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Abstract

We study the optimal dynamics of an AK economy where population is uniformly distributed along the unit circle. Locations only differ in initial capital endowments. Spatio-temporal capital dynamics are described by a parabolic partial differential equation. The application of the maximum principle leads to necessary but non-sufficient first-order conditions. Thanks to the linearity of the production technology and the special spatial setting considered, the value-function of the problem is found explicitly, and the (unique) optimal control is identified in feedback form. Despite constant returns to capital, we prove that the spatio-temporal dynamics, induced by the willingness of the planner to give the same (detrended) consumption over space and time, lead to convergence in the level of capital across locations in the long-run.

Key words: Economic Growth, Spatial Dynamics, Optimal Control, Partial-Differential Equations

JEL Classification: C60, O11, R11, R12, R13

1. Introduction

Optimal and market allocation of economic activity across space has always been a central issue in economic theory from the seminal work of Hotelling (1929). Recently, some authors have studied the optimal spatial allocation of economic activity in dynamic settings with capital accumulation. To our knowledge, Brito (2004) is the first attempt to fully characterize the corresponding optimal spatio-temporal dynamics, followed by Brock and Xepapadeas (2008) and Boucekkine *et al.* (2009). This research is surveyed by Desmet and Rossi-Hansberg (2010).

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Factor mobility turns out to be crucial: Brito and Boucekkine *et al.* consider frictionless capital mobility while Brock and Xepapadeas invoke a spatial externality without capital mobility. In the former, the production function exhibits decreasing returns: capital flows from regions with low marginal productivity of capital to regions with high marginal productivity. As a result, capital spatio-temporal dynamics are shown to be driven by a partial differential equation (PDE) of the form:

$$\frac{\partial k}{\partial t}(t,z) - \frac{\partial^2 k}{\partial z^2}(t,z) = F\left(k(t,z), z\right) - c(t,z), \tag{1}$$

where z is the spatial position (which could be a position in the real line as in Boucekkine *et al.*, 2009, or in the unit circle as in this article). Per capita consumption and capital, $c(t,z)$ and $k(t,z)$ respectively vary in time and space. The production function $F(\cdot,\cdot)$ can depend explicitly on space. The specific nature of the equation comes from the term $\frac{\partial^2 k}{\partial z^2}(t,z)$ which captures capital flows across space as explained in Section 2: this makes the problem infinite-dimensioned.

Both Brito and Boucekkine *et al.* have attempted to solve spatial Ramsey models where capital follows equation (1). Both have used a straight line as a model of space and have used an adapted maximum principle to derive the corresponding first-order conditions, in particular the adjoint equation which is a PDE too. As explained in Boucekkine *et al.*, the maximum principle yields an ill-posed system of PDE equations. The problem arises from the specific generated adjoint equation which, coupled with the associated transversality condition, does not allow to prove neither the existence nor uniqueness of an optimal control (ill-posedness). In particular, potential multiplicity of solutions is the key problem faced by both Brito and Boucekkine *et al.*, who have ended up restricting the model and/or the set of optimal solutions to get rid of this problem: Boucekkine *et al.* restrict utility functions to be linear and Brito identifies a special type of solutions (called *travelling waves*).

In this paper, we consider the *AK* production function case and we model space as a circle. Even if the linear production function simplifies the adjoint equation and the space is bounded, the problem mentioned above still remains as we will show. Precisely, we show that ill-posedness is due to the fact that the first-order optimality conditions found by Boucekkine *et al.*, specified for our AK problem, are necessary but not sufficient to determine the optimal solution; as a result, other "irrelevant" solutions to these conditions do emerge. This makes a big difference with respect to the standard finite-dimensioned AK model (without space) where the first-order conditions are also sufficient. The key tool to reach these results is the use of a dynamic programming method well adapted to the infinite dimensionality of the problem. After rewriting the problem in a suitable infinite dimensional space, we exploit the linearity of the production function and the spatial setting (that's the circle as a compact manifold without boundary) to identify an explicit value function, which in turn allows us to solve the problem in feedback form and then to find explicitly the optimal control. The methodology is described in the last paragraph of Section 2, Appendix A gives the related details. We prove that the unique solution to the dynamic programming problem does satisfy the first-order optimality conditions, hence the necessity of the latter. Moreover, we prove that these conditions do have other solutions which are not solutions to the original optimal control problem, implying that these conditions are not sufficient. Another point of potential methodological interest is the proof of Theorem 3.3 where we make use of Fourier series to explore the asymptotic behavior of the solutions. It can be of interest in infinite dimensional control problems beyond the particular case analyzed in this paper. We give details of this approach in Appendix B.

A second set of contributions concerns the full analytical characterization of optimal spatio-temporal dynamics in the AK case. Individuals are distributed along the unit circle with possibly unequal initial capital endowments. How would optimal spatio-temporal dynamics alter this initial capital distribution? In the standard AK model, no transition dynamics set in because of constant returns to scale. We obtain a striking result: spatio-temporal dynamics set in and lead to the convergence of time detrended capital stocks across space to the same common value whatever the initial spatial distribution of capital. Convergence here is not the result of decreasing returns but is exclusively due to spatio-temporal dynamics. Our main finding can also be related to the literature on economic growth with heterogeneous initial endowments (see Chatterjee, 1994). This said, this paper highlights the specific nature of spatial dynamics, an aspect omitted in the latter literature.

This note is organized as follows. Section 2 sketches the model. Section 3 displays the analytical results allowed by the dynamic programming model. Section 4 uses the solution obtained by the dynamic programming method to characterize ill-posedness in the sense of Boucekkine *et al.* in the model under scrutiny. Section 5 provides with complementary numerical illustrations.

2. The model

We assume that individuals are distributed homogeneously along the unit circle in the plane, which we denote by $\mathbb{T}$. Using polar coordinates $\mathbb{T}$ can be described as the set of spatial parameters θ in $[0, 2\pi]$ with $\theta = 0$ and $\theta = 2\pi$ being identified.

