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Abstract

Even if its roots are much older, random sets theory has been considered as an academic area, part of stochastic geometry, since Matheron [7]. Random sets theory was first applied in some fields related to engineering sciences like geology, image analysis and expert systems (see Goutsias et al. [4]), and recently in non-parametric statistics (Koshevoy et al. [5]) or also (see Molchanov [9]) in economic theory (for instance in finance and game theory) and in econometrics (for instance in linear models with interval-valued dependent or independent variables). We apply in this paper random sets theory to decision making. Our main result states that under a kind of vNM condition decision making for an arbitrary random set lottery reduces to decision making for a single-valued random set lottery, and the latter set is the set-valued expectation of the former random set. Through experiments in a laboratory, we observe consistency of decision making for ordering random sets with fixed act and varied random sets.

Keywords:
Knightian decision maker, Choquet-type decision maker, Random sets, Set-valued expectation.
JEL: D81, D01, C91, C10.

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1. Introduction

Suppose that the set over which an agent has to make a choice can expand or restrict in a random manner. How does this agent behave? The idea that a set can be random is caught by the mathematical notion of *random set*. It was mainly used in integral geometry where random set is considered as a pointed process. But since the eighties it has been used in statistics (see Vitale [11], Koshevoy at al. [5], Molchanov [7]). For instance, in signal treatment, if you take a grid of pixels, some of these pixels may be randomly colored in black and white, and so the resulting picture is a random set. In inference statistics, a random set is a confidence region for an estimated parameter. Recently, random sets have been used in econometrics and finance (see Molchanov [9]).

We study in this paper the risk evaluation of random set lotteries. It turns out that decision theory considers a particular random set lotteries of the form of constant maps or, equivalently, single-valued random sets lotteries. Namely, Savage addressed his theory to such random set lotteries. Specifically, the space-state is a single-valued random set within the Savage approach.

Ellsberg, having criticized the Savage axioms, considered a lottery with a non single-valued random set. Recall that in his experiment, there is an urn which contains red, blue and yellow balls, such that the red balls constitute one third and there is no prior information about the proportion of the blue balls.

Let us explain how to model such an urn as a random set. For that we consider a two elements probability space $\Omega = \{\omega_1, \omega_2\}$ with the Boolean algebra 2^Ω and a distribution $P(\omega_1) = \frac{1}{3}$, $P(\omega_2) = \frac{2}{3}$. Consider a 3-dimensional vector space $\mathbb{R}^3$ and denote its basis vectors by $r = (1, 0, 0)$, $b = (0, 1, 0)$ and $c = (0, 0, 1)$. Then a random set R sends ω_1 to the point r and ω_2 to the segment $[b, c] := \{\alpha b + (1 - \alpha)c \ 0 \leq \alpha \leq 1\}$.

Such a random set is not single-valued random set since, either with probability $\frac{1}{3}$ it might be a vector r, or with probability $\frac{2}{3}$ it might be the segment $[b, c]$.

In the Ellsberg paradox, a Decision Maker (DM) has to compare four "lotteries": A - to get \$100 if he will pick a red ball; B - to get \$100 if he picks a blue ball; C - to get \$100 if he picks red or yellow ball, and D - to get \$100 if he picks blue or yellow ball.

Experiments report that people prefer A to B rather than B to A, and D to C rather than vise versa.

In terms of the random set R, DM has to compare the following linear function on $\mathbb{R}^3$, which is specified by the values at vectors r, b and c by:

$$\begin{array}{lll} u_1(r) = \$100, & u_1(y) = \$0, & u_1(b) = \$0; \\ u_2(r) = \$0, & u_2(y) = \$100, & u_2(b) = \$0; \\ u_3(r) = \$100, & u_3(y) = \$0, & u_3(b) = \$100; \\ u_4(r) = \$0, & u_4(y) = \$100, & u_4(b) = \$100. \end{array} \tag{1}$$

The main purpose of this paper is to present a theory of decision making for non-constant random sets.

2. More explanations about random sets lotteries

Let us explain in more detail what is a random set and a random set lottery in a general case. Let us consider a customary set up for decision theory. There are a state space S endowed with σ-field of measurable set $\mathcal{S}$, the space X of measurable functions on S with respect to $\mathcal{S}$, the dual space X^* identified with the set of signed measures on $\mathcal{S}$, and the space $\mathcal{F}(X^*)$ of closed subsets of X^* endowed with the fit-to-hit topology. For a finite set S, we get the following data: $X \cong \mathbb{R}^S \cong X^*$, and $\mathcal{F}(X^*)$ is the space of closed subsets of $\mathbb{R}^S$ endowed with the fit-to-hit topology (see Matheron [7]).

We accept the point of view of mathematicians, and a *random set* is a probability space $(\Omega, \mathcal{A}, P)$ and a measurable mapping $R : \Omega \to \mathcal{F}(X^*)$ (with respect to $\mathcal{A}$). A *random set lottery* is a pair (R, x) constituted from a random set R and an act $x \in X$. We will define more precisely this concept latter.

We study the preferences of DM on the set of random lotteries.

Knightian's decision theory: Consider the most closed situation studied in the literature, the Knightian decision theory. For that on has to consider a constant random set Id_M sending every $\omega \in \Omega$ to the same convex closed set $M \subset X^*$, $Id_M(\omega) = M$, $\omega \in \Omega$. For such a convex subset $M \subset X^*$, the lotteries are specified by acts $x \in X$. A Knightian's DM orders acts by the rule $x \preceq_M y$ if, for any $\mu \in M$, there holds $\int_S x(s)\mu(ds) \leq \int_S y(s)\mu(ds)$. The ordering $\preceq_M$ is not a total ordering.

Given $M \subset X^*$, we consider different completions of $\preceq_M$, alike Savage type, naive Choquet type, or a Choquet type.

