 0$ and the other assumes that $DEN < 0$.

7.5.1 Case 1: $DEN > 0$

First, the condition $D < 1$ holds iff $\sigma > \sigma_D$ with $\sigma_D > 0$ iff $\xi_{cl} < \xi_{cl}^D$ and $\xi_{cl}^D > 0$ iff $\xi_{lw} < \rho/(1 - \beta_1)$, where

$$\sigma_D \equiv \frac{[\rho - (1 - \beta_1)\xi_{lw}]\varepsilon_{1,11} + \rho\beta_1\xi_{cl}}{\frac{1}{s}(1 - \beta_1)\varepsilon_{1,11}} \quad (41)$$

$$\xi_{cl}^D \equiv -\frac{1}{\rho\beta_1}[\rho - (1 - \beta_1)\xi_{lw}]\varepsilon_{1,11} \quad (42)$$

This implies that $D < 1$ is verified if either (i) $\xi_{lw} < \rho/(1 - \beta_1)$ and either $\xi_{cl} < \xi_{cl}^D$ and $\sigma > \sigma_D$ or $\xi_{cl} > \xi_{cl}^D$ and all $\sigma > 0$; or (ii) $\xi_{lw} > \rho/(1 - \beta_1)$ and $\xi_{cl} > 0$ and all $\sigma > 0$. Therefore, the condition $D < 1$ holds under $DEN > 0$ if

1. whenever $\xi_{lw} < 0$ and either $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma^*$; or $\xi_{cl} > \xi_{cl}^D$ and $\sigma < \sigma^*$;
2. whenever $0 < \xi_{lw} < \rho$ and either $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma^*$; or $\xi_{cl} > \xi_{cl}^D$ and $\sigma < \sigma^*$.
3. whenever $\xi_{lw} > \rho$ and $\xi_{cl} > \xi_{cl}^*$ and all $\sigma < \sigma^*$.

Second, the condition $T < 1 + D$ holds iff $\xi_{cl} < 1$. Given that $DEN > 0$, the condition $T < 1 + D$ holds is verified if $\sigma < \sigma^*$ and either $\xi_{lw} < 0$ and $0 < \xi_{cl} < 1$ or $\xi_{lw} < -\rho/\varepsilon_{1,11}$ and $\xi_{cl}^* < \xi_{cl} < 1$.

Third, the condition $T > -1 - D$ holds iff $\sigma < \sigma_{AB}$ with $\sigma_{AB} > 0$ iff $\xi_{cl} > \xi_{cl}^{AB}$; and $\xi_{cl}^{AB} > 0$ iff $\xi_{lw} > \xi_{lw}^{AB}$ and $\xi_{lw}^{AB} > 0$ iff $\varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$, where

$$\sigma_{AB} \equiv -\frac{2\varepsilon_{1,11}[(1 + \beta_1)\xi_{lw} - \rho] - \rho(1 - \beta_1) + \rho(1 + \beta_1)\xi_{cl}}{2\frac{1}{s}(1 + \beta_1)\varepsilon_{1,11}} \quad (43)$$

$$\xi_{cl}^{AB} \equiv -\frac{2\varepsilon_{1,11}[(1 + \beta_1)\xi_{lw} - \rho] - \rho(1 - \beta_1)}{\rho(1 + \beta_1)} \quad (44)$$

$$\xi_{lw}^{AB} \equiv \frac{\rho}{(1 + \beta_1)\varepsilon_{1,11}} \left[\varepsilon_{1,11} + \frac{1}{2}(1 - \beta_1) \right] \quad (45)$$

Then taking the conditions under which $DEN > 0$, we consider

$$\begin{aligned} \sigma_{AB} - \sigma^* &= \frac{1}{2} \frac{s\rho}{\varepsilon_{1,11}} (\xi_{cl} - \hat{\xi}_{cl}) \\ \hat{\xi}_{cl} &\equiv -\frac{2}{1 + \beta_1} \left[\varepsilon_{1,11} + \frac{1}{2}(1 - \beta_1) \right] \end{aligned} \quad (46)$$

and

$$\xi_{cl}^* - \hat{\xi}_{cl} = \frac{\varepsilon_{1,11}}{\rho} (\hat{\xi}_{lw} - \xi_{lw})$$

$$\hat{\xi}_{lw} \equiv \frac{2\rho}{1+\beta_1} \left[\varepsilon_{1,11} + \frac{1}{2}(1-\beta_1) \right] \frac{1}{\varepsilon_{1,11}} \quad (47)$$

Whenever $\xi_{lw} < 0$, the condition $T > -1 - D$ is verified if

- (B.1) $\varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and all $\xi_{lw} < 0$ and either $0 < \xi_{cl} < \hat{\xi}_{cl}$ and $\sigma < \sigma^*$ or $\xi_{cl} > \hat{\xi}_{cl}$ and $\sigma < \sigma_{AB}$;
- (B.2) $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw} < \xi_{lw}^{AB} < 0$ and all $\xi_{cl} > 0$ and $\sigma < \sigma_{AB}$;
- (B.3) $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < 0$ and $\xi_{cl} > \xi_{cl}^{AB}$ and $\sigma < \sigma_{AB}$.

Whenever $\xi_{lw} > 0$, the condition $T > -1 - D$ is verified if

- (D1) $\varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $0 < \xi_{lw} < \xi_{lw}^{AB}$ and all $\xi_{cl} > \xi_{cl}^*$ and $\sigma < \min\{\sigma_{AB}, \sigma^*\}$;
- (D2) $\varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$ and $\xi_{cl} > \xi_{cl}^*$ and $\sigma < \min\{\sigma_{AB}, \sigma^*\}$;
- (D3) $\varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw} > \hat{\xi}_{lw}$ and $\xi_{cl} > \xi_{cl}^{AB}$ and $\sigma < \sigma_{AB}$;
- (D4) $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and all $\xi_{lw} > 0$ and $\xi_{cl} > \xi_{cl}^{AB}$ and $\sigma < \sigma_{AB}$.

Subcase (1.1): $\xi_{lw} < 0$

In order to characterize the conditions of local indeterminacy, we start by determining the intersection of the conditions under which both $D < 1$ and $T < 1 + D$ are simultaneously met. Since

$$\begin{aligned} \xi_{cl}^D - 1 &= -\varepsilon_{1,11} \frac{1-\beta_1}{\rho\beta_1} (\tilde{\xi}_{lw} - \xi_{lw}) \\ \tilde{\xi}_{lw} &\equiv \frac{\rho}{1-\beta_1} \frac{\varepsilon_{1,11} + \beta_1}{\varepsilon_{1,11}} \end{aligned} \quad (48)$$

then $D < 1$ and $T < 1 + D$ hold if either of the following conditions holds:

- (A.1) $-1 < \varepsilon_{1,11} < -\beta_1$ and all $\xi_{lw} < 0$ and $-\varepsilon_{1,11} < \xi_{cl} < 1$ and $\sigma_D < \sigma < \sigma^*$;
- (A.2) $\varepsilon_{1,11} > -\beta_1$ and $\xi_{lw} < \tilde{\xi}_{lw}$ and $-\varepsilon_{1,11} < \xi_{cl} < 1$ and $\sigma_D < \sigma < \sigma^*$;
- (A.3) $\varepsilon_{1,11} > -\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw} < 0$ and $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma^*$;
- (A.4) $\varepsilon_{1,11} > -\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw} < 0$ and $\xi_{cl}^D < \xi_{cl} < 1$ and $\sigma < \sigma^*$.

Now we consider the intersection of the conditions (A.1) – (A.4) with the conditions (B.1) – (B.3).