Our assumption of a non-growing and spatially homogeneous population distribution is made for convenience. Considering an heterogeneous spatial population distribution would lead to introduce population density in the social Benthamite welfare function, which will disable the simple analytical solution derived in this paper. Second, the choice of the unit circle to represent space is not innocuous. It has indeed two important geometric properties: it is *compact* and *without boundary*. Both conditions are essential to avoid the specification of boundary conditions, to simplify the form of the partial differential equation that drives the system and then to simplify the form of the Hamilton-Jacobi-Bellman (HJB) equation. Thanks to that simple form we are able to apply the dynamic programming finding an explicit solution of the HJB equation and solving the problem. See Remark B.2 for other details.

The law of motion of capital in time and space is adopted from the related literature. At a given point $(t, \theta) \in [0, \infty) \times \mathbb{T}$, physical capital $k(t, \theta)$ evolves according to

$$\frac{\partial k}{\partial t}(t, \theta) = Ak(t, \theta) - c(t, \theta) - \tau(t, \theta). \tag{2}$$

A represents the level of technology, which we assume to be constant over time and space. The production function is AK at any point in space. $c(t, \theta)$ and $\tau(t, \theta)$ stand for consumption and net trade balance at $(t, \theta) \in [0, \infty) \times \mathbb{T}$ respectively. Capital depreciation is zero everywhere. Finally, we assume there is no adjustment or transportation costs when moving capital from a location to another. Such costs traditionally generate non-instantaneous adjustment, we switch off this dynamics engine to rely on spatial dynamics.

The trade balance over an arc $B = \overrightarrow{\theta_1\theta_2}$ of the circle, with $\theta_1 < \theta_2$, is equal to what enters at θ_1 minus what goes out at θ_2, in formulas: $\int_{\theta_1}^{\theta_2} \tau(t, \theta)\, \mathrm{d}\theta = -\left(\frac{\partial k}{\partial \theta}(t, \theta_2) - \frac{\partial k}{\partial \theta}(t, \theta_1)\right)$. Thanks to the Lagrange form of the reminder in the Taylor expansion, last expression is equal to $-(\theta_2 - \theta_1)\frac{\partial^2 k(t,\bar{\theta})}{\partial \theta^2}$ for some $\bar{\theta}$ within $[\theta_1, \theta_2]$. Letting θ_2 to θ_1 we get $\tau(t, \theta) = -\frac{\partial^2 k(t,\theta)}{\partial \theta^2}$. Using the latter expression for τ in (2) gives

$$\begin{cases} \dfrac{\partial k}{\partial t}(t,\theta) = \dfrac{\partial^2 k(t,\theta)}{\partial \theta^2} + Ak(t,\theta) - c(t,\theta), \quad \forall t \geq 0, \quad \forall \theta \in \mathbb{T} \\ k(t,0) = k(t,2\pi), \quad \forall t \geq 0 \\ k(0,\theta) = k_0(\theta), \quad \forall \theta \in [0,2\pi]. \end{cases} \tag{3}$$

Provided an initial distribution of physical capital $k_0(\cdot)$ on $\mathbb{T}$, the policy maker has to choose a control $c(\cdot,\cdot)$ to maximize the following functional

$$J(k_0, c(\cdot,\cdot)) := \int_0^{+\infty} e^{-\rho t} \int_0^{2\pi} \frac{c(t,\theta)^{1-\sigma}}{1-\sigma} \, \mathrm{d}\theta \, \mathrm{d}t \tag{4}$$

The value function of our problem starting from k_0 is defined as

$$V(k_0) := \sup_{c(\cdot,\cdot)} J(k_0, c(\cdot,\cdot)) \tag{5}$$

where the supremum is taken over the controls that ensure the capital to remain non-negative at every time and at every point of the space.

The method employed involves regular enough functions $k(\cdot,\cdot)$, $c(\cdot,\cdot)$, so that for any time $t \in [0,+\infty)$ the functions $k(t,\cdot)$, $c(t,\cdot)$ of the space variable can be considered as elements of the Hilbert space $L^2(\mathbb{T})$. $L^2(\mathbb{T})$ is the set of the functions $f\colon \mathbb{T} \to \mathbb{R}$ s.t. $\int_0^{2\pi} |f(\theta)|^2 \, \mathrm{d}\theta < +\infty$. This simplifying feature allows to apply dynamic programming techniques in $L^2(\mathbb{T})$ which are similar in spirit to those employed in the finite dimensional case. We write and solve the Hamilton-Jacobi-Bellman (HJB) equation in $L^2(\mathbb{T})$ and we use its solution to find the optimal control in feedback form. The method is detailed in Appendix A.

3. Spatial dynamics in the AK model: analytical results

We start characterizing a crucial property of the optimal solution: as in the pre-existing AK frameworks, the planner chooses a constant consumption level over time and space, and all aggregate variables grow at a constant growth rate from $t = 0$.

Theorem 3.1. *Suppose that*

$$A(1-\sigma) < \rho \tag{6}$$

and consider $k_0 \in L^2(\mathbb{T})$, a positive initial distribution of physical capital. Define

$$\eta := \frac{\rho - A(1-\sigma)}{2\pi\sigma}. \tag{7}$$

Provided that the trajectory $k^*(t,\theta)$, driven by the feedback control (constant in θ)

$$c^*(t,\theta) = \eta \int_0^{2\pi} k^*(t,\varphi) \, \mathrm{d}\varphi \tag{8}$$

remains positive, $c^*(t,\theta)$ is the unique optimal control of the problem. Moreover the value function of the problem is finite and can be written explicitly as

$$V(k_0) = \alpha \left(\int_0^{2\pi} k_0(\theta) \, \mathrm{d}\theta \right)^{1-\sigma} \tag{9}$$

where

$$\alpha = \frac{1}{1-\sigma}\left(\frac{\rho - A(1-\sigma)}{2\pi\sigma}\right)^{-\sigma}. \tag{10}$$

Proof. See Appendix B. $\square$

The theorem deserves some comments. First, the parametric condition (6) is exactly the one needed in the standard AK theory to assure that the objective function is finite. It is also needed here to guarantee that the value function is finite. Indeed $V(k_0)$ is given explicitly by (9). Notice that $V(k_0)$ depends only on the aggregate initial capital stock. Two different capital distributions having the same aggregate value will yield the same value function, and therefore the same optimal consumption rule (here given by (8)). The optimal solution features a kind of pooling of resources, which is hardly surprising given the social optimum setting considered. Finally, one has to notice that both the value function and the feedback control are identical to those of the standard AK model once we normalize for population size, equal to 2π. We now extract the optimal solutions for the aggregate capital stock and the induced optimal consumption rule at any point (t, θ).