Savage decision theory: DM has to consider the subset $M \subset X^*$ as the DM state space, and an act is a functional $x \in X$[1], and thus the mapping $x : M \to \mathbb{R}$ is well-defined. Savage's DM has to choose a measure Π on M and the ordering $x \preceq_{M,\Pi} y$ is defined by $\int_M(\int_S x(s)\mu(ds))\Pi(d\mu) \leq \int_M(\int_S y(s)\mu(ds))\Pi(d\mu)$. The ordering $\preceq_{M,\Pi}$ is total, additive, and it extends $\prec_M$, that is:

$$x \preceq_M y \quad \Rightarrow \quad x \prec_{M,\Pi} y \tag{2}$$

Naive Choquet decision theory: the state of spaces is a subset $M \subset X^*$, an act is a functional $x \in X$, and thus the mapping $x : M \to \mathbb{R}$ is well-defined. DM has to choose a subset of measure $N \subset M$ and the ordering $x \preceq_{M,N} y$ is defined by $\min_{\mu \in N}(\int_S x(s)\mu(ds)) \leq \min_{\mu \in N}(\int_S y(s)\mu(ds))$. The ordering $\preceq_{M,N}$ is total, comonotone additive, and it extends $\preceq_M$, that is:

$$x \prec_M y \quad \Rightarrow \quad x \prec_{M,N} y \tag{3}$$

Choquet-type DM: The state of space is M. DM chooses a convex closed set C from the set $\mathcal{PM}(M)$ of closed subsets of the set of measures on M. Then the ordering $x \preceq_{M,C} y$ on X is defined by:

$$\min_{\Pi \in C} \int_M (\int_S x(s)\mu(ds))\Pi(d\mu) \leq \min_{\Pi' \in C} \int_M (\int_S y(s)\mu(ds))\Pi'(d\mu) \tag{4}$$

The ordering $\preceq_{M,C}$ is total, comonotone additive, and it extends $\preceq_M$.

For a general random set lottery (R, x) we can expect that a DM makes a decision for each pair $(R(\omega), x)$ according to his type, and then "aggregates" it over Ω due to probability distribution P on $\mathcal{A}$. Such an aggregation can

[1]The space X is a subset the total space of acts on M, that is the set of functions from M to $\mathbb{R}$. Thus, we consider the restriction of an ordering of DMs of Savage's or Choquet's-types to $X^* \times X$. An important feature of any act of X is that, for any subset $M \subset X^*$, it belongs to the set of acts for M.

be defined directly for a specific single set random lottery corresponding to the set-valued expectation of the random set R,

$$E(R) := \int_{\Omega} R(\omega)P(d\omega). \tag{5}$$

The random set expectation $E(R)$ is a closed convex subset of X^*.

For example, for the random set from the Ellsberg paradox, the expectation set is the segment $[\frac{1}{3}r + \frac{2}{3}b, \frac{1}{3}r + \frac{2}{3}g]$.

The Knightian ordering $x \succ_R y$ holds if, for any $\mu \in E(R)$, we have

$$\int_S x(s)\mu(ds) \geq \int_S y(s)\mu(ds) \tag{6}$$

Now we explain our main result on an example of the Savage-type DM.

The Savage-type evaluation of a random set lottery R can be considered as two possible completions of the Knightian DM, for each $R(\omega)$, and for the expectation $E(R)$, respectively.

For the latter case, a DM chooses a distribution Π on the expectation set $E(R)$, and this leads to the ordering $x \succ_{R,\Pi} y$ if there holds

$$\int_{E(R)} (\int_S x(s)\mu(ds))\Pi(d\mu) \geq \int_{E(R)} (\int_S y(s)\mu(ds))\Pi(d\mu) \tag{7}$$

In the former case, for each $\omega \in \Omega$, a DM make choice of a distribution Π_ω in $R(\omega)$. Then the order is defined from

$$\begin{array}{ll} \int_\Omega(\int_{\mu \in R(\omega)} \int_S F(s)\mu(ds))\Pi_\omega(d\mu))P(d\omega) & \geq \\ \int_\Omega(\int_{\mu \in R(\omega)} \int_S G(s)\mu(ds))\Pi_\omega(d\mu))P(d\omega) & \end{array} \tag{8}$$

Our main result (Theorem 1) states, that for measurable (wrt $\mathcal{A}$) family Π_ω, $\omega \in \Omega$, and if, for every fixed act x, DM's ordering satisfies a kind of vNM setting with respect to the space of random sets, there exists an expectation measure $E(\Pi)$, such that there holds

$$\int_\Omega (\int_{\mu \in R(\omega)} \int_S x(s)\mu(ds))\Pi_\omega(d\mu))P(d\omega) = \int_{E(R)} (\int_S x(s)\mu(ds))E(\Pi)(d\mu) \tag{9}$$

Analogous commutations take place for naive Choquet and Choquet type DM's.

This result means that under vNM setting with respect to random sets and a given act, decision making for a random set can be performed for the set-valued expectation of the random set.

Here is an example of a non single-valued random set lottery with expectation set being parallelogram. Namely, there is an urn which contains red, green, yellow and blue balls. There are $\frac{1}{3}$ either red or green balls and $\frac{2}{3}$ either yellow or blue. The prizes are: red - \$100, yellow - \$60, blue - \$40, and green - \$20.

As a random set lottery it is set as follows. Consider a 4 elements set $S = \{r, g, y, b\}$. Then $X \cong \mathbb{R}^4$ and $X^* \cong \mathbb{R}^4$.

Consider the following measures in X^*, $r = (1, 0, 0, 0)$, $g = (0, 1, 0, 0)$, $y = (0, 0, 1, 0)$, $b = (0, 0, 0, 1)$, $A = (\frac{1}{3}, 0, \frac{2}{3}, 0)$, and $B = (0, \frac{1}{3}, 0, \frac{1}{3})$.

