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Cristina Di Girolami, Giorgio Fabbri and Francesco Russo

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The covariation for Banach space valued processes and applications

Cristina Di Girolami [*], Giorgio Fabbri[†] and Francesco Russo[‡]

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Abstract

This article focuses on a new concept of quadratic variation for processes taking values in a Banach space B and a corresponding covariation. This is more general than the classical one of Métivier and Pellaumail. Those notions are associated with some subspace χ of the dual of the projective tensor product of B with itself. We also introduce the notion of a convolution type process, which is a natural generalization of the Itô process and the concept of $\bar{\nu}_0$-semimartingale, which is a natural extension of the classical notion of semimartingale. The framework is the stochastic calculus via regularization in Banach spaces. Two main applications are mentioned: one related to Clark-Ocone formula for finite quadratic variation processes; the second one concerns the probabilistic representation of a Hilbert valued partial differential equation of Kolmogorov type.

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1 Introduction and motivations

The notion of covariation and quadratic variation are fundamental in stochastic calculus related to Brownian motion and semimartingales. However, they also play a role in stochastic calculus for non-semimartingales.

In the whole paper a fixed strictly positive time $T > 0$ will be fixed. Given a real continuous process $X = (X_t)_{t\in[0,T]}$, there are two classical definitions of quadratic variation related to it, denoted by $[X]$. The first one, inspired to [23], says that, when it exists, $[X]_t$ is a continuous process being the limit, in probability, of $\sum_{i=0}^{n-1}(X_{t_{i+1}\wedge t} - X_{t_i\wedge t})^2$ where $0 = t_0 < t_1 < \ldots < t_n = T$ is element of a sequence of subdivisions whose mesh $\max_{i=1}^{n-1}(t_{i+1} - t_i)$ converges to zero. The second one, less known, is based on stochastic calculus via regularization; it characterizes $[X]$ as the continuous process such that $[X]_t$ is the limit in probability for every $t \in [0, T]$, when $\varepsilon \to 0$, of $\frac{1}{\varepsilon}\int_0^t (X_{s+\varepsilon} - X_s)^2 ds, t \in [0, T]$. In all the known

*Laboratoire Manceau de Mathématiques, Faculté des Sciences et Techniques, Université du Maine, Département de Mathématiques, Avenue Olivier Messiaen, 72085 Le Mans Cedex 9 (France) E-mail: Cristina.Di_Girolami@univ-lemans.fr

†EPEE, UniversitÈ d?Evry-Val-d?Essonne (TEPP, FR-CNRS 3126), DÈpartement d'Economie, 4 bd. FranÁois Mitterrand, 91025 Evry cedex, France. E-mail: giorgio.fabbri@univ-very.fr

‡ENSTA ParisTech, Unité de Mathématiques appliquées, 828, boulevard des Maréchaux, F-91120 Palaiseau (France). E-mail: francesco.russo@ensta-paristech.fr

examples both definitions give the same result. We will use here the second formulation, which looks operational and simple. If $[X]$ exists than X is said **finite quadratic variation** process. A real process X such that $[X] \equiv 0$ is called **zero quadratic variation process**; we also say in this case that X has a zero quadratic variation. If X is a (continuous) semimartingale, $[X]$ is the classical bracket. Consequently, if W is the real Brownian motion then $[W]_t = t$.

In Section 5 we remind a suitable notion of quadratic variation for a process $\mathbb{X}$ with values in a Banach space B, [17, 14]. This has significant infinite dimensional applications but it also has motivations in the study of real stochastic processes with finite quadratic variation, even for Brownian motion and semimartingales.

Indeed, the class of real finite quadratic variation processes is quite rich even if many important fractional type processes do not have this property. Below we enumerate a list of such processes. All the considered processes will be continuous for simplicity. A survey of stochastic calculus via regularization which focuses on covariation is [45].

1. A bounded variation process has zero quadratic variation.

2. A semimartingale with decomposition $S = M + V$, M being a local martingale and V a bounded variation process is a finite quadratic variation process with $[S] = [M]$.

3. A fractional Brownian motion $X = B^H, 0 < H < 1$ has finite quadratic variation if and only if $H \geq \frac{1}{2}$. If $H > \frac{1}{2}$, it is a zero quadratic variation process.

4. An important subclass of finite quadratic variation processes is constituted by **Dirichlet** processes, which should more properly be called **Föllmer-Dirichlet**, since they were introduced by H. Föllmer [24]; they were later further investigated by J. Bertoin, see [2]. An a $(\mathscr{F}_t)$-**Dirichlet** process admits a (unique) decomposition of the form $X = M + A$, where M is an $(\mathscr{F}_t)$-local martingale and A is a zero quadratic variation (such that $A_0 = 0$ a.s.). In this case $[X] = [M]$. It is simple to produce Dirichlet processes X with the same quadratic variation as Brownian motion. Consider for instance $X = W + A$ where W is a classical Brownian motion and A has zero quadratic variation. In general we postulate that $A_0 = 0$ a.s. so that the mentioned decomposition is unique.

5. Another interesting example is the bifractional Brownian motion, introduced first by [29]. Such a process X depends on two parameters $0 < H < 1, 0 < K \leq 1$ and it is often denoted by $B^{H,K}$. If $HK > \frac{1}{2}$ then $B^{H,K}$ has zero quadratic variation. If $K = 1$, that process is a fractional Brownian motion with parameter H. A singular situation produces when $HK = \frac{1}{2}$. In that case X is a finite quadratic variation process and $[X]_t = 2^{1-K}t$. That process is neither a semimartingale nor a Dirichlet process, see [40]. In particular not all the finite quadratic variation processes are Dirichlet processes.

A simple link from real valued processes to Banach valued processes is the following. Let $0 < \tau \leq T$. Let $X = (X_t, t \in [0, T])$ be a real continuous process, that we naturally prolongate for $t \leq 0$ setting $X_t = X_0$ and $X_t = X_T$ if $t \geq T$. The process $X(\cdot)$ defined by $\mathbb{X} = X(\cdot) = \{X_t(u) := X_{t+u}; u \in [-\tau, 0]\}$, constitutes the τ-memory of process X. The natural state space for $\mathbb{X}$ is the non-reflexive separable space $B = C([-\tau, 0])$. $X(\cdot)$ is the so called *window* process associated with X (of width $\tau > 0$). If X is a Brownian motion (resp. semimartingale, diffusion, Dirichlet process), then $X(\cdot)$ will be called window Brownian motion (resp. window semimartingale, window diffusion, window Dirichlet process).

If $X = W$ is a classical Wiener process, $\mathbb{X} = X(\cdot)$ has no natural quadratic variation, in the sense of Dinculeanu or Métivier and Pellaumail, see Subsection 5.2. However it will possess a more general quadratic variation called χ-quadratic variation, which is related to a specific sub-Banach space χ of $(B\hat{\otimes}_\pi B)^*$.

A first natural application of our covariational calculus is motivated as follows. If $h \in L^2(\Omega)$, the martingale representation theorem states the existence of a predictable process $\xi \in L^2(\Omega \times [0,T])$ such that $h = \mathbb{E}[h] + \int_0^T \xi_s dW_s$. If $h \in \mathbb{D}^{1,2}$ in the sense of Malliavin calculus, see for instance [34, 31], the celebrated **Clark-Ocone** formula says $\xi_s = \mathbb{E}\left[D_s^m h | \mathscr{F}_s\right]$ where D^m is the Malliavin gradient. So

$$h = \mathbb{E}[h] + \int_0^T \mathbb{E}\left[D_s^m h | \mathscr{F}_s\right] dW_s. \tag{1}$$

A.S. Ustunel [48] obtains a generalization of (1) when $h \in L^2(\Omega)$, making use of the predictable projections of a Wiener distributions in the sense of S. Watanabe [51].
A natural question is the following: is Clark-Ocone formula *robust* if the law of $X = W$ is not anymore the Wiener measure but X is still a finite quadratic variation process even not necessarily a semimartingale? Is there a reasonable class of random variables h for which a representation of the type $h = H_0 +'' \int_0^T \xi_s dX_s''$, $H_0 \in \mathbb{R}$, ξ adapted? Since X is a not a semimartingale, previous integral has of course to be suitably defined, in the spirit of a limit of Riemann-Stieltjes non-anticipating sum. We decided however to interpret the mentioned integral as a forward integral in the regularization method, see Section 3. We denote it as $\int_0^T \xi_s d^- X_s$.
So let us suppose that $X_0 = 0$, $[X]_t = t$ and $\tau = T$ for simplicity. We look for a reasonably rich class of functionals $G : C([-T,0]) \longrightarrow \mathbb{R}$ such that the r.v. $h := G(X_T(\cdot))$ admits a representation of the type

$$h = G_0 + \int_0^T \xi_s d^- X_s, \tag{2}$$

provided that $G_0 \in \mathbb{R}$ and ξ adapted process with respect to the canonical filtration of X. The idea is to express $h = G(X_T(\cdot))$ as $u(T, X_T(\cdot))$ or in some cases

$$h = G(X_T(\cdot)) = \lim_{t \uparrow T} u(t, X_t(\cdot)),$$

where $u \in C^{1,2}\left([0,T[\times C([-T,0])\right)$ solves an infinite dimensional partial differential equation, and (2) holds with $\xi_t = Du(t,\eta)(\{0\}), t \in]0,T[$. At this point we will have $h = u(0, X_0(\cdot)) + \int_0^T \xi_s d^- X_s$, recalling that $D\,u : [0,T] \times C([-T,0]) \longrightarrow (C([-T,0])^* = \mathcal{M}([-T,0])$. This is the object of Section 7.3. A first step in this direction was done in [16] and more in details in Chapter 9 of [15].

A second interesting application concerns convolution processes, see Section 5.3. Consider H and U two separable Hilbert spaces and a C_0-semigroup (e^{tA}) on H, see Sections 2.2 for definitions and references. Let $\mathbb{W}$ be an U-values Q-Wiener process for some positive bounded operator Q on U. Let $\sigma = (\sigma_t, t \in [0,T])$ and $b = (b_t, t \in [0,T])$ two suitable predictable integrands, see Section 2.3 for details. An H-valued convolution process has the following form:

$$\mathbb{X}_t = e^{tA} x_0 + \int_0^t e^{(t-r)A} \sigma_r d\mathbb{W}_r + \int_0^t e^{(t-r)A} b_r dr, t \in [0,T], \tag{3}$$

for some $x_0 \in H$. Convolution type processes are an extension of Itô processes, which appear when A vanishes. Mild solutions of infinite dimensional evolution equations are in natural way convolution processes. They have no scalar quadratic variation even if driven by a one-dimensional Brownian motion. Still it can be proved that they admit a χ-quadratic variation for some suitable space χ, see Proposition 5.21.
Another general concept of processes that we will introduce is the one of $\bar{\nu}_0$-semimartingales. An H valued process $\mathbb{X}$ is said $\bar{\nu}_0$-semimartingale if there is Banach space $\bar{\nu}_0$ including H (or in which H is continuously

injected) so that $\mathbb{X}$ is the sum of an H-valued local martingale and a bounded variation $\bar{\nu}_0$-valued process. A convolution process will be shown to be a $\bar{\nu}_0$-semimartingale, where the dual $\bar{\nu}_0^*$ equals $D(A^*)$, see item 2. of Proposition 5.21.

Let us come back for a moment to real valued processes. A real process X is called **weak Dirichlet** (with respect to a given filtration), if it can be written as the sum of a local martingale and a process A such that $[A, N] = 0$ for every continuous local martingale. A significant result of F. Gozzi and F. Russo, see [28], is the following. If $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is of class $C^{0,1}$, then $Y_t = f(t, X_t), t \in [0, T]$ is a weak Dirichlet process. A similar result, in infinite dimension, is obtained replacing the process X with its associated window $X(\cdot)$, see [14]. The notion of Dirichlet process extends to the infinite dimensional framework via the notion of *ν-weak Dirichlet process*, see Definition 5.9. An interesting example of ν-weak Dirichlet process is given, once more, by convolution processes, see Proposition 5.21.

Generalizing the result of [28], it can be proved that, given $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ of class $C^{0,1}$ and being $\mathbb{X}$ a suitable ν-weak Dirichlet process with finite χ-quadratic variation, where χ is Chi-subspace associated with ν, then $Y_t = u(t, \mathbb{X}_t)$ is a real weak Dirichlet process. Moreover its (Fukushima-Dirichlet type) decomposition is provided in Theorem 6.7. That theorem can be seen as a substitution-tool of Itô's formula if u is not smooth and is a key tool for the application we provide in Section 8. Examples of such ν-weak Dirichlet processes are convolution type processes, or more generally $\bar{\nu}_0$-semimartingales, see Proposition 5.15, item 2. In Section 8, we study the solution of a non-homogeneous Kolmogorov equation and we provide a uniqueness result for the related solution. The proof of the result is based on the a representation result for (strong) solutions of the Kolmogorov equation that is obtained thanks to the uniqueness of the decomposition of a real weak Dirichlet process. The uniqueness result covers cases that, as far as we know, were not yet included in the literature. For instance, in our results, the initial datum g of the Kolmogorov equation is asked be continuous but we do not require any boundedness assumption on it. This kind of problem cannot be studied if the problem is approached, as in [4, 26], looking at the properties of the transition semigroup on the space $C_b(H)$ (resp. on $B_b(H)$) of continuous and bounded (resp. bounded) functions defined on H, because, in this case, the initial datum always needs to be bounded. More details are contained in Section 8. In the same spirit, further applications to stochastic verification theorems, in which the Kolmogorov type equation, is replaced by an Hamilton-Jacobi-Bellman equation, can be realized, see for instance [22].

2 Preliminaries

2.1 Functional analysis background

Given an underlying Banach space B (resp. Hilbert space H), $|\cdot|_B$ (resp. $|\cdot|_H$) will generally denote the norm related to B (resp. H). However, if the considered norm is clear we will often only indicate it by $|\cdot|$. Even the associated inner product with $|\cdot|_H$ will be indicated by $\langle \cdot, \cdot \rangle_H$ or simply by $\langle \cdot, \cdot \rangle$. Given an element a of a Hilbert space H, we generally denote by a^*, the corresponding element of H^* via Riesz identification. We will use the identity ${}_{H^*}\langle a^*, b \rangle_H = \langle a, b \rangle_H = \langle a, b \rangle_H$ without comments. Let B_1, B_2 be two separable real Banach spaces. We denote with $B_1 \otimes B_2$ the algebraic tensor product defined as the set of the elements of the form $\sum_{i=1}^n x_i \otimes y_i$, for some positive integer n where x_i and y_i are respectively elements of B_1 and B_2. The product $\otimes$: $B_1 \times B_2 \to B_1 \otimes B_2$ is bilinear.

A natural norm on $B_1 \otimes B_2$ is the projective norm π: for all $u \in B_1 \otimes B_2$, we denote with $\pi(u)$ the norm

$$\pi(u) := \inf \left\{ \sum_{i=1}^{n} |x_i|_{B_1} |y_i|_{B_2} \ : \ u = \sum_{i=1}^{n} x_i \otimes y_i \right\}.$$

This belongs to the class of the so-called *reasonable norms* $|\cdot|$, in particular verifying $|x_1 \otimes x_2| = |x_1|_{B_1}|x_2|_{B_2}$, if $x_1 \in B_1, x_2 \in B_2$. We denote with $B_1 \hat{\otimes}_\pi B_2$ the Banach space obtained as completion of $B_1 \otimes B_2$ for the norm π, see [46] Section 2.1. We remark that its topological dual $(B_1 \hat{\otimes}_\pi B_2)^*$ is isomorphic to the space of continuous bilinear forms $\mathcal{B}i(B_1, B_2)$ of continuous bilinear forms, equipped with the norm $\| \cdot \|_{B_1,B_2}$ where $\|\Phi\|_{B_1,B_2} = \sup_{\substack{a_1 \in B_1, a_2 \in B_2 \\ |a_1|_{B_1}, |a_2|_{B_2} \le 1,}} |\Phi(a_1, a_2)|$.

Lemma 2.1. Let B_1 and B_2 be two separable, reflexive real Banach spaces. Given $a^* \in B_1^*$ and $b^* \in B_2^*$ we can associate to $a^* \otimes b^*$ the elements $j(a^* \otimes b^*)$ of $(B_1 \otimes B_2)^*$ acting as follows on a generic element $u = \sum_{i=1}^n x_i \otimes y_i \in B_1 \otimes B_2$:

$$\langle j(a^* \otimes b^*), u \rangle = \sum_{i}^{n} \langle a^*, x_i \rangle \langle b^*, y_i \rangle .$$

$j(a^* \otimes b^*)$ extends by continuity to the whole $B_1 \otimes B_2$ and its norm in $(B_1 \otimes B_2)^*$ equals $|a^*|_{B_1^*} |b^*|_{B_2^*}$. In particular if ν_i is a (dense) subspaces of $B_i^*, i = 1, 2$, then the projective tensor product $\nu_1 \hat{\otimes}_\pi \nu_2$ can be seen as a subspace of $(B_1 \hat{\otimes}_\pi B_2)^*$.

Proof. See [22] Lemma 2.4. □

Remark 2.2. We remark that $B_1 \hat{\otimes}_\pi B_2$ fails to be Hilbert even if B_1 and B_2 are Hilbert spaces. It is not even reflexive space. Fore more information about tensor topologies, we refer e.g. to [46].

Let us consider now two separable Banach spaces B_1 and B_2. With $C(B_1; B_2)$, we symbolize the set of the locally bounded continuous B_2-valued functions defined on B_1. This is a Fréchet type space with the seminorms

$$\|u\|_r := \sup \{ |u(x)|_{B_2} \ : \ x \in B_1, \ with \, |x|_{B_1} \le r \} \tag{4}$$

for $r \in \mathbb{N}^*$.

If $B_2 = \mathbb{R}$ we will often simply use the notation $C(B_1)$ instead of $C(B_1; \mathbb{R})$. Similarly, given a real interval I, typically $I = [0, T]$ or $I = [0, T[$, we use the notation $C(I \times B_1; B_2)$ for the set of the continuous B_2-valued functions defined on $I \times B_1$ while we use the lighter notation $C(I \times B_1)$ when $B_2 = \mathbb{R}$. Eventually a function $(t, \eta) \mapsto u(t, \eta) \in C(I \times B_1)$ will be said to belong to $C^{1,2}(I \times B_1)$ if $\partial_t u(t, \eta)$ belongs to $C(I \times B_1)$ while (denoted with D and D^2 the derivatives w.r.t. the variable $\eta \in B_1$) $Du(t, \eta)$ belongs to $C(I \times B_1; {B_1}^*)$ and $D^2 u(t, \eta)$ to $C(I \times B_1; \mathcal{B}i(B_1, B_1))$.

We denote by $\mathcal{L}(B_1; B_2)$ the space of linear bounded maps from B_1 to B_2. It is of course a Banach space and we will denote by $\| \cdot \|_{\mathcal{L}(B_1;B_2)}$ the corresponding norm. We will often indicate in the sequel by a double bar, i.e. $\| \cdot \|$ the norm of an operator or more generally of a function. As a particular case, if we denote U, H two separable Hilbert spaces, $\mathcal{L}(U; H)$ will be the space of linear bounded maps from U to H. If $U = H$, we set $\mathcal{L}(U) := \mathcal{L}(U; U)$. $\mathcal{L}_2(U; H)$ will be the set of *Hilbert-Schmidt* operators from U to H and $\mathcal{L}_1(H)$ (resp. $\mathcal{L}_1^+(H)$) will be the space of (non-negative) *nuclear* operators on H. For details about the notions of Hilbert-Schmidt and nuclear operator, the reader may consult [46], Section 2.6 and [8] Appendix C. If $T \in \mathcal{L}_2(U; H)$ and $T^* : H \to U$ is the adjoint operator, then $TT^* \in \mathcal{L}_1(H)$ and the Hilbert-Schmidt norm of T gives $\|T\|^2_{\mathcal{L}_2(U;H)} = \|TT^*\|_{\mathcal{L}_1(H)}$. We recall that, for a generic element $T \in \mathcal{L}_1(H)$ and given a basis $\{e_n\}$ of H the sum $\sum_{n=1}^{\infty} \langle Te_n, e_n \rangle$ is absolutely convergent and independent of the chosen basis $\{e_n\}$. It is called *trace* of T and denoted with $\mathrm{T}r(T)$. $\mathcal{L}_1(H)$ is a Banach space and we denote by $\|\cdot\|_{\mathcal{L}_1(H)}$ the corresponding norm. If T is non-negative then $\mathrm{T}r(T) = \|T\|_{\mathcal{L}_1(H)}$ and in general we have the inequalities

$$|\mathrm{T}r(T)| \leq \|T\|_{\mathcal{L}_1(H)}, \quad \sum_{n=1}^{\infty} |\langle Te_n, e_n \rangle| \leq \|T\|_{\mathcal{L}_1(H)}, \tag{5}$$

see Proposition C.1, [8]. As a consequence, if T is a non-negative operator, the relation below

$$\|T\|^2_{\mathcal{L}_2(U;H)} = \mathrm{T}r(TT^*). \tag{6}$$

will be very useful in the sequel.