Proposition 3.2. *Under the same assumptions of Theorem 3.1 the aggregate capital* $K(t) := \int_0^{2\pi} k(t,\theta)\,\mathrm{d}\theta$*, along the optimal trajectories, is*

$$K(t) = K(0)e^{\beta t} \tag{11}$$

where

$$\beta := \left[\frac{A-\rho}{\sigma}\right] \tag{12}$$

and $K(0) := \int_0^{2\pi} k_0(\theta)\,\mathrm{d}\theta$.

Proof. See Appendix B. $\square$

Notice that the optimal aggregate capital stock evolves exactly as in the standard AK model: it grows at the same standard growth rate, $\beta = \frac{A-\rho}{\sigma}$, from $t = 0$. From (8), one can then use (11) to obtain that the optimal control is $c^*(t,\theta) = \eta K(0)e^{\beta t}$. This is a crucial property of the model, the planner will choose the same (detrended) consumption level for all individuals whatever their location and generation. Thus, the aggregate variables have no transition dynamics just like in the standard AK theory. Nonetheless, if the planner is willing to keep (detrended) consumption constant over time and space, he has to move capital over time and space accordingly. What could be the induced optimal spatio-temporal dynamics? The next theorem gives the answer to this appealing question.

Theorem 3.3. *Assume that the hypotheses of Theorem 3.1 are satisfied. Suppose that*

$$\rho < A(1-\sigma) + \sigma. \tag{13}$$

Then, along the optimal trajectory, the detrended capital

$$k_D(t,\theta) := \frac{k(t,\theta)}{e^{\beta t}} \tag{14}$$

converges uniformly (and a fortiori *pointwise), as a function of θ, to the constant function $\frac{K(0)}{2\pi}$ when t tends to infinity. In other words*

$$\lim_{t\to\infty}\left(\sup_{\theta\in\mathbb{T}}\left|k_D(t,\theta)-\frac{K(0)}{2\pi}\right|\right)=0.$$

Proof. See Appendix B. □

Theorem 3.3 is the main result of this paper: it shows that though no transition dynamics occur for aggregate variables, spatio-temporal dynamics do set in, ultimately ensuring convergence in capital over space.[1] The spatio-temporal dynamics induced by the willingness of the planner to give the same consumption over space and time lead to equalize the capital level across locations in the long-run. Hence, these dynamics do eliminate the initial inequalities in capital endowments in our spatial AK model. As mentioned before, this result is striking in many respects, notably with respect to the traditional theory of convergence, which typically builds on decreasing returns. AK technologies are usually associated with divergence. If one has in mind the existing literature on the link between growth and inequalities where individuals have heterogeneous initial endowments, then our result could be interpreted at first glance as symmetrical to the properties highlighted by Chatterjee (1994) under decreasing returns. This said, the spatial dynamics entailed in our model are specific. They are specific *per se* because they are generated by a generic diffusion state equation with no counterpart in the standard growth theory. Our results are also original in the sense that thanks to our explicit solution, we identify a threshold value for the discount rate, that is $\rho<\bar{\rho}=A(1-\sigma)+\sigma$, above which anything can happen. Notice however, that this condition is very largely satisfied in realistic parameterizations of the model. The next section provides a complementary numerical investigation.

4. Ill-posedness, the maximum principle and dynamic programming

Section 3 neatly illustrates the working of the adapted dynamic programming method. Thanks to the linearity of the technology and the spatial setting considered (that's a compact manifold without boundary), the (unique) solution to the HJB equation has been explicitly found, which has allowed to characterize the (unique) optimal control in feedback form. How can this neat finding be related to the negative ill-posedness problem reported by Boucekkine *et al.* on the use of the maximum principle? If one tries to address the optimal control problem using the maximum principle, the resulting set of first-order conditions would be (with $q(t,\theta)$ the adjoint variable): (i) the state equation (3), (ii) the maximum condition $q(t,\theta)=e^{-\rho t}c(t,\theta)^{-\sigma}$, (iii) the adjoint equation $\frac{\partial q(t,\theta)}{\partial t}=-\frac{\partial^2 q(t,\theta)}{\partial\theta^2}-Aq(t,\theta)$ and (iv) the transversality condition $\lim_{t\to\infty}q(t,\theta)=0$ for all $\theta\in[0,2\pi]$. These are indeed the conditions found in by Boucekkine *et al.* (2009) specified for the AK case on the circle. The problem arises from the adjoint equation (iii). In the standard (non-spatial) AK model, its counterpart is given by the ordinary differential equation $\dot{q}(t)=-Aq(t)$, which solution is $q(0)e^{-At}$. The standard way to solve the optimal control problem in this case is to identify $q(0)$ using the transversality condition. In the infinite-dimensioned spatial case one would like to do the same thing taking a generic $q(0,\theta)$, looking at the evolution of the adjoint variable driven by (iii) and selecting the $q(0,\theta)$ consistently with the transversality

[1] Formally, we prove uniform convergence of the spatial distribution of capital by showing convergence of the corresponding Fourier series coefficients.

condition (iv). Unfortunately, this is no longer possible in this case for the same reason detailed in Boucekkine *et al.* (2009): ill-posedness shows up especially as the transversality condition is no longer enough to select a unique solution.[2]