The intersection of (A1) and (B1): Since $\varepsilon_{1,11} > -1$ then $\hat{\xi}_{cl} < -\varepsilon_{1,11} < 1$. Further, we know that $\sigma_{AB} < \sigma^*$ iff $\xi_{cl} > \hat{\xi}_{cl}$ and $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$, where

$$\xi_{cl}^{ABD} \equiv \left[(1 - \beta_1)^2 - 4\beta_1\varepsilon_{1,11} \right] / (1 + \beta_1)^2 \quad (49)$$

with $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < 1$. As a result, local indeterminacy occurs whenever $-1 < \varepsilon_{1,11} < -\beta_1$ and all $\xi_{lw} < 0$ and $\xi_{cl}^{ABD} < \xi_{cl} < 1$ and $\sigma_D < \sigma < \sigma_{AB}$.

The intersection of (A2) and (B1): Since $\varepsilon_{1,11} > -1$ then we have $\hat{\xi}_{cl} < -\varepsilon_{1,11} < 1$. In addition, we note that $\sigma_{AB} < \sigma^*$ for $\xi_{cl} > \hat{\xi}_{cl}$ and $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$ with $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < 1$. As a result, local indeterminacy occurs whenever $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} < \tilde{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < 1$ and $\sigma_D < \sigma < \sigma_{AB}$.

The intersection of (A2) and (B2): Note that for all $\varepsilon_{1,11} > -\beta_1$ we have $[4\beta_1(\varepsilon_{1,11} + 1) - (1 + \beta_1)(1 - \beta_1)] > 0$ and thus $\tilde{\xi}_{lw} < \xi_{lw}^{AB}$. Further, since $\hat{\xi}_{cl} < -\varepsilon_{1,11}$ for all $\varepsilon_{1,11} > -\beta_1$, we have $\sigma_{AB} < \sigma^*$ for all $-\varepsilon_{1,11} < \xi_{cl} < 1$ and $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$ with $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < 1$. As a result, local indeterminacy occurs whenever $\varepsilon_{1,11} > -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} < \tilde{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < 1$ and $\sigma_D < \sigma < \sigma_{AB}$.

The intersection of (A2) and (B3): As shown in the preceding case, for all $\varepsilon_{1,11} > -\beta_1$, we have $[4\beta_1(\varepsilon_{1,11} + 1) - (1 + \beta_1)(1 - \beta_1)] > 0$ and thus $\tilde{\xi}_{lw} < \xi_{lw}^{AB}$. This implies that the intersection is empty.

The intersection of (A3) and (B1): Since $\varepsilon_{1,11} > -1$ then we have $\hat{\xi}_{cl} < -\varepsilon_{1,11} < 1$. In addition, we have $\sigma_{AB} < \sigma^*$ for all $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$ with $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < 1$ and $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < \xi_{cl}^D$. As a result, local indeterminacy occurs whenever $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} > \tilde{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma_{AB}$.

The intersection of (A3) and (B2): As shown above, for all $\varepsilon_{1,11} > -\beta_1$, we have $[4\beta_1(\varepsilon_{1,11} + 1) - (1 + \beta_1)(1 - \beta_1)] > 0$ and thus $\tilde{\xi}_{lw} < \xi_{lw}^{AB}$. In addition, we have $\sigma_{AB} < \sigma^*$ for all $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$ with $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$ and $\xi_{cl}^{ABD} > \xi_{cl}^D$ iff $\xi_{lw} > \ddot{\xi}_{lw}$, where

$$\ddot{\xi}_{lw} \equiv \frac{\rho(1 + 3\beta_1)}{(1 + \beta_1)^2} \left(\varepsilon_{1,11} + \frac{\beta_1(1 - \beta_1)}{3\beta_1 + 1} \right) \frac{1}{\varepsilon_{1,11}} \quad (50)$$

with $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1(1 - \beta_1) / (3\beta_1 + 1)$. Since $-\frac{1}{2}(1 - \beta_1) < -\beta_1(1 - \beta_1) / (3\beta_1 + 1)$ and $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$, then local indeterminacy occurs for $\sigma_D < \sigma < \sigma_{AB}$

and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and either $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$; or $\varepsilon_{1,11} > -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$ (since $\xi_{lw}^{AB} < \ddot{\xi}_{lw}$). After making some simplification, local indeterminacy occurs if $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma_{AB}$.

The intersection of (A3) and (B3): For $\varepsilon_{1,11} > -\beta_1$, we have $\tilde{\xi}_{lw} < \xi_{lw}^{AB}$. Further, $\xi_{cl}^{AB} < \xi_{cl}^D$ iff $\xi_{lw} < \ddot{\xi}_{lw}$, with $\xi_{lw}^{AB} < \ddot{\xi}_{lw}$ and $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1(1-\beta_1)/(3\beta_1+1)$ with $-\frac{1}{2}(1-\beta_1) < -\beta_1(1-\beta_1)/(3\beta_1+1)$. Then $\xi_{cl}^{AB} < \xi_{cl}^D$ is satisfied if either $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\xi_{lw} > \xi_{lw}^{AB}$; or $\varepsilon_{1,11} > -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\xi_{lw}^{AB} < \xi_{lw} < \ddot{\xi}_{lw}$ since $\xi_{lw}^{AB} < \ddot{\xi}_{lw}$. Further, $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ iff $\xi_{lw} > \dot{\xi}_{lw}$ where

$$\dot{\xi}_{lw} \equiv \frac{\rho}{2\varepsilon_{1,11}} \frac{3+\beta_1}{1+\beta_1} \left(\varepsilon_{1,11} + \frac{1-\beta_1}{3+\beta_1} \right) \quad (51)$$

with $\dot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\frac{1-\beta_1}{3+\beta_1}$. Since $-\frac{1}{2}(1-\beta_1) < -(1-\beta_1)/(3+\beta_1)$ and $\xi_{lw}^{AB} < \dot{\xi}_{lw} < \ddot{\xi}_{lw}$. First, for all $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$, we have $\sigma_{AB} < \sigma^*$. In addition, $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$ with $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$ and $\xi_{cl}^{ABD} < \xi_{cl}^D$ for $\xi_{lw} < \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1(1-\beta_1)/(3\beta_1+1)$. Thus we have $\xi_{lw}^{AB} < \xi_{lw} < \ddot{\xi}_{lw}$. This implies that in all the above cases, we have $\xi_{lw} < \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} \leq 0$, then $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < \xi_{cl}^D$ and $\xi_{cl}^{ABD} > \xi_{cl}^{AB}$. Therefore, local indeterminacy occurs for $\sigma_D < \sigma < \sigma_{AB}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ if either $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\beta_1(1-\beta_1)/(3\beta_1+1)$ and $\xi_{lw} > \xi_{lw}^{AB}$; or $\varepsilon_{1,11} > -\beta_1(1-\beta_1)/(3\beta_1+1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \dot{\xi}_{lw}$.

As a result, local indeterminacy occurs if $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \min\{0, \ddot{\xi}_{lw}\}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma_{AB}$.

The intersection of (A4) and (B1): Note that $\hat{\xi}_{cl} < 1$ and $\hat{\xi}_{cl} < \xi_{cl}^D$ and $\sigma_{AB} < \sigma^*$ for $\xi_{cl} > \hat{\xi}_{cl}$, then local indeterminacy occurs whenever $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw} > \tilde{\xi}_{lw}$ and $\xi_{cl}^D < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

The intersection of (A4) and (B2): As shown above, for all $\varepsilon_{1,11} > -\beta_1$, we have $\tilde{\xi}_{lw} < \xi_{lw}^{AB}$. As a result, local indeterminacy occurs whenever $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw} < \xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^D < \xi_{cl} < 1$ and $0 < \sigma < \min\{\sigma^*, \sigma_{AB}\}$.