Observe that every element $u \in H\hat{\otimes}_\pi H$ is isometrically associated with an element T_u in the space of nuclear operators $\mathcal{L}_1(H)$. The identification (which is in fact an isometric isomorphism) associates to any element u of the form $\sum_{i=1}^{\infty} a_n \otimes b_n$ in $H\hat{\otimes}_\pi H$ the nuclear operator T_u defined as

$$T_u(x) := \sum_{i=1}^{\infty} \langle x, a_n \rangle \, b_n, \tag{7}$$

see for instance [46] Corollary 4.8 Section 4.1 page 76.

We recall that, to each element φ of $(H\hat{\otimes}_\pi H)^*$, we can associate a bilinear continuous map B_φ and a linear continuous operator $L_\varphi : H \to H$ such that

$$\langle L_\varphi(x), y \rangle = B_\varphi(x, y) = \varphi(x \otimes y) \qquad \text{for all } x, y \in H, \tag{8}$$

see [46], the discussion before Proposition 2.11 Section 2.2. at page 24. One can prove the following, see [22] Proposition 2.6 or [15] Proposition 6.6.

Proposition 2.3. Let $u \in H\hat{\otimes}_\pi H$ and $\psi \in (H\hat{\otimes}_\pi H)^*$ with associated maps $T_u \in \mathcal{L}_1(H), L_\varphi \in \mathcal{L}(H; H)$. Then

$${}_{(H\hat{\otimes}_\pi H)^*}\langle \varphi, u \rangle_{H\hat{\otimes}_\pi H} = Tr\left(T_u L_\varphi\right).$$

Proposition 2.4. Let $g : [0, T] \to \mathcal{L}_1^+(H)$ measurable such that

$$\int_0^T \|g(r)\|_{\mathcal{L}_1(H)} dr < \infty. \tag{9}$$

Then $\int_0^T g(r)dr \in \mathcal{L}_1^+(H)$ and its trace equals $\int_0^T \mathrm{T}r(g(r))dr$.

Proof. $\int_0^T g(r)dr \in \mathcal{L}_1(H)$ by the the first inequality of (5) and by Bochner integrability property. Clearly the mentioned integral is a non-negative operator. The remainder follows quickly from the relation between the trace and the $\mathcal{L}_1(H)$ norm that we have recalled above; indeed if (e_n) is an orthonormal basis,

$$\sum_{n=1}^{N} \left\langle \int_0^T g(r)\, dr\, e_n, e_n \right\rangle = \int_0^T \sum_{n=1}^{N} \langle g(r)e_n, e_n \rangle \, dr$$

and we can pass to the limit thanks to (5), (9) and Lebesgue's dominated convergence theorem. $\square$

2.2 General probabilistic framework

In the whole paper we will fix $T > 0$. $(\Omega, \mathcal{F}, \mathbb{P})$ will be a fixed probability space and $\mathcal{P}$ will denote the predictable σ-field on $\Omega \times [0,T]$. $(\mathscr{F}_t) = (\mathscr{F}_t, t \in [0,T])$ will be a filtration fulfilling the usual conditions. If B is a Banach space, $\mathcal{B}(B)$ will denote its Borel σ-algebra. A B-valued random variable C is integrable if $\mathbb{E}(|C|)$ is finite and the quantity $\mathbb{E}(C)$ exists as an element in B. It fulfills in particular the Pettis property: $\varphi(\mathbb{E}(C)) = \mathbb{E}(\varphi(C))$ for any $\varphi \in B^*$.

Given a σ-algebra $\mathcal{G}$, the random element $\mathbb{E}(C|\mathcal{G}) : \Omega \to B$ denotes the conditional expectation of C with respect to $\mathcal{G}$. The concept of conditional expectation for B-valued random elements, when B is a separable Banach space, are recalled for instance in [8] Section 1.3. In particular, for every $\varphi \in B^*$ we have $\mathbb{E}\left(\Psi_{B^*}\langle \varphi, C\rangle_B\right) = \mathbb{E}\left(\Psi_{B^*}\langle \varphi, \mathbb{E}(C|\mathcal{G})\rangle_B\right)$, for any bounded r.v. $\mathcal{G}$-measurable Ψ.

A stochastic process will stand for an application $[0,T] \times \Omega \to B$, which is measurable with respect to the σ-fields $\mathcal{B}([0,T]) \otimes \mathcal{F}$ and $\mathcal{B}(B)$. If B is infinite dimensional, the processes are indicated by bold letters $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$. Given a Banach space B, a process $\mathbb{X} : ([0,T] \times \Omega, \mathcal{F}) \to B$ is said to be **strongly (Bochner) measurable** if it is the limit of $\mathcal{F}$- measurable countably-valued functions. By default, a process $[0,T] \times \Omega \to B$, which is measurable with respect to the σ-fields $\mathcal{P}$ and $\mathcal{B}(\mathbb{R})$ is said to be **predictable** with respect to the given filtration $(\mathscr{F}_t, t \in [0,T])$. A priori, such a process will also be considered as strongly measurable. Any cadlag or caglad process is strongly measurable.

Let H, U be separable Hilbert spaces, $Q \in \mathcal{L}(U)$ be a positive, self-adjoint operator and define $U_0 := Q^{1/2}(U)$. This is again a separable Hilbert space. Even if not necessary we suppose Q to be injective, which avoids formal complications. We endow U_0 with the scalar product $\langle a, b\rangle_{U_0} := \left\langle Q^{-1/2}a, Q^{-1/2}b\right\rangle$. $Q^{1/2} \colon U \to U_0$ is an isometry, see e.g. [8] Section 4.3. Assume that $\mathbb{W}^Q = \{\mathbb{W}^Q_t : 0 \le t \le T\}$ is an U-valued $\mathscr{F}$-Q-Wiener process (with $\mathbb{W}^Q_0 = 0$, $\mathbb{P}$ a.s.). The notion of Q-Wiener process and $(\mathscr{F}_t)$-Q-Wiener process were defined for example in [8] Chapter 4, see also [25] Chapter 2.1. We recall that $\mathcal{L}_2(U_0; H)$ stands for the Hilbert space of the Hilbert-Schmidt operators from U_0 to H.

An U-valued process $\mathbb{M}\colon [0,T] \times \Omega \to U$ is called $(\mathscr{F}_t)$**-martingale** if, for all $t \in [0,T]$, $\mathbb{M}$ is $(\mathscr{F}_t)$-adapted with $\mathbb{E}\left[|\mathbb{M}_t|_U\right] < +\infty$ and $\mathbb{E}\left[\mathbb{M}_s|\mathscr{F}_t\right] = \mathbb{M}_t$ for all $0 \le t \le s \le T$. In the sequel, the reference to the filtration $(\mathscr{F}_t, t \in [0,T])$ will be often omitted. The mention "adapted", "predictable" etc... we will always refer to *with respect to the filtration* $\{\mathscr{F}_t\}_{t \ge 0}$. An U-valued martingale $\mathbb{M}$ is said to be **square integrable** if $\mathbb{E}\left[|\mathbb{M}_T|^2_U\right] < +\infty$. A Q-Wiener process is a square integrable martingale. We denote with $\mathcal{M}^2(0,T;U)$ the linear space of square integrable martingales indexed by $[0,T]$ with values in U, i.e. of measurable processes $\mathbb{M} : [0,T] \times \Omega \to U$ such that $E(|\mathbb{M}_T|^2_U) < \infty$. In particular for $\mathbb{M} \in \mathcal{M}^2(0,T;U)$, the quantity

$$|\mathbb{M}|_{\mathcal{M}^2(0,T;U)} := \left(\mathbb{E} \sup_{t \in [0,T]} |\mathbb{M}_t|^2_U\right)^{1/2}$$

is finite. Moreover, it defines a norm and $\mathcal{M}^2(0,T;U)$ endowed with it is a Banach space as stated in [8]Proposition 3.9. An U-valued process $\mathbb{M}\colon [0,T] \times \Omega \to U$ is called **local martingale** if there exists a non-decreasing sequence of stopping times $\tau_n \colon \Omega \to [0,T] \cup \{+\infty\}$ such that $\mathbb{M}_{t \wedge \tau_n}$ for $t \in [0,T]$ is a martingale and $\mathbb{P}\left[\lim_{n \to \infty} \tau_n = +\infty\right] = 1$. All the considered martingale and local martingales will be supposed to be continuous.

Given a continuous local martingale $\mathbb{M}\colon [0,T] \times \Omega \to U$, the process $|\mathbb{M}|^2$ is a real local sub-martingale, see Theorem 2.11 in [30]. The increasing predictable process, vanishing at zero, appearing in the Doob-Meyer decomposition of $|\mathbb{M}|^2$ will be denoted by $([\mathbb{M}]^{\mathbb{R},cl}_t, t \in [0,T])$. It is of course uniquely determined and continuous.

A B-valued process $\mathbb{A}$ is said to be a **bounded variation process** or **to have bounded variation** if almost every trajectory has bounded variation i.e. if, for almost all ω, the supremum of $\sum_{i=1}^{N} |\mathbb{A}_{t_{i-1}}(\omega) - \mathbb{A}_{t_i}(\omega)|_B$ over all the possible subdivisions $0 = t_0 < \ldots < t_N$, $N \in \mathbb{N}^*$, is finite. If $B = U$ is a Hilbert space, following [32], Definition 23.7, we say that an U-valued process $\mathbb{X}$ is a **semimartingale** if $\mathbb{X}$ can be written as $\mathbb{X} = \mathbb{M} + \mathbb{A}$ where $\mathbb{M}$ is a local martingale and $\mathbb{A}$ a bounded variation process. The total variation function process associated with $\mathbb{A}$ is defined similarly as for real valued processes and it is denoted by $t \mapsto \|\mathbb{A}_t\|$.

2.3 The Hilbert space valued Itô stochastic integral

We recall here some basic facts about the Hilbert space valued Itô integral, which was made popular for instance by G. Da Prato and J. Zabczyk, see [8, 9]. More recent monographs on the subject are [25, 38].

Let H and U be two separable Hilbert spaces. We adopt the notations that we have introduced in previous subsection 2.2. $\mathcal{I}_{\mathbb{M}}(0,T;U,H)$ will be the set of the processes $\mathbb{X}\colon [0,T] \times \Omega \to \mathcal{L}(U;H)$ that are strongly measurable from $([0,T] \times \Omega, \mathscr{P})$ to $\mathcal{L}(U;H)$ and such that

$$|\mathbb{X}|_{\mathcal{I}_{\mathbb{M}}(0,T;U,H)} := \left(\mathbb{E} \int_0^T \|\mathbb{X}_r\|^2_{\mathcal{L}(U;H)} d[\mathbb{M}]^{\mathbb{R},cl}_r \right)^{1/2} < +\infty.$$

$\mathcal{I}_{\mathbb{M}}(0,T;U,H)$ endowed with the norm $|\cdot|_{\mathcal{I}_{\mathbb{M}}(0,T;U,H)}$ is a Banach space. The linear map

$$\begin{cases} I\colon \mathcal{I}_{\mathbb{M}}(0,T;U,H) \to \mathcal{M}^2(0,T;H) \\ \mathbb{X} \mapsto \int_0^T \mathbb{X}_r d\mathbb{M}_r \end{cases}$$

is a contraction, see e.g. [32] Section 20.4 above Theorem 20.5. As illustrated in [30] Section 2.2 (above Theorem 2.14), the stochastic integral w.r.t. $\mathbb{M}$ extends to the integrands $\mathbb{X}$ which are strongly measurable from $([0,T] \times \Omega, \mathscr{P})$ to $\mathcal{L}(U;H)$ and such that

$$\int_0^T \|\mathbb{X}_r\|^2_{\mathcal{L}(U;H)} d[\mathbb{M}]^{\mathbb{R},cl}_r < +\infty \qquad a.s. \tag{10}$$

We denote by $\mathcal{J}^2(0,T;U,H)$ such a family of integrands w.r.t. $\mathbb{M}$.

We have the following standard fact, see e.g. [30] Theorem 2.14.

Proposition 2.5. Let $\mathbb{M}$ be a continuous U-valued $(\mathcal{F}_t)$-local martingale, $\mathbb{X}$ a process verifying (10). Then $\mathbb{N}_t = \int_0^t \mathbb{X}_r d\mathbb{M}_r, t \in [0,T]$, is an $(\mathscr{F}_t)$-local martingale with values in H.

Consider now the case when the integrator $\mathbb{M}$ is a Q-Wiener process, with values in U, where Q be again a positive injective and self-adjoint operator in $Q \in \mathcal{L}(U)$, see Section 2.2. We consider U_0 with its inner product as before. By (6) we can easily prove that, given $A \in \mathcal{L}_2(U_0;H)$, we have $\|A\|^2_{\mathcal{L}_2(U_0;H)} = \mathrm{Tr}\left(AQ^{1/2}(AQ^{1/2})^*\right)$. Let $\mathbb{W}^Q = \{\mathbb{W}^Q_t : 0 \leq t \leq T\}$ be an U-valued $(\mathscr{F}_t)$-Q-Wiener process with $\mathbb{W}^Q_0 = 0$, $\mathbb{P}$ a.s. In this case the Itô-type integral with respect to $\mathbb{W}$ extends to a larger class, see Chapter 4.2 and 4.3 of [8]. If $\mathbb{Y}$ is a predictable process with values in $\mathcal{L}_2(U_0;H)$ with some integrability properties, then the Itô-type integral of $\mathbb{Y}$ with respect to $\mathbb{W}$, i.e. $\int_0^t \mathbb{Y}_r d\mathbb{W}_r, t \in [0,T], t \in [0,T]$, is well-defined.

Proposition 2.6. Let $\mathbb{M}$ be a process of the form

$$\mathbb{M}_t = \int_0^t \mathbb{Y}_r d\mathbb{W}^Q_r, t \in [0,T], \tag{11}$$

where $\mathbb{Y}$ be an $\mathcal{L}(U;H)\cap\mathcal{L}_2(U_0;H)$-valued predictable process such that

$$\int_0^T \mathrm{T}r[\mathbb{Y}_r Q^{1/2}(\mathbb{Y}_r Q^{1/2})^*]dr < \infty \text{ a.}s. \tag{12}$$

Then $\mathbb{M}$ is a H-valued local martingale. Moreover we have the following.

(i) If $\mathbb{X}$ is an H-valued predictable process such that

$$\int_0^T \langle \mathbb{X}_r, \mathbb{Y}_r Q^{1/2}(\mathbb{Y}_r Q^{1/2})^*\mathbb{X}_r\rangle_H dr < \infty,\ a.s., \tag{13}$$

then the process

$$N_t = \int_0^t \langle \mathbb{X}_r, d\mathbb{M}_r\rangle_H, t\in[0,T], \tag{14}$$

is a real local martingale. If the expectation of (13) is finite, then N is a square integrable martingale.

(ii) If, for some separable Hilbert space E, $\mathbb{K}$ is a $\mathcal{L}(H,E)$-valued, $(\mathscr{F}_t)$-predictable process such that

$$\int_0^T \mathrm{T}r[\mathbb{K}_r\mathbb{Y}_r Q^{1/2}(\mathbb{K}_r\mathbb{Y}_r Q^{1/2})^*]dr < \infty \text{ a.}s., \tag{15}$$

then the E-valued Itô-type stochastic integral $\int_0^t \mathbb{K}d\mathbb{M}, t\in[0,T]$, is well-defined, it is a local martingale and it equals $\int_0^t \mathbb{K}\mathbb{Y}d\mathbb{W}^Q$.

Proof. The results above are a consequence of [8] Section 4.7. at least when the expectations of (12), (13) and (15) are finite. In particular the first part is stated in Theorem 4.12 of [8]. Otherwise, on proceeds by localizations, via stopping arguments. □

Remark 2.7. In the sequel we will also denote previous integral by $\int_0^t\langle \mathbb{X}_r, d\mathbb{M}_r\rangle_H, t\in[0,T]$, or by $\int_0^t {}_{H^*}\langle \mathbb{X}_r^*, d\mathbb{M}_r\rangle_H, t\in[0,T]$, using the Riesz identification.

3 Finite dimensional calculus via regularization

3.1 Integrals and covariations

This theory has been developed in several papers, starting from [41, 42]. A survey on this subject is given in [45]. The formulation is light, efficient when the integrator is a finite quadratic variation process, but it extends to many integrator processes whose paths have a p-variation with $p < 2$. Integrands are allowed to be anticipative and the integration theory and calculus appears to be close to a pure pathwise approach even though there is still a probability space behind. The theory clearly allows non-semimartingales integrators. Let now X (resp. Y) be a real continuous (resp. a.s. integrable) process, both indexed by $t\in[0,T]$.

Definition 3.1. Suppose that, for every $t\in[0,T]$, the following limit

$$\int_0^t Y_r d^- X_r := \lim_{\epsilon\to 0}\int_0^t Y_r \frac{X_{r+\epsilon}-X_r}{\epsilon}dr \tag{16}$$

exists in probability. If the obtained random function admits a continuous modification, that process is denoted by $\int_0^\cdot Y d^- X$ and called **(proper) forward integral of Y with respect to X.**

Definition 3.2. If the limit (16) exists in probability for every $t \in [0, T[$ and $\lim_{t\to T} \int_0^t Y d^- X$ exists in probability, the limiting random variable is called the **improper forward integral of** Y **with respect to** X and it is still denoted by $\int_0^T Y d^- X$.

As we mentioned, the covariation is a crucial notion in stochastic calculus via regularization.

Definition 3.3. The **covariation of** X **and** Y is defined by

$$[X, Y]_t = ([Y, X]_t) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t (X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s) ds, t \in [0, T],$$

if the limit exists in probability for every $t \in [0, T]$, provided that the limiting random function admits a continuous version. If $X = Y$, X is said to be **finite quadratic variation process** and we set $[X] := [X, X]$. A vector $(X^1, \ldots, X^n)$ of real processes is said to admit **all its mutual brackets** if $[X^i, X^j], 1 \leq i, j \leq n$, exist.

One natural question arises. What is the link between the regularization and discretization techniques of Föllmer ([23]) type? Let Y be a cadlag process. One alternative method could be to define $\int_0^T Y dX$ as the limit of

$$\sum_{i=0}^{n-1} Y_{t_i}(X_{t_{i+1}} - X_{t_i}), N \in N^*,$$

when the mesh $\max_{i=0}^{N-1}(t_{i+1} - t_i)$ of the subdivision

$$0 = t_0 < \ldots < t_N = T \tag{17}$$

converges to zero. A large part of calculus via regularization could be essentially translated in that formal language via discretization. However, even if it is not essential, we decided to keep going on with regularization methods. First, because that approach is direct and analytically efficient. Second, in many contexts the class of integrands is larger. Let us just fix one simple example: the Wiener integral with respect to Brownian motion. Let $g \in L^2([0, T])$ and W be a classical Wiener process; $\int_0^t g d^- W, t \in [0, T]$ exists and equals Wiener-Itô integral $\int_0^t g dW, t \in [0, T]$. However the discretizations limit of $\sum_{i=0}^{n-1} g(t_i)(W_{t_{i+1}} - W_{t_i})$ may either not exist, or depend on the sequences of subdivisions. Indeed, as an example, let us choose $g = 1_{\mathbb{Q} \cap [0,T]}$ where $\mathbb{Q}$ is the set of rational numbers. If, all the t_i elements of subdivision (17) where irrational (except for the extremities), then the limit would be zero, as for the Itô-Wiener integral, being $g = 0$ a.e. If on the contrary, all of the t_i are rational, then the limit is $W_T - W_0$.

In the proposition below we list some properties relating Itô calculus and forward calculus, see e.g. [45].

Proposition 3.4. Suppose that M is a local continuous martingale and Y cadlag and predictable. Let V be a bounded variation process. Let S^1, S^2 be $(\mathscr{F}_t)$-semimartingales with decomposition $S^i = M^i + V^i$, $i = 1, 2$, where $M^i, i = 1, 2$ are $(\mathscr{F}_t)$-continuous local martingales and V^i continuous adapted bounded variation processes. We have the following.