Using dynamic programming, one gets rid of the adjoint equation (and the transversality condition), and there is some hope to overcome ill-posedness. We show that this approach is conclusive in Section 3. So how to interpret the ill-posedness problem described above? Of course, as in finite dimension, the adjoint variable $q(t, \theta)$ is connected to the value function, V, as follows: $q(t, \theta) = e^{-\rho t}\nabla V(k(t))(\theta)$, where ∇V is the Gâteaux derivative of V (more details in Appendix A). One can study the dynamics of q since the value function is fully identified. We have

$$q(t, \theta) = e^{-\rho t}\nabla V(k(t))(\theta) = e^{-\rho t}\alpha(1-\sigma)\langle k(t), \mathbb{1}\rangle^{-\sigma}\,\mathbb{1}(\theta) = \left(\frac{\rho - A(1-\sigma)}{2\pi\sigma}K(0)\right)^{-\sigma}\mathbb{1}(\theta)e^{-At}.$$

One can directly see that such a q satisfy the adjoint equation (iii): indeed $\frac{\partial^2 q(t,\theta)}{\partial\theta^2} = 0$ since q is constant in θ and $\frac{\partial q(t,\theta)}{\partial t} = -A\left(\frac{\rho - A(1-\sigma)}{2\pi\sigma}K(0)\right)^{-\sigma}\mathbb{1}(\theta)e^{-At} = -Aq(t, \theta)$. Clearly this q also satisfies the transversality condition (iv). This proves the necessity of the first-order conditions put forward by Boucekkine *et al.* (2009).

We now straightforwardly show that these first-order conditions are not sufficient in the infinite horizon case spatial AK model considered here.[3] Indeed, it is enough to observe that the the adjoint variable (iii), together with the transversality condition (iv) admits more than one solution, for example all the functions of the form $c\mathbb{1}(\theta)e^{-At}$ for some real constant c satisfy both. Thus the first-order conditions found above are only necessary and not sufficient to determine the optimum. All in all, our spatial AK model analysis allows to reach the conclusion that the ill-posedness problem highlighted by Boucekkine *et al.* recovers the occurrence of necessary but non-sufficient first-order conditions from the application of the maximum principle.

It is worth pointing out here that the ill-posedness problem solved in this paper is not due to the spatial setting adopted: as clearly explained in Remark (B.2) in Appendix B, the spatial setting may complicate the search for an explicit solution to the HJB but is not ultimately responsible for ill-posedness. The main source of the latter is the conjunction of the infinite-dimensionality of the problem, coming from the PDE governing the state equation, and the infinite time horizon. The maximum principle applied to the same type of PDEs does not suffer from ill-posedness if time horizons were finite (see for example Barbu and Precupanu, 2012, Chapter 4). However, not all infinite time horizons optimal control problems of parabolic PDEs suffer from ill-posedness: this problem may not occur for example if the functional involved are quadratic or similar (see Faggian, 2008). Unfortunately, and to the best of our knowledge, there is no general result on necessary and sufficient conditions for the maximum principle in the infinite time horizon case for infinite dimensional optimization problems. Much remains to do for this general class of problems.

[2] We'd reach exactly the same conclusion if instead of considering Hilbert spaces of functions on $[0,\ 2\pi]$, we'd go through the maximum principle with periodic functions on the whole real space.

[3] Boucekkine *et al.* (2009) invoke a sufficient condition theorem due to Gozzi and Tessitore (1998). However, this rather natural concavity-based sufficiency theorem is established in the finite time horizon case (see also Barbu and Precupanu, 2012, Theorem 4.5, page 243), not in the infinite horizon case.

5. Spatial dynamics in the AK model: computational results

We perform a numerical exercise to shed more light on the transitional dynamics of the spatial AK-model. To this end, we use the explicit optimal dynamics of the detrended capital in the form of Fourier series (see the appendix). There are three key parameters in our modeling: ρ, A and σ. In the AK-model $A = Y/K$ and consequently we choose $A = 1/3$ as a reasonable ratio output to physical capital. Then we fix $\rho = 0.07$ and $\sigma = 0.8$.

Table 1: Parameters values

Total Factor Productivity	A	1/3
Time discount rate	ρ	0.07
Intertemporal elasticity of substitution	σ	0.8

Provided that the set $\{\rho, A, \sigma\}$ satisfies both conditions[4] (6) and (13), Theorem 3.3 proves that detrended capital at any point (t, θ) converges to the constant function $\frac{K(0)}{2\pi}$ when t tends to infinity. To illustrate this result, we study the case of an economy made of two regions, the first region, $[0, \pi]$ is initially endowed with twice the capital of the second region, $[\pi, 2\pi]$, namely

$$k_0(\theta) = \begin{cases} 20, \theta \in [0, \pi], \\ 10, \theta \in [\pi, 2\pi]. \end{cases}$$

In the standard AK-model the optimal trajectory instantaneously adjusts to the optimal consumption and production plan. This behavior does no longer hold spatially when physical capital is allowed to move across space as figures 1 and 2 show. Capital moves from rich locations towards poor ones: adjustment is not instantaneous since it takes time for capital to achieve its final location. Besides, any point is constrained in the amount of capital it can send by its (optimal) consumption path, from $t \geq 0$. As capital moves, a wave appears and it brings about the emergence of a temporary production agglomeration in the rich region. Simultaneously, a depressed area is formed in the center of the poor region. When capital moves, all locations in the rich region send capital to the poor region to reach the optimal path. The depressed area appears since capital moves from $t = 0$, also within the poor region. With time, both the agglomeration and depression lose force and we can observe a complete convergence to the spatially homogenous steady state value for detrended capital from $t = 10$. The graphs show physical capital across space. Figure 1 shows the optimal trajectory of detrended physical capital for $(\theta, t) \in [0, 2\pi] \times [0, 8]$[5]. Figure 2 illustrates the importance of condition (13). For this example $\rho = A(1 - \sigma) + \sigma = 0.8667$ and we can see that detrended capital does not converge to a spatially homogenous distribution. A noteworthy aspect is that detrended physical capital does converge to a steady state, and it does so very fast.

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[4]For $A = 1/3$ and $\sigma = 0.8$, the range of values for ρ which satisfies conditions (6) and (13) is $]0.0667, 0.8667[$.

[5]Convergence towards the steady state would have been slower if transportation costs, adjustment costs, capital depreciation or trade barriers had been present.