The intersection of (A4) and (B3): First, we have $\tilde{\xi}_{lw} < \xi_{lw}^{AB}$ for all $\varepsilon_{1,11} > -\beta_1$. Further, $\xi_{cl}^{AB} > \xi_{cl}^D$ iff $\xi_{lw} > \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} > \xi_{lw}^{AB}$ and $\xi_{cl}^{AB} < 1$ since $\xi_{lw} < 0$. Then local indeterminacy occurs for $\sigma < \min\{\sigma^*, \sigma_{AB}\}$ if $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and either $\xi_{lw}^{AB} < \xi_{lw} < \dot{\xi}_{lw}$ and $\xi_{cl}^D < \xi_{cl} < 1$; or $\xi_{lw} > \dot{\xi}_{lw}$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$.

Subcase (1.2): $\xi_{lw} > 0$

The conditions $D < 1$ and $T < 1 + D$ are simultaneously met in either of the following cases:

(C1) whenever $\sigma_D < \sigma < \sigma^*$ if

(C1.1) either $-1 < \varepsilon_{1,11} < -\beta_1$ and either $\xi_{lw} < \tilde{\xi}_{lw}$ and $-\varepsilon_{1,11} < \xi_{cl} < 1$ or $\tilde{\xi}_{lw} < \xi_{lw} < \rho$ and $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$;

(C1.2) or $\varepsilon_{1,11} > -\beta_1$ and $0 < \xi_{lw} < \rho$ and $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$.

(C2) whenever $\sigma < \sigma^*$ if

(C2.1) either $-1 < \varepsilon_{1,11} < -\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw} < \rho$ and $\xi_{cl}^D < \xi_{cl} < 1$;

(C2.2) or $\varepsilon_{1,11} > -\beta_1$ and $0 < \xi_{lw} < \rho$ and $\xi_{cl}^D < \xi_{cl} < 1$.

(C3) whenever $\sigma < \sigma^*$ if $\xi_{cl}^* < \xi_{cl} < 1$ and

(C3.1) either $-1 < \varepsilon_{1,11} < -(1 - \beta_1)$ and $\rho < \xi_{lw} < -\rho/\varepsilon_{1,11}$;

(C3.2) or $\varepsilon_{1,11} > -(1 - \beta_1)$ and $\rho < \xi_{lw} < \rho/(1 - \beta_1)$;

(C3.3) or $\varepsilon_{1,11} > -(1 - \beta_1)$ and $\rho/(1 - \beta_1) < \xi_{lw} < -\rho/\varepsilon_{1,11}$.

Local indeterminacy occurs if the conditions $D < 1$, $T < 1 + D$ and $T > -1 - D$ hold. Thus we should now consider the intersection of the conditions (C1) – (C3) and the conditions (D1) – (D4).

The intersection of (C1.1) and (D1): First, $\tilde{\xi}_{lw} > \xi_{lw}^{AB}$ iff $\varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$, with $-1 < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < -\beta_1$. As $\varepsilon_{1,11} > -1$, we have $\tilde{\xi}_{lw} < \xi_{lw}^{AB} < \rho < -\rho/\varepsilon_{1,11}$ and thus $\xi_{cl}^* < -\varepsilon_{1,11}$ for $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$. Then we have $\xi_{cl}^* < -\varepsilon_{1,11} < 1$.

Now, consider at first $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11} < 1$. Therefore, we have $\sigma_{AB} < \sigma^*$ for $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$ and $-\varepsilon_{1,11} < \xi_{cl} < 1$. Moreover, $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$. However, since $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < 1$ and $\xi_{cl}^{ABD} > \xi_{cl}^D$ iff $\xi_{lw} > \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1(1 - \beta_1)/(3\beta_1 + 1)$ with $-\beta_1 < -\beta_1(1 - \beta_1)/(3\beta_1 + 1)$. This implies that $\ddot{\xi}_{lw} > 0$ for all $-\beta_1 < \varepsilon_{1,11} < -\beta_1$ and $\xi_{lw} < \xi_{lw}^{AB} < \ddot{\xi}_{lw}$. Therefore, $\xi_{cl}^{ABD} < \xi_{cl}^D$ for $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$ and $-\beta_1 < \varepsilon_{1,11} < -\beta_1$.

As a result, local indeterminacy occurs for $\sigma_D < \sigma < \sigma_{AB}$ if

1. either $-1 < \varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$ and $\xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^{ABD} < \xi_{cl} < 1$;
2. or $-\beta_1 < \varepsilon_{1,11} < -\beta_1$ and $\xi_{lw} < \tilde{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < 1$;
3. or $-\beta_1 < \varepsilon_{1,11} < -\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$.

The intersection of (C1.2) and (D1): We have $\xi_{lw}^{AB} < \rho$ and for all $\xi_{lw} < \rho$, we have $\xi_{cl}^* < -\varepsilon_{1,11} < \xi_{cl}^D$. Further, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11}$ which is sufficient to conclude that we have $\sigma_{AB} < \sigma^*$. In addition, $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$. Then $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$ and $\xi_{cl}^{ABD} > \xi_{cl}^D$ iff $\xi_{lw} > \ddot{\xi}_{lw}$ with $\xi_{lw}^{AB} < \ddot{\xi}_{lw}$. This implies that $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < \xi_{cl}^D$ for $\xi_{lw} < \xi_{lw}^{AB}$.

As a result, local indeterminacy occurs whenever $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma_{AB}$.

The intersection of (C1.1) and (D2): First, we have $\tilde{\xi}_{lw} < \hat{\xi}_{lw} < \rho$ and $\xi_{lw}^{AB} < \rho$. Further, $\tilde{\xi}_{lw} > \xi_{lw}^{AB}$ iff $\varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$ with $-1 < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < -\beta_1$. In addition, $\xi_{cl}^* < -\varepsilon_{1,11}$ for $\xi_{lw} < \rho$. Then, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11} < 1$. Therefore, we have $\sigma_{AB} < \sigma^*$. Moreover, $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$. However, we have $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$ and $\xi_{cl}^{ABD} > \xi_{cl}^D$ iff $\xi_{lw} > \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1(1 - \beta_1)/(3\beta_1 + 1)$. We have $\ddot{\xi}_{lw} > 0$ for $-1 < \varepsilon_{1,11} < -\beta_1$ and $\hat{\xi}_{lw} < \ddot{\xi}_{lw}$. Therefore, $\xi_{cl}^{ABD} < \xi_{cl}^D$ and thus $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < \xi_{cl}^D$.

As a result, local indeterminacy appears for $\sigma_D < \sigma < \sigma_{AB}$ if

1. either $-1 < \varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw} < \hat{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$;
2. or $-\beta_1 < \varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$;
3. or $-1 < \varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$ and $\xi_{lw}^{AB} < \xi_{lw} < \tilde{\xi}_{lw}$ and $-\varepsilon_{1,11} < \xi_{cl} < 1$.

The intersection of (C1.2) and (D2): First, we have $\xi_{lw}^{AB} < \hat{\xi}_{lw} < \rho$ and $\xi_{cl}^* < -\varepsilon_{1,11} < \xi_{cl}^D$ for $\xi_{lw} < \rho$. In addition, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11}$ which is sufficient to conclude that $\sigma_{AB} < \sigma^*$. In addition, $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$. Then $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$ and $\xi_{cl}^{ABD} < \xi_{cl}^D$ iff $\xi_{lw} < \ddot{\xi}_{lw}$. Since $\xi_{lw}^{AB} < \hat{\xi}_{lw} < \ddot{\xi}_{lw}$, then $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < \xi_{cl}^D$ for $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$. As a result, local indeterminacy occurs whenever $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma_{AB}$.