1. M is a finite quadratic variation process and $[M]$ is the classical bracket $\langle M \rangle$.
2. $\int_0^\cdot Y d^- M$ exists and it equals Itô integral $\int_0^\cdot Y dM$.
3. Let us suppose V to be continuous and Y cadlag (or vice-versa); then $[V] = [Y, V] = 0$. Moreover $\int_0^\cdot Y d^- V = \int_0^\cdot Y dV$, is the **Lebesgue-Stieltjes integral**.

4. $[S^i]$ is the classical bracket and $[S^i] = \langle M^i \rangle$.
5. $[S^1, S^2]$ is the classical bracket and $[S^1, S^2] = \langle M^1, M^2 \rangle$.
6. If S is a continuous semimartingale and Y is cadlag and adapted, then $\int_0^{\cdot} Y d^- S = \int_0^{\cdot} Y dS$ is again an Itô integral.

Coming back to the general calculus we state the **integration by parts** formula, see e.g. item 4. of Proposition [45].

Proposition 3.5. Let X and Y be continuous processes. Then

$$Y_t X_t = Y_0 X_0 + \int_0^t Y d^- X + \int_0^t X d^- Y + [X, Y]_t,$$

provided that two of the three previous integrals or covariation exist. If X is a continuous bounded variation process, then $\int_0^t X d^- Y = Y_t X_t - Y_0 X_0 - \int_0^t Y dX$.

The kernel of calculus via regularization is Itô formula. It is a well-known result in the semimartingales theory, but it also extends to the framework of finite quadratic variation processes. Here we only remind the one-dimensional case, in the form of a Itô chain rule. It is essentially a consequence of Proposition 4.3 of [44].

Theorem 3.6. Let $F : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $F \in C^{1,2}\left([0, T[\times \mathbb{R}\right)$ and X be a finite quadratic variation process. We set $Y_t = F(t, X_t), t \in [0, T]$. Let $Z = (Z_t, t \in [0, T])$ be an a.s. bounded process. We have

$$\int_0^t Z_r d^- Y_r = \int_0^t Z_r \partial_r F(r, X_r) dr + \int_0^t Z_r \partial_x F(r, X_r) d^- X_r + \frac{1}{2} \int_0^t Z_r \partial^2_{x\,x} F(r, X_r) d[X]_r, \tag{18}$$

in the following sense: if the first (resp. the third) integral exists then the third (resp. the first) exists and formula (18) holds.

Taking $Z = 1$, comes out the natural Itô formula below.

Proposition 3.7. With the same assumptions of Theorem 3.6 we have

$$\int_0^t \partial_x F(r, X_r) d^- X_r = F(t, X_t) - F(0, X_0) - \int_0^t \partial_r F(r, X_r) dr - \frac{1}{2} \int_0^t \partial^2_{x\,x} F(r, X_r) d[X]_r.$$

Theorem 6.3 will extend the formula above to the case of Banach space valued integrators.

An adaptation of Proposition 11 of [45] and Proposition 2.2 of [27] gives the following. Given a real interval I and $h : I \to \mathbb{R}$ be a bounded variation function, we denote by $\|h\|_{\mathrm{v}ar}$ the total variation of h.

Proposition 3.8. Let I be a real interval and $f, g : [0, T] \times I \to \mathbb{R}$. Let X and Y be two real processes such that (X, Y) admits all its mutual brackets and $f(t, \cdot)$ and $g(t, \cdot)$ have bounded variation for any $t \in [0, T]$. We suppose moreover that $t \mapsto \|f(t, \cdot)\|_{\mathrm{v}ar}$, $t \mapsto \|g(t, \cdot)\|_{\mathrm{v}ar}$ are bounded.
Then $[f(\cdot, X), g(\cdot, Y)]_t = \int_0^t \partial_x f(s, X_s) \partial_x g(s, Y_s) d[X, Y]_s$.

Below we introduce the notion of weak Dirichlet process which was introduced in [21] and [28].

Definition 3.9. A real process $X \colon [0, T] \times \Omega \to \mathbb{R}$ is called **weak Dirichlet process** if it can be written as

$$X = M + A, \tag{19}$$

where

(i) M is a local martingale,

(ii) A is a process such that $[A, N] = 0$ for every continuous local martingale N and $A_0 = 0$.

Proposition 3.10. 1. The decomposition described in Definition 3.9 is unique.

2. A real continuous semimartingale S is a weak Dirichlet process.

Proof. 1. is stated in Remark 3.5 of [28]. 2. is obvious since a bounded variation process V is a zero quadratic variation process by item 3. of Proposition 3.4. □

3.2 The deterministic calculus via regularization.

An useful particular case arises when Ω is a singleton, i.e. when the calculus becomes deterministic. We will essentially concentrate in the definite integral on an interval $J =]a, b]$, where $a < b$ are two real numbers. Typically, in our applications we will consider $a = -T$ or $a = -t$ and $b = 0$. That integral will be a real number, instead of functions.

We start with a convention. If $f : [a, b] \to \mathbb{R}$ is a cadlag function, we extend it naturally to another cadlag function f_J on real line setting

$$f_J(x) = \begin{cases} f(b) & : \quad x > b \\ f(x) & : \quad x \in [a, b] \\ 0 & : \quad x < a. \end{cases}$$

If g is finite Borel measure on $[0, T]$, we define the **deterministic forward integral** $\int_{]a,b]} g(dx) d^- f(x)$ (or simply $\int_{]a,b]} g d^- f$) as the limit of $\int_{]a,b]} \frac{g(ds)}{\varepsilon} (f_J(s + \varepsilon) - f_J(s))$, when $\varepsilon \downarrow 0$, provided it exists. In most of the cases g will be absolutely continuous whose density will be still denoted by the same letter. A similar definition can be provided for the (deterministic) covariation of $[f, g]$ of two (continuous) functions f and g defined on some interval I. Without restriction of generality, we suppose that $0 \in I$. We set $[g, f](x),\ x \in I$, the pointwise limit (if it exists), when $\varepsilon \to 0$ of

$$\int_0^x (g(r + \varepsilon) - g(r))(f(r + \varepsilon) - f(r)) \frac{dr}{\varepsilon}, x \in I.$$

If $g = f$, we also denote it with $[f]$.

Remark 3.11. The following statements follows directly from the definition and are left to the reader. The reader may consult [43] for similar considerations. By default, the bounded variations function will be considered as cadlag.

1. If f has bounded variation then $\int_{]a,b]} g(s) d^- f(s)$ is the classical Lebesgue-Stieltjes integral $\int_{]a,b]} g df$. In particular, if $g = 1$, $\int_a^b g(s) d^- f(s) = f(b) - f(a)$.

2. If g has bounded variation, the following integration by parts formula holds: $\int_{]a,b]} g(s) d^- f(s)$ equals $g(b-) f(b) - \int_{]a,b]} f(s) dg(s)$.

3. A deterministic version of Theorem 3.6 can be easily stated, with respect to integrals of the type $\int_{]a,b]}$ instead of $\int_0^t$.

Besides $B = C([-T,0])$, we introduce another Banach space. Given a continuous function $g : [-T,0] \to \mathbb{R}$ we define the 2-**regularization variation** by $|g|_{2,var} := \sup_{0<\varepsilon<1} \int_{-T}^{0} (g(s+\varepsilon) - g(s))^2 \frac{ds}{\varepsilon}$. We define by V_2 the space of $g \in B$ such that $|g|_{2,var}$ is finite. If $\eta \in C([-T,0])$, we denote $|\eta|_\infty := \sup_{x\in[-T,0]} |\eta(x)|$.

Proposition 3.12. The functional $g \mapsto |g|_\infty| + |g|_{2,var}$ is a norm on V_2. Moreover V_2, equipped with that norm, is a Banach space.

Proof. To prove that $|\cdot|_{2,var}$ is a norm, the only non-obvious property consists in establishing the triangle inequality. This follows because of the triangle inequality related to the $L^2([-T,0])$-norm. It remains to show that any Cauchy sequence in V_2 converges to an element of V_2. Let (g_n) be such a sequence. Since $C([-T,0])$ is a Banach space, there is $g \in C([-T,0])$ such that g_n converges uniformly to g. Let $M > 0$. Since (g_n) is a Cauchy sequence with respect to $|\cdot|_{2,var}$, there is N such that if $n, m \geq N$, with

$$\int_{-T}^{0} ((g_n - g_m)(r+\varepsilon) - (g_n - g_m)(r))^2 \frac{dr}{\varepsilon} \leq M,$$

for every $0 < \varepsilon < T$. Let us fix $0 < \varepsilon < T$. Choosing $m = N$ in previous expression and letting n go to ∞ it follows that

$$\begin{aligned} \int_{-T}^{0} (g(r+\varepsilon) - g(r))^2 \frac{dr}{\varepsilon} &\leq 2\int_{-T}^{0} ((g - g_N)(r+\varepsilon) - (g - g_N(r)))^2 \frac{dr}{\varepsilon} \\ &+ 2\int_{-T}^{0} (g_N(r+\varepsilon) - g_N(r))^2 \frac{dr}{\varepsilon} \\ &\leq M + \int_{-T}^{0} (g_N(r+\varepsilon) - g_N(r))^2 \frac{dr}{\varepsilon}. \end{aligned}$$

Taking the supremum on $0 < \varepsilon < T$, we get that $|g|_{2,var}$ is finite and the result follows. ☐

V_2 is a Banach subspace of B. Given a continuous function $\psi : [0,T] \to \mathbb{R}$ be a continuous increasing function such that $\psi(0) = 0$, we denote by $V_{2,\psi}$ the space of functions $\eta : [-T,0] \to \mathbb{R}$ such that $[\eta]$ exists and equals ψ.

Proposition 3.13. $V_{2,\psi}$ *is a closed subspace of* V_2.

Proof. Let (g_n) be a sequence in $V_{2,0}$ i.e. such that $[g_n](x), x \in [-T,0]$ exists and equals ψ. We suppose that g_n converges to g in V_2. Now, for fixed $\varepsilon > 0$, $x \in [-T,0]$ we consider

$$I_\psi(\varepsilon, x) := -\int_{x}^{0} dr\, (g(r+\varepsilon) - g(r))^2 \frac{dr}{\varepsilon} - \psi(x). \tag{20}$$

We want to prove that for every $x \in \mathbb{R}$, $I_\psi(\varepsilon, x)$ converges to ψ, when $\varepsilon \to 0+$. The left-hand side of (20) is bounded by $4I_1(\varepsilon, N, x) + 4I_2(\varepsilon, N, x) + 4I_3(N, x)$, where, for $x \in [-T,0], N \in \mathbb{N}^*$,

$$\begin{aligned} I_1(\varepsilon, N, x) &= \left| \int_{x}^{0} ((g - g_N)(r+\varepsilon) - (g - g_N(r))^2 \frac{dr}{\varepsilon} \right| \\ I_2(\varepsilon, N, x) &= \left| \int_{x}^{0} ((g_N(r+\varepsilon) - g_N(r)))^2 \frac{dr}{\varepsilon} - [g_N](x) \right| \\ I_3(N, x) &= |[g_N](x) - \psi(x)|. \end{aligned} \tag{21}$$

We fix $x \in [-T, 0]$. Since $g_N \in V_{2,\psi}$ then $[g_N] = [g] = \psi$ and $I_3(N, x)$ equals zero. Let $\delta > 0$ and N such that for every $0 < \varepsilon < T$, $I_1(\varepsilon, N, x) \le \delta$. Choose ε_0 such that $I_2(\varepsilon, N, x) \le \delta$ if $0 < \varepsilon < \varepsilon_0$. Consequently for $0 < \varepsilon < \varepsilon_0$, then we have $I_\psi(\varepsilon, x) \le 2\delta$. This shows that $I_\psi(\varepsilon, x)$ converges to zero and so $V_{2,\psi}$ is a closed subspace of V_2. □

4 About infinite dimensional classical stochastic calculus

4.1 Generalities

Infinite dimensional stochastic calculus is an important tool for studying properties related to stochastic evolution problems, as stochastic partial differential equations, stochastic functional equations, as delay equations. When the evolution space is Hilbert a lot of work was performed, see typically the celebrated monograph of G. Da Prato and J. Zabczyk [8], in particular Section 2.3 mentions the corresponding notion of stochastic integral. An alternative, similar approach, is the one related to random fields, see e.g. [50] and [11]. Infinite dimensional stochastic calculus has been also developed in the framework of Gelfand triples, used for instance in [36]. Contributions exist also for Banach space valued stochastic integrals, see [3, 13, 12, 49], where the situation is more involved than in the Hilbert framework: the so-called reproducing kernel space cannot be described as $Im(Q^{1/2})$, as in Section 2.3, and the notion of Hilbert-Schmidt operator has to be substituted with the one of γ-radonifying.
The aim of our approach is to try to introduce suitable techniques which allow to treat typical infinite dimensional processes similarly to finite-dimensional diffusions. As we mentioned, stochastic process with values in infinite dimensional space will be indicated by a bold letter of the type $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ and so on. Let B a separable Banach space and $\mathbb{X}$ be a B-valued process. Consider $F : B \longrightarrow \mathbb{R}$ be of class C^2 in the Fréchet sense. One may ask what could be a good Itô formula in this framework. We are interested in an Itô type expansion of $F(\mathbb{X})$, keeping in mind that, classically, Itô formulae contain an integral term involving second order type derivatives and a quadratic variation. We first introduce some classical notions of quadratic variation very close to those of the literature, see [18, 33, 32], but in the spirit of calculus via regularization. Those above mentioned authors introduce in fact two quadratic variations: the *real* and the *tensor* quadratic variation. The definition below is a reformulation in terms of regularization of the *real* quadratic variation of $\mathbb{X}$. We prefer here, to avoid possible confusions, to replace the denomination *real* with *scalar*.

Definition 4.1. Consider a separable Banach spaces B. We say that a strongly measurable process $\mathbb{X}\colon [0,T] \times \Omega \to B$ admits a **scalar quadratic variation** if, for any $t \in [0,T]$, the limit, for $\epsilon \searrow 0$ of

$$[\mathbb{X}, \mathbb{X}]^{\epsilon,\mathbb{R}}_t := \int_0^t \frac{|\mathbb{X}_{r+\epsilon} - \mathbb{X}_r|^2_B}{\epsilon} dr,$$

exists in probability and it admits a continuous version. The limit process is called **scalar quadratic variation** of $\mathbb{X}$ and it is denoted with $[\mathbb{X}, \mathbb{X}]^{\mathbb{R}}$.

In Definition 1.4 of [17] the authors introduce the following definition.

Definition 4.2. Consider two separable Banach spaces B_1 and B_2. Suppose that either B_1 or B_2 is different from $\mathbb{R}$. Let $\mathbb{X}\colon [0,T] \times \Omega \to B_1$ and $\mathbb{Y}\colon [0,T] \times \Omega \to B_2$ be two strongly measurable processes. We say that $(\mathbb{X}, \mathbb{Y})$ admits a **tensor covariation** if the limit, for $\epsilon \searrow 0$ of the $B_1 \hat{\otimes}_\pi B_2$-valued processes

$$[\mathbb{X}, \mathbb{Y}]^{\otimes,\epsilon} := \int_0^{\cdot} \frac{(\mathbb{X}_{r+\epsilon} - \mathbb{X}_r) \otimes (\mathbb{Y}_{r+\epsilon} - \mathbb{Y}_r)}{\epsilon} dr$$

exists in the ucp sense (i.e. uniform convergence in probability). The limit process is called **tensor covariation** of $(\mathbb{X}, \mathbb{Y})$ and is denoted with $[\mathbb{X}, \mathbb{Y}]^{\otimes}$. The tensor covariation $[\mathbb{X}, \mathbb{X}]^{\otimes}$ is called **tensor quadratic variation** of $\mathbb{X}$ and denoted with $[\mathbb{X}]^{\otimes}$.

Remark 4.3. Let $\mathbb{X}, \mathbb{Y}$ be strongly measurable processes defined on $[0,T] \times \Omega$ with values respectively on B_1 and B_2. We have the following.

1. If $\mathbb{X}$ has a zero scalar quadratic variation and $\mathbb{Y}$ has a scalar quadratic variation then $[\mathbb{X}, \mathbb{Y}]^{\otimes} = [\mathbb{Y}, \mathbb{X}]^{\otimes} = 0$. Moreover $\mathbb{X} + \mathbb{Y}$ has a scalar quadratic variation and $[\mathbb{X} + \mathbb{Y}]^{\mathbb{R}} = [\mathbb{Y}]^{\mathbb{R}}$;

2. If $\mathbb{X}$ is a bounded variation process then $\mathbb{X}$ admits a zero scalar quadratic variation.

3. Let $\mathbb{M}$ be a local martingale with values in a separable Hilbert space H. Then it has scalar quadratic variation.

4. If $B = \mathbb{R}^n$ the space $B\hat{\otimes}_\pi B$ is associated with the space of $n \times n$ real matrices, as follows. Let $(e_i, 1 \le i \le n)$ be the canonical orthonormal basis of $\mathbb{R}^n$. A matrix $A = (a_{ij})$ is naturally associated with the element $\sum_{i,j=1}^n a_{ij} e_i \otimes e_j$. $\mathbb{X} = (X^1, \ldots, X^n)$ admits all its mutual covariations if and only if $\mathbb{X}$ admits a tensor quadratic variation. Moreover $[\mathbb{X}, \mathbb{X}] = \sum_{i,j=1}^n [X_i, X_j] e_i \otimes e_j$.

Items 1. and 2. are easy to establish. Item 3. is stated in Remark 4.9 of [22]. Item 4. constitutes an easy exercise, but it was stated in Section 6.2.1 of [15].

Let us consider now $F : B \longrightarrow \mathbb{R}$ be of class C^2. In particular $DF : B \longrightarrow \mathcal{L}(B; \mathbb{R}) := B^*$ and $D^2F : B \longrightarrow \mathcal{L}(B; B^*) \cong \mathcal{B}i(B, B) \cong (B\hat{\otimes}_\pi B)^*$ are continuous. As a first attempt, we expect to obtain an Itô formula type expansion of the following type.

$$F(\mathbb{X}_t) = F(\mathbb{X}_0) +'' \int_0^t {}_{B^*}\langle DF(\mathbb{X}_s), d\mathbb{X}_s \rangle_B {}'' + \frac{1}{2} \int_0^t {}_{(B\hat{\otimes}_\pi B)^*}\langle D^2F(\mathbb{X}_s), d[\mathbb{X}]_s \rangle_{B\hat{\otimes}_\pi B}. \tag{22}$$

This supposes of course that the tensor covariation $[\mathbb{X}, \mathbb{X}]^{\otimes}$ exists and it has bounded variation. A reasonable sufficient condition for this demands that the scalar quadratic variation $[\mathbb{X}, \mathbb{X}]^{\mathbb{R}}$ exists. A formal proof of the Itô formula, inspired from the one-dimensional case could be the following. Let $\varepsilon > 0$. We have

$$\int_0^t \frac{F(\mathbb{X}_{s+\epsilon}) - F(\mathbb{X}_s)}{\epsilon} ds \xrightarrow[\epsilon \to 0]{ucp} F(\mathbb{X}_t) - F(\mathbb{X}_0), t \in [0,T].$$

By a Taylor's expansion the left-hand side equals the sum of

$$\int_0^t {}_{B^*}\langle DF(\mathbb{X}_s), \frac{\mathbb{X}_{s+\epsilon} - \mathbb{X}_s}{\epsilon} \rangle_B ds \; + \int_0^t {}_{(B\hat{\otimes}_\pi B)^*}\langle D^2F(\mathbb{X}_s), \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2}{\epsilon} \rangle_{B\hat{\otimes}_\pi B} ds + R(\epsilon, t),$$

where $R(\varepsilon, \cdot)$ converges ucp to zero. Consequently, from previous formal proof, requires a good notion of quadratic variation. Moreover the first (stochastic) integral needs to be defined. The following natural obstacles problems appear.

- In many interesting cases mentioned at the beginning of Section 4.1, $\mathbb{X}$ is not a semimartingale, and it has not even a scalar and tensor quadratic variations.

- Stochastic integration, when the integrator takes values in a Banach space is not an easy task.

4.2 Tensor covariation and operator-valued covariation

In Definition 4.2 we introduced the notion of tensor covariation in the spirit of Metivier Pellaumail. Before proceeding and introducing the more general definition of χ-*covariation* we devote some space recalling another (somehow classical) definition used for example by several authors in stochastic calculus in Hilbert spaces, as Da Prato and Zabczyk.