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Appendices

A. The Hilbert space setting and description of the method

A.1. The optimal control problem in infinite dimensions

This appendix provides the definitions and the properties used in the proofs of the theorems in the paper. As announced at the end of Section 2, the dynamics of the model are described in the space $L^2(\mathbb{T})$. $L^2(\mathbb{T})$ is a Hilbert space whose elements are functions. The scalar product of f and g in $L^2(\mathbb{T})$ is defined as $\langle f, g\rangle := \int_0^{2\pi} f(\theta)g(\theta)\,\mathrm{d}\theta$. The norm in $L^2(\mathbb{T})$ is given by $|f|_{L^2(\mathbb{T})} := \langle f, f\rangle^{1/2} = \left(\int_0^{2\pi} (f(\theta))^2\ \mathrm{d}\theta\right)^{1/2}$, and the distance between two elements $f, g \in L^2(\mathbb{T})$ is equal to $|f - g|_{L^2(\mathbb{T})}$.

We introduce the operator G on $L^2(\mathbb{T})$ defined as

$$G(f) = \frac{\partial^2 f}{\partial\theta^2}.$$

Since we cannot define the second derivative on all the functions of $L^2(\mathbb{T})$, we need to introduce a subset of $L^2(\mathbb{T})$ on which G is well defined: it is the *domain* of G and it is denoted with $D(G)$. The functions that belong to $D(G)$ are those whose first and second derivatives are in $L^2(\mathbb{T})$. $D(G)$ is often denoted by $H^2(\mathbb{T})$ and it is a natural choice for us because it is made exactly of functions f for which $G(f)$ remains in $L^2(\mathbb{T})$, which is the space in which we want to work.

Next we define the expression e^{tG}: given a distribution $k_0 \in L^2(\mathbb{T})$, which is a function of the space position θ, the expression $e^{tG}k_0$ denotes the unique solution of

$$\begin{cases} \frac{\partial k}{\partial t}(t,\theta) = \frac{\partial^2 k}{\partial \theta^2}(t,\theta) = G(k(t,\theta)) \\ k(t,0) = k(t,2\pi) \\ k(0,\theta) = k_0(\theta), \quad \forall \theta \in \mathbb{T}, \end{cases} \tag{A.1}$$

computed at time t. For a fixed t the expression $e^{tG}k_0$ is a function of the space variable θ and $e^{tG}k_0(\theta) = k(t,\theta)$ where $k(\cdot,\cdot)$ is the solution of (A.1).[6] e^{tG} has some important properties, the first is that $e^{tG}k_0 \in L^2(\mathbb{T})$ for all $t \geq 0$. e^{tG} is a "semigroup" because it satisfies the following "semigroup property": for all $k_0 \in L^2(\mathbb{T})$ and $t, s \geq 0$, $e^{(t+s)G}k_0 = e^{sG}\left(e^{tG}k_0\right)$.[7] e^{tG} is said to be "generated" by G (and it justifies the presence of G in the notation "e^{tG}"). This means that, if $k_0 \in D(G)$, then for all $t \geq 0$, $\lim_{h\to 0} \frac{e^{(t+h)G}k_0 - e^{tG}k_0}{t} = Ge^{tG}k_0$. This fact is not surprising if one looks at the first line of (A.1). Indeed, since $k(t,\theta) = e^{tG}k_0(\theta)$, the left side is exactly $\frac{\partial k}{\partial t}(t,\theta) := \lim_{h\to 0} \frac{k(t+h,\theta)-k(t,\theta)}{t} = \lim_{h\to 0} \frac{e^{(t+h)G}k_0 - e^{tG}k_0}{t}$ and the right side is $Gk(t,\theta) = Ge^{tG}k_0(\theta)$. This implies that $\frac{\mathrm{d}e^{Gt}k_0}{\mathrm{d}t} = Ge^{Gt}k_0$. The former property justifies the exponential notation: we write "$e^{Gt}k_0$" because when we take the derivative in t, it behaves exactly as a standard exponential function.

Since G is by definition identifiable with $\frac{\partial^2}{\partial \theta^2}$ and $\dot{k}$ is the time derivative, the state equation (3) can be rewritten as an evolution equation in $L^2(\mathbb{T})$:

$$\begin{cases} \dot{k}(t) = Gk(t) + Ak(t) - c(t), \\ k(0) = k_0. \end{cases} \tag{A.2}$$

$k(t)$ is an element of $L^2(\mathbb{T})$ for all $t \in [0,+\infty)$, *i.e.*, $k(t)$ is a function of space. Hence we can compute $k(t)$ at the spatial point $\theta \in \mathbb{T}$ and we write $k(t)(\theta)$, which is in fact capital at time t in the point of the space θ. The same holds for $c(t)$. The set of admissible controls is[8] $\mathcal{U}_{k_0} := \{c \in L^2_{\mathrm{loc}}(\mathbb{R}^+; L^2(\mathbb{T})) : c(t)(\theta),\ k(t)(\theta) \geq 0 \text{ for all } (t,\theta) \in \mathbb{R} \times \mathbb{T}\}$ so that we can rewrite the value function more formally as $V(k_0) := \sup_{c(\cdot)\in\mathcal{U}_{k_0}} J(k_0, c(\cdot))$.

Once we have chosen $c(\cdot) \in \mathcal{U}_{k_0}$ for a given initial distribution $k_0 \in L^2(\mathbb{T})$, with an abuse of notation we denote by $k(\cdot)$ the solution of (A.2), the same letter we used for the solution of (3). This is justified because $k(t)(\theta) = k(t,\theta)$, where the latter is the solution of (3). Since we do not require the initial distribution nor the control to be regular, the solution of (3) has to be understood in some generalized form: it does not need it to be C^2 in the space variable nor C^1 in the time variable, (see Lions and Magenes (1972) Section 15.1 page 78). In this sense a unique solution exists and it belongs to $C([0,+\infty); L^2(\mathbb{T}))$ (see Bensoussan *et al.*, 2007, Proposition 3.4, Chapter II-1 page 136).