The intersection of (C1.1) and (D3): In this case, we have $(3\beta_1 - 1) > 0$ and $\tilde{\xi}_{lw} < \hat{\xi}_{lw} < \rho$. Note that $\xi_{cl}^{AB} < \xi_{cl}^D$ iff $\xi_{lw} < \ddot{\xi}_{lw}$, with $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1(1 - \beta_1)/(3\beta_1 + 1)$. Since $-1 < -\beta_1 < -\frac{1}{2}(1 - \beta_1) < -\beta_1(1 - \beta_1)/(3\beta_1 + 1)$, then $\ddot{\xi}_{lw} > 0$, for all $-1 < \varepsilon_{1,11} < -\beta_1$, with $\hat{\xi}_{lw} < \ddot{\xi}_{lw} < \rho$ and thus $\xi_{cl}^{AB} < \xi_{cl}^D$ is verified for all $\hat{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$. In addition, we have $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ iff $\xi_{lw} > \dot{\xi}_{lw}$ where $\dot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -(1 - \beta_1)/(3 + \beta_1)$. However, since $-\beta_1(1 - \beta_1)/(3\beta_1 + 1) < -(1 - \beta_1)/(3 + \beta_1)$, then $\dot{\xi}_{lw} > 0$, for all $-1 < \varepsilon_{1,11} < -\beta_1$, with $\hat{\xi}_{lw} < \dot{\xi}_{lw} < \ddot{\xi}_{lw}$.

We also observe that $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$. First, we have $\xi_{cl}^{ABD} < \xi_{cl}^D$ and $\xi_{cl}^{ABD} > \xi_{cl}^{AB}$ for all $\xi_{lw} < \dot{\xi}_{lw}$. Further, $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$. Moreover, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11}$ which is sufficient to conclude that in both cases, we have $\sigma_{AB} < \sigma^*$.

As a result, local indeterminacy occurs for $\sigma_D < \sigma < \sigma_{AB}$ if $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and either $-1 < \varepsilon_{1,11} < -\beta_1$ and $\hat{\xi}_{lw} < \xi_{lw} < \dot{\xi}_{lw}$; or $-1 < \varepsilon_{1,11} < -\beta_1$ and $\dot{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$.

The intersection of (C1.2) and (D3): Since $\hat{\xi}_{lw} < \rho$ and $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ iff $\xi_{lw} > \dot{\xi}_{lw}$ where $\dot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -(1 - \beta_1) / (3 + \beta_1)$, and $\xi_{cl}^{AB} > \xi_{cl}^D$ iff $\xi_{lw} > \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1(1 - \beta_1) / (3\beta_1 + 1)$. In this case, we have $\dot{\xi}_{lw} > 0$ and $\ddot{\xi}_{lw} > 0$ for $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$. Further, $\hat{\xi}_{lw} < \dot{\xi}_{lw} < \ddot{\xi}_{lw} < \rho$, then this gives rise to the following cases:

1. For $\hat{\xi}_{lw} < \xi_{lw} < \dot{\xi}_{lw}$ then $\xi_{cl}^{AB} < -\varepsilon_{1,11}$ and $\xi_{cl}^{AB} < \xi_{cl}^D$;
2. For $\dot{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ then $-\varepsilon_{1,11} < \xi_{cl}^{AB} < \xi_{cl}^D$;
3. For $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ then $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ and $\xi_{cl}^{AB} > \xi_{cl}^D$.

Moreover, $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$. As $\xi_{cl}^{ABD} < \xi_{cl}^D$ and $\xi_{cl}^{AB} < \xi_{cl}^{ABD}$ for all $\xi_{lw} < \dot{\xi}_{lw}$ and $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$, local indeterminacy occurs for $\sigma_D < \sigma < \sigma_{AB}$ if $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\hat{\xi}_{lw} < \xi_{lw} < \dot{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$.

The intersection of (C1.2) and (D4): Since $\dot{\xi}_{lw} < \ddot{\xi}_{lw} < \rho$, the following cases arise:

1. For $-\frac{1}{2}(1 - \beta_1) < \varepsilon_{1,11} < -(1 - \beta_1) / (3 + \beta_1)$, then $\xi_{cl}^{AB} < -\varepsilon_{1,11}$ for $\xi_{lw} < \dot{\xi}_{lw}$ and $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ for $\xi_{lw} > \dot{\xi}_{lw}$. Further, $\xi_{cl}^{AB} < \xi_{cl}^D$ for $\xi_{lw} < \dot{\xi}_{lw}$ and $\xi_{cl}^{AB} > \xi_{cl}^D$ for $\xi_{lw} > \dot{\xi}_{lw}$. Since $\dot{\xi}_{lw} < \ddot{\xi}_{lw} < \rho$, then
 - (a) whenever $\xi_{lw} < \dot{\xi}_{lw}$, then $\xi_{cl}^{AB} < -\varepsilon_{1,11}$ and $\xi_{cl}^{AB} < \xi_{cl}^D$;
 - (b) whenever $\dot{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ then $-\varepsilon_{1,11} < \xi_{cl}^{AB} < \xi_{cl}^D$;
 - (c) whenever $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ then $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ and $\xi_{cl}^{AB} > \xi_{cl}^D$.
2. For $-(1 - \beta_1) / (3 + \beta_1) < \varepsilon_{1,11} < -\beta_1(1 - \beta_1) / (3\beta_1 + 1)$ then $-\varepsilon_{1,11} < \xi_{cl}^{AB} < \xi_{cl}^D$ for $\xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{AB} > \xi_{cl}^D$ for $\xi_{lw} > \ddot{\xi}_{lw}$.
 - (a) whenever $\xi_{lw} < \ddot{\xi}_{lw}$ then $-\varepsilon_{1,11} < \xi_{cl}^{AB} < \xi_{cl}^D$;
 - (b) whenever $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ then $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ and $\xi_{cl}^{AB} > \xi_{cl}^D$.
3. For $\varepsilon_{1,11} > -\beta_1(1 - \beta_1) / (3\beta_1 + 1)$ then $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ and $\xi_{cl}^{AB} > \xi_{cl}^D$ for $\xi_{lw} < \rho$.

In addition, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11}$ and $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} > -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} < \tilde{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma_{AB}$.

The intersection of (C2.1) with (D1): First, we have $\xi_{lw}^{AB} < \rho$ and $\tilde{\xi}_{lw} > \xi_{lw}^{AB}$ iff $\varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$ with $-1 < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < -\beta_1$. Since $\xi_{lw}^{AB} < \rho$ then $\xi_{cl}^* < \xi_{cl}^D < 1$. In addition, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11}$ which is sufficient to conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs if $-[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < \varepsilon_{1,11} < -\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^D < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

The intersection of (C2.1) with (D2): Since $\tilde{\xi}_{lw} < \hat{\xi}_{lw}$ and $\xi_{lw}^{AB} < \hat{\xi}_{lw} < \rho$ and $\tilde{\xi}_{lw} > \xi_{lw}^{AB}$ iff $\varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$ with $-1 < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < -\beta_1$. Note also that for $\xi_{lw} < \rho$, we have $\xi_{cl}^* < \xi_{cl}^D < 1$. Finally, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11}$ and thus $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs for $\sigma < \sigma_{AB}$ if $\xi_{cl}^D < \xi_{cl} < 1$ and either $-1 < \varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}$; or $-[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < \varepsilon_{1,11} < -\beta_1$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$.

The intersection of (C2.1) with (D3): Since $\tilde{\xi}_{lw} < \hat{\xi}_{lw} < \rho$. In addition, we have $\xi_{cl}^{AB} > \xi_{cl}^D$ for $\xi_{lw} > \tilde{\xi}_{lw}$. Since $-\frac{1}{2}(1 - \beta_1) < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$, then $\tilde{\xi}_{lw} > 0$ with $\hat{\xi}_{lw} < \tilde{\xi}_{lw} < \rho$. This implies that $\xi_{cl}^{AB} < \xi_{cl}^D$ for $\hat{\xi}_{lw} < \xi_{lw} < \tilde{\xi}_{lw}$ and $\xi_{cl}^{AB} > \xi_{cl}^D$ for $\xi_{lw} < \xi_{lw} < \rho$. Further, $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^Z$ where

$$\xi_{lw}^Z \equiv -\frac{\rho}{\varepsilon_{1,11}} \frac{\beta_1 - \varepsilon_{1,11}}{1 + \beta_1} \quad (52)$$

with $\rho < \xi_{lw}^Z$. This implies that $\xi_{cl}^{AB} < 1$ for $\hat{\xi}_{lw} < \xi_{lw} < \rho$. Finally, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^D$ for all $\xi_{lw} < \xi_{lw}^Z$. Therefore, we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs for $\sigma < \sigma_{AB}$ if $-1 < \varepsilon_{1,11} < -\beta_1$ and either $\hat{\xi}_{lw} < \xi_{lw} < \tilde{\xi}_{lw}$ and $\xi_{cl}^D < \xi_{cl} < 1$; or $\tilde{\xi}_{lw} < \xi_{lw} < \rho$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$.