A function $x(r) = 100$, $x(g) = 20$, $x(y) = 60$, $x(b) = 40$.

Consider the following random set. Set $\Omega = \{\omega_1, \omega_2\}$ and $P(\omega_1) = \frac{1}{3}$, $P(\omega_2) = \frac{2}{3}$. Then define the random set $R(\omega_1) = [r, g]$, $R(\omega_2) = [y, b]$ being random segments ($[r, g] := \{\alpha r + (1 - \alpha) g \, 0 \leq \alpha \leq 1\}$).

The expectation of the random set R, is the parallelogram $E(R_1) = \frac{1}{3}[r, g] + \frac{2}{3}[y, b]$ with following vertices $A = \frac{1}{3}r + \frac{2}{3}y$, $C = \frac{1}{3}r + \frac{2}{3}b$, $D = \frac{1}{3}g + \frac{2}{3}y$, and $B = \frac{1}{3}g + \frac{2}{3}b$.

To state our main result we need some technical preparations.

3. Random sets

Let S be a state space and let X be the set of measurable functions on S wrt σ-field of measurable set $\mathcal{S}$. Then the dual space X^* is identified with the set of signed measures on $\mathcal{S}$. We let $\mathcal{F}(X^*)$ denote the set of closed convex subsets of X^*, $\mathcal{K}(X^*)$ denote the set of compact subsets of X^*, and $\mathcal{G}(X^*)$ denote the set of open subsets of X^*.

A mapping $R : \Omega \to \mathcal{F}(X^*)$ is *measurable* if, for any compact set $K \in \mathcal{K}(X^*)$ and any finite collection of open sets $G_i \in \mathcal{G}(X^*)$, $i = 1, \ldots, k$, the set $\{\omega : R(\omega) \cap K = \emptyset, R(\omega) \cap G_1 \neq \emptyset, \ldots, R(\omega) \cap G_k \neq \emptyset\}$ belongs to $\mathcal{A}$. Moreover, the values $P\{\omega : R(\omega) \cap K = \emptyset, R(\omega) \cap G_1 \neq \emptyset, \ldots, R(\omega) \cap G_k \neq \emptyset\}$ form a probability distribution on the σ-algebra $\sigma_{\mathcal{F}}$ of subsets of $\mathcal{F}(X^*)$ spanned by the sets of the form $\{F \in \mathcal{F}(X^*) : F \cap K = \emptyset, F \cap G_1 \neq \emptyset, \ldots, F \cap G_k \neq \emptyset\}$, $K \in \mathcal{K}(X^*)$ and $G_i \in \mathcal{G}(X^*)$, $i = 1, \ldots, k$, $k = 1, \ldots$.

However, capacity functionals are usually of use in random set theory (see Appendix).

Definition 1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, then a measurable map $R : \Omega \to \mathcal{F}(X^*)$ is said to be a* random set.

For a finite set S, the space of measurable functions X is the Euclidean space $\mathbb{R}^S$, the dual space of signed measure, X^* is the set of linear functionals on $\mathbb{R}^S$ and is isomorphic to $\mathbb{R}^S$ (non-canonically). The set $\mathcal{F}(\mathbb{R}^S)$ is the set of closed subsets of $\mathbb{R}^S$, and the corresponding σ-algebra is the fit-to-hit topology, that is in the above defined σ-algebra $\sigma_{\mathcal{F}}$ we have to consider usual closed, open and compact sets in $\mathbb{R}^S$ (see Matheron [7] or Molchanov [8]).

For a random closed set R, we can consider its set-valued expectation $E(R) \subset X^*$. The expectation $E(R)$ is a convex set. The set-valued expectation can be defined in two ways and we will demonstrate them.

For compact convex random sets we can, following to the Aumann integration ([1]), define the set of selections

$$\mathcal{R} := \{r(\omega) \in R(\omega) \,|\, r \text{ is measurable wrt } \mathcal{A}\}.$$

For a finite S, the set $\mathcal{R}$ is the collection of random vectors of $\mathbb{R}^S$, which, for each ω, belongs to $R(\omega)$. Then, for a convex and compact-valued random set R, the Aumann expectation is the set of expectations of such random vectors,

$$E(R) := \{\int_{\Omega} r(\omega) P(d\omega) \,|r \in \mathcal{R}\} \tag{10}$$

There is another implicit definition of the expectation via support functions. Namely, the expectation of a random set R is a convex closed set $E(R)$ given implicitly via a support function taking the following form

$$\phi_{E(R)}(q) = \int_{\Omega} \phi_{R(\omega)}(q) P(d\omega), \quad q \in X, \tag{11}$$

where $\phi_A(q) = \sup_{a \in A} q(a)$ denotes the support function to a set $A \subset X^*$ (For a finite set S, and a subset $A \subset \mathbb{R}^S$ and a vector $q \in \mathbb{R}^S$, the value $q(a)$ is the usual scalar product.)

Note, that due to Lyapunov's theorem, for compact convex random sets, this definition of the set-valued expectation coincides with the Aumann integration.

Here are some examples of random sets and expectations.

1. The simplest example of a random set is a random vector in X^*, its usual expectation is a vector and it coincides with the set-valued expectation.
2. The constant random set Id_M which sends Ω to some fixed convex closed set $M \subset X^*$ has the set valued expectation $E(Id_M) = M$.
3. Let D be a random vector in $\mathbb{R}^S$, then the random interval $[0, D]$ is a random set. Expectations of such a random segment $E([0, D])$ is called *zonoid* and such sets are of importance in Statistics and Finance (see Koshevoy *et al.* [5]).

4. DM and Random sets lotteries

Definition 2. *A* random set lottery *is a pair* (R, x) *where* R *is a convex closed set,* $R : \Omega \to \mathcal{F}(X^*)$*, and* $x \in X$ *is a act.*[2]

We assume that DM has a weak order preference. We assume also that the unit ball is compact in X^* (that is the case for a finite S), we are then able to endow the space of random sets with the weak-topology wrt the Hausdorff distance.