Let H be a separable Hilbert spaces and $\mathbb{M}, \mathbb{N}$ two H-valued continuous local martingales.The first covariation was denoted by $[\mathbb{M}, \mathbb{N}]^{\otimes}$, the second one will be denoted by $[\mathbb{M}, \mathbb{N}]^{cl}$. A first difference arises by the fact $[\mathbb{M}, \mathbb{N}]^{\otimes}$ takes values in $H\hat{\otimes}_\pi H$ and $[\mathbb{M}, \mathbb{N}]^{cl}$ lives in $\mathcal{L}_1(H)$.

We remind from Section 2.1 that every element $u \in H\hat{\otimes}_\pi H$ is isometrically associated with an element T_u in the space of nuclear operators $\mathcal{L}_1(H)$, so it makes sense to compare Definition 4.2 and the definition of operator-valued covariation.

Definition 4.4. Let $\mathbb{X}$ and $\mathbb{Y}$ be two H-valued continuous processes. We say that $(\mathbb{X}, \mathbb{Y})$ admits an **operator-valued covariation**, denoted by $[\mathbb{X}, \mathbb{Y}]^{cl}$, if there exists a bounded variation process $\mathbb{V}$ with values in $\mathcal{L}_1(H)$, denoted by $[\mathbb{X}, \mathbb{Y}]^{cl}$, such that, for every $a, b \in H$, the covariation (in the sense of regularization) of $\langle a, \mathbb{X}\rangle$ and $\langle b, \mathbb{Y}\rangle$ equals $\langle \mathbb{V}a, b\rangle$. In the sequel we will also use the notation $[\mathbb{X}, \mathbb{Y}]^{cl}(a, b) := \langle \mathbb{V}a, b\rangle$. In other words the continuous linear functional $[\mathbb{X}, \mathbb{Y}]^{cl}(a, \cdot)$ is Riesz-identified to $\mathbb{V}a$. We will of course identify without further mention $\mathbb{V}(a)$ and $[\mathbb{X}, \mathbb{Y}]^{cl}(a, \cdot)$.

Remark 4.5.

1. If $\mathbb{X}$ and $\mathbb{Y}$ are local martingales then $\langle a, \mathbb{X}\rangle$ and $\langle b, \mathbb{Y}\rangle$ are real local martingales and previous covariations in the sense of regularization are classical covariations of martingales, see Proposition 2.4 (3) of [43] and item 5. of Proposition 3.4.
2. If $\mathbb{X} = \mathbb{Y}$ is a local martingale and $\mathbb{V}$ is $[\mathbb{X}, \mathbb{X}]^{cl}$, then, by Doob-Meyer decomposition, $\mathbb{V}$ fulfills the following property. For every $a, b \in H$, we have $\langle a, \mathbb{X}\rangle\langle b, \mathbb{X}\rangle - \langle \mathbb{V}a, b\rangle$ is a local martingale and obviously $\langle \mathbb{V}.(a), a\rangle \geq 0$ is a non-negative increasing process for every $a \in H$; in particular, for every $t \in [0, T]$, $\mathbb{V}_t$ is a non-negative map of $\mathcal{L}_1(H)$.
3. Proposition 3.12 in [8] states that for a continuous square integrable martingale $\mathbb{X}$, the quadratic variation exists (and is unique). By stopping arguments, this can be extended to every local martingale $\mathbb{X}$.

The proposition below illustrates some relations between the tensor covariation and the operator-valued covariation.

Proposition 4.6.

1. The operator-valued covariation is unique.
2. If $(\mathbb{X}, \mathbb{Y})$ admits a tensor covariation then, it also have an operator-valued covariation and, after the identification above between $H\hat{\otimes}_\pi H$ and $\mathcal{L}_1(H)$, they are equal.
   In particular, for every $a \in H, b \in H$, $\langle j(a^* \otimes b^*), [\mathbb{X}, \mathbb{Y}]^{\otimes}\rangle = [\langle \mathbb{X}, a\rangle, \langle \mathbb{Y}, b\rangle]$.
3. If $\mathbb{X}$ and $\mathbb{Y}$ are local martingales then they admit a scalar, tensor and operator-valued covariations.

Proof. Let $\varepsilon > 0$. Taking into account Lemma 2.1, choosing $\varphi \in (H\hat{\otimes}_\pi H)^*$ of the type $\varphi = j(a^* \otimes b^*)$ where $a, b, \in H$, we have

$$
\begin{aligned}
&{}_{(H\hat{\otimes}_\pi H)^*}\left\langle \varphi, \frac{1}{\varepsilon}\int_0^t (\mathbb{X}_{s+\varepsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\varepsilon} - \mathbb{Y}_s) ds \right\rangle_{H\hat{\otimes}_\pi H} \\
&= \frac{1}{\varepsilon}\int_0^t ds \ {}_{(H\hat{\otimes}_\pi H)^*}\langle \varphi, (\mathbb{X}_{s+\varepsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\varepsilon} - \mathbb{Y}_s)\rangle_{H\hat{\otimes}_\pi H} = \frac{1}{\varepsilon}\int_0^t ds \langle \mathbb{X}_{s+\varepsilon} - \mathbb{X}_s, a\rangle\langle \mathbb{Y}_{s+\varepsilon} - \mathbb{Y}_s, b\rangle.
\end{aligned}
\tag{23}
$$

So the first expression of the equality above converges if the covariation of the real processes $\langle \mathbb{X}, a\rangle$ and $\langle \mathbb{Y}, b\rangle$ exists.

1. Let be two $\mathcal{L}_1(H)$- valued processes $\mathbb{V}^1, \mathbb{V}^2$ verifying

$$\langle \mathbb{V}^i(a), b\rangle = [\langle \mathbb{X}, a\rangle, \langle \mathbb{Y}, b\rangle], i = 1, 2,$$

   for every $a, b \in H$. Let $\mathbb{U}^i$ be the associated process with values in $H\hat{\otimes}_\pi H$ in the sense of the usual isomorphism (7) between $H\hat{\otimes}_\pi H$ and $\mathcal{L}^1(H)$. Then, taking into account (23), for every $t \in [0, T]$ we have $\langle \varphi, \mathbb{U}^1_t\rangle = \langle \varphi, \mathbb{U}^2_t\rangle$, for every φ in $(H\hat{\otimes}_\pi H)^*$ of the type $\varphi = j(a^* \otimes b^*)$. Since, by Lemma 4.17 of [22], the algebraic tensor product $H^* \otimes H^*$ is weakly-star dense in $(H\hat{\otimes}_\pi H)^*$, the uniqueness property $\mathbb{U}^1 = \mathbb{U}^2$ holds.

2. Suppose that $[\mathbb{X}, \mathbb{Y}]^{\otimes}$ exists. Let $a, b \in H$ and set $\varphi = j(a^* \otimes b^*)$. If $[\mathbb{X}, \mathbb{Y}]^{\otimes}$ exists, by (23), then $\langle \varphi, [\mathbb{X}, \mathbb{Y}]^{\otimes}\rangle = [\langle \mathbb{X}, a\rangle, \langle \mathbb{Y}, b\rangle]$. We set now $[\mathbb{X}, \mathbb{Y}]^{cl}(a, b) = [\langle \mathbb{X}, a\rangle, \langle \mathbb{Y}, b\rangle]$. By the usual isomorphism (7), between $H\hat{\otimes}_\pi H$ and $\mathcal{L}_1(H)$, according to the convention in Definition 4.4, $a \mapsto [\mathbb{X}, \mathbb{Y}]^{cl}(a, \cdot)$ defines an $\mathcal{L}_1(H)$-valued process.

3. If $\mathbb{M}$ and $\mathbb{N}$ are local martingales, then $\mathbb{M}$ and $\mathbb{N}$ admit a scalar quadratic because of Proposition 1.7 of [17]. Moreover $(\mathbb{M}, \mathbb{N})$ admits a tensor covariation by Lemma 4.16 of [22]. By previous item, it also admits an operator-valued covariation.

□

We specify now our result for some particular Hilbert valued martingales, namely for the Brownian martingales. The framework is the same we used in Subsection 2.3.

Proposition 4.7. Let U and H two separable real Hilbert spaces. Let Q be a positive self-adjoint, injective operator in $\mathcal{L}(U)$. We set $U_0 := Q^{1/2}(U)$ and we consider $\mathbb{W}^Q = \{\mathbb{W}^Q_t : 0 \le t \le T\}$ an U-valued Q-Wiener process with $\mathbb{W}^Q_0 = 0$, $\mathbb{P}$ a.s. Let us suppose that $(\mathscr{F}_t)$ is the canonical filtration generated by $\mathbb{W}^Q$ and consider a predictable $\mathcal{L}_2(U_0, H)$-valued process (Φ_t) such that

$$\int_0^T \mathrm{Tr}[\Phi_r Q^{1/2}(\Phi_r Q^{1/2})^*]dr < \infty \qquad \mathbb{P} - a.s. \tag{24}$$

and the process $\mathbb{M}$ defined as $\mathbb{M}_t = \int_0^t \Phi_r d\mathbb{W}^Q{}_r, t \in [0, T]$. We have the following.

1. $[\mathbb{M}, \mathbb{M}]^{cl}_t = \int_0^t Q^\Phi_r dr$ where

$$Q^\Phi_t = (\Phi_t Q^{1/2})(\Phi_t Q^{1/2})^*.$$

2. $[\mathbb{M}, \mathbb{M}]^{\otimes}_t$ is characterized by $\langle j(a^* \otimes b^*), [\mathbb{M}, \mathbb{M}]^{\otimes}_t\rangle = \int_0^t \langle a, Q^\Phi_s b\rangle ds$ for any $a, b \in H$.

3. For every $\varphi \in (H\hat{\otimes}_\pi H)^*$, we have

$$_{(H\hat{\otimes}_\pi H)^*}\langle \varphi, [\mathbb{M}, \mathbb{M}]^{\otimes}_t\rangle_{H\hat{\otimes}_\pi H} = \int_0^t \mathrm{Tr}(L_\varphi Q^\Phi_r)dr. \tag{25}$$

Proof. 1. It is a consequence of Theorem 4.12 in [8], where the result is stated under the hypothesis that the expectation of (24) is finite. It can be extended to the general case with a stopping argument.

2. From Proposition 2.6 we know that $\mathbb{M}$ is an H-valued local martingale. So by item 3. of Proposition 4.6, $\mathbb{M}$ admits both a tensor and an operator-valued quadratic variation; thanks to item 2. of the same proposition, they coincide once we have identified $H\hat{\otimes}_\pi H$ with $\mathcal{L}^1(H)$. Item 2. of Proposition 4.6 describes also the relation between the two and gives the evaluations $[\mathbb{M},\mathbb{M}]^{\otimes}_t(a^*\otimes b^*)$. Since, by Lemma 4.17 of [22], the algebraic tensor product $H^*\otimes H^*$ is weakly-star dense in $(H\hat{\otimes}_\pi H)^*$ the evaluation of $[\mathbb{M},\mathbb{M}]^{\otimes}_t$ on $j(a^*\otimes b^*)$ characterizes $[\mathbb{M},\mathbb{M}]^{\otimes}_t$.

3. It follows from Proposition 2.3 and Proposition 2.4.

□

Lemma 4.8. Under the same assumptions of Proposition 4.7 we consider an $\mathcal{L}(H)$-valued process $(\mathbb{Y}_t)$ such that

$$\int_0^T \mathrm{T}r\left(\mathbb{Y}_r\Phi_r Q^{1/2}(\mathbb{Y}_r\Phi_r Q^{1/2})^*\right)dr<\infty \qquad \mathbb{P}-a.s. \tag{26}$$

Denote by $\mathbb{J}_t$ the element of $(H\hat{\otimes}_\pi H)^*$ corresponding to $\mathbb{Y}_t$ through the isomorphism described in (7). Then

$$\int_0^t {}_{(H\hat{\otimes}_\pi H)^*}\langle \mathbb{J}_r, d[\mathbb{M},\mathbb{M}]^{\otimes}_r\rangle_{H\hat{\otimes}_\pi H}dr=\int_0^t \mathrm{T}r\left[Y_r(\Phi_r Q^{1/2})(\Phi_r Q^{1/2})^*\right]dr.$$

Proof. It is a consequence of item 3. of Proposition 4.7 and by Lemma 4.9 below. □

Lemma 4.9. Consider L and T in the sense of the lines before Proposition 2.3. Let $\dot{G}:[0,T]\to\mathcal{L}(H)$, and $\dot{g}:[0,T]\to(H\hat{\otimes}_\pi H)^*$ such that for every $r\in[0,T]$, $\dot{G}(r)=L_{\dot{g}(r)}$ is Lebesgue-Bochner integrable on $[0,T]$. We define $G:[0,T]\to\mathcal{L}(H)$, by $G(t)=\int_0^t\dot{G}(r)dr$ and $g(t)=\int_0^t\dot{g}(r)dr$. Let $J:[0,T]\to\mathcal{L}_1(H)$ and $j:[0,T]\to H\hat{\otimes}_\pi H$ such that for every $r\in[0,T]$, $J(r)=T_{j(r)}$.

If $\int_0^T\|J(r)\dot{G}(r)\|_{\mathcal{L}_1(H)}dr<\infty$, then

$$\int_0^t {}_{(H\hat{\otimes}_\pi H)^*}\langle \dot{g}(r), j(r)\rangle_{H\hat{\otimes}_\pi H}dr=\int_0^t \mathrm{T}r\left(\dot{G}(r)J(r)\right)dr<\infty.$$

Proof. The proof follows first showing the equality for step functions j (resp. J), and then passing to the limit. □

5 Notion of χ-covariation

5.1 Basic definitions

We introduce now a more general notion of covariation (and quadratic variation) than the ones discussed before, which are essentially only suitable for semimartingale processes. The basic concepts were introduced in [15, 17, 14]. The notion of χ-quadratic variation and χ-covariation is based on the notion of Chi-subspace. Let B, B_1, B_2 be separable Banach spaces.

Definition 5.1. A Banach subspace χ continuously injected into $(B_1 \hat{\otimes}_\pi B_2)^*$ will be called **Chi-subspace** (of $(B_1 \hat{\otimes}_\pi B_2)^*$). In particular it holds

$$\| \cdot \|_\chi \geq \| \cdot \|_{(B_1 \hat{\otimes}_\pi B_2)^*}. \tag{27}$$

Typical examples of Chi-subspaces are the following.

1. Let ν_1 (resp. ν_2) be a dense subspace of B_1^* (resp. B_2^*) then a typical Chi-subspace (of $(B_1 \hat{\otimes}_\pi B_2)^*$) is the topological projective tensor product of ν_1 with ν_2, denoted by $\nu_1 \hat{\otimes}_\pi \nu_2$. This is naturally embedded in $(B_1 \hat{\otimes}_\pi B_2)^*$ as recalled in Lemma 2.1.

2. In particular, if ν_0 is dense subspace of B^*, then $\chi := \nu_0 \hat{\otimes}_\pi \mathbb{R}$ is a Chi-subspace of $(B \hat{\otimes}_\pi \mathbb{R})^*$, which can be naturally identified with B^*. By a slight abuse of notations one could say that ν_0 is a Chi-subspace of B^*.

3. Let B is a separable Hilbert space H and A is a generator of a C_0-semigroup on H, see [20] and [37] Chapter 1 for a complete treatment of the subject. Denote with $D(A)$ and $D(A^*)$ respectively the domains of A and A^* endowed with the graph norm, see again [37] Chapter 1 or [20] Chapter II. Then a typical Chi-subspace of $(H \hat{\otimes}_\pi H)^*$ can be obtained setting $\chi := \nu_0 \hat{\otimes}_\pi \nu_0$ and $\nu_0 = D(A^*)$ endowed with its the graph norm.

4. If $B = C([-\tau, 0])$, then χ could be the space $\mathcal{M}([-\tau, 0]^2)$ of finite signed measures on $[-\tau, 0]^2$. Other examples of χ-subspaces are given in Section 5.2.

5. It is not difficult to see that a direct sum of Chi-subspaces is a Chi-subspace. This produces further examples of Chi-subspaces, see Proposition 3.16 of [17].

Let $\mathbb{X}$ be a B_1-valued and $\mathbb{Y}$ be a B_2-valued process. We suppose $\mathbb{X}$ to be continuous. Let χ be a Chi-subspace of $(B_1 \hat{\otimes} B_2)^*$. We denote by $\mathcal{C}([0,T])$ space of real continuous processes equipped with the ucp topology. If $\varepsilon > 0$, we denote by $[\mathbb{X}, \mathbb{Y}]^\epsilon$ be the application

$$[\mathbb{X}, \mathbb{Y}]^\epsilon : \chi \longrightarrow \mathcal{C}([0,T])$$

defined by

$$\phi \mapsto \left(\int_0^t {}_\chi\langle \phi, \frac{J\left((\mathbb{X}_{r+\epsilon} - \mathbb{X}_r) \otimes (\mathbb{Y}_{r+\epsilon} - \mathbb{Y}_r) \right)}{\epsilon} \rangle_{\chi^*} \, dr \right)_{t \in [0,T]},$$

where $J : B_1 \hat{\otimes}_\pi B_2 \to (B_1 \hat{\otimes}_\pi B_2)^{**}$ is the canonical injection between a Banach space and its bidual (omitted in the sequel).

Definition 5.2. $(\mathbb{X}, \mathbb{Y})$ **admits a χ-covariation** if

(H1) For all $(\epsilon_n) \downarrow 0$ it exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \frac{\left\| (\mathbb{X}_{r+\epsilon_{n_k}} - \mathbb{X}_r) \otimes (\mathbb{Y}_{r+\epsilon_{n_k}} - \mathbb{Y}_r) \right\|_{\chi^*}}{\epsilon_{n_k}} dr \quad < \infty \quad a.s.$$

(H2) There exists a process, denoted by $[\mathbb{X}, \mathbb{Y}]^\chi : \chi \longrightarrow \mathcal{C}([0,T])$ such that

$$[\mathbb{X}, \mathbb{Y}]^\epsilon(\phi) \xrightarrow[\epsilon \to 0]{ucp} [\mathbb{X}, \mathbb{Y}](\phi), \quad \forall \, \phi \in \chi.$$

(H3) There is a χ^*-valued bounded variation process $\widetilde{[\mathbb{X},\mathbb{Y}]}^\chi : [0,T]\times\Omega \to \chi^*$, such that $\widetilde{[\mathbb{X},\mathbb{Y}]}^\chi{}_t(\phi) = [\mathbb{X},\mathbb{Y}]^\chi(\phi)_t, \forall t\in[0,T]$ a.s. for all $\phi \in \chi$.

Definition 5.3. If $B = B_1 = B_2$, and $\mathbb{X} = \mathbb{Y}$, we say that $\mathbb{X}$ has a **χ-quadratic variation**, if $(\mathbb{X},\mathbb{X})$ admits a χ-covariation.

Definition 5.4. When $(\mathbb{X},\mathbb{Y})$ admits a χ-covariation, the χ^*-valued process $\widetilde{[\mathbb{X},\mathbb{Y}]}$ (which is indeed a modification of $[\mathbb{X},\mathbb{Y}]$) will be called **χ-covariation** of $(\mathbb{X},\mathbb{Y})$. If $\mathbb{X}$ admits a quadratic variation, the χ^*-valued process $\widetilde{[\mathbb{X},\mathbb{X}]}$, also denoted by $\widetilde{[\mathbb{X}]}$, is called **quadratic variation** of $\mathbb{X}$.

Remark 5.5.

1. $\widetilde{[\mathbb{X}]}^\chi$ will be the quadratic variation intervening in the second order derivative term of Itô's formula stated in Theorem 6.3, which will make formula (22) rigorous.
2. For every fixed $\phi\in\chi$, the real processes $((\widetilde{[\mathbb{X},\mathbb{Y}]}^\chi)(\phi), t\in[0,T])$ and $([\mathbb{X},\mathbb{Y}]^\chi(\phi)_t, t\in[0,T])$, are indistinguishable.
3. The χ^*-valued process $\widetilde{[\mathbb{X},\mathbb{Y}]}$ is weakly star continuous, i.e. $\widetilde{[\mathbb{X}]}(\phi)$ is continuous for every fixed $\phi\in\chi$, see [17] Remark 3.10 1.

A particular situation arises when $\chi = (B_1\hat{\otimes}_\pi B_2)^*$.

Definition 5.6.