The intersection of (C2.2) with (D1): We have $\xi_{lw}^{AB} < \rho$. Further, for $\xi_{lw} < \rho$, we have $\xi_{cl}^* < \xi_{cl}^D < 1$. In addition, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^D$ for all $\xi_{lw} < \xi_{lw}^Z$ with $\rho < \xi_{lw}^Z$. Therefore, we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs if $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^D < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

The intersection of (C2.2) with (D2): Since $\xi_{lw}^{AB} < \hat{\xi}_{lw} < \rho$. In addition, for $\xi_{lw} < \rho$, we have $\xi_{cl}^* < \xi_{cl}^D$. Further, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^D$ for all $\xi_{lw} < \xi_{lw}^Z$ with $\rho < \xi_{lw}^Z$. Therefore, we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs if $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$ and $\xi_{cl}^D < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

The intersection of (C2.2) with (D3): Since $\hat{\xi}_{lw} < \rho$. In addition, $\xi_{cl}^{AB} > \xi_{cl}^D$ for $\xi_{lw} > \ddot{\xi}_{lw}$. Since $-\frac{1}{2}(1-\beta_1) < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$, then $\ddot{\xi}_{lw} > 0$ with $\hat{\xi}_{lw} < \ddot{\xi}_{lw} < \rho$ and thus $\xi_{cl}^{AB} < \xi_{cl}^D$ for $\hat{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{AB} > \xi_{cl}^D$ for $\xi_{lw} < \xi_{lw} < \rho$. Further, $\xi_{cl}^{AB} > 1$ for $\xi_{lw} > \xi_{lw}^Z$ with $\rho < \xi_{lw}^Z$. Therefore, $\xi_{cl}^{AB} < 1$ whenever $\hat{\xi}_{lw} < \xi_{lw} < \rho$. To determine $\min\{\sigma_{AB}, \sigma^*\}$, as previously shown, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^D$ for all $\xi_{lw} < \xi_{lw}^Z$ with $\rho < \xi_{lw}^Z$. Therefore, we conclude that $\sigma_{AB} < \sigma^*$ in both cases. As a result, local indeterminacy occurs for $\sigma < \sigma_{AB}$ if $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and either $\hat{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^D < \xi_{cl} < 1$; or $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ and $\xi_{cl}^{AB} < \xi_{cl}^{AB} < 1$.

The intersection of (C2.2) with (D4): We have $\xi_{cl}^{AB} > \xi_{cl}^D$ iff $\xi_{lw} > \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} < \rho$. Then $\xi_{cl}^{AB} < \xi_{cl}^D$ for $\xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{AB} > \xi_{cl}^D$ for $\xi_{lw} < \xi_{lw} < \rho$. In addition, we have $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^Z$ with $\rho < \xi_{lw}^Z$. Then $\xi_{cl}^{AB} < 1$ for all $\xi_{lw} < \rho$. This implies three cases:

1. For $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\xi_{lw} < \ddot{\xi}_{lw}$ then $\xi_{cl}^{AB} < \xi_{cl}^D$ and $\xi_{cl}^{AB} < 1$;
2. For $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ then $\xi_{cl}^D < \xi_{cl}^{AB} < 1$;
3. For $\varepsilon_{1,11} > -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ then $\ddot{\xi}_{lw} < 0$ and $\xi_{cl}^D < \xi_{cl}^{AB} < 1$.

As previously shown, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^D$ for all $\xi_{lw} < \xi_{lw}^Z$ with $\rho < \xi_{lw}^Z$. Therefore, we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs for $\sigma < \sigma_{AB}$ if

1. either $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $0 < \xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^D < \xi_{cl} < 1$;
2. or $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$;
3. or $\varepsilon_{1,11} > -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $0 < \xi_{lw} < \rho$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$.

The intersection of (C3.1) and (D3): Since $\hat{\xi}_{lw} < \rho$ and $\xi_{cl}^{AB} > \xi_{cl}^*$ iff $\xi_{lw} > \hat{\xi}_{lw}$ with $\hat{\xi}_{lw} < \rho$. Therefore, $\xi_{cl}^{AB} > \xi_{cl}^*$ for $\rho < \xi_{lw} < -\rho/\varepsilon_{1,11}$. Further, $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^Z$ with $\rho < \xi_{lw}^Z < -\rho/\varepsilon_{1,11}$. Then for $\rho < \xi_{lw} < \xi_{lw}^Z$ we have $\xi_{cl}^* < \xi_{cl}^{AB} < 1$ and for $\xi_{lw}^Z < \xi_{lw} < -\rho/\varepsilon_{1,11}$ we have $\xi_{cl}^{AB} > 1$ and $\xi_{cl}^{AB} > \xi_{cl}^*$. As previously shown, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^*$ whenever $\xi_{lw} > \hat{\xi}_{lw}$. Since $\hat{\xi}_{lw} < \rho$, we conclude that $\sigma_{AB} < \sigma^*$ in both cases. As a result, local indeterminacy occurs if $-1 < \varepsilon_{1,11} < -(1-\beta_1)$ and $\rho < \xi_{lw} < \xi_{lw}^Z$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

The intersection of (C3.2) and (D3): As $-(1-\beta_1) < -\frac{1}{2}(1-\beta_1)$ and $\hat{\xi}_{lw} < \rho$. Then for this range of ξ_{lw} , we have $\xi_{cl}^{AB} > \xi_{cl}^*$. Further, $\rho < \xi_{lw}^Z < \rho$

$/(1 - \beta_1)$ whenever $-(1 - \beta_1) < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$. Finally, it remains to determine $\min\{\sigma_{AB}, \sigma^*\}$. As previously shown, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^*$ whenever $\xi_{lw} > \hat{\xi}_{lw}$. Then we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs if $-(1 - \beta_1) < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\rho < \xi_{lw} < \xi_{lw}^Z$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

The intersection of (C3.2) and (D4): First, $\xi_{cl}^{AB} < 1$ iff $\xi_{lw} < \xi_{lw}^Z$, with $\rho < \xi_{lw}^Z < \rho/(1 - \beta_1)$. In addition, we have $\xi_{cl}^{AB} > \xi_{cl}^*$ iff $\xi_{lw} > \hat{\xi}_{lw}$ with $\hat{\xi}_{lw} < \rho$. Finally, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^*$ whenever $\xi_{lw} > \hat{\xi}_{lw}$. Since $\hat{\xi}_{lw} < \rho$, we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} > -\frac{1}{2}(1 - \beta_1)$ and $\rho < \xi_{lw} < \xi_{lw}^Z$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

The intersection of (C3.3) and (D3): Since $-(1 - \beta_1) < -\frac{1}{2}(1 - \beta_1)$ and $\hat{\xi}_{lw} < \rho/(1 - \beta_1)$ and $\xi_{cl}^{AB} > \xi_{cl}^*$ whenever $\xi_{lw} > \hat{\xi}_{lw}$ and thus the intersection $(\xi_{cl}^* < \xi_{cl} < 1) \cap (\xi_{cl} > \xi_{cl}^{AB})$ is non-empty if and only if $\xi_{cl}^{AB} < 1$ which requires that $\xi_{lw} < \xi_{lw}^Z$. Since $\xi_{lw}^Z < \rho/(1 - \beta_1)$, this implies that $\xi_{cl}^{AB} < 1$ can NOT be satisfied. As a result, there is no room for local indeterminacy.