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[6] Another way to describe the action of the the semigroup e^{tG} is the following: given $k_0 \in L^2(\mathbb{T})$ we denote by $\tilde{k}_0 \colon \mathbb{R} \to \mathbb{R}$ it "extension" (by 2π-periodicity) on the real line i.e. $\tilde{k}_0(s) := k_0\left(2\pi\left\{\frac{s}{2\pi}\right\}\right)$ where $\left\{\frac{s}{2\pi}\right\}$ denotes the fractional part of $\frac{s}{2\pi}$. Then one has that $e^{tG}k_0(\theta) = \int_{-\infty}^{+\infty} \frac{1}{(4\pi t)^{1/2}} e^{-(r-\theta)^2/4t}\tilde{k}_0(r)\,\mathrm{d}r$

[7] More details are provided in Chapter IX, Yosida (1980).

[8] $L^2_{\mathrm{loc}}(\mathbb{R}^+; L^2(\mathbb{T}))$ is the space of real locally square-integrable functions. Formally

$$L^2_{\mathrm{loc}}(\mathbb{R}^+; L^2(\mathbb{T})) := \left\{c \colon \mathbb{R}^+ \times \mathbb{T} \to \mathbb{R} \ : \ \int_0^M \int_0^{2\pi} |c(t)(\theta)|^2 \,\mathrm{d}\theta\,\mathrm{d}t < +\infty \text{ for all } M > 0\right\}.$$

Observe that for every continuous function $f \in L^2(\mathbb{T})$ we have $f(0) = f(2\pi)$, since $\theta = 0$ and $\theta = 2\pi$ represent the same point on $\mathbb{T}$. Consequently the boundary condition $k(t, 0) = k(t, 2\pi)$ is already included in the definition of the domain of G and the boundary condition required in the PDE version of the state equation disappears when we rewrite the equation in (A.2).

We define function $\mathbb{1} \in L^2(\mathbb{T})$ as the constant (in the space variable) equal to 1:

$$\begin{cases} \mathbb{1}\colon \mathbb{T} \to \mathbb{R} \\ \mathbb{1}(\theta) \equiv 1. \end{cases}$$

We can use $\mathbb{1}$ to rewrite the functional (4) as

$$J(k_0, c(\cdot,\cdot)) := \int_0^{+\infty} e^{-\rho t} \int_0^{2\pi} \frac{c(t,\theta)^{1-\sigma}}{1-\sigma}\, \mathrm{d}\theta\, \mathrm{d}t = \int_0^{+\infty} e^{-\rho t} \langle \mathbb{1}, U(c(t)) \rangle\, \mathrm{d}t, \tag{A.3}$$

where $U(f)\colon \mathbb{T} \to \mathbb{R}$ is the function $U(f)(\theta) = \frac{f(\theta)^{1-\sigma}}{1-\sigma}$ for a given $f \in L^2(\mathbb{T})$.

A.2. The method: dynamic programming approach in infinite dimensions

Summarizing we have rewritten the problem in the infinite dimensional $L^2(\mathbb{T})$ setting and we need to maximize (A.3) subject to (A.2). We employ the dynamic programming method in $L^2(\mathbb{T})$ so we need to write and solve an infinite dimensioned Hamilton-Jacobi-Bellman (HJB) equation in the infinite dimensional $L^2(\mathbb{T})$ setting. Then we use the solution to find the optimal control in feedback form.[9]

Before writing the HJB equation of the problem we need to introduce the concept of "Gâteaux derivative". If we have a regular function from $\mathbb{R}^N \to \mathbb{R}$, then its gradient at a certain point is an element of $\mathbb{R}^N$. The same thing happens if we want to compute the gradient (the Gâteaux derivative) of a (regular) function $v\colon L^2(\mathbb{T}) \to \mathbb{R}$: it can be computed in $k \in L^2(\mathbb{T})$ and it is an element of $L^2(\mathbb{T})$ denoted with $\nabla v(k)$. Hence we can compute $\nabla v(k)$ at a certain space-point θ and we get a real number. Since $\nabla v(k) \in L^2(\mathbb{T})$, we can compute $G\nabla v(k)$ if $\nabla v(k)$ is in $D(G)$, *i.e.* if it is regular enough as a function of θ. More details on Gâteaux derivatives can be found for example in the book by Li and Yong (1995), page 44 (in the discussion after Theorem 2.19).

Observe that, similarly to what happens in the standard finite dimensional case, we expect that the solution of the HJB equation is equal to the value function of the infinite dimensional problem. In other words the unknown $v(k)$ of the HJB equation will be equal to the value function $V(k)$ introduced above. This fact will be formally proved in Appendix B, Step 3 in the proof of Theorem 3.1.

The HJB equation of the problem is defined as

$$\rho v(k) = \langle k, G\nabla v(k) \rangle + A \langle k, \nabla v(k) \rangle + \sup_{c \in L^2(\mathbb{T};\mathbb{R}^+)} \{ -\langle c, \nabla v(k) \rangle + \langle \mathbb{1}, U(c) \rangle \}. \tag{A.4}$$

The HJB equation (A.4) is infinite dimensional, this means that its solution is a function $v\colon L^2(\mathbb{T}) \to \mathbb{R}$, *i.e.*, for every $k \in L^2(\mathbb{T})$, $v(k)$ is a real number, being k in itself a function from $\mathbb{T}$ to $\mathbb{R}$. (A.4) is formally obtained in analogy with the HJB equation in the dynamic programming for finite dimensional problems. On the left hand side we have the discount rate times the unknown v computed at point k.

On the right hand side of (A.4) there is the supremum on c of the scalar product of the right side of the state equation (A.2) with the differential of v computed at point k, $\nabla v(k)$, plus the utility. We can write $\langle k, G\nabla v(k) \rangle$ instead of $\langle Gk, \nabla v(k) \rangle$, because G is self-adjoint: for f and g in $D(G)$, $\langle Gf, g \rangle = \langle f, Gg \rangle$: G.

Next we need to find an explicit solution v to this differential equation. This is part of the results in Appendix B. We also show that this solution is indeed the value function of the problem, and we use it to give an explicit expression of the optimal control of the problem in feedback form.