Consider a constant random set, $Id_M(\omega) = M$ for some convex closed $M \subset X^*$.

In such a case, a DM faces a problem to compare elements of X given a set $M \subset X^*$.

Then there are the following recipes:

- A Knightian DM constructs the ordering on X defined by $x \preceq_M y$ if, for any $\mu \in M$, there holds $\int_S x(s)\mu(ds) \leq \int_S y(s)\mu(ds)$. The ordering $\preceq_M$ is not a total ordering.

- A Savage-type DM has to consider M as the new state of space, and choose a measure Π on M. Then the ordering $x \preceq_{M,\Pi} y$ is defined by $\int_M(\int_S x(s)\mu(ds))\Pi(d\mu) \leq \int_M(\int_S y(s)\mu(ds))\Pi(d\mu)$. The ordering $\preceq_{M,\Pi}$ is total, additive, and it extends $\preceq_M$.

[2] It could occur that $X \subset (X^*)^*$, but we are interested in acts from X.

- A Naive Choquet-type DM has to consider M as the new state space and choose a subset of measure $N \subset M$. Then the ordering $x \preceq_{M,N} y$ is defined by $\min_{\mu \in N}(\int_S x(s)\mu(ds)) \leq \min_{\mu \in N}(\int_S y(s)\mu(ds))$. The ordering $\preceq_{M,N}$ is total, comonotone[3] additive, and it extends $\preceq_M$.

- A Choquet DM chooses a convex closed set C from the set $\mathcal{PM}(M)$ of closed subsets of the set of measures on M. Then the ordering $x \preceq_{M,C} y$ on X is defined by $\min_{\Pi \in C} \int_M (\int_S x(s)\mu(ds))\Pi(d\mu) \leq$ $\min_{\Pi' \in C} \int_M (\int_S y(s)\mu(ds))\Pi'(d\mu)$. The ordering $\preceq_{M,C}$ is total, comonotone additive, and it extends $\preceq_M$.

- A strong Knightian DM constructs the ordering on X defined by $x \prec^s_M y$ if there holds $\max_{\mu \in M} \int_S x(s)\mu(ds) \leq \min_{\mu \in M} \int_S y(s)\mu(ds)$. The ordering $\prec^s_M$ is not a total ordering, and is stronger than $\preceq_M$.

For a general random set R, we can either reduce the problem to the constant random set with the expectation set $E(R)$ or perform an aggregation of DMs on each $R(\omega)$, $\omega \in \Omega$.

Since Knightian decision theory, for a given single-valued random set Id_M, offers a non total ordering of acts of X, the aggregation can be performed for the unanimity case. Namely, we have:

Proposition 1. *Let $R : \Omega \to \mathcal{F}(X^*)$ be a random convex set, and let x and y be acts such that, for every $\omega \in \Omega$, there holds $x \preceq_{R(\omega)} y$. Then $x \preceq_{E(R)} y$.*

Proof. Assume firstly, that R is a random convex compact set. In such a case, due to Aumann integration, we have that every element of $E(R)$ takes the form $E(r)$ for some selections $r \in \mathcal{R}$. Since x and y are linear functionals on X^* and, for every $\omega \in \Omega$, there holds $x \preceq_{R(\omega)} y$, we obtain that there holds $x(E(r)) \leq y(E(r))$, and the proposition follows.

For a closed random set R, we can use the second definition of the expectation. There are two cases: either for all q there holds $Arg \max_{m \in R(\omega)} q(m)$ belongs to $R(\omega)$, and then the same line of arguments as above yields the proposition, or, for some q, the latter set is empty. In the latter case, for such a q, choose a sequence $m_i(\omega) \in R(\omega)$ so that there holds $\lim q(m_i(\omega)) = max_{m \in R(\omega)} q(m)$. For each element of the sequence, there holds $x(m_i(\omega)) \leq$

[3] Functions f and g are said to be *comonotone*, if $\mathrm{Argmax}_A f = \mathrm{Argmax}_A g$ and $\mathrm{Argmin}_A f = \mathrm{Argmin}_A g$, for any closed set A of the support of X^*.

$y(m_i(\omega))$, and therefore $x(\int_\Omega m_i(\omega)P(d\omega)) \leq \int_\Omega y(m_i(\omega)P(d\omega))$. That implies the proposition. ■

For the total orderings like Savage-type or Choquet-type, an aggregation is possible under an analogue of vNM setting for a fixed x.

Suppose that a DM has a total ordering on the set of random set lotteries (R, x), and suppose that, given x, the preference satisfies vNM setting, that is under standard continuous assumptions (there is a Hausdorff metric on random sets wrt capacity functionals, see Appendix), there holds

$$\begin{array}{ll} (R_1, x) \preceq (R_2, x), (Q_1, x) \preceq (Q_2, x) & \Rightarrow \\ (\alpha R_1 + (1-\alpha)Q_1, x) \preceq (\alpha R_2 + (1-\alpha)Q_2, x) & \end{array} \tag{12}$$

for any $0 \leq \alpha \leq 1$, and $\alpha R_1 + (1-\alpha)Q_1$ is the random set being the convex combination of two mappings. (Similarly like in probability theory, we can consider random sets on universal state space Ω and algebra $\mathcal{A}$.)

For the cases of the Savage-type, the Choquet-type and naive Choquet-type DMs, for each ω, choose a measure Π_ω on the set of measures on $R(\omega)$, a subset Π_ω from the set of measures on $R(\omega)$, and a subset $N(\omega) \subset R(\omega)$, respectively, such that the corresponding functions are measurable wrt $\mathcal{A}$.