The intersection of (C3.3) and (D4): First, we have $\xi_{cl}^{AB} > \xi_{cl}^*$ whenever $\xi_{lw} > \hat{\xi}_{lw}$ with $\hat{\xi}_{lw} < \rho/(1 - \beta_1)$. Then $\xi_{cl}^{AB} < 1$ iff $\xi_{lw} < \xi_{lw}^Z$. Since $\varepsilon_{1,11} > -\frac{1}{2}(1 - \beta_1)$ then $\rho/(1 - \beta_1) < \xi_{lw}^Z < -\rho/\varepsilon_{1,11}$. Further, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^*$ whenever $\xi_{lw} > \hat{\xi}_{lw}$. Since $\hat{\xi}_{lw} < \rho/(1 - \beta_1)$, we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} > -\frac{1}{2}(1 - \beta_1)$ and $\rho/(1 - \beta_1) < \xi_{lw} < \xi_{lw}^Z$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

7.5.2 Case 2: $DEN < 0$

Taking the conditions under which $DEN < 0$ is verified, the condition $D < 1$ is satisfied if

1. For $\xi_{lw} < 0$ and $\xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^* < \sigma < \sigma_D$.
2. For $0 < \xi_{lw} < \rho$ and $\xi_{cl} < \xi_{cl}^*$ and $\sigma < \sigma_D$.
3. For $\rho < \xi_{lw} < \rho/(1 - \beta_1)$ and $\xi_{cl} < \xi_{cl}^D$ and $\sigma < \sigma_D$.
4. For $0 < \xi_{lw} < \rho$ and $\xi_{cl}^* < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^* < \sigma < \sigma_D$.

The condition $T < 1 + D$ holds if

1. For $\xi_{lw} < 0$ and $\xi_{cl} > 1$ and $\sigma > \sigma^*$
2. For $0 < \xi_{lw} < -\rho/\varepsilon_{1,11}$ and $\xi_{cl}^* < \xi_{cl} < 1$ and $\sigma > \sigma^*$
3. For $\xi_{lw} > -\rho/\varepsilon_{1,11}$ and either $1 < \xi_{cl} < \xi_{cl}^*$ and all $\sigma > 0$ or $\xi_{cl} > \xi_{cl}^*$ and $\sigma > \sigma^*$

Then both conditions $D < 1$ and $T < 1 + D$ are simultaneously satisfied if

- (S1) Whenever $\varepsilon_{1,11} < -1$ and $\xi_{lw} < 0$ and $1 < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^* < \sigma < \sigma_D$.
- (S2) Whenever $\varepsilon_{1,11} < -1$ and $-\rho/\varepsilon_{1,11} < \xi_{lw} < \rho$ and $1 < \xi_{cl} < \xi_{cl}^*$ and $\sigma < \sigma_D$.
- (S3) Whenever $\varepsilon_{1,11} < -1$ and $\rho < \xi_{lw} < \tilde{\xi}_{lw}$ and $1 < \xi_{cl} < \xi_{cl}^D$ and $\sigma < \sigma_D$.
- (S4) Whenever $\varepsilon_{1,11} < -1$ and EITHER $0 < \xi_{lw} < -\rho/\varepsilon_{1,11}$ and $\xi_{cl}^* < \xi_{cl} < 1$ and $\sigma^* < \sigma < \sigma_D$; OR $-\rho/\varepsilon_{1,11} < \xi_{lw} < \rho$ and $\xi_{cl}^* < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^* < \sigma < \sigma_D$.
- (S5) Whenever $\varepsilon_{1,11} > -1$ and $0 < \xi_{lw} < \rho$ and $\xi_{cl}^* < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^* < \sigma < \sigma_D$.

Now consider the conditions under which $T > -1 - D$ holds: First, whenever $\xi_{lw} < 0$, then $T > -1 - D$ is verified if $\sigma > \max\{\sigma^*, \sigma_{AB}\}$. Second, whenever $\xi_{lw} > 0$, then $T > -1 - D$ holds if

- (R1) For $\varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl} < \xi_{cl}^*$ and $\sigma > \sigma_{AB}$.
- (R2) For $\varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} < \xi_{lw}^{AB}$ and either $\xi_{cl}^* < \xi_{cl} < \hat{\xi}_{cl}$ and $\sigma > \sigma_{AB}$; or $\xi_{cl} > \hat{\xi}_{cl}$ and $\sigma > \sigma^*$.
- (R3) For $\varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$, and either $\xi_{cl} < \xi_{cl}^{AB}$ and all $\sigma > 0$; or $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^*$ and $\sigma > \sigma_{AB}$; or $\xi_{cl} > \xi_{cl}^*$ and $\sigma > \max\{\sigma_{AB}, \sigma^*\}$.
- (R4) For $\varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} > \hat{\xi}_{lw}$, and either $\xi_{cl} < \xi_{cl}^*$ and all $\sigma > 0$; or $\xi_{cl}^* < \xi_{cl} < \xi_{cl}^{AB}$ and $\sigma > \sigma^*$; or $\xi_{cl} > \xi_{cl}^{AB}$ and $\sigma > \sigma^*$.
- (R5) For $\varepsilon_{1,11} > -\frac{1}{2}(1 - \beta_1)$ and $0 < \xi_{lw} < \hat{\xi}_{lw}$ and either $\xi_{cl} < \xi_{cl}^{AB}$ and all $\sigma > 0$; or $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^*$ and $\sigma > \sigma_{AB}$; or $\xi_{cl} > \xi_{cl}^*$ and $\sigma > \max\{\sigma_{AB}, \sigma^*\}$.
- (R6) For $\varepsilon_{1,11} > -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} > \hat{\xi}_{lw}$ and either $0 < \xi_{cl} < \xi_{cl}^*$ and all $\sigma > 0$; or $\xi_{cl}^* < \xi_{cl} < \xi_{cl}^{AB}$ and $\sigma > \sigma^*$; or $\xi_{cl} > \xi_{cl}^{AB}$ and $\sigma > \sigma^*$.

Case (2.1): $\xi_{lw} < 0$

We have $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} > -\varepsilon_{1,11}$ since $\varepsilon_{1,11} < -1$. This implies that $\sigma_{AB} > \sigma^*$ for all $-1 < \xi_{cl} < -\varepsilon_{1,11}$. Further, $\sigma_{AB} < \sigma_D$ iff $\xi_{cl} < \xi_{cl}^{ABD}$. Since $1 < \xi_{cl}^{ABD} < -\varepsilon_{1,11}$, then local indeterminacy occurs if $\varepsilon_{1,11} < -1$ and $\xi_{lw} < 0$ and $1 < \xi_{cl} < \xi_{cl}^{ABD}$ and $\sigma_{AB} < \sigma < \sigma_D$.

Case (2.2): $\xi_{lw} > 0$

We consider in this case the intersection of (S2) – (S5) and (R1) – (R6).

The intersection of (S2) and (R1): Note that $\xi_{lw}^{AB} > -\rho/\varepsilon_{1,11}$ iff $\varepsilon_{1,11} < -\frac{1}{2}(3 + \beta_1)$. Further, we have $-\frac{1}{2}(3 + \beta_1) < -1$ and $\xi_{lw}^{AB} < \rho$. Moreover, $\sigma_{AB} < \sigma_D$ iff $\xi_{cl} < \xi_{cl}^{ABD}$, with $\xi_{cl}^{ABD} > 1$ and $\xi_{cl}^{ABD} > \xi_{cl}^*$ iff $\xi_{lw} < \xi_{lw}^F$. Since $\xi_{lw}^F > -\rho/\varepsilon_{1,11}$ and $\xi_{lw}^F > \xi_{lw}^{AB}$, where

$$\xi_{lw}^F \equiv \frac{\rho}{(1 + \beta_1)^2} \left[4\beta_1 - \frac{1}{\varepsilon_{1,11}} (1 - \beta_1)^2 \right] \quad (53)$$

Then we have $\xi_{cl}^{ABD} > \xi_{cl}^*$ for all $-\rho/\varepsilon_{1,11} < \xi_{lw} < \xi_{lw}^{AB}$. This implies that $\sigma_{AB} < \sigma_D$ for all $1 < \xi_{cl} < \xi_{cl}^*$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} < -\frac{1}{2}(3 + \beta_1)$ and $-\rho/\varepsilon_{1,11} < \xi_{lw} < \xi_{lw}^{AB}$ and $1 < \xi_{cl} < \xi_{cl}^*$ and $\sigma_{AB} < \sigma < \sigma_D$.