We devote Remark B.2 to the description of the elements of the model which are necessary to apply our approach as for example whether the geometric peculiarities of the circle play a role. This is particularly important if one wants to try to apply the same method to different situations and models.

[9] This technique has already been successfully applied to the optimal control of another class of infinite-dimensioned dynamic equations, namely delayed differential equations, *e.g.* by Fabbri and Gozzi (2008).

B. Proofs

Proof of Theorem 3.1. We prove the theorem using dynamic programming. The proof is presented in some steps: first, we find an explicit solution to the HJB equation. In the second step, we prove the feasibility of the induced consumption trajectory. Finally in the third step, we prove the optimality of this consumption trajectory showing at the same time that the explicit solution we found is the value function of the system, *i.e.*, $v(k_0) = V(k_0)$ where $V(k_0)$ is defined in (5).

Step 1*: We find an explicit solution of the HJB equation (A.4) on the open set* $\Omega := \left\{k \in L^2(\mathbb{T}) \ : \ \langle k, \mathbb{1}\rangle > 0\right\}$.

We look for a continuous solution v with continuous differential, $v \in C^1(\Omega)$. This implies that v changes continuously when its argument changes continuously w.r.t. the distance in $L^2(\mathbb{T})$ and that its differential $\nabla v\colon \Omega \to L^2(\mathbb{T})$ moves continuously in the $L^2(\mathbb{T})$ distance when its argument changes continuously w.r.t. the $L^2(\mathbb{T})$ distance. But we ask more: we need the term $\langle k, G\nabla v(k)\rangle$ to change continuously when we change continuously k w.r.t. the $L^2(\mathbb{T})$-norm. In general it is not obvious because G is a second derivative. Hence we require $\nabla v(k) \in D(G)$ and $G\nabla v\colon \Omega \to L^2(\mathbb{T})$ to be continuous for all $k \in \Omega$.

Once we have made all these requirements about regularity of the solution, all terms in (A.4) can be computed. We say that v is a solution of (A.4) if it is regular and it solves (A.4) at all points of Ω.

We look for a solution of (A.4) of the following form: $v(k) = \alpha \langle k, \mathbb{1}\rangle^{1-\sigma}$ for some positive real number α, so that $\nabla v(k) = \alpha(1-\sigma)\langle k, \mathbb{1}\rangle^{-\sigma}\mathbb{1}$. Note that $\nabla v(k) \in D(G)$ and $G\nabla v\colon \Omega \to L^2(\mathbb{T})$ is continuous for all $k \in \Omega$. Substituting in (A.4) we obtain:

$$\rho\alpha \langle k, \mathbb{1}\rangle^{1-\sigma} = \alpha(1-\sigma)\langle k, \mathbb{1}\rangle^{-\sigma}\langle k, G\mathbb{1}\rangle + A\alpha(1-\sigma)\langle k, \mathbb{1}\rangle^{-\sigma}\langle k, \mathbb{1}\rangle + \sup_{c\in L^2(\mathbb{T};\mathbb{R}^+)} \left\{-\alpha(1-\sigma)\langle k, \mathbb{1}\rangle^{-\sigma}\langle c, \mathbb{1}\rangle + \langle \mathbb{1}, U(c)\rangle\right\}.$$

Observing that $G\mathbb{1} = 0$ and that the supremum is attained when $c = (\alpha(1-\sigma))^{-1/\sigma}\langle k, \mathbb{1}\rangle\mathbb{1}$, the expression above becomes:

$$\rho\alpha \langle k, \mathbb{1}\rangle^{1-\sigma} = A\alpha(1-\sigma)\langle k, \mathbb{1}\rangle^{1-\sigma} - 2\pi\alpha(1-\sigma)(\alpha(1-\sigma))^{-1/\sigma}\langle k, \mathbb{1}\rangle^{1-\sigma} + 2\pi\frac{\left[(\alpha(1-\sigma))^{-1/\sigma}\langle k, \mathbb{1}\rangle\right]^{1-\sigma}}{1-\sigma}.$$

From it we obtain $\rho = A(1-\sigma) - 2\pi(1-\sigma)(\alpha(1-\sigma))^{-1/\sigma} + 2\pi(\alpha(1-\sigma))^{-1/\sigma}$, so there exists a solution of the requested form when $\alpha = \frac{1}{1-\sigma}\left(\frac{\rho-A(1-\sigma)}{2\pi\sigma}\right)^{-\sigma}$. Before passing to step 2 we make an observation that will be useful later: given an admissible control $c(\cdot)$, the related trajectory $k(\cdot)$ is given by the solution of (A.2). Hence, at every time and at every point θ of the space, $k(\cdot)$ remains below the solution of $\dot{\bar{k}}(t) = G\bar{k}(t) + A\bar{k}(t)$ with $\bar{k}(0) = k_0$. In particular, for all $t \geq 0$, $\langle \bar{k}(t), \mathbb{1}\rangle \geq \langle k(t), \mathbb{1}\rangle$. $\bar{k}(t)$ can be expressed as $\bar{k}(t) = e^{tA}e^{tG}k_0$ so that $\langle \bar{k}(t), \mathbb{1}\rangle = e^{tA}\langle \bar{k}_0, \mathbb{1}\rangle$. We show how to prove this last equality, see the explanations after (B.8) and (B.10). This means that for every choice of $c(\cdot)$ we have

$$\left|e^{-\rho t}v(k(t))\right| = e^{-\rho t}\alpha\langle k(t), \mathbb{1}\rangle^{1-\sigma} \leq e^{-\rho t}\alpha\left\langle \bar{k}(t), \mathbb{1}\right\rangle^{1-\sigma} = e^{-\rho t}e^{tA(1-\sigma)}\left\langle \bar{k}_0, \mathbb{1}\right\rangle^{1-\sigma} \xrightarrow{t\to\infty} 0 \tag{B.1}$$

where we obtain the last limit thanks to hypothesis (6).