Then under the vNM setting, *the Savage-type DM* evaluates the pair (R, x) as

$$\int_\Omega (\int_{\mu \in R(\omega)} \int_S x(s)\mu(ds))\Pi_\omega(d\mu))P(d\omega); \tag{13}$$

the Choquet-type DM as

$$\int_\Omega (\min_{\Pi_\omega \in C} \int_{R(\omega)} (\int_S x(s)\mu(ds))\Pi_\omega(d\mu)))P(d\omega); \tag{14}$$

the naive Choquet-type DM evaluates this pair as

$$\int_\Omega (\min_{\mu \in N(\omega)} \int_S x(s)\mu(ds))P(d\omega). \tag{15}$$

To evaluate pairs (R, x) and (R', y) the choice of a measure on a subset in identical sets has to be the same, that is, for ω and $\omega' \in \Omega$, there holds $R(\omega) = R'(\omega')$.

Definition 3. *A collection of measures Π_ω, $\omega \in \Omega$ is* measurable wrt $\mathcal{A}$, *if, for any Borel set $A \subset X$, the function $\Pi_\omega(A)$ is $\mathcal{A}$-measurable.*

Consider the case of a finite state space S.

Theorem 1. *Let S be finite, the mapping Π_ω sending Ω to measures on X is measurable, and a selection $N(\omega) \subset R(\omega)$ is a random set. Then we have the following commutations:*

- *Savage type DM: there exists a measure Π_R on the expectation $E(R)$ such that there holds*

$$\begin{aligned} \textstyle\int_\Omega(\int_{\mu\in R(\omega)}\int_S x(s)\mu(ds))\Pi_\omega(d\mu))P(d\omega) &= \\ \textstyle\int_{\mu\in E(R)}\int_S x(s)\mu(ds))\Pi_R(d\mu) \end{aligned} \tag{16}$$

- *Choquet type DM: there exists a set of measure Π_E of the set of measures $\mathcal{C}(E(R))$ on the expectation set $E(R)$ such that there holds*

$$\begin{aligned} \textstyle\int_\Omega(\min_{\Pi_\omega\in C}\int_{R(\omega)}(\int_S x(s)\mu(ds))\Pi_\omega(d\mu)))P(d\omega) &= \\ \textstyle\min_{\Pi_E\in C(E(R))}\int_{E(R)}(\int_S x(s)\mu(ds))\Pi_E(d\mu) \end{aligned} \tag{17}$$

- *naive Choquet type DM: there exists a subset $N(E)$ of the expectation $E(R)$ such that there holds*

$$\int_\Omega(\min_{\mu\in N(\omega)}\int_S x(s)\mu(ds))P(d\omega) = \int_\Omega(\min_{\mu\in N(E)}\int_S x(s)\mu(ds)) \tag{18}$$

Proof. The simplest case is the naive Choquet type DM. in such a case, $N(E)$ is the expectation of the random set $N(\omega)$, $\omega \in \Omega$. For other cases, we consider the following construction of measures on the sum of sets. Let A and B be two subsets of X^* which are domains of measures μ and ν on the corresponding algebras in A and B. Then consider the product measure $\mu \times \nu$ on the product $A \times B$ endowed with the product algebra. Consider the sum mapping $\pi : A \times B \to A + B$. The direct image $\pi_*(\mu \times \nu)$ defines a measure on $A + B$. For any $0 \le \alpha \le 1$, the same construction defines a measure on $\alpha A + (1-\alpha)B$ for the mapping $\pi_\alpha : A \times B \to \alpha A + (1-\alpha)B$, $\pi_\alpha(a,b) = \alpha a + (1-\alpha)b$. Because of this construction, given a measurable mapping $\omega \to \Pi_\omega$, we obtain a measure $\Pi(P)$ supported on the expectation set $E(R)$. For this measure $\Pi(P)$ the claimed equality holds.

Analogous construction for set of measures (as in the Choquet case) defines the set of measure Π_E of the set of measures on $E(R)$. For this set Π_E the claimed equality holds. ■

5. Experiment

We propose the following tests at the Experimental Laboratory of the University of Paris 1 (France) in order to understand the distribution of types of DMs which have to compare the random set lotteries (see Appendix for details). In other words, we tried through these experiments to reveal the consistency of DM with ordering random sets with fixed act and varied random sets.

Here are the tests we propose. Let us remark that in all tests the potential gain or loss are expressed in terms of ECU (Experimental Currency Unit) with 10 ECU = EUR 1.

Tests B1 and B2. B1: An urn contains white and black balls in unknown proportion. Withdrawing a black ball DM gains ECU 100, otherwise zero. B2: An urn contains white and black balls also in unknown proportion, but withdrawing a black ball DM gains ECU 1000, otherwise zero.

We ask for money evaluation for EUR 1 for the former random set lottery and EUR 10 for the latter one.

For both lotteries, the state space $S = \{s\}$ is a singleton, then $X \cong \mathbb{R}$, the dual space is $X^* \cong \mathbb{R}$, and consider $R = Id_{[0,1]}$ the single segment random set, and $x = 100$ for the former one and $y = 1000$ for the latter one.

The results show that 44 participants from 85 have a behavior different from the Savage- or Choquet-types, since only 35 accepted the money evaluation for both and 6 rejected for both.

This makes us think that 50% of the population is not SEU or CEU decision makers.

Test C. There are two urns. Urn 1 contains white and black balls in unknown proportion. Withdrawing a black ball DM gains ECU 1000, otherwise zero. Urn 2 also contains white and black balls in unknown proportion, but withdrawing a black ball DM gains ECU 90, otherwise ECU 10.

These lotteries can be modeled in two ways. Either the same random set and two different acts or two different random set and the same act.

For modeling with the same random set, consider $S = \{s_1, s_2\}$ is a two-elements set, then $X \cong \mathbb{R}^2$, the dual space is $X^* \cong \mathbb{R}^2$, and consider $R = Id_{[(0,1),(1,0)]}$ the single segment random set, and $x((0,1)) = 1000$, $x((1,0)) = 0$ for the former lottery and $y((0,1)) = 90$, and $y((1,0)) = 10$ for the latter one.