The intersection of (S2) and (R3): We have $\xi_{lw}^{AB} > -\rho/\varepsilon_{1,11}$ iff $\varepsilon_{1,11} < -\frac{1}{2}(3 + \beta_1)$ with $\xi_{lw}^{AB} < \rho$ and $-\rho/\varepsilon_{1,11} < \rho < \hat{\xi}_{lw}$. This gives rise to the following subcases:

Subcase 1: For $\varepsilon_{1,11} < -\frac{1}{2}(3 + \beta_1)$, then $-\rho/\varepsilon_{1,11} < \xi_{lw}^{AB} < \rho$ and $-\rho/\varepsilon_{1,11} < \rho < \hat{\xi}_{lw}$. Further, $\xi_{cl}^{AB} < \xi_{cl}^*$ for all $\xi_{lw} < \hat{\xi}_{lw}$. Moreover, $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^Z$ where $\xi_{lw}^Z < \xi_{lw}^Z < \hat{\xi}_{lw}$ and $-\rho/\varepsilon_{1,11} < \xi_{lw}^Z < \rho$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} < -\frac{1}{2}(3 + \beta_1)$ and $\xi_{lw}^Z < \xi_{lw} < \rho$ and $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^*$ and $\sigma < \sigma_D$.

Subcase 2: Note that $\sigma_{AB} < \sigma_D$ iff $\xi_{cl} < \xi_{cl}^{ABD}$, with $\xi_{cl}^{ABD} > 1$ and $\xi_{cl}^{ABD} > \xi_{cl}^{AB}$ iff $\xi_{lw} < \check{\xi}_{lw}$. We are in the case in which $\varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and thus $\check{\xi}_{lw} > 0$ with $\check{\xi}_{lw} > \rho$ and $\check{\xi}_{lw} > \xi_{lw}^{AB}$. Therefore, $\xi_{cl}^{ABD} > \xi_{cl}^{AB}$ for all $\xi_{lw}^{AB} < \xi_{lw} < \rho$ and $\xi_{cl}^{ABD} > \xi_{cl}^*$ iff $\xi_{lw} < \xi_{lw}^F$ with $\xi_{lw}^{AB} < \xi_{lw}^F < \rho$; and $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^Z$ with $\xi_{lw}^{AB} < \xi_{lw}^Z < \rho$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} < -\frac{1}{2}(3 + \beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \rho$ and $\max\{1, \xi_{cl}^{AB}\} < \xi_{cl} < \min\{\xi_{cl}^*, \xi_{cl}^{ABD}\}$ and $\sigma_{AB} < \sigma < \sigma_D$.

Subcase 3: For $-\frac{1}{2}(3 + \beta_1) < \varepsilon_{1,11} < -1$, we have $\xi_{lw}^{AB} < -\rho/\varepsilon_{1,11} < \rho < \hat{\xi}_{lw}$. Further, $\xi_{cl}^{AB} < \xi_{cl}^*$ for all $\xi_{lw} < \hat{\xi}_{lw}$. Moreover, $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^Z$ with $-\rho/\varepsilon_{1,11} < \xi_{lw}^Z < \rho$. As a result, local indeterminacy occurs if $-\frac{1}{2}(3 + \beta_1) < \varepsilon_{1,11} < -1$ and $\xi_{lw}^Z < \xi_{lw} < \rho$ and $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^*$ and $\sigma < \sigma_D$.

The intersection of (S3) and (R3): First, we have $\xi_{lw}^{AB} < \rho < \hat{\xi}_{lw}$ and $[4\beta_1\varepsilon_{1,11} + \beta_1\beta_1 + 4\beta_1 - 1] < 0$ (since $\varepsilon_{1,11} < -1$) then $\xi_{lw} > \xi_{lw}^{AB}$ and $\tilde{\xi}_{lw} > \hat{\xi}_{lw}$. Moreover, $\xi_{cl}^D < \xi_{cl}^*$ since $\xi_{lw} > \rho$. In addition, we have $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^Z$. However, since $\xi_{lw}^Z < \rho$, this implies that $\xi_{cl}^{AB} > 1$ for all $\rho < \xi_{lw} < \hat{\xi}_{lw}$. Further, $\xi_{cl}^{AB} > \xi_{cl}^D$ iff $\xi_{lw} > \check{\xi}_{lw}$, with $\check{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{1+3\beta_1}$, with $-1 < -\frac{\beta_1(1-\beta_1)}{1+3\beta_1}$. Then $\check{\xi}_{lw} > 0$ for $\varepsilon_{1,11} < -1$, with $\rho < \check{\xi}_{lw} < \hat{\xi}_{lw}$. As a result, local indeterminacy occurs if

1. either $\varepsilon_{1,11} < -1$ and $\rho < \xi_{lw} < \ddot{\xi}_{lw}$ and either $1 < \xi_{cl} < \xi_{cl}^{AB}$ and $\sigma < \sigma_D$; or $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^{ABD}$ and $\sigma_{AB} < \sigma < \sigma_D$;
2. or $\varepsilon_{1,11} < -1$ and $\ddot{\xi}_{lw} < \xi_{lw} < \hat{\xi}_{lw}$ and $1 < \xi_{cl} < \xi_{cl}^D$ and $\sigma < \sigma_D$.

The intersection of (S3) and (R4): Since $\rho < \hat{\xi}_{lw} < \tilde{\xi}_{lw}$ and $\xi_{cl}^D < \xi_{cl}^*$ whenever $\xi_{lw} > \rho$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} < -1$ and $\hat{\xi}_{lw} < \xi_{lw} < \tilde{\xi}_{lw}$ and $1 < \xi_{cl} < \xi_{cl}^D$ and $\sigma < \sigma_D$.

The intersection of (S4) and (R2): We have $\xi_{lw}^{AB} < \rho$ and $-\rho/\varepsilon_{1,11} < \xi_{lw}^{AB}$ iff $\varepsilon_{1,11} < -\frac{1}{2}(3 + \beta_1)$. Since $\hat{\xi}_{cl} > -\varepsilon_{1,11} > 1$ and $\xi_{cl} < \hat{\xi}_{cl}$ which implies that $\sigma_{AB} > \sigma^*$. In addition, $\sigma_{AB} < \sigma_D$ iff $\xi_{cl} < \xi_{cl}^{ABD}$ with $\xi_{cl}^{ABD} > 1$. Further, since $\xi_{lw}^F > \xi_{lw}^{AB} > -\rho/\varepsilon_{1,11}$, then $\xi_{cl}^{ABD} > \xi_{cl}^*$ for $-\rho/\varepsilon_{1,11} < \xi_{lw} < \xi_{lw}^{AB}$. Thus $\xi_{cl}^* < \xi_{cl}^{ABD} < -\varepsilon_{1,11}$. As a result, local indeterminacy occurs for $\sigma_{AB} < \sigma < \sigma_D$ if

1. For $\varepsilon_{1,11} < -\frac{1}{2}(3 + \beta_1)$ and either $\xi_{lw} < -\rho/\varepsilon_{1,11}$ and $\xi_{cl}^* < \xi_{cl} < 1$; or $-\rho/\varepsilon_{1,11} < \xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^* < \xi_{cl} < -\varepsilon_{1,11}$;
2. For $-\frac{1}{2}(3 + \beta_1) < \varepsilon_{1,11} < -1$ and $\xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^* < \xi_{cl} < 1$.