- We say that $(\mathbb{X},\mathbb{Y})$ admits a **global covariation** if it admits a χ-covariation with $\chi = (B_1\hat{\otimes}_\pi B_2)^*$. In this case we will omit the mention χ in $\widetilde{[\mathbb{X},\mathbb{Y}]}^\chi$ and $[\mathbb{X},\mathbb{Y}]^\chi$.
- The modification $\widetilde{[\mathbb{X},\mathbb{X}]}$, which is a $(B\hat{\otimes}_\pi B)^{**}$-valued process is also called **global quadratic variation** of $\mathbb{X}$.

Remark 5.7. The following statements are easy to establish, see Remarks 4.8 and 4.10 of [22].

1. If $\mathbb{X}$ has zero scalar quadratic variation then $\mathbb{X}$ has a zero tensor quadratic variation and $\mathbb{X}$ has a zero global quadratic variation.
2. If $\mathbb{X}$ and $\mathbb{Y}$ have a scalar quadratic variation and $(\mathbb{X},\mathbb{Y})$ has a tensor covariation, then $(\mathbb{X},\mathbb{Y})$ admit a global covariation. Moreover $\widetilde{[\mathbb{X},\mathbb{Y}]} = [\mathbb{X},\mathbb{Y}]^\otimes$, where the equality holds in $B_1\hat{\otimes}_\pi B_2$.
3. If $(\mathbb{X},\mathbb{Y})$ admits a global covariation, then they it admits a χ-covariation for every Chi-subspace χ. Moreover $\widetilde{[\mathbb{X},\mathbb{Y}]}^\chi_t(\phi) = \widetilde{[\mathbb{X},\mathbb{Y}]}_t(\phi)$, for every $t\in[0,T], \phi\in\chi$.

Proposition 5.8. Let $\mathbb{X}^i = \mathbb{M}^i + \mathbb{V}^i, i = 1,2$ be two semimartingales with values in B_i. Let χ any Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$. Then $(\mathbb{X}^1,\mathbb{X}^2)$ admits a χ-covariation and
$\widetilde{[\mathbb{X}^1,\mathbb{X}^2]}^\chi(\phi) = {}_{H\hat{\otimes}_\pi H}\langle[\mathbb{M}^1,\mathbb{M}^2]^\otimes, \phi\rangle_{(H\hat{\otimes}_\pi H)^*}, \ \forall\phi\in\chi.$

Proof. By item 2. of Remark 4.3, $\mathbb{V}$ has a zero scalar quadratic variation. By Proposition 4.6 3. $(\mathbb{M}_1,\mathbb{M}_2)$ admits a tensor covariation. By item 1. of Remark 4.3 and the by linearity of tensor covariation it follows that $[\mathbb{X}^1,\mathbb{X}^2]^\otimes = [\mathbb{M}^1,\mathbb{M}^2]^\otimes$. Again by point 1. of Remark 4.3, $\mathbb{X}^1$ and $\mathbb{X}^2$ have a scalar quadratic variation. Again by Remark 5.7 2., $(\mathbb{X}^1,\mathbb{X}^2)$ admits a global quadratic variation and so the result follows by Remark 5.7 3. □

Indeed the notion of global covariation is closely related to the weak-$*$ convergence in $(B\hat{\otimes}_\pi B)^{**}$. If the probability space Ω were a singleton, i.e. in the deterministic case, if $\mathbb{X}$ admits a χ-quadratic variation then

$$[\mathbb{X},\mathbb{X}]^\epsilon_t \xrightarrow[\epsilon\to 0]{w^*} \widetilde{[\mathbb{X},\mathbb{X}]}_t, \forall t \in [0,T].$$

As we mentioned, the notion of weak Dirichlet process admits a generalization to the Banach space case.

Definition 5.9. Let $\mathbb{V},\mathbb{X}$ be two B-valued continuous processes and ν_0 be a dense subspace of B^*. We set $\nu = \nu_0 \otimes \mathbb{R}$.

1. $\mathbb{V}$ is said $(\mathscr{F}_t)-\nu$-martingale orthogonal process if for any real $(\mathscr{F}_t)$-local martingale N we have $[\mathbb{V},N]^\nu = 0$

2. $\mathbb{X}$ is said $(\mathscr{F}_t)-\nu$-weak Dirichlet if it is the sum of an a $(\mathscr{F}_t)$-local martingale $\mathbb{M}$ and an $(\mathscr{F}_t)-\nu$-martingale orthogonal process.

Remark 5.10.

1. If $B=\mathbb{R}$, then any $(\mathscr{F}_t)-\nu$-weak Dirichlet (resp. $(\mathscr{F}_t)-\nu$-martingale orthogonal) process is a real $(\mathscr{F}_t)$-weak Dirichlet (resp. $(\mathscr{F}_t)$-martingale orthogonal) process.

2. The notions of Dirichlet, weak Dirichlet, ν-weak Dirichlet process depend on an underlying filtration $(\mathscr{F}_t)$. When not necessary it will be omitted. We will speak about Dirichlet (resp. weak Dirichlet, ν-weak Dirichlet process) instead of $(\mathscr{F}_t)$-Dirichlet (resp. $(\mathscr{F}_t)$-weak Dirichlet, $(\mathscr{F}_t)$-ν-weak Dirichlet process).

Remark 5.11. Let H be a separable Hilbert space and ν_0 be a Banach space continuously embedded in H^*. We set $\chi = \nu_0\hat{\otimes}_\pi\nu_0,\ \nu = \nu_0 \otimes \mathbb{R}$. A zero χ-quadratic variation process is a ν-weak orthogonal process. This was the object of Proposition 4.29 in [22].

We introduce below the useful notion of $\bar{\nu}_0$-semimartingale.

Definition 5.12. Let $(\mathbb{S}_t, t\in[0,T])$ be an H-valued progressively measurable process and a Banach space $\bar{\nu}_0$ in which H is continuously embedded. $\mathbb{S}$ is said $\bar{\nu}_0$-**semimartingale** (or more precisely $\bar{\nu}_0-(\mathscr{F}_t)$-semimartingale) if it is the sum of a local martingale $\mathbb{M}$ and a process $\mathbb{A}$ which finite variation as $\bar{\nu}_0$-valued process.

Proposition 5.13.

1. An H-valued $\bar{\nu}_0$-semimartingale is a semimartingale as $\bar{\nu}_0$-valued process.

2. The decomposition of a $\bar{\nu}_0$-semimartingale is unique, if for instance we prescribe that $\mathbb{A}_0 = 0$ a.s.

Proof.

1. Indeed an H-valued martingale is clearly a $\bar{\nu}_0$-valued martingale and consequently, by stopping arguments, an H-valued local martingale is a $\bar{\nu}_0$-semimartingale.

2. It follows by the decomposition of a $\bar{\nu}_0$-valued semimartingale.

□

The uniqueness of the decomposition of a $\bar{\nu}_0$-semimartingale allows to define an extension of Itô integral, that will still denoted in the same way.

Definition 5.14. Let H, E be separable Hilbert spaces. Let $\bar{\nu}_0$ be a Banach space in which H is continuously injected and $\mathbb{S} = \mathbb{M} + \mathbb{A}$ be a H-valued which is a $\bar{\nu}_0$-semimartingale. Suppose that $(\mathbb{Y}_t)$ is a progressively measurable, such that

$$\int_0^T \|\mathbb{Y}_r\|^2_{\mathcal{L}(H,E)} d[\mathbb{M}]^{\mathbb{R},cl}_r + \int_0^T \|\mathbb{Y}_r\|_{\mathcal{L}(\bar{\nu}_0,E)} d\|A\|_r < \infty, \tag{28}$$

where $r \mapsto \|\mathbb{A}(r)\|$ is the total variation function of $r \mapsto \mathbb{A}(r)$. We denote by $\int_0^t \mathbb{Y}_s d\mathbb{S}_s := \int_0^t \mathbb{Y}_s d\mathbb{M}_s + \int_0^t \mathbb{Y}_s d\mathbb{A}_s, t \in [0,T]$.

Proposition 5.15. Let H be a separable Hilbert space, continuously embedded in a Banach space $\bar{\nu}_0$. Let $\mathbb{S} = \mathbb{M} + \mathbb{A}$ be an H-valued process which is a $\bar{\nu}_0$-semimartingale. We set $\nu_0 = \bar{\nu}_0^*$. We set $\chi = \nu_0 \hat{\otimes}_\pi \nu_0$.

1. $\mathbb{A}$ admits a zero χ-quadratic variation.
2. $[\mathbb{M}, \mathbb{A}]^\chi = 0$.
3. $\mathbb{S}$ is a $\nu_0 \hat{\otimes}_\pi \mathbb{R}$-weak Dirichlet process.
4. $\mathbb{S}$ has a χ-quadratic variation. Moreover $\widetilde{[\mathbb{S},\mathbb{S}]}^\chi(\phi) = {}_{(H\hat{\otimes}_\pi H)^*}\langle \phi, [\mathbb{M},\mathbb{M}]^{\otimes} \rangle_{H\hat{\otimes}_\pi H}$.

Proof. 1. Observe that, thanks to Lemma 3.18 in [17], it will be enough to show that

$$I(\epsilon) := \frac{1}{\epsilon} \int_0^T |(\mathbb{A}(r+\epsilon) - \mathbb{A}(r)) \otimes^2 |_{\chi^*} dr \xrightarrow{\epsilon \to 0} 0, \qquad \text{in probability.} \tag{29}$$

In fact, identifying χ^* with the space of bounded bilinear functions on ν_0, i.e. $\mathcal{B}i(\nu_0, \nu_0)$, recalling that $\nu_0 = \bar{\nu}_0^*$, the left-hand side of (29) gives

$$\begin{aligned} I(\epsilon) &= \frac{1}{\epsilon} \int_0^T \sup_{|\phi|_{\nu_0}, |\psi|_{\nu_0} \le 1} |\langle (\mathbb{A}(r+\epsilon) - \mathbb{A}(r)), \phi \rangle \langle (\mathbb{A}(r+\epsilon) - \mathbb{A}(r)), \psi \rangle| \, dr \\ &\le \int_0^T \|\mathbb{A}(r+\epsilon) - \mathbb{A}(r)\|^2_{\bar{\nu}_0} dr. \end{aligned}$$

Since $\mathbb{A}$ is an $\bar{\nu}_0$-valued bounded variation process, previous quantity converges to zero, by Remark 4.3 2.

2. It follows by very close arguments. In particular, an adaptation of Lemma 3.18 of [17] shows that it will be enough to show that

$$J(\epsilon) := \frac{1}{\epsilon} \int_0^T |(\mathbb{A}(r+\epsilon) - \mathbb{A}(r)) \otimes \mathbb{M}(r+\epsilon) - \mathbb{M}(r))|_{\chi^*} dr \xrightarrow{\epsilon \to 0} 0, \qquad \text{in probability.} \tag{30}$$

Then we use the fact that $\mathbb{M}$ is a $\bar{\nu}_0$-local martingale and therefore, by item 3. of Proposition 4.6, it has a scalar quadratic variation, as $\bar{\nu}_0$-valued process.

3. follows by Remark 5.11.

4. Indeed the bilinerarity of the χ-covariation and items 1. and 2. imply that $[\mathbb{S},\mathbb{S}]^\chi = [\mathbb{M},\mathbb{M}]^\chi$. The result follows then by Proposition 5.8.

□

Below we will state examples of processes having a χ-quadratic variation.

5.2 Window processes

Let $B = C([-\tau, 0])$, for some $\tau > 0$, $X = (X_t, t \in [0,T])$ be a real process and $\mathbb{X} = (X_t(\cdot), t \in [0,T])$, the corresponding **window process**, i.e. such that $X_t(x) = X_{t+x}, x \in [-\tau, 0]$. We start with some basic examples.

Proposition 5.16. If X has Hölder continuous paths with parameter $\gamma > \frac{1}{2}$ then $X(\cdot)$ has a zero scalar quadratic variation and therefore a global quadratic variation.

Proof. It follows directly from the definition and the Hölder path property. □

A typical example of such processes is fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ or the bifractional Brownian motion with parameters H, K and $HK > \frac{1}{2}$, see for instance [29, 40]. By Proposition 4.7 [17], the window of a classical Wiener process has no scalar quadratic variation so no global quadratic variation since condition (H1) cannot be fulfilled. For this reason, it is important to investigate if it has a χ-quadratic variation for a suitable subspace χ of $(B\hat{\otimes}_\pi B)^*$. In the framework of window processes, typical examples of χ are the following.

1. $\mathcal{M}([-T, 0]^2)$ equipped with the total variation norm.
2. $L^2([-\tau, 0]^2)$.
3. $\mathcal{D}_{0,0} = \{\mu(dx, dy) = \lambda\, \delta_0(dx) \otimes \delta_0(dy)\}$.
4. Let $\mathcal{D}_0$ be the one-dimensional space of measures obtained as multiple of the Dirac measure δ_0. The following linear subspace of $\mathcal{M}([-T, 0]^2)$ given by $\mathcal{D}_{0,0} \oplus L^2([-T, 0])\otimes\mathcal{D}_0 \oplus \mathcal{D}_0\otimes L^2([-T, 0]) \oplus L^2([-T, 0]^2)$. This is a Banach space, equipped with a self-explained sum of three norms. By the lines above Remark 3.5 in [17], that space is the Hilbert tensor product $(\mathcal{D}_0 \oplus L^2)\hat{\otimes}_h^2$.
5. $Diag := \{\mu(dx, dy) = g(x)\delta_y(dx)dy; g \in L^\infty([-T, 0])\}$.
6. The direct sum χ_0 of the spaces defined in 4. and 5. is a Chi-subspace. We remind item 5. at the beginning of Section 5.2.

Remark 5.17. The window Brownian motion $W(\cdot)$ does not have a χ -quadratic variation for $\chi = \mathcal{M}([-\tau, 0]^2)$. This follows because the bidual of $C([-\tau, 0]^2)$ is isometrically embedded into its bidual, and the window Brownian motion has no scalar quadratic variation. In particular condition (H1) of the χ-covariation cannot be fulfilled.

In all the other cases a classical Wiener process has a χ-quadratic variation. Indeed this also extends to the case of a generic finite quadratic variation process. The proposition below is the consequence of Propositions 4.8 and 4.15 of [17] and the fact that the direct sum of Chi-subspaces is a Chi-subspace.

From now on, in this section, for simplicity we set $\tau = T$.

Proposition 5.18. Let X be a finite quadratic variation process. Then $X(\cdot)$ has a χ_0-quadratic variation. Moreover, for $\mu \in \chi_0$, we have

$$[X(\cdot)]_t(\mu) = \int_{D_t} d\mu(x, y)[X]_{t-x}dx,$$

where D_t is the diagonal $\{(x, y) \in [-T, 0]^2 | -t \le x = y \le 0\}$.

In fact a Chi-subspace will plays the role of a *suitable* subspace of $(B\hat{\otimes}_\pi B)^*$, in which lives the second Fréchet derivative of a functional $F : B \to \mathbb{R}$ is forced to live, in view of expanding $F(\mathbb{X})$ via a Itô type formula of the type (22).

Example 5.19. Here are some typical particular cases of elementary functionals whose second derivatives belong to some Chi-spaces mentioned above. The details of the verification are left to the reader.

1. $F(\eta) = f(\eta(0))$ where $f : \mathbb{R} \to \mathbb{R}$ is of class C^2. Then $D^2F(\eta) \in \mathcal{D}_{0,0}$ for every $\eta \in B$.

2. $F(\eta) = \left(\int_{-T}^0 \eta(s)ds\right)^2$. Then, for every $\eta \in B$, $D^2F(\eta) \in (\mathcal{D}_0 \oplus L^2)\hat{\otimes}_h^2$.

3. $F(\eta) = \int_{-T}^0 \eta(s)^2 ds$. In this case $D^2F(\eta) \in Diag$ for every $\eta \in B$.

5.3 Convolution type processes

Let H be a separable Hilbert space. Those processes, taking values in H, are the natural generalization of Itô type processes. Let A be the generator of a C_0-semigroup on H. Denote again with $D(A)$ and $D(A^*)$ respectively the domains of A and A^* endowed with the graph norm.

Let U_0, U be separable Hilbert spaces according to Sections 2.2 and 2.3. Let $\mathbb{W}$ be a Q-Wiener process with values in U where $Q \in \mathcal{L}(U)$ a positive, injective and self-adjoint operator and define again $U_0 := Q^{1/2}(U)$ endowed with the scalar product $\langle a, b\rangle_{U_0} := \left\langle Q^{-1/2}a, Q^{-1/2}b\right\rangle$. Let $\sigma = (\sigma_t, t \in [0,T])$ with paths a.s. in $\mathcal{L}_2(U_0; H)$ and $b = (b_t, t \in [0,T])$ with paths taking values in H being predictable such that

$$\mathbb{P}\left[\int_0^T \left(\|\sigma_t\|^2_{\mathcal{L}_2(U_0;H)} + |b_t|\right) dt < \infty\right] = 1. \tag{31}$$

Let $x_0 \in H$.

Definition 5.20. A continuous process of the type

$$\mathbb{X}_t = e^{tA}x_0 + \int_0^t e^{(t-r)A}\sigma_r d\mathbb{W}_r + \int_0^t e^{(t-r)A} b_r dr, \tag{32}$$

is said **convolution type process** (related to A).

e^{tA} stands of course for the C_0-semigroup associated with A. Clearly if $A = 0$, the semigroup is the identity, then a convolution type process is a Itô type process. Natural examples of convolution processes are given by mild solutions of stochastic PDEs, see for instance [8] Chapter 7 or [25] Chapter 3.1.

Proposition 5.21. Let $\mathbb{X}$ be a convolution type process as in (32) and

$$\nu_0 = D(A^*) \subset H^*, \quad \chi = \nu_0 \hat{\otimes}_\pi \nu_0.$$

The following properties hold.

1. $\mathbb{X}$ admits a decomposition of the type $\mathbb{M} + \mathbb{V}$ where

$$\mathbb{M}_t = x_0 + \int_0^t \sigma_r d\mathbb{W}_r, \quad \mathbb{V}_t = \int_0^t b_r dr + \mathbb{A}_t,\ t \in [0,T],$$

where $\mathbb{A}$ is a progressively measurable process such that

$${}_H\langle \mathbb{A}_t, \phi\rangle_{H^*} = \int_0^t {}_H\langle \mathbb{X}_r, A^*\phi\rangle_{H^*} dr,\ \forall \phi \in \nu_0. \tag{33}$$

2. Let $\bar{\nu}_0$ be the dual of $D(A^*)$, $\bar{\nu}_0$ contains H since $D(A^*)$ and H are Hilbert spaces and then reflexive. Then $\mathbb{X}$ is an $\bar{\nu}_0$-semimartingale with decomposition $\mathbb{M} + \mathbb{V}$, $\mathbb{M}$ being the local martingale part.

3. The process $\mathbb{A}$ appearing in 1. admits a χ-zero quadratic variation.

4. $\mathbb{X}$ admits a χ-quadratic variation given

$$\widetilde{[\mathbb{X},\mathbb{X}]}^{\chi}(\varphi) = \int_0^t \mathrm{Tr}\left(L_\varphi(\sigma_r Q^{\frac{1}{2}})(\sigma_r Q^{\frac{1}{2}})^*\right) dr, \quad \varphi \in \chi, \tag{34}$$

where L_φ was defined in (8).

Proof. 1. This follows by Theorem 12, [35], see also lemma 5.1 [22].

2. The process $\mathbb{A}$ can be considered as a $\bar{\nu}_0$-valued process. From (33), it follows that, for $0 \le s \le t \le T$ and $\phi \in D(A^*)$, using (33), we have

$${}_H\left\langle \mathbb{A}_t - \mathbb{A}_s, \phi \right\rangle_{H^*} = \int_s^t ds \ {}_H\langle \mathbb{X}_r, A^*\phi\rangle_{H^*},$$

so the $|\cdot|_{\bar{\nu}_0}$ norm of $\mathbb{A}_t - \mathbb{A}_s$ is estimated by

$$\sup_{\substack{\phi\in D(A^*)\\ |\phi|_{D(A^*)}\le 1}} |\left\langle \mathbb{A}_t - \mathbb{A}_s, \phi\right\rangle| \le \int_s^t ds \sup_{\substack{\phi\in D(A^*)\\ |\phi|_{D(A^*)}\le 1}} {}_H|\left\langle \mathbb{X}_r, A^*\phi\right\rangle_{H^*}| \le \int_s^t |\mathbb{X}_r|_H dr.$$

Previous inequalities show that $\mathbb{A}$ has a total variation as $\bar{\nu}_0$-valued process which is bounded by $\int_0^T dr|\mathbb{X}_r|_H$. Since the $\bar{\nu}_0$-norm is dominated by the H-norm and $\int_0^\cdot b_r dr$ is an H-valued bounded variation process, then $\mathbb{V}$ is also a bounded variation $\bar{\nu}_0$-valued process. Finally $\mathbb{X}$ is a $\bar{\nu}_0$-semimartingale.