Note, that the acts x and y are comonotone and 58.8% prefer y to x.

One another hand, we can model these lotteries differently. Namely, consider a singleton $S = \{s\}$, then $X \cong \mathbb{R}$, the dual space is $X^* \cong \mathbb{R}$.

For the former one we set $R = Id_{[0,1]}$ the single segment random set, and $u = 1000$, and for the latter one we have another single segment random set $R' = Id_{[\frac{1}{100}, \frac{9}{100}]}$.

From such a point of view, 58.8% prefer R' to R.

Test D. The urn 1 is the same as in Test C, but gains are EUR 95 for the black ball and ECU 5 for the white ball; the urn 2 is identical to that in Test C.

These urns can be presented by the same function $u = 100$ and different random sets $R_1 = Id_{[0.05,0.95]}$ and $R_2 = Id_{[0.1,0.9]}$ for the former and latter lotteries, or with the same single segment random set $R = Id_{[(0,1),(1,0)]}$, and different acts $f((0,1)) = 95$, $x((1,0)) = 5$ for the former lottery and $y((0,1)) = 90$, and $y((1,0)) = 10$ for the latter one.

For that case, 80% prefer the former one to the latter.

Test E. Case 1. An urn contains red, green, yellow and blue balls. There are $\frac{1}{3}$ either red or green balls and $\frac{2}{3}$ either yellow or blue. The gains are: red - ECU 100, yellow - ECU 60, blue - ECU 40, and green - ECU 20.

Case 2. Urn 1 contains $\frac{1}{3}$ black balls to which we assign the gain ECU 100 and $\frac{2}{3}$ white balls we assigned the gain ECU 60. Urn 2 also contains also $\frac{1}{3}$ black balls but to which assign the gain ECU 20 and $\frac{2}{3}$ white balls we assigned the gain ECU 40.

In the second case, urn 1 or urn 2 is chosen randomly.

For consider a 4 elements set $S = \{r, g, y, b\}$. Then $X \cong \mathbb{R}^4$ and $X^* \cong \mathbb{R}^4$.

Consider the following measures in X^*, $r = (1, 0, 0, 0)$, $g = (0, 1, 0, 0)$, $y = (0, 0, 1, 0)$, $b = (0, 0, 0, 1)$, $A = (\frac{1}{3}, 0, \frac{2}{3}, 0)$, and $B = (0, \frac{1}{3}, 0, \frac{1}{3})$.

A function $x(r) = 100$, $x(g) = 20$, $x(y) = 60$, $x(b) = 40$.

Consider the following two random sets. Set $\Omega = \{\omega_1, \omega_2\}$. Then, for $P_1(\omega_1) = \frac{1}{3}$, $P_1(\omega_2) = \frac{2}{3}$, define the random set $R_1(\omega_1) = [r, g]$, $R_1(\omega_2) = [y, b]$. Another random set is $R_2 := Id_{[A,B]}$.

RS-expected utility for the case 1: $\frac{1}{3}[20, 100] + \frac{2}{3}[40, 60] = [\frac{100}{3}, \frac{220}{3}]$.

RS-expected utility for the case 2: $[\frac{1}{3}20 + \frac{2}{3}40, \frac{1}{3}100 + \frac{2}{3}60] = [\frac{100}{3}, \frac{220}{3}]$.

Now, consider the expectation of first random R_1, $E(R_1) = \frac{1}{3}[r, g] + \frac{2}{3}[y, b]$ is the parallelogram with vertices $A = \frac{1}{3}r + \frac{2}{3}y$, $C = \frac{1}{3}r + \frac{2}{3}b$, $D = \frac{1}{3}g + \frac{2}{3}y$, and $B = \frac{1}{3}g + \frac{2}{3}b$.

Thus, $R_2 = Id_{[A,B]}$ is a diagonal of $Id_{E(R_1)}$, that is the second lottery corresponds to the smallest set, and there holds $\min_{R_1} x = \min_{R_2} x$ and $\max_{R_1} x = \max_{R_2} x$.

In the experiment, 15.3% are indifferent between these random sets lotteries, and 50% prefer R_1 to R_2.

Thus, this multidimensional random set shows a difference from the univariate DM behavior wrt inclusion of sets.

6. Conclusion

The question we address is to know whether the concept of random sets could be useful in decision theory. The answer to this question seems to be positive. Indeed we show, see theorem 1, that under vNM setting wrt random sets and a given act, decision making for a random set can be performed for the set-valued expectation of the random set.

Appendix

Appendix A. Capacity functionals

A random set R gives rise to the following *capacity* (or hitting) *functional*

$$T_R(K) = P(\{\omega : R(\omega) \cap K \neq \emptyset\}),$$

defined for K ranging over closed subsets of $\mathbb{R}^n$.

The following properties of this capacity functional are easy to check.

1) T_R is upper semi-continuous on $\mathcal{F}$, that is $T(K_n) \to T(K)$ with $K_n \downarrow K$.

2) The functionals given by

$$\begin{aligned} S_1(K; K_1) &= T_R(K \cup K_1) - T_R(K) \\ &\cdots \\ S_m(K; K_1, \ldots, K_m) &= S_{m-1}(K; K_1, \ldots, K_{m-1}) - S_{m-1}(K \cup K_m; K_1, \ldots, K_{m-1}) \end{aligned}$$

are non-negative for all $m \geq 1$ and K, $K_i \in \mathcal{F}$.

Example 1. *Let $R = (-\infty, X]$ be a random set in $\mathbb{R}^1$, where X is a random variable. Then $T_R(K) = \mathrm{P}(\{X > \inf K\})$ for $K \in \mathcal{F}$.*

Example 2. *Let $R = \{X\}$ be a random singleton in $\mathbb{R}^n$. Then $T_R(K) = \mathrm{P}(X \in K)$ is the probability distribution of X.*

It is not difficult to prove that the capacity functional T_R is additive if and only if R is a random singleton. Moreover, in this case S_m vanishes at $K = K_1 \cap \ldots \cap K_m$, $m \geq 1$ (the inclusion-exclusion formula for additive measures).