The intersection of (S4) and (R3): First, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} > \xi_{cl}^*$ for $\xi_{lw} < \hat{\xi}_{lw}$. Further, since $\xi_{lw}^{AB} < \rho$ and $\hat{\xi}_{lw} > \rho > -\rho/\varepsilon_{1,11}$. In addition, $\xi_{lw}^{AB} > -\rho/\varepsilon_{1,11}$ iff $\varepsilon_{1,11} < -\frac{1}{2}(3 + \beta_1)$. Since $\hat{\xi}_{cl} > -\varepsilon_{1,11} > 1$ and $\xi_{cl} < \hat{\xi}_{cl}$, then $\sigma_{AB} > \sigma^*$. In addition, $\sigma_{AB} < \sigma_D$ iff $\xi_{cl} < \xi_{cl}^{ABD}$ with $\xi_{cl}^{ABD} < -\varepsilon_{1,11}$ and $\xi_{cl}^{ABD} > 1 > \xi_{cl}^*$. As a result, local indeterminacy occurs for $\sigma_{AB} < \sigma < \sigma_D$ if

1. For $\varepsilon_{1,11} < -\frac{1}{2}(3 + \beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \rho$ and $\xi_{cl}^* < \xi_{cl} < \xi_{cl}^{ABD}$;
2. For $-\frac{1}{2}(3 + \beta_1) < \varepsilon_{1,11} < -1$ and either $\xi_{lw}^{AB} < \xi_{lw} < -\rho/\varepsilon_{1,11}$ and $\xi_{cl}^* < \xi_{cl} < 1$; or $-\rho/\varepsilon_{1,11} < \xi_{lw} < \rho$ and $\xi_{cl}^* < \xi_{cl} < \xi_{cl}^{ABD}$.

The intersection of (S5) and (R2): We have $\hat{\xi}_{lw} < \rho$ and $\xi_{lw}^{AB} < \rho$ and $\xi_{cl}^* < \hat{\xi}_{cl} < -\varepsilon_{1,11}$. Further, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ and $\sigma_{AB} < \sigma_D$ iff $\xi_{cl} < \xi_{cl}^{ABD}$. However, since $\xi_{cl}^{ABD} > -\varepsilon_{1,11}$, then $\sigma_{AB} < \sigma_D$ is satisfied for all $\xi_{cl}^* < \xi_{cl} < \hat{\xi}_{cl}$. As a result, local indeterminacy occurs if $-1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $0 < \xi_{lw} < \xi_{lw}^{AB}$ and either $\xi_{cl}^* < \xi_{cl} < \hat{\xi}_{cl}$ and $\sigma_{AB} < \sigma < \sigma_D$; or $\hat{\xi}_{cl} < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^* < \sigma < \sigma_D$.

The intersection of (S5) and (R3): First, we have $-1 < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw}^{AB} < \rho$ and $\hat{\xi}_{lw} < \rho$. Further, note that $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\xi_{cl}^* < \hat{\xi}_{cl} < -\varepsilon_{1,11}$ for $\xi_{lw} < \hat{\xi}_{lw}$. This gives rise to two cases:

1. $\max\{\sigma_{AB}, \sigma^*\} = \sigma_{AB}$ for all $\xi_{cl}^* < \xi_{cl} < \hat{\xi}_{cl}$. Further, $\sigma_{AB} < \sigma_D$ iff $\xi_{cl} < \xi_{cl}^{ABD}$ with $\xi_{cl}^{ABD} > -\varepsilon_{1,11}$. Therefore, local indeterminacy occurs if $-1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$ and $\xi_{cl}^* < \xi_{cl} < \hat{\xi}_{cl}$ and $\sigma_{AB} < \sigma < \sigma_D$.
2. $\max\{\sigma_{AB}, \sigma^*\} = \sigma^*$ for all $\hat{\xi}_{cl} < \xi_{cl} < -\varepsilon_{1,11}$. Then local indeterminacy occurs if $-1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$ and $\hat{\xi}_{cl} < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^* < \sigma < \sigma_D$.

The intersection of (S5) and (R4): First, we have $\hat{\xi}_{lw} < \rho$ and $\xi_{cl}^{AB} > \xi_{cl}^*$ for all $\xi_{lw} > \hat{\xi}_{lw}$. In addition, we have $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ iff $\xi_{lw} > \hat{\xi}_{lw}$ with $\dot{\xi}_{lw} > 0$ for all $-1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\hat{\xi}_{lw} < \dot{\xi}_{lw} < \rho$. Further, we have $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$. Since $\hat{\xi}_{cl} < \xi_{cl}^*$ for $\xi_{lw} > \hat{\xi}_{lw}$, then with $\sigma_{AB} < \sigma^*$ for all $\xi_{cl} > \xi_{cl}^*$ and thus $\max\{\sigma_{AB}, \sigma^*\} = \sigma^*$. As a result, local indeterminacy occurs if $-1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\hat{\xi}_{lw} < \xi_{lw} < \rho$ and $\xi_{cl}^* < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^* < \sigma < \sigma_D$.

The intersection of (S5) and (R5): Further, we have $\hat{\xi}_{lw} < \rho$ and $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$. Since $\hat{\xi}_{cl} > \xi_{cl}^*$ for $\xi_{lw} < \hat{\xi}_{lw}$ and $\hat{\xi}_{cl} < -\varepsilon_{1,11}$. In addition, we have $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$ with $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$. Thus $\sigma_{AB} < \sigma_D$ for all $\hat{\xi}_{cl} < \xi_{cl} < -\varepsilon_{1,11}$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} > -\frac{1}{2}(1 - \beta_1)$ and $0 < \xi_{lw} < \hat{\xi}_{lw}$ and either $\xi_{cl}^* < \xi_{cl} < \hat{\xi}_{cl}$ and $\sigma_{AB} < \sigma < \sigma_D$; or $\hat{\xi}_{cl} < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^* < \sigma < \sigma_D$.

The intersection of (S5) and (R6): We first have $\hat{\xi}_{lw} < \rho$ and $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ iff $\xi_{lw} > \hat{\xi}_{lw}$ with $\dot{\xi}_{lw} > 0$ if $\varepsilon_{1,11} < -\frac{1-\beta_1}{3+\beta_1}$ and $\xi_{lw} < 0$ if $\varepsilon_{1,11} > -\frac{1-\beta_1}{3+\beta_1}$. As a result, local indeterminacy occurs for all $\sigma^* < \sigma < \sigma_D$ if $\varepsilon_{1,11} > -\frac{1}{2}(1 - \beta_1)$ and $\hat{\xi}_{lw} < \xi_{lw} < \rho$ and $\xi_{cl}^* < \xi_{cl} < -\varepsilon_{1,11}$.

7.6 Proof of Proposition 4

Consider the critical value $\sigma = \sigma_{AB}$ which solves $T(\sigma_{AB}) = -1 - D(\sigma_{AB})$. Obviously, $\sigma_{AB} > 0$ and well-defined under the conditions in Proposition 3, except (1.1). Consider the case (1.2) in Proposition 3, one can easily verify that $\xi_{lw}\varepsilon_{1,11} + \rho\xi_{lc} < 0$ and $\xi_{lc} - 1 > 0$ and $(1 + \beta_1)[2\xi_{lw}\varepsilon_{1,11} + \rho\xi_{lc}] - 2\rho[\varepsilon_{1,11} + \frac{1}{2}(1 - \beta_1)] > 0$. This implies that $T(0) < 1 + D(0)$ and $T(0) < -1 - D(0)$ and so the locus (T, D) for σ is very closed to zero lies in the saddle-path stability region.

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