3. follows from item 1. of Proposition 5.15.

4. follows from item 4. of Proposition 5.15 and item 3. of Proposition 4.7.

□

6 Stochastic calculus

6.1 Banach space valued forward integrals

Let U, H be separable Hilbert spaces and B, E be separable Banach spaces.

Definition 6.1. Let $(\mathbb{Y}_t, t \in [0,T])$ be a strongly measurable process taking values in $\mathcal{L}(B,E)$ and $\mathbb{X} = (\mathbb{X}_t, t \in [0,T])$, be a B-valued continuous process and the following. $\int_0^T \|\mathbb{Y}_r\|_{\mathcal{L}(B,E)} dr < \infty$. a.s. We suppose the following.

- $\lim_{\varepsilon\to 0} \int_0^t \mathbb{Y}_r \frac{\mathbb{X}_{r+\varepsilon} - \mathbb{X}_r}{\varepsilon} dr$ exists in probability for any $t \in [0,T]$.
- Previous limit random function admits a continuous version.

In this case, we say that the **forward integral** of $\mathbb{Y}$ with respect to $\mathbb{X}$, denoted by $\int_0^\cdot \mathbb{Y} d^-\mathbb{X}$ exists.

Remark 6.2. 1. If $E=\mathbb{R}$ than we often denote $\int_0^{\cdot} \mathbb{Y}_r d\mathbb{X}_r = \int_0^{\cdot} {}_{B^*}\langle \mathbb{Y}_r, d\mathbb{X}_r\rangle_B$.

2. If $\mathbb{X}=\mathbb{V}$ is a continuous bounded variation process, and $\mathbb{Y}$ is an a.s. bounded strongly measurable process having at most countable number jumps (as for instance cadlag or caglad), then $\int_0^{\cdot} \mathbb{Y}_r d^-\mathbb{X}_r$ exists and it equals the Bochner-Lebesgue integral $\int_0^{\cdot} \mathbb{Y}_r d\mathbb{X}_r$.
If $\mathbb{X}$ is a.s. is absolutely continuous with derivative $r\mapsto \dot{X}_r$, then, whenever $\int_0^T \|Y_r\|_{\mathcal{L}(B,E)}|\dot{X}_r|_B dr < \infty$ a.s. $\int_0^{\cdot} \mathbb{Y}_r d^-\mathbb{X}_r$ exists and it equals the same Bochner integral as before.
In both cases, the proof is similar to the case when the processes are real-valued, see e.g. Proposition 1.1 in [42] making use of stochastic Fubini's theorem, i.e. Theorem 4.18 of [8].

3. Suppose that $B=H$ and $E=U$. Let $\mathbb{X}=\mathbb{M}$ be an $(\mathscr{F}_t)$-local martingale and $\mathbb{Y}$ be a predictable process such that $\int_0^T \|\mathbb{Y}_r\|^2_{\mathcal{L}(U,H)} d[M]^{\mathbb{R},cl}_r < \infty$ a.s. Then $\int_0^t \mathbb{Y}_r d^-\mathbb{M}_r$ exists and it equals the Itô type integral $\int_0^t \mathbb{Y}_r d\mathbb{M}_r$, see Theorem 3.6 in [22].

4. Suppose that $\mathbb{M}=\mathbb{W}$ is a Q-Wiener process with values in a separable Hilbert space H and $\mathbb{Y}$ is a predictable process such that $\int_0^T \mathrm{Tr}((\mathbb{Y}_r Q^{\frac{1}{2}})(\mathbb{Y}_r Q^{\frac{1}{2}})^* dr < \infty$ a.s. Then the forward integral $\int_0^t \mathbb{Y}_r d^-\mathbb{W}_r, t\in[0,T]$ exists and it equals the Itô integral $\int_0^t \mathbb{Y}_r d\mathbb{W}_r, t\in[0,T]$, see Theorem 3.4 in [22].

5. A consequence of the previous two items is the following. If $\mathbb{X}=\mathbb{M}+\mathbb{V}$ is an H-valued semimartingale, and $\mathbb{Y}$ is a cadlag predictable process, then $\int_0^t \mathbb{Y} d^-\mathbb{X}, t\in[0,T]$, exists and it is the sum $\int_0^t \mathbb{Y} d\mathbb{M} + \int_0^t \mathbb{Y} d\mathbb{V}$.

6.2 Itô formulae

We can now state the following Banach space valued Itô's formula, see Theorem 5.2 of [17].

Theorem 6.3. Let B a separable Banach space, χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ and let $\mathbb{X}$ a B-valued continuous process admitting a χ-quadratic variation. Let $F:[0,T]\times B\longrightarrow \mathbb{R}$ be $C^{1,2}$ Fréchet such that

$$D^2F:[0,T]\times B\longrightarrow \chi\subset (B\hat{\otimes}_\pi B)^* \quad \text{continuously.}$$

Then for every $t\in[0,T]$ the forward integral

$$\int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), d^-\mathbb{X}_s\rangle_B \tag{35}$$

exists and the following formula holds.

$$F(t,\mathbb{X}_t) = F(0,\mathbb{X}_0) + \int_0^t \partial_r F(r,\mathbb{X}_r)dr + \int_0^t {}_{B^*}\langle DF(r,\mathbb{X}_r), d^-\mathbb{X}_r\rangle_B + \frac{1}{2}\int_0^t {}_{\chi}\langle D^2F(r,\mathbb{X}_r), d\widetilde{[\mathbb{X}]}_r\rangle_{\chi^*}. \tag{36}$$

The assumption that the second derivatives to lives in a suitable χ-space can be relaxed in some situations, see for instance Proposition 6.4.

Proposition 6.4. Let H be a separable Hilbert space. Let ν_0 be a dense subset of H^*. We set $\chi=\nu_0\hat{\otimes}_\pi\nu_0$. Let $\mathbb{X}$ be a χ-finite quadratic variation H-valued process. Let $F\colon[0,T]\times H\to\mathbb{R}$ of class $C^{1,2}$ such that $(t,x)\to DF(t,x)$ is continuous from $[0,T]\times H$ to ν_0. Suppose moreover the following assumptions.

(i) There exists a (cadlag) bounded variation process $C\colon [0,T]\times\Omega\to H\hat{\otimes}_\pi H$ such that, for all t in $[0,T]$ and $\phi\in\chi$,

$$ {}_{H\hat{\otimes}_\pi H}\langle C_t(\cdot),\phi\rangle_{(H\hat{\otimes}_\pi H)^*} = [\mathbb{X},\mathbb{X}]^\chi_t(\phi)(\cdot) \qquad a.s. $$

(ii) For every continuous function $\Gamma\colon [0,T]\times H\to\nu_0$ the integral

$$ \int_0^t \langle \Gamma(r,\mathbb{X}_r), d^-\mathbb{X}_r\rangle \tag{37} $$

exists. Then

$$ \begin{aligned} F(t,\mathbb{X}_t) &= F(0,\mathbb{X}_0) + \int_0^t {}_{H^*}\langle \partial_x F(r,\mathbb{X}_r), d^-\mathbb{X}_r\rangle_H \\ &+ \frac{1}{2}\int_0^t {}_{(H\hat{\otimes}_\pi H)^*}\langle D^2F(r,\mathbb{X}_r), dC_r\rangle_{H\hat{\otimes}_\pi H} + \int_0^t \partial_r F(r,\mathbb{X}_r)dr. \end{aligned} \tag{38} $$

Proof. In Theorem 5.4 of [22], the result is formulated for the particular case $\nu_0 = D(A^*)$ where A is the generator of a C_0-semigroup; the arguments to extend the result to the case of a generic ν_0 are the same. □

Remark 6.5. Clearly

$$ (t,\omega)\mapsto (\phi\mapsto {}_{\chi^*}\langle C_t(\omega),\phi\rangle_\chi) = \widetilde{[\mathbb{X},\mathbb{X}]^\chi}_t(\omega), $$

are indistinguishable processes with values in χ^*, if we identify χ^* as a space which contains the bidual of $H\hat{\otimes}_\pi$ and therefore $H\hat{\otimes}_\pi$ itself.

A consequence of previous proposition is a natural Itô formula for convolution type processes.

Proposition 6.6. Let $\mathbb{X}$ be a convolution type process as in Definition 5.20 with σ and b verifying (31). Assume that $F\in C^{1,2}([0,T]\times H)$ with $DF\in C([0,T]\times H, D(A^*))$. Then, for every $t\in[0,T]$,

$$ \begin{aligned} F(t,\mathbb{X}_t) &= F(0,\mathbb{X}_0) + \int_0^t \partial_r F(r,\mathbb{X}_r)dr \\ &+ \int_0^t \langle DF(r,\mathbb{X}_r), b_r\rangle\, dr + \int_0^t \langle DF(r,\mathbb{X}_r), \sigma_r d\mathbb{W}^Q_r\rangle \\ &+ \int_0^t \langle A^*DF(r,\mathbb{X}_r),\mathbb{X}_r\rangle\, dr + \frac{1}{2}\int_0^t \mathrm{Tr}\left[\left(\sigma_r Q^{1/2}\right)\left(\sigma_r Q^{1/2}\right)^* D^2F(r,\mathbb{X}_r)\right] dr, \qquad \mathbb{P}-a.s. \end{aligned} \tag{39} $$

where for $(r,\eta)\in[0,T]\times H$, again we associate $D^2F(r,\eta)$, which is in principle an element of $\mathcal{B}i(H,H)$, with a map in $\mathcal{L}(H)$, as in (8).

Proof. It is a consequence of Proposition 6.4 using Proposition 5.21 as follow. Let $\chi = \nu_0\hat{\otimes}_\pi\nu_0$ with $\nu_0 = D(A^*)$.

Indeed, thanks item 4. of Proposition 5.21, $\mathbb{X}$ admits a χ-quadratic variation. Consider the decomposition $\mathbb{M}+\mathbb{V}$ defined in item 1. of Proposition 5.21. We first check that hypothesis (ii) of Proposition 6.4 is satisfied. $\int_0^t\langle\Gamma(r,\mathbb{X}_r), d^-\mathbb{M}_r\rangle$ (resp. $\int_0^t\langle\Gamma(r,\mathbb{X}_r), d^-\int_0^\cdot b_r dr\rangle$) exists and it equals the Itô type integral

$$ \int_0^t \langle\Gamma(r,\mathbb{X}_r), d\mathbb{M}_r\rangle \tag{40} $$

(resp.

$$\int_0^t \langle \Gamma(r, \mathbb{X}_r), b_r \rangle dr). \tag{41}$$

This happens because of items 2. and 3. of Remark 6.2. Consequently $\int_0^t \langle \Gamma(r, \mathbb{X}_r), d^- \mathbb{X}_r \rangle$ exists if

$$\int_0^t \left\langle \Gamma(r, \mathbb{X}_r), d^- \mathbb{A}_r \right\rangle, \qquad t \in [0, T], \tag{42}$$

exists and it equals the sum of (40), (41) and (42). We recall that $\mathbb{A}$ was defined in Proposition 5.21.
So let us show that (42) exists. For every $t \in [0, T]$, using (33), we evaluate limit of its ϵ-approximation, using item 1. of Proposition 5.21.

$$\begin{aligned} \frac{1}{\epsilon} \int_0^t \langle \Gamma(r, \mathbb{X}_r), \mathbb{A}_{r+\epsilon} - \mathbb{A}_r \rangle \, dr &= \frac{1}{\epsilon} \int_0^t \int_r^{r+\epsilon} \langle \mathbb{X}_u, A^* \Gamma(r, \mathbb{X}_r) \rangle \, du dr \\ &= \frac{1}{\epsilon} \int_0^t \int_{u-\epsilon}^u \langle \mathbb{X}_u, A^* \Gamma(r, \mathbb{X}_r) \rangle \, dr du \xrightarrow{\epsilon \to 0} \int_0^t \langle \mathbb{X}_u, A^* \Gamma(u, \mathbb{X}_u) \rangle \, du. \end{aligned} \tag{43}$$

The validity of Hypothesis (i) comes out setting $C_t = \int_0^t (\sigma_r Q^{\frac{1}{2}})(\sigma_r Q^{\frac{1}{2}})^* dr$. It holds because $\mathbb{X}$ is a $\bar{\nu}_0$-semimartingale, taking into account Proposition 5.15 4. and item 1. of Proposition 4.7. Expression (39) results now from (38). The first integral of the right-hand side of (38) gives the second, third and fourth integrals of (39). In particular the second and the third ones are obtained differentiating $\mathbb{M}$ and $\int_0^\cdot b_r dr$, using Remark 6.2 2., 3. and Proposition 2.6. The fourth integral comes from (43) choosing $\Gamma = DF$. Finally the last integral in (39) comes from the third addendum of (38), taking into account (34) and Lemma 4.9 with $j = C$ and $\dot{g}(r) = D^2 F(r, \mathbb{X}_r)$. □

The theorem below operates as a substitute of a non-smooth Itô formula. It is a stability of ν-weak Dirichlet processes, which was the object of Theorem 4.2 of [22].

Theorem 6.7. Let H be a separable Hilbert space. Let ν_0 be a dense subset of H^*. We set $\nu = \nu_0 \otimes \mathbb{R}$ and $\chi = \nu_0 \hat{\otimes}_\pi \nu_0$. Let $F \in C^{0,1}([0, T] \times H)$ such that DF is continuous from $[0, T] \times H$ to ν_0. Let $\mathbb{X} = \mathbb{M} + \mathbb{V}$ be a $(\mathscr{F}_t) - \nu$-weak Dirichlet process and we suppose that $\mathbb{X}$ has a χ-quadratic variation.
Then $(F(t, \mathbb{X}_t))$ is a real $(\mathscr{F}_t)$-weak Dirichlet process with martingale part M^u where

$$M_t^F = F(0, \mathbb{X}_0) + \int_0^t \langle DF(r, \mathbb{X}_r), d\mathbb{M}_r \rangle_H.$$

7 Calculus with respect to window processes

Let X be a real (continuous) process such that $[X]_t \equiv \psi(t)$ with $\psi(t) = \sigma^2 t$, $\sigma \geq 0$. Let $B = C([-T, 0])$. Let $u : [0, T] \times B \to \mathbb{R})$ of class $C^{0,1}([0, T[\times B$. For $t \in [0, T[, \eta \in B$, we set

$$D^{\delta_0} u(t, \eta) = Du(t, \eta)(0), \;\; D^\perp u(t, \eta) = Du(t, \eta) - Du(t, \eta)(0).$$

In this section, for $t \in [0, T]$, we will also denote

$$D_t := \left\{ (x, y) \in [-t, 0]^2 | x = y \right\}. \tag{44}$$

7.1 The case of *vanilla* random variables

A window diffusion X is naturally related to an infinite dimensional Kolmogorov type equation. But in fact this link remains valid when the diffusion is a non-semimartingale with the same quadratic variation. Let us concentrate on the case of a (non-necessarily semimartingale) process X such that $[X]_t = \sigma^2 t, t \in [0,T]$, for $\sigma \geq 0$.

In order to motivate the discussion, we start with the simple representation a r.v. of the type $h = f(X_T)$ where $f : \mathbb{R} \to \mathbb{R}$ is continuous with polynomial growth. We suppose the existence of $v \in C^{1,2}([0,T[\times\mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$ such that

$$\begin{cases} \partial_t v(t,x) + \frac{\sigma^2}{2}\partial^2_{xx} v(t,x) = 0 \\ v(T,x) = f(x). \end{cases}$$

Then

$$h := f(X_T) = v(0, X_0) + \int_0^T \partial_x v(s, X_r) d^- X_r,$$

where previous integral is an improper forward integral. That result appeared in [47], [52]. The proof can be easily formulated through Proposition 3.7. Later on, generalizations were performed in the case of Asiatic options and other classes in [7, 1] and [6], which also considers r.v. of the type $h = f(X_{t_0}, \ldots, X_{t_N})$, for $0 = t_0 < \ldots < t_N = T$.

The natural question concerns the validity of a similar formula when h is path dependent.

Previous toy model can be revisited using infinite dimensional calculus via regularization.

Proposition 7.1. We set again $B = C([-T,0])$ and $\eta \in B$ and we define $G : B \longrightarrow \mathbb{R}$, by $G(\eta) := f(\eta(0))$ and $u : [0,T] \times B \longrightarrow \mathbb{R}$, by $u(t,\eta) := v(t,\eta(0))$. Then $u \in C^{1,2}\left([0,T[\times B;\mathbb{R}\right) \cap C^0\left([0,T] \times B;\mathbb{R}\right)$ and it solves

$$\begin{cases} \partial_t u(t,\eta) & + & \frac{\sigma^2}{2}\langle D^2 u\,(t,\eta)\ ,\ 1_{D_t}\rangle = 0,\ (t,\eta) \in [0,T| \times B, \\ u(T,\eta) & = & G(\eta),\ \eta \in B. \end{cases} \tag{45}$$

Proof. The final condition is obviously verified since $u(T,\eta) = v(T,\eta(0)) = f(\eta(0)) = G(\eta)$ for all $\eta \in B$. Moreover u is obviously of class $C^{1,2}([0,T[\times B) \cap C^0([0,T] \times B)$ and $\partial_t u\,(t,\eta) = \partial_t v\,(t,\eta(0))$; also $Du\,(t,\eta) = \partial_x v\,(t,\eta(0))\,\delta_0$ and $D^2 u\,(t,\eta) = \partial^2_{x\,x} v\,(t,\eta(0))\,\delta_0 \otimes \delta_0$. Finally $\partial_t u\,(t,\eta) + \frac{\sigma^2}{2} D^2 u\,(t,\eta)(D_t) = 0$. □

Suppose that $u : [0,T] \times B \to \mathbb{R}$ is of class $C^{0,1}([0,T[\times B)$. A quantity which will play a role in the sequel is the deterministic forward integral $\int_{]-t,0]} D^\perp_{dx} u(t,\eta) d^- \eta(x)$, see Section 3.2.

Suppose that for a given (t,η), $D^\perp_{dx} u(t,\eta)$ is absolutely continuous, we denote by $x \mapsto D^{\rm ac}_x u(t,\eta)$ the corresponding derivative. If moreover $x \mapsto D^{\rm ac}_x u(t,\eta)$ has bounded variation, then previous deterministic integral exists and,

$$\int_{]-t,0]} D^\perp_{dx} u(t,\eta) d^- \eta(x) = \eta(0) D^{ac}_x u(t,\eta)(\{0\}) - \int_{]-t,0]} \eta(x) D^{ac}_{dx} u(t,\eta(x)),$$

because of Remark 3.11 2.

In the toy model mentioned above, that integral is clearly zero since $D^\perp u$ is identically zero.

7.2 Itô formulae for window processes

The Itô formula stated in Theorem 6.3 can be particularized for the case when $\mathbb{X} = X(\cdot)$ is a window process (with $\tau = T$), associated with a finite quadratic variation process X. We recall that $\mathbb{X}$ admits a χ_0-quadratic variation, where χ_0 is the Chi-space of signed measures on $[-T,0]^2$ introduced in item 6. before Proposition 5.18. In particular Theorem 6.3 applies, so that integral (35) exists and it it decomposes in the sum

$$\int_0^t D^{\delta_0} F(r, \mathbb{X}_r) d^- X_r + \int_0^t {}_{B^*}\langle D^\perp F(r, \mathbb{X}_r), d^- \mathbb{X}_r \rangle_B, \tag{46}$$

where $D^{\delta_0}F$ and $D^\perp F$ were defined at the beginning of Section 7, provided that at least one of the two addends exist.

Remark 7.2. 1. The second term is the limit in probability of the expression

$$\int_0^t dr \int_{-r}^0 D^\perp_{dx} F(r, X_r) \frac{X_{r+x+\varepsilon} - X_{r+x}}{\varepsilon}, \tag{47}$$

when ε goes to zero.

2. If X is a semimartingale, then the first integral is the Itô integral $\int_s^t D^{\delta_0} F(r, \mathbb{X}_r) dX_r$. Consequently the second one is forced to exist.

3. If X is not necessarily a semimartingale, sufficient conditions for its existence can be provided. Suppose that the deterministic quadratic variation of almost all path of X exists. In particular $[X]$ exists as an increasing real process. In this case a sufficient condition for the existence of the second integral in (46), is the realization of following Condition related to F. We recall that the space $V_{2,\psi}$, for a fixed increasing continuous function $\psi : [0,T] \to \mathbb{R}$ such that $\psi(0) = 0$, was defined at Section 3.2. B denotes here $C([-T,0])$.