Example 3. *Let $R = B_1(X)$ be the unit ball with random center X. Then $T_R(K) = \mathrm{P}(X \in (K + B_1(0)))$, where $A + B = \{a + b \,|\, a \in A, b \in B\}$, A, $B \subset \mathbb{R}^n$. Note that this functional does not satisfy additivity.*

Example 4. *Let w_t, $t \geq 0$, be a Wiener process and let R be its set of zeros $R = \{t \geq 0 \,:\, w_t = 0\}$. Then $T_R([a, b]) = 1 - 2/\pi \arcsin \sqrt{a/b}$.*

Choquet's theorem (see the below result due to Matheron [7]) states that a capacity functional which satisfies conditions 1) and 2) determines uniquely the distribution of a random closed set:

Result 1 (Matheron [7]). *Let $T : \mathcal{F}(\mathbb{R}^n) \to [0,1]$. There exists a unique random closed set R in $\mathbb{R}^n$ with the capacity functional T such that $\mathrm{P}(R \cap K \neq \emptyset) = T(K)$ iff T satisfies conditions 1) and 2).*

The hitting functional is well-known as well as another functional, the *containment functional*, which takes the following form $t_R(K) = P(\{\omega : R(\omega) \subset K\})$, $K \in \mathcal{F}$. This functional is upper semi-continuous and has non-positive functions S^m, these functions are similar to S_m where $\cup$ is replaced by $\cap$. Specifically, the following holds: 2')

$$\begin{aligned} S^1(K;K_1) &= t_R(K \cap K_1) - t_R(K) \\ &\cdots \\ S^m(K;K_1,\ldots,K_m) &= S^{m-1}(K;K_1,\ldots,K_{m-1}) - S^{m-1}(K \cap K_m;K_1,\ldots,K_{m-1}) \end{aligned}$$

are non-positive for all $m \geq 1$ and K, $K_i \in \mathcal{F}$.

The following result is due to Vitale [11]:

Result 2 (Vitale [11]). *Let $t : \mathcal{F} \to [0,1]$. There exists a unique random closed compact set R in $\mathbb{R}^n$ with the capacity functional t such that $\mathrm{P}(R \subset K) = t(K)$ iff t satisfies conditions 1) and 2').*

For any random singleton R, we have $T_R = t_R$, and vice versa. The difference $T_R - t_R$ is called *uncertainty*.

Note (see Matheron [7]) that a random closed set R is convex if and only if its hitting functional T is C-additive, that is if

$$T_R(K \cup K') + T_R(K \cap K') = T_R(K) + T_R(K')$$

for all convex compacts K and K' such that $K \cup K'$ is a convex compact.

In short, to set a random convex compact R is equivalent to setting a measurable map from a probability space to the set of convex compact of $\mathbb{R}^n$, or else to set the hitting functional T_R, or to set the containment functional t_R, or a probability distribution on the σ-algebra $\sigma_{\mathcal{F}}$[4].

[4]The relation between the probability distribution on $\sigma_{\mathcal{F}}$ of a random set R and the

Appendix B. The Experiment

Before we present the results, let us introduce the data set.

Appendix B.1. The data set

The experiments were made at the Experimental Economics Laboratory of the University of Paris I Pantheon-Sorbonne. 85 individuals (mostly students) have participated to the experiment.

The individuals are randomly selected from the Lab data base. The figure of 85 is mainly a consequence of our budget of about EUR 2000. The individuals got an email from the Lab to participate in an experiment.

For 10 minutes, we explain to the individuals through a PowerPoint document the purpose of our study. The individuals know that the choice they will make will determine their final gain in such a way that we expect them to particpate in the tests seriously. In all tests the potential gain or loss are expressed in terms of ECU (Experimental Currency Unit) with 10 ECU = EUR 1. Each participant has an initial endowment of 160 ECU (that is, EUR 16).

We explain to the individuals that the experiment includes five tests B1, B2, C, D, E and that in all but two (tests B1 and B2) the tests require the choice between two situations: situation 1 and situation 2. We explain to the individuals that they can choose either situation 1, or situation 2 or they can be indifferent to both options. B1 and B2 are different from the tests C, D and E in the sense that the individuals have to say whether they agree to bet respectively EUR 1 and EUR 10. There are also two other tests named "Example" and "Test A". The purpose of the Example test is to permit the individuals to get familiar with the computer sessions and the test A is a control test. The results of neither tests are reported here.

It is usual in experimental economics to provide some financial incentives to the individuals in order to avoid a random choice from them (some papers show however that is there is no need to do so). As we said above, in order to

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containment functional t_R for an empirical random set $R_\mu : (2^N, 2^{2^N}, \mu) \to \mathcal{F}(\Delta(N)) \subset \mathcal{F}(\mathbb{R}^N)$ is known as the *Möbious transform*. Here N is a finite set, $\Delta(N)$ is the simplex $\{x \in \mathbb{R}^N \,|\, x_i \geq 0, \sum_{i \in N} x_i = 1\}$, and $R_\mu(A) = \Delta(A) = \{x \in \mathbb{R}^N \,|\, x_i \geq 0, \sum_{i \in A} x_i = 1\}$, $A \subseteq N$.

incite the individuals to answe the questions seriously, we explain them from the beginning that that the computer will randomly choose one test among the five tests other than B1 and B2 and that they will play the game that is included in this test. Their gain or loss in this game will be added to their initial endowment of EUR 16 in order to have their final payment (which is of course never negative). Moreover the individuals always have to play test B1 and B2. Their gain or loss in the two games is also added to their initial endowment.