   Definition 7.3. A continuous function $u : [0,T] \times B \to \mathbb{R}$ of class $C^{0,1}([0,T[\times B)$ is said to fulfill **Condition** (C) (related to ψ) if the following holds.

   (a) For each $t \in [0,T], \eta \in V_{2,\psi}$, the deterministic integral

   $$\int_{]-t,0]} D^\perp_x u(t,\eta) d^- \eta(x) \tag{48}$$

   exists.

   (b) For any $\varepsilon > 0, t \in [0,T], \eta \in B$, we denote

   $$I(t,\eta,\varepsilon) := \int_{-t}^0 D^\perp_x u(t,\eta) \frac{\eta(x+\varepsilon) - \eta(x)}{\varepsilon} dx. \tag{49}$$

   We suppose the existence of $J : [0,T] \times V_{2,\psi} \to \mathbb{R}_+$ such that $|I(t,\eta,\varepsilon)| \le J(t,\eta), \forall \eta \in V_{2,\psi}$ and such that for each compact K of B included in $V_{2,\psi}$, $\int_0^T \sup_{\eta \in K} J(s,\eta) ds < \infty$.

   As far as last point is concerned we remark that relatively compact subsets of B are very tiny. Sufficient conditions for the validity of Condition (C) will be given below. Clearly this condition implies the existence of the second integral of (46). In fact the set $\{X_s(\cdot), s \in [-T,0]\}$ is compact in B and included in $V_{2,\psi}$. If Condition (C) is verified then

   $$|I(s, X_s(\cdot), \varepsilon)| \le \sup_{\eta \in K} J(s,\eta),$$

   where $K = K(\omega) = \{X_s(\cdot), s \in [-T,0]\}$. The result follows by Lebesgue dominated convergence theorem.

A sufficient condition for the realization of Condition (C) is given below. This will be a consequence of integration by parts and Itô chain rule (18) in Proposition 3.6, expressed in the context of deterministic calculus via regularization.

Lemma 7.4. Suppose the existence of continuous maps $F_i : [0,T] \times B \times [-T,0] \to \mathbb{R}$, $G_i : [0,T] \times [-T,0] \times \mathbb{R} \to \mathbb{R}$, $1 \le i \le N$, such that $D^{\perp}_{dx} u(t,\eta)$ is absolutely continuous and $D^{\perp}_{dx} u(t,\eta) = D^{ac}_{x} u(t,\eta) dx = \sum_{i=1}^{N} F_i(t,\eta,x) G_i(t,x,\eta(x)) dx$, with the following properties for any subset K of $V_{2,\psi}$ such that

$$\sup_{\eta \in K} \|\eta\|_{2,\psi} < \infty \text{ and it is a compact subset of } B. \tag{50}$$

For every $1 \le i \le N$, for any such K, we suppose the following.

- For any $(t,\eta) \in [0,T] \times V_{2,\psi}$, $F_i(t,\eta,\cdot)$ has bounded variation.
- $(t,\eta) \mapsto \|F_i(t,\eta,\cdot)\|_{var}$ is bounded on $[0,T] \times K$.
- $G_i \in C^{0,1}([0,T] \times [-T,0] \times \mathbb{R})$.

Then u fulfills Condition (C) with respect to ψ.

Proof. By additivity we can reduce to the case $N = 1$ and we set $F = F_1$, $G = G_1$. Let K be a subset of $V_{2,\psi}$ such that (50) is fulfilled.
Let $F : [0,T] \times V_{2,\psi} \times [-T,0] \to \mathbb{R}$ be measurable such that for every t and η we suppose that $x \mapsto F(t,\eta,x)$ has bounded variation and $(t,\eta) \mapsto \|F_i(t,\eta,\cdot)\|_{var}$ is bounded on $[0,T] \times K$. Let $G : [0,T] \times [-T,0] \times \mathbb{R} \to \mathbb{R}$ of class $C^{0,1,1}([0,T] \times [-T,0] \times \mathbb{R})$. Let $\eta \in V_{2,\psi}$ and set $\tilde{G} : [0,T] \times [-T,0] \times \mathbb{R} \to \mathbb{R}$ the primitive defined by $\tilde{G}(t,x,y) = \int_0^y G(t,x,\tilde{y}) d\tilde{y}$. By formula (18) in Proposition 3.6 and Remark 3.11 3., we obtain

$$\begin{aligned}\int_{]-t,0]} F(t,\eta,x) G(t,x,\eta(x)) d^-\eta(x) &= \int_{]-t,0]} F(t,\eta,x) d^-_x \tilde{G}(t,x,\eta(x)) - \frac{1}{2} \int_{]-t,0]} F(t,\eta,x) \partial_{\eta(x)} G(t,x,\eta(x)) d[\eta](x) \\ &- \int_{]-t,0]} F(t,\eta,x) \partial_x \tilde{G}(t,x,\eta(x)) dx,\end{aligned} \tag{51}$$

provided that the first integral after the equality symbol is well-defined. By Remark 3.11 2., that integral equals

$$F(t,\eta,0-)\tilde{G}(t,0,\eta(0)) - \int_{]-t,0]} F(t,\eta,dx) \tilde{G}(t,x,\eta(x));$$

consequently item (a) of Condition (C) is fulfilled.
In the sequel of the proof, for $\eta \in B$, we denote by $R_K(t,\eta,\varepsilon)$ a quantity such that for every $0 < \varepsilon < 1$, $\sup_{t \in [0,T], \eta \in K} |R(t,\eta,\varepsilon)| \le C(T,K)$, where $C(T,K)$ only depend on T and K. We denote

$$K_0 := \bigcup_{\eta \in K} Im(\eta),$$

which is clearly a compact subset of $\mathbb{R}$. We need to control the quantity

$$\int_{-t}^{0} F(t,\eta,x) G(t,x,\eta(x)) \frac{\eta(x+\varepsilon) - \eta(x)}{\varepsilon} dx. \tag{52}$$

We set again $\tilde{G}(t,x,y) = \int_0^y G(t,x,\tilde{y})d\tilde{y}$ for $t \in [0,T]$ and $x \in [-T,0], y \in \mathbb{R}$, so that $\frac{\partial \tilde{G}}{\partial y}(t,x,y) = G(t,x,y)$. By Taylor expansion (52) equals

$$I_1(t,\eta,\varepsilon) - I_2(t,\eta,\varepsilon) - I_3(t,\eta,\varepsilon) + R_K(t,\eta,\varepsilon),$$

where

$$I_1(t,\eta,\varepsilon) := \int_{-t}^0 \frac{dx}{\varepsilon} F(t,\eta,x) \left(\tilde{G}(t,x+\varepsilon,\eta(x+\varepsilon)) - \tilde{G}(t,x,\eta(x)) \right),$$

$$I_2(t,\eta,\varepsilon) := \int_{-t}^0 \frac{dx}{\varepsilon} F(t,\eta,x) \frac{\partial \tilde{G}}{\partial x}(t,x,\eta(x)),$$

$$I_3(t,\eta,\varepsilon) := \frac{1}{2} \int_{-t}^0 \frac{dx}{\varepsilon} F(t,\eta,x) \int_0^1 da \frac{\partial \tilde{G}}{\partial z}(t,x+a\varepsilon,\eta(x)+a(\eta(x+\varepsilon)-\eta(x)))\left(\eta(x+\varepsilon)-\eta(x)\right)^2.$$

$I_1(t,\eta,\varepsilon)$ equals

$$\int_{-t}^0 \frac{dx}{\varepsilon} \left(F(t,\eta,x) - F(t,\eta,x-\varepsilon)\right) \tilde{G}(t,x,\eta(x)) + R_K(t,\eta,\varepsilon)$$
$$= \int_{[-t,0]} \frac{F(t,\eta,dy)}{\varepsilon} \int_y^{y+\varepsilon} dx\, \tilde{G}(t,x,\eta(x)) + R_K(t,\eta,\varepsilon). \quad (53)$$

Consequently, for $0 < \varepsilon < 1$ we have

$$|I_1(t,\eta,\varepsilon)| \le \sup_{\substack{x\in[-T,0]\\ y\in K_0\\ t\in[0,T]}} |\tilde{G}(t,x,y)| \sup_{\eta\in K, t\in[0,T]} \|F(t,\eta,\cdot)\|_{var} + \sup_{\substack{\eta\in K\\ 0<\varepsilon<1\\ t\in[0,T]}} |R_K(t,\eta,\varepsilon)| =: C_1;$$

$I_2(t,\eta,\varepsilon)$ can be handled in similar (but easier) way to I_1. There is a constant C_2 such that

$$\sup_{\substack{x\in[-T,0]\\ \eta\in K\\ t\in[0,T]}} |I_2(t,\eta,\varepsilon)| \le C_2.$$

Concerning $I_3(t,\eta,\varepsilon)$, for $0 < \varepsilon < 1$, we have

$$|I_3(t,\eta,\varepsilon)| \le \sup_{\substack{x\in[-T,0]\\ y\in K_0\\ t\in[0,T]}} \left| \frac{\partial \tilde{G}}{\partial y}(t,x,y) \right| \sup_{\substack{x\in[-T,0]\\ \eta\in K}} |F(t,\eta,x)| \left(\sup_{\eta\in K} \|\eta\|_{2,\psi} \right),$$

which is bounded because of (50). Finally item (b) of condition (C) is also fulfilled. □

7.3 An infinite dimensional PDE

In this subsection again B will stand for $C([-T,0])$. We are interested here in a class of functionals $G : B \longrightarrow \mathbb{R}$ such that the r.v. $h := G(X_T(\cdot))$ admits a representation

$$h = G_0 + \int_0^T \xi_s d^- X_s, \quad (54)$$

where G_0 is a real number and ξ is adapted with respect to the canonical filtration $(\mathscr{F}_t)$ of X. If X is a classical Wiener process, and h belongs to some suitable Malliavin type Sobolev space, then $G_0 = E(h)$ and Clark-Ocone formula says that ξ in (54) is given by $\xi_t = \mathbb{E}(D_t^m h|\mathscr{F}_t), t \in [0,T]$.

In this section we want to show that the replication of a random variable $h = G(X(\cdot))$, is robust with respect to the quadratic variation of X, for a large class of G; the fact that the underlying process is distributed according to Wiener measure is not so relevant. We are indeed interested in a representation (54), which formulates G_0 and ξ through two functionals of X, which do not depend on the specific model of X such that $[X]_t \equiv \sigma^2 t, t \in [0,T]$.

The methodology for expressing a *Clark-Ocone type formula* for finite quadratic variation processes consists in two steps.

1. We need to choose a functional $u : [0,T] \times B \longrightarrow \mathbb{R}$ which solves the infinite dimensional PDE (56) with final condition G.

2. Using an Itô type formula we establish a representation form (54).

The proposition below represents the second step of the procedure.

Below ψ will stand for $\psi(t) \equiv \sigma^2 t$, for some $\sigma \geq 0$.

Proposition 7.5. Let X a process such that a.s. the limit $[X,X]$ in Definition 3.3 holds a.s. and gives ψ. Let $u : [0,T] \times B \longrightarrow \mathbb{R}$ be a function of class $C^{1,2}\left([0,T[\times B\right) \cap C^0\left([0,T] \times B\right)$. For $(t,\eta) \in [0,T] \times B$, we decompose $Du(t,\eta) = D^{\delta_0}u(t,\eta)\delta_0 + D_{dx}^{\perp}u(t,\eta)$. We symbolize again through the Chi-space χ_0 of signed measures $\mathcal{M}([-T,0]^2)$ defined in item 6. in Section 5.2. We suppose the following.

1. u fulfills Condition (C) and we denote
$$I(u)(t,\eta) := \int_{]-t,0]} D_{dx}^{\perp}u(t,\eta)d^-\eta(x),\ (t,\eta) \in [0,T] \times V_{2,\psi}. \tag{55}$$

2. For all $t \in [0,T], \eta \in V_{2,\psi}$, $D^2u(t,\eta) \in \chi_0$ and the map $(t,\eta) \mapsto D^2u(t,\eta)$ is continuous with respect to the topologies of $[0,T] \times B$ and χ_0.

3. u solves the solving the infinite dimensional PDE
$$\begin{cases} \partial_t u(t,\eta) + I(u)(t,\eta) + \frac{\sigma^2}{2}\langle D^2 u\,(t,\eta)\ ,\ 1_{D_t}\rangle = 0,\ (t,\eta) \in [0,T] \times V_{2,\psi}, \\ u(T,\cdot) = G. \end{cases} \tag{56}$$

Then representation (54) holds with $G_0 = u(0, X_0(\cdot))$ and $\xi_s = D^{\delta_0}u(s, X_s(\cdot))$.

Remark 7.6. The condition on X implies that X is a finite quadratic variation process and $[X]_t = \sigma^2 t$.This is a little bit stronger but it is fulfilled in almost the known models where $[X] = \psi$. A typical X with this a.s. property is the sum of a Wiener process and a Hölder process V with respect to an index $\gamma > \frac{1}{2}$.

Proof. The proof the proposition is a consequence of Theorem 6.3 and of the considerations following the statement of Condition (C). In particular we remark that for all $t \in [0,T]$, a.s. we have
$$I(u)(t, X_t(\cdot)) = \int_0^t {}_{B^*}\langle D^{\perp}F(r,\mathbb{X}_r), d^-\mathbb{X}_r\rangle_B. \tag{57}$$

□

Coming back to the two steps mentioned at the beginning of Section 7.3, Theorem 9.41 and Theorem 9.53 of [15] give some sufficient conditions to solve (56). This constitutes step 1. This can be done for instance in the two following cases.

1. G has a smooth Fréchet dependence on $L^2([-T, 0])$.
2. $h := G(X_T(\cdot)) = f\left(\int_0^T \varphi_1(s)d^- X_s, \ldots, \int_0^T \varphi_n(s)d^- X_s\right)$,
   - $f : \mathbb{R}^n \to \mathbb{R}$ continuous with linear growth
   - $\varphi_i \in C^2([0, T]; \mathbb{R}), 1 \leq i \leq N$.

Remark 7.7. Suppose that $X = W$. There are cases where the methodology developed here is operational and the classical Clark-Ocone formula does not apply. For instance in some cases h may be allowed even not to belong to $L^1(\Omega)$ and a fortiori $h \notin \mathbb{D}^{1,2}$ or $h \notin L^2(\Omega)$, see for instance Proposition 9.55 in [15].

Remark 7.8.

1. Remark that our representation theorems also holds when $\sigma = 0$.
2. In a work in preparation, the authors extend the present theory to the case when X is replaced with the window of a generic diffusion process.
3. The present approach was developed at the same time and independently than functional Itô's calculus of B. Dupire, R. Cont, D. Fournié, see e.g. [19, 5].

8 Applications to study of Kolmogorov equations

In this section we illustrate how to use the tools of stochastic calculus via regularization in the study of solutions of forward Kolmogorov equations (i.e. Fokker-Planck equations) related to an evolution problem in infinite dimensions, for instance a stochastic PDE. Kolmogorov equations in infinite dimension constitute a classical field of study, they appear for example in quantum field theory and in stochastic reaction-diffusion. We do not have here the ambition of summarizing the existing literature but only to describe how the development of the theory we have described in previous sections can help to treat some cases that are not covered by the existing literature.

We are interested in studying a class of Kolmogorov equation associated to an evolution equation of the form (3) using the strong solution approach. In other words we will define the solution of the Kolmogorov equation using approximating sequences, see Definition 8.7. The main results of the section are the following:

(i) We provide, first of all, in Theorem 8.8, a probabilistic representation of strong solutions $(t, \eta) \to v(t, \eta)$ of the Kolmogorov equation decomposing it into two stochastic terms: the evaluation of the initial datum of the Kolmogorov equation along the trajectory of a reversed evolution equation and a stochastic integral term depending on the first derivative of v.

(ii) In Proposition 8.9 we show that a strong solution of the Kolmogorov equation is also a mild solution. The definition of mild solution will be recalled in (64). As a corollary we get the uniqueness of the strong solution.

With respect to similar contributions in this sense (see e.g. [4, 26, 10]) we are able to prove the uniqueness of the strong solution in cases in which the stochastic evolution equation connected to the Kolmogorov equation is not homogeneous and in which the regularity of the solution $(t, \eta) \mapsto v(t, \eta)$ is only requested to be $C^{0,1}$ with $Dv(t,\eta) \in C([0,T] \times H; D(A^*))$, where A is the generator of the C_0-semigroup appearing in the infinite dimensional stochastic evolution equation, see e.g. (3), related to the Kolmogorov equation. More details about comparison with existing results are given in Example 8.10 and Remark 8.11.

8.1 The setting

Let H be a separable Hilbert space and A be the generator of a C_0-semigroup on H, see Section 5.3. We denote again with $D(A)$ and $D(A^*)$ respectively the domains of A and A^* endowed with the graph norm. Let fix again $T > 0$.

Let us again consider $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space and $(\mathscr{F}_t)_{t\geq 0}$ a filtration on it satisfying the usual conditions. Assume that U, U_0 are separable Hilbert spaces, $Q \in \mathcal{L}(U)$ is a positive, injective and self-adjoint operator as in Section 2.2 or 5.3 and define $U_0 := Q^{1/2}(U)$. Let again $\mathbb{W}^Q = \{\mathbb{W}^Q_t : 0 \leq t < +\infty\}$ be an U-valued $(\mathscr{F}_t)$-Q-Wiener process, with $\mathbb{W}^Q_0 = 0$, $\mathbb{P}$ a.s.

We consider two functions b and σ as follows.

Hypothesis 8.1. *$b\colon [0,T] \times H \to H$ is a continuous function and satisfies, for some $C > 0$,*

$$
\begin{aligned}
&|b(t,\eta) - b(t,\gamma)| \leq C|\eta - \gamma|,\\
&|b(t,\eta)| \leq C(1 + |\eta|),
\end{aligned}
$$

for all $\eta, \gamma \in H$, $t \in [0,T]$. $\sigma\colon [0,T] \times H \to \mathcal{L}_2(U_0; H)$ is continuous and, for some $C > 0$,

$$
\begin{aligned}
&\|\sigma(t,\eta) - \sigma(t,y)\|_{\mathcal{L}_2(U_0;H)} \leq C|\eta - \gamma|,\\
&\|\sigma(t,\eta)\|_{\mathcal{L}_2(U_0;H)} \leq C(1 + |\eta|),
\end{aligned}
$$

for all $\eta, \gamma \in H$, $s \in [0,T]$.

Remark 8.2. Observe that, thanks to the definition of norm on U_0, the hypothesis $\|\sigma(t,\eta)\|_{\mathcal{L}_2(U_0;H)} \leq C(1 + |\eta|)$ implies

$$
\|\sigma(t,\eta)Q^{1/2}\|_{\mathcal{L}_2(U;H)} \leq C(1 + |\eta|), \ (t,\eta) \in [0,T] \times H
$$

and then

$$
\left\| \left(\sigma(t,\eta)Q^{1/2}\right) \left(\sigma(t,\eta)Q^{1/2}\right)^* \right\|_{\mathcal{L}_1(H)} \leq C^2(1 + |\eta|)^2, (t,\eta) \in [0,T] \times H.
$$

For $\eta \in H$, we consider the equation

$$
\begin{cases}
d\mathbb{X}_t = (A\mathbb{X}_t + b(t,\mathbb{X}_t))\,dt + \sigma(t,\mathbb{X}_t)d\mathbb{W}^Q_t,\\
\mathbb{X}_0 = \eta.
\end{cases}
\tag{58}
$$

The solution of (58) is understood in the mild sense, so an H-valued predictable continuous process $\mathbb{X}$ is said to be a **mild solution** of (58) if

$$
\mathbb{P}\left(\int_0^T (|\mathbb{X}_r| + |b(r,\mathbb{X}_r)| + \|\sigma(r,\mathbb{X}_r)\|^2_{\mathcal{L}_2(U_0;H)})dr < +\infty \right) = 1
$$

and

$$\mathbb{X}_t = e^{tA}\eta + \int_0^t e^{(t-r)A} b(r, \mathbb{X}_r) dr + \int_0^t e^{(t-r)A} \sigma(r, \mathbb{X}_r) d\mathbb{W}^Q_r \tag{59}$$

$\mathbb{P}$-a.s. for every $t \in [0,T]$,

Thanks to Hypothesis 8.1, standard results about stochastic infinite dimensional evolution equation, see e.g. Theorem 3.3 of [25], ensure that there exists a unique solution $\mathbb{X}$ of (58), which admits a continuous modification. So for us, the solution $\mathbb{X}$ can always be considered as a continuous process.