Appendix B.2. Some informations about the participants

The first table provides some information about the status of the individuals (student or not).

Table 1.

Status	**Frequency**	**Percent**
Student	79	92.94
Non student	6	7.06
Total	85	100

The second table provides some information about the distribution of the data set according to the gender of the individuals.

Table 2.

Sex	**Frequency**	**Percent**
F	40	47.06
M	45	52.94
Total	85	100

The third and fourth tables provide some information about the distribution of the data set according to the age of the individuals. In the fourth table the "young" category includes the subjects under 25.

Table 3. Age

Mean	**SD**	**Min**	**Max**
24.97	6.69	18	52

Table 4.

Age Category	**Frequency**	**Percent**
Young	58	68.24
Old	27	31.76
Total	85	100

Tables 5 and 6 provide some information about the distribution of the data set according to the PSC of the father and of the mother in three categories respectively: executive/employee or blue collar worker/unemployed or retired.

Table 5.

PSC of the father	**Frequency**	**Percent**
Executive	45	52.94
Employee or Blue collar	25	29.41
Unemployed or Retired	15	17.65
Total	85	100

Table 6.

PSC of the mother	**Frequency**	**Percent**
Executive	19	22.35
Employee or Blue collar	41	48.24
Unemployed or Retired	25	29.41
Total	33	100

Finally we provide in table 7 the distribution of the data set according to whether the individual found the experiment interesting or not ("Interest" variable).

Table 7.

Interest	**Frequency**	**Percent**
No	4	4.71
Yes	81	95.29
Total	85	100

Appendix B.3. The results

Table 8 includes the answers of the participants concerning the tests B1, B2, C, D and E.

Table 8.

Obs	**Test-B1**	**Test-B2**	**Test-C**	**Test-D**	**Test-E**
1	Yes	No	1	1	1
2	Yes	No	1	1	2
3	Yes	Yes	1	3	3
4	Yes	Yes	1	3	3
5	Yes	No	1	2	1
6	Yes	No	1	1	1
7	Yes	Yes	1	2	1
8	Yes	No	1	1	1
9	Yes	No	2	3	1
10	Yes	No	2	1	1
11	No	No	2	3	1
12	Yes	No	1	1	2
13	Yes	No	1	1	3
14	Yes	Yes	2	1	1
15	Yes	No	2	1	2
16	No	No	1	3	2
17	No	No	2	1	1
18	Yes	Yes	1	1	2
19	Yes	No	1	1	3
20	Yes	Yes	1	1	1
21	Yes	No	1	1	1
22	Yes	No	1	1	3
23	Yes	No	1	1	2
24	No	No	2	1	1
25	Yes	No	1	1	1
26	Yes	Yes	2	1	3
27	Yes	Yes	1	1	2
28	Yes	Yes	1	1	1
29	Yes	No	1	1	3
30	Yes	No	2	1	1

Obs	**Test-B1**	**Test-B2**	**Test-C**	**Test-D**	**Test-E**
31	Yes	No	1	1	1
32	Yes	No	1	1	2
33	Yes	No	1	2	1
34	Yes	Yes	1	1	1
35	Yes	No	2	1	1
36	Yes	No	1	2	2
37	Yes	Yes	2	1	1
38	Yes	Yes	1	2	1
39	Yes	No	2	1	2
40	Yes	No	1	3	3
41	Yes	Yes	1	1	1
42	Yes	No	1	1	1
43	Yes	No	2	1	1
44	Yes	Yes	1	1	3
45	Yes	No	2	1	2
46	Yes	No	2	1	2
47	Yes	No	1	1	1
48	Yes	No	2	1	2
49	Yes	No	2	1	1
50	Yes	No	2	1	2
51	Yes	No	1	1	2
52	Yes	Yes	2	1	2
53	Yes	No	2	1	1
54	No	Yes	2	1	3
55	Yes	Yes	3	3	2
56	Yes	Yes	1	1	1
57	Yes	Yes	1	1	2
58	Yes	Yes	1	3	1
59	Yes	Yes	1	1	2
60	Yes	No	3	1	3

Obs	**Test-B1**	**Test-B2**	**Test-C**	**Test-D**	**Test-E**
61	Yes	Yes	1	1	2
62	Yes	No	1	2	3
63	No	Yes	2	1	1
64	Yes	No	1	1	2
65	Yes	Yes	2	1	1
66	Yes	Yes	1	1	1
67	Yes	No	1	1	1
68	No	No	2	1	2
69	Yes	No	1	1	1
70	Yes	Yes	1	1	2
71	Yes	Yes	1	1	2
72	Yes	No	1	1	1
73	No	Yes	2	3	1
74	Yes	Yes	1	1	3
75	No	No	2	1	1
76	Yes	Yes	2	1	2
77	No	Yes	1	1	1
78	Yes	Yes	1	1	2
79	Yes	Yes	2	1	2
80	Yes	No	2	1	1
81	Yes	No	2	3	2
82	Yes	No	2	1	2
83	Yes	Yes	2	1	1
84	Yes	No	2	1	1
85	No	Yes	1	2	1

Table 9 provides the frequency of the answers to the tests.

Table 9.

Test-B1	**Frequency**	**Percent**
Answer=Yes	74	87.06
Answer=No	11	12.94
Total	85	100
Test-B2	**Frequency**	**Percent**
Answer=Yes	35	41.18
Answer=No	50	58.82
Total	85	100
Test-C	**Frequency**	**Percent**
Answer=1	50	58.82
Answer=2	3	38.82
Answer=3	2	2.35
Total	85	100
Test-D	**Frequency**	**Percent**
Answer=1	68	80
Answer=2	7	8.24
Answer=3	10	11.76
Total	85	100
Test-E	**Frequency**	**Percent**
Answer=1	43	50.29
Answer=2	29	34.12
Answer=3	13	15.29
Total	85	100

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