8.2 The Kolmogorov equation

Let $g\colon H \to \mathbb{R}$ be a continuous and bounded function. We introduce now the following non-homogeneous Kolmogorov equation.

$$\begin{cases} -\partial_t v + \langle A^* Dv, \eta\rangle + \frac{1}{2}\mathrm{T}r\left[\sigma(t,\eta)\sigma^*(t,\eta)D^2 v\right] + \langle Dv, b(t,\eta)\rangle = 0 \qquad (t,\eta) \in [0,T] \times H, \\ v(0,\eta) = g(\eta), \qquad \eta \in H. \end{cases} \tag{60}$$

In the above equation, given $(t,\eta) \in [0,T] \times H$, given $v : [0,T] \times H \times H$, $Du(t,\eta)$ (resp. $D^2 v(t,\eta)$) is the Fréchet (resp. second Fréchet) derivative of v w.r.t. to the second variable η; it is identified with elements of H (resp. with a symmetric bounded operator on H, taking into account the identification (8)). $\partial_t v$ is the derivative w.r.t. the time variable.

We recall that the spaces $C([0,T] \times H), C(H), C([0,T] \times H; D(A^*))$ are Fréchet type spaces if equipped with the topology defined by the seminorms (4). We denote with $\mathscr{L}_0$ the operator on $C([0,T] \times H)$ defined as

$$\begin{cases} D(\mathscr{L}_0) := \left\{\varphi \in C^{1,2}([0,T] \times H) \ : \ D\varphi \in C([0,T] \times H; D(A^*))\right\} \\ \mathscr{L}_0(\varphi)(t,\eta) := -\partial_t\varphi(t,\eta) + \langle A^* D\varphi(t,\eta), \eta\rangle + \frac{1}{2}\mathrm{T}r\left[\sigma(t,\eta)\sigma^*(t,\eta)D^2\varphi(t,\eta)\right]. \end{cases} \tag{61}$$

Using this notation, (60) can be rewritten as

$$\begin{cases} \mathscr{L}_0(v(t,\cdot)) + \langle Dv, b(t,\eta)\rangle = 0, \qquad (t,\eta) \in [0,T] \times H, \\ v(0,\eta) = g(\eta) \qquad \eta \in H. \end{cases} \tag{62}$$

8.3 Mild, strict and strong solutions

We recall here three different definitions of solution of the Kolmogorov equation, see e.g. [10] for more details. Assume that Hypothesis 8.1 is verified. Fix $s \in (0,T]$. By the same arguments as those at the end of Section 8.1, the equation below has a unique mild solution $\mathbb{Y}^s$ on $[0,s]$:

$$\begin{cases} d\mathbb{Y}^s_t = (A\mathbb{Y}_t + b(s-t, \mathbb{Y}^s_t))\, dt + \sigma(s-t, \mathbb{Y}^s_t) d\mathbb{W}^Q_t, \qquad t \in [0,s], \\ \mathbb{Y}^s_0 = \eta. \end{cases} \tag{63}$$

We will be in fact mainly interested in its value at point s.

Definition 8.3. [Mild solution of the Kolmogorov equation].

We call **mild solution** of the Kolmogorov equation (60) the function

$$V(s,\eta) := \mathbb{E}\left[g(\mathbb{Y}^s_s)\right], \tag{64}$$

where $\mathbb{Y}^s$ is the solution of (63).

Remark 8.4. Whenever b and σ does not depend explicitly on time we have $\mathbb{Y}^s_s = \mathbb{X}_s$, where $\mathbb{X}$ is the solution of (58), so the definition given above reduces to the mild solution given in [10] Section 6.5 page 122. In this case the mild solution can be expressed in terms of the transition semigroup $(P_t, t > 0)$ corresponding to (58). More precisely one has $V(t,\eta) := P_t(g)(\eta)$, where, for any $t \in]0,T]$, and for any bounded, measurable function $\phi \colon H \to \mathbb{R}$, P_t is characterized as

$$(P_t\ \phi)(\eta) = \mathbb{E}[\phi(\mathbb{X}_t)]. \tag{65}$$

We recall, in a slightly more general situation, the notion of strict and strong solutions. Let consider $h \in C([0,T]\times H)$, $g \in C(H)$ and the Cauchy problem

$$\begin{cases} (\mathscr{L}_0(v) + h)(t,\eta) = 0, & (t,\eta) \in [0,T]\times H, \\ v(0,\eta) = g(\eta), & \eta \in H. \end{cases} \tag{66}$$

Moreover, for any $s \in (0,T]$, we consider the following Kolmogorov equation with final datum:

$$\begin{cases} \partial_t u(t,\eta) + \langle A^* Du(t,\eta), x\rangle + \frac{1}{2}\mathrm{Tr}\left[\sigma(t,x)\sigma^*(t,x)D^2 u\right] + h(t,\eta) = 0, & (t,\eta) \in [0,s)\times H, \\ u(s,\eta) = g(\eta), & \eta \in H. \end{cases} \tag{67}$$

Introducing the new notation

$$\begin{cases} D(\mathscr{L}^s_0) := \left\{\varphi \in C^{1,2}([0,s]\times H) \ :\ D\varphi \in C([0,s]\times H; D(A^*))\right\}, \\ \mathscr{L}^s_0(\varphi)(t,\eta) := \partial_t\varphi(t,\eta) + \langle A^* D\varphi(t,\eta), \eta\rangle + \frac{1}{2}\mathrm{Tr}\left[\sigma(t,\eta)\sigma^*(t,\eta)D^2\varphi(t,\eta)\right], \end{cases} \tag{68}$$

the equation (67) can be rewritten as

$$\begin{cases} (\mathscr{L}^s_0(u) + h)(t,\eta) = 0, & (t,\eta) \in [0,s[\times H, \\ u(s,\eta) = g(\eta), & \eta \in H. \end{cases} \tag{69}$$

Remark 8.5. Observe that the sign in front of ∂_t are opposite in (60) and (67).

Definition 8.6. [Strict solution of the Kolmogorov equation].

Consider $h \in C([0,T]\times H)$ and $g \in C(H)$. We say that $v \in C^{1,2}([0,T]\times H)$ (resp. $u \in C([0,s]\times H)$) is a **strict solution** of (66) (resp. of (69)) if $v \in D(\mathscr{L}_0)$ (resp. if $u \in D(\mathscr{L}^s_0)$) and (66) (resp. (69)) is satisfied.

Definition 8.7 (Strong solution of the Kolmogorov equation). .

Let $h \in C([0,T] \times H)$ and $g \in C(H)$. We say that $v \in C^{0,1}([0,T] \times H)$ with $Dv \in C([0,T] \times H; D(A^*))$ (resp. $u \in C^{0,1}([0,s] \times H)$ with $Du \in C([0,s] \times H; D(A^*))$) is a **strong solution** of (66) (resp. of (69)) if there exist three sequences $\{v_n\} \subseteq D(\mathscr{L}_0)$ (resp. $\{u_n\} \subseteq D(\mathscr{L}_0^s)$), $\{h_n\} \subseteq C([0,T] \times H)$ (resp. $C([0,s] \times H)$) and $\{g_n\} \subseteq C(H)$ fulfilling the following.

(i) For any $n \in \mathbb{N}$, v_n (resp. u_n) is a strict solution of the problem

$$\begin{cases} \mathscr{L}_0(v_n)(t,\eta)) + h_n(t,\eta) = 0, & (t,\eta) \in [0,T] \times H, \\ v_n(0,\eta) = g_n(\eta) & \eta \in H. \end{cases} \tag{70}$$

$$\text{(resp. of} \begin{cases} (\mathscr{L}_0^s(u_n) + h_n)(t,\eta) = 0, & (t,\eta) \in [0,s[\times H, \\ u_n(s,\eta) = g_n(\eta), & \eta \in H. \end{cases}) \tag{71}$$

(ii) The following convergences hold:

$$\begin{cases} v_n \to v & \text{in } C([0,T] \times H), \\ h_n \to h & \text{in } C([0,T] \times H), \\ g_n \to g & \text{in } C(H), \end{cases} \qquad \left(resp. \begin{cases} u_n \to u & \text{in } C([0,s] \times H), \\ h_n \to h & \text{in } C([0,s] \times H), \\ g_n \to g & \text{in } C(H). \end{cases} \right)$$

8.4 Decomposition for strong solutions of the Kolmogorov equation

Theorem 8.8. Consider $g \in C(H)$. Assume that Hypothesis 8.1 is satisfied. Suppose that $v \in C^{0,1}([0,T] \times H)$ with $Dv \in C(H; D(A^*))$ is a strong solution of (66). Then, given $s \in (0,T]$ and $\eta \in H$, we have

$$v(s,\eta) = g(\mathbb{Y}_s^s) - \int_0^s \left\langle Dv(s-r, \mathbb{Y}_r^s), \sigma(s-r, \mathbb{Y}_r^s) d\mathbb{W}_r^Q \right\rangle, \tag{72}$$

where $\mathbb{Y}^s$ is the solution of (63).

Proof. We denote by v_n the sequence of smooth solutions of the approximating problems prescribed by Definition 8.7, which converges to v. We fix $s > 0$ and we observe that $t \mapsto u(t,\eta) := v(s-t,\eta)$ is a strong solution of

$$\begin{cases} \partial_t u + \langle A^* Du, \eta \rangle + \frac{1}{2} \mathrm{Tr} \left[\sigma(s-t,\eta) \sigma^*(s-t,\eta) D^2 u \right] + \langle Du, b(s-t,\eta) \rangle = 0, \\ u(s,\eta) = g(\eta), \end{cases} \tag{73}$$

in the sense of Definition 8.7 (in the case of (71)) if we use, as a approximating sequence, $u_n(t,\eta) := v_n(s-t,\eta)$. Thanks to Proposition 6.6, every u_n verifies, for $t \in [0,s]$,

$$\begin{aligned} u_n(t, \mathbb{Y}_t^s) &= u_n(0,\eta) + \int_0^t \partial_r u_n(r, \mathbb{Y}_r^s) dr \\ &+ \int_0^t \langle A^* Du_n(r, \mathbb{Y}_r^s), \mathbb{Y}_r^s \rangle \, dr + \int_0^t \langle Du_n(r, \mathbb{Y}_r^s), b(s-r, \mathbb{Y}_r^s) \rangle \, dr \\ &+ \frac{1}{2} \int_0^t \mathrm{Tr} \left[\left(\sigma(s-r, \mathbb{Y}_r^s) Q^{1/2} \right) \left(\sigma(s-r, \mathbb{Y}_r^s) Q^{1/2} \right)^* D^2 u_n(s-r, \mathbb{Y}_r^s) \right] dr \\ &\qquad + \int_0^t \left\langle Du_n(r, \mathbb{Y}_r^s), \sigma(s-r, \mathbb{Y}_r^s) d\mathbb{W}_r^Q \right\rangle. \qquad \mathbb{P} - a.s. \end{aligned} \tag{74}$$

Since u_n is a strict solution of (73) the expression above gives, for $t \in [0, s]$,

$$u_n(t, \mathbb{Y}_t^s) = u_n(0, \eta) + \int_0^t \langle Du_n(r, \mathbb{Y}_r^s), \sigma(s - r, \mathbb{Y}_r^s) dW_r \rangle . \tag{75}$$

Define, for $t \in [0, s]$,

$$M_t^n := u_n(t, \mathbb{Y}_t^s) - u_n(0, \eta). \tag{76}$$

$(M^n)_{n \in \mathbb{N}}$ is a sequence of real local martingales (vanishing at zero). Since, thanks to Theorem 7.4 of [8] one has

$$\mathbb{E} \sup_{t \in [0,s]} \left(1 + |\mathbb{Y}_t^s|^N \right) < +\infty \qquad \text{for any } N \geq 1,$$

M^n converges ucp, thanks to the definition of strong solution, to

$$M_t := u(t, \mathbb{Y}_t^s) - u(0, \eta). \tag{77}$$

Since the space of real continuous local martingales equipped with the ucp topology is closed (see e.g. Proposition 4.4 of [28]) then M is a continuous local martingale.

Now set $\nu_0 = D(A^*)$, $\chi = \nu_0 \hat{\otimes}_\pi \nu_0$ and we show how the theory developed in the previous sections can help us here. Proposition 5.21 2. ensures that $\mathbb{Y}^s$ is a $\bar{\nu}_0$-semimartingale with $\bar{\nu}_0$ being the dual of $D(A^*)$. By Proposition 5.15 3., it is a $\nu_0 \hat{\otimes}_\pi \mathbb{R}$-weak Dirichlet process with decomposition $\mathbb{M} + \mathbb{A}$ where $\mathbb{M}$ is the local martingale defined by $\mathbb{M}_t = \eta + \int_0^t \sigma(s - r, \mathbb{Y}_r^s) d\mathbb{W}_r^Q$ and $\mathbb{A}$ is a $\nu_0 \hat{\otimes}_\pi \mathbb{R}$-martingale-orthogonal process. Moreover $\mathbb{X}$ has a finite χ-quadratic variation by Proposition 5.21 item 4.

Theorem 6.7 and Proposition 2.6 (ii) ensures that the process $u(\cdot, \mathbb{Y}_\cdot^s)$ is a real weak Dirichlet process whose local martingale part being equal to

$$N_t = u(0, \eta) + \int_0^t \left\langle Du(r, \mathbb{X}_r), \sigma(s - r, \mathbb{Y}_r^s) d\mathbb{W}_r^Q \right\rangle . \tag{78}$$

Observe that 13 is satisfied thanks to 8.2 and the continuity of Dv, $\mathbb{X}$ and $\mathbb{Y}^s$.

By item 1. of Proposition 3.10 the decomposition of a real weak Dirichlet process is unique so, identifying (77) with (78), for any $t \in [0, s]$, we get

$$u(t, \mathbb{Y}_t^s) = u(0, \eta) + \int_0^t \left\langle Du(r, \mathbb{Y}_r^s), \sigma(s - r, \mathbb{Y}_r^s) d\mathbb{W}_r^Q \right\rangle . \tag{79}$$

Since $v(s, \eta) = u(0, \eta)$, for any $t \in [0, s]$, by (79) it yields

$$v(s, \eta) = u(0, \eta) = u(t, \mathbb{Y}_t^s) - \int_0^t \left\langle Du(r, \mathbb{Y}_r^s), \sigma(s - r, \mathbb{Y}_r^s) d\mathbb{W}_r^Q \right\rangle . \tag{80}$$

In particular, for $t = s$, since $u(s, \cdot) = g$ by (73), it follows

$$\begin{aligned} v(s, \eta) &= u(0, \eta) = g(\mathbb{Y}_s^s) - \int_0^s \left\langle Du(r, \mathbb{Y}_r^s), \sigma(s - r, \mathbb{Y}_r^s) d\mathbb{W}_r^Q \right\rangle \\ &= g(\mathbb{Y}_s^s) - \int_0^s \left\langle Dv(s - r, \mathbb{Y}_r^s), \sigma(s - r, \mathbb{Y}_r^s) d\mathbb{W}_r^Q \right\rangle , \end{aligned}$$

which concludes the proof. □

We are now able to establish uniqueness of the solution of the Kolmogorov equation.

Proposition 8.9. Assume that Hypotheses 8.1 are satisfied and that g is a continuous function from H to $\mathbb{R}$. Let $v \in C^{0,1}([0,T] \times H)$ with $Dv \in C(H; D(A^*))$ be a strong solution of (60). Let v such that Dv has most polynomial growth in the η variable. Then the following holds.

(i) The expectation appearing in (64) makes sense and it is finite; consequently the function V is well-defined.

(ii) $v = V$ on $[0,T] \times H$.

Proof. Thanks to Theorem 8.8, we can write, for any $s \in (0,T]$,

$$v(s,\eta) + \int_0^s \left\langle Dv(s-r, \mathbb{Y}_r^s), \sigma(s-r, \mathbb{Y}_r^s) d\mathbb{W}_r^Q \right\rangle = g(\mathbb{Y}_s^s). \tag{81}$$

Observe that, by Theorem 7.4 in [8], all the momenta of $\sup_{r\in[0,t]} |\mathbb{Y}_r^s|$ are finite. On the other hand Dv has polynomial growth, then, recalling Remark 8.2, for $t \in [0,s]$,

$$\mathbb{E} \int_0^t \left\langle Dv(s-r, \mathbb{Y}_r^s), \left(\sigma(s-r, \mathbb{Y}_r^s) Q^{1/2}\right) \left(\sigma(s-r, \mathbb{Y}_r^s) Q^{1/2}\right)^* Dv(s-r, \mathbb{Y}_r^s) \right\rangle dr$$

is less or equal to, for all $t \in [0,s]$,

$$\mathbb{E} \int_0^t C \left(1 + |\mathbb{Y}_r^s|^N\right) dr$$

for some constants C and N and then, thanks again to Theorem 7.4 of [8] is finite.

Consequently, by Proposition 2.6 (i)

$$t \mapsto \int_0^t \left\langle Dv(s-r, \mathbb{Y}_r^s), \sigma(s-r, \mathbb{Y}_r^s) d\mathbb{W}_r^Q \right\rangle, \qquad t \in [0,s],$$

is a true martingale vanishing at 0. Consequently, for any $t \in [0,s]$, its expectation is zero. In the left-hand side of (81) we have a deterministic value and a random variable with zero-expectation, so the expectation of the right-hand side is well-defined and equals $v(s,\eta)$. In particular we have

$$v(s,\eta) = \mathbb{E}\left[g(\mathbb{Y}_s^s))\right]$$

which concludes the proof. □

Example 8.10. Whenever b and σ do not depend directly on the time and then the Kolmogorov equation is homogeneous, if $\mathbb{X}$ is the solution of (58) and $\mathbb{Y}^s$ the solution of (63) we have

$$\mathbb{X} = \mathbb{Y}^s \qquad \text{on } [0,s],$$

for any $s \in]0,T]$. So in particular $v(s,\eta) = V(s,\eta) = P_s(g)(\eta)$ where (P_t) is the transition semigroup associated to (58). In this case Proposition 8.9 gives a result similar to that of Theorem 7.6.2 Chapter 7 of [10]. In that case the authors do not use a strong solution approach. The two results have different hypotheses; in fact the one contained in [10] requires that v is in twice differentiable with locally uniformly continuous derivatives in the η variable while our result require the C^1 regularity and that $Dv(t,\eta) \in C([0,T] \times H; D(A^*))$.

Remark 8.11. The technique we have presented here can easily be adapted to treat other cases. One is the case in which $b \equiv 0$ and the function h appearing in (66) is a generic continuous function.

In this case the uniqueness result can be formulated as follows: any strong solution with the regularity required by Proposition 8.9 can be expressed as

$$v(s,\eta) = \mathbb{E}\left[g(\mathbb{Y}_s^s) + \int_0^s h(s-r, \mathbb{Y}_r^s) dr \right].$$

Whenever σ does not depend directly on the time the expression above can be rewritten as

$$v(s,\eta) = \mathbb{E}\left[g(\mathbb{X}_s) + \int_0^s h(s-r, \mathbb{X}_r) dr \right].$$

So the existence of $\mathbb{E}\left[g(\mathbb{X}_s)\right]$ implies the existence of $\mathbb{E}\left[\int_0^s h(s-r, \mathbb{X}_r)dr\right]$ and vice-versa. When one of the two exists (e.g. if g of h are bounded or have polynomial growing) we can write latter expression as

$$P_s(g)(\eta) + \int_0^s P_r(h(s-r,\cdot))(\eta) dr.$$

Then v is the mild solution used for example (in the particular case σ being the identity) in [26]. In that paper, the author uses a strong solution approach, introducing a series of functional spaces that allow to deal with a possible singularity at time 0 (that we do not have here), but he does not explicitly provide a uniqueness result.

Observe that in [26, 4] the problem is approached by studying the properties of the transition semigroup defined in (65) on the space $C_b(H)$ of the continuous bounded function (or in some cases, on the space $B_b(H)$ of bounded function) defined on H introducing a new notion of semigroup (see also [39]). This kind of methodology *structurally* requires the initial datum g to belong to $C_b(H)$ (or $B_b(H)$) and then Kolmogorov equations with unbounded initial datum cannot be studied.

Remark 8.12. The ideas we used here to prove the relation between strong and mild solutions of the Kolmogorov equations can be used to study second order Hamilton-Jacobi-Bellman equation related to optimal control problems driven by stochastic PDEs and provide consequently verification theorems. This kind of approach is used for example in Section 6 of [22].

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