Step 2*: We prove that the feedback control provided by the solution is admissible.*

The feedback control provided by the solution is

$$\begin{cases} \phi\colon L^2(\mathbb{T}) \to L^2(\mathbb{T}) \\ \phi(k) := \arg\max_{c\in L^2(\mathbb{T})}\left\{-\alpha(1-\sigma)\langle k, \mathbb{1}\rangle^{-\sigma}\langle c, \mathbb{1}\rangle + \langle \mathbb{1}, U(c)\rangle\right\} = (\alpha(1-\sigma))^{-1/\sigma}\langle k, \mathbb{1}\rangle\mathbb{1} = \eta\langle k, \mathbb{1}\rangle\mathbb{1}, \end{cases} \tag{B.2}$$

where $\eta = \frac{\rho-A(1-\sigma)}{2\pi\sigma}$. The related trajectory is, by definition, the solution of the following integral (mild) equation

$$k(t) = e^{tA}e^{Gt}k_0 - \int_0^t e^{(t-s)A}e^{(t-s)G}\eta\langle k(s), \mathbb{1}\rangle\mathbb{1}\,\mathrm{d}s. \tag{B.3}$$

It is not an explicit form since the unknown k appears in both sides given that the optimal control is specified in feedback form, *i.e.*, as a function of the state k. One can prove that such an equation has a unique solution that we call $k^*(t)$ (see for example Bensoussan *et al.*, 2007, Proposition 3.4, Chapter II-1 page 136). The control we want to prove to be admissible is $c^*(t) := \phi(k^*(t))$, for all $t \geq 0$. Since by hypothesis $k^*(t)(\theta)$ remains positive, then $c^*(t)$ remains positive too and then it is admissible.

Step 3*: We prove that the feedback control is optimal proving at the same time that the solution of the HJB equation we found is indeed the value function.*

$c^*(\cdot)$ is an optimal control if for any other admissible control $\tilde{c}(\cdot)$ we have $J(k_0, c^*(\cdot)) \geq J(k_0, \tilde{c}(\cdot))$. Let us call $\tilde{k}(\cdot)$ the trajectory related to the admissible control $\tilde{c}(\cdot)$ and let us denote by $w(t,k)\colon \mathbb{R} \times L^2(\mathbb{T}) \to \mathbb{R}$ the function $w(t,k) := e^{-\rho t} v(k)$. We have:

$$\begin{aligned} v(k_0) - w(T, \tilde{k}(T)) = w(t, \tilde{k}(0)) - w(T, \tilde{k}(T)) &= -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t} w(t, \tilde{k}(t))\, \mathrm{d}t \\ &= \int_0^T e^{-\rho t} \left[\rho v(\tilde{k}(t)) - \left\langle G\tilde{k}(t) + A\tilde{k}(t) - \tilde{c}(t), \nabla v(\tilde{k}(t))\right\rangle\right] \mathrm{d}t. \end{aligned} \tag{B.4}$$

Using the regularizing properties of the heat semigroup one can prove that $\tilde{k}(t) \in D(G)$, for all $t > 0$. Passing to the limit in (B.4) as $t \to \infty$ and using (B.1) we have

$$v(k_0) = \int_0^{+\infty} e^{-\rho t} \left[\rho v(\tilde{k}(t)) - \left\langle A\tilde{k}(t) - \tilde{c}(t), \nabla v(\tilde{k}(t))\right\rangle - \left\langle \tilde{k}(t), G\nabla v(\tilde{k}(t))\right\rangle\right] \mathrm{d}t, \tag{B.5}$$

$$\begin{aligned} \text{and then } v(k_0) - J(k_0, \tilde{c}(\cdot)) &= \int_0^{+\infty} e^{-\rho t}\Big[\Big(\rho v(\tilde{k}(t)) - \left\langle A\tilde{k}(t), \nabla v(\tilde{k}(t))\right\rangle - \left\langle \tilde{k}(t), G\nabla v(\tilde{k}(t))\right\rangle\Big) \\ &\quad + \Big(\left\langle \tilde{c}(t), \nabla v(\tilde{k}(t))\right\rangle - \langle \mathbb{1}, U(\tilde{c}(t))\rangle\Big)\Big] \mathrm{d}t \\ &= \int_0^{+\infty} e^{-\rho t}\left[\left(\sup_{c \in L^2(\mathbb{T};\mathbb{R}^+)} \left\{-\left\langle c, \nabla v(\tilde{k}(t))\right\rangle + \langle \mathbb{1}, U(c)\rangle\right\}\right) - \left(-\left\langle \tilde{c}(t), \nabla v(\tilde{k}(t))\right\rangle + \langle \mathbb{1}, U(\tilde{c}(t))\rangle\right)\right] \mathrm{d}t \geq 0, \end{aligned} \tag{B.6}$$

where we used that v is a solution of (A.4). (B.6) shows that $v(k_0) - J(k_0, \tilde{c}(\cdot)) \geq 0$. Furthermore, the same expression also implies that $v(k_0) - J(k_0, c^*(\cdot)) = 0$ since $c^*(\cdot)$ is defined using the feedback defined in (B.2). Hence, for all admissible $\tilde{c}$, $v(k_0) - J(k_0, \tilde{c}(\cdot)) \geq 0 = v(k_0) - J(k_0, c^*(\cdot))$ so that $J(k_0, \tilde{c}(\cdot)) \leq J(k_0, c^*(\cdot))$, implying c^*'s optimality. In particular, since $v(k_0) = J(k_0, c^*(\cdot)) = 0$ and c^* is an optimal control, $v(k_0)$ is the value function at k_0. The uniqueness of the optimal control follows from standard convexity considerations. □

Proof of Proposition 3.2. We use that the optimal trajectory solves the mild equation (B.3). Along the optimal trajectories we have

$$k^*(t) = e^{At}e^{Gt}k_0 - \int_0^t e^{(t-s)A}e^{(t-s)G}c^*(s)\,\mathrm{d}s = e^{At}e^{Gt}k_0 - \int_0^t e^{(t-s)A}e^{(t-s)G}\eta\,\langle k^*(s), \mathbb{1}\rangle\,\mathbb{1}\,\mathrm{d}s. \tag{B.7}$$

Both sides are elements of $L^2(\mathbb{T})$ so one can take the scalar product of both with $\mathbb{1} \in L^2(\mathbb{T})$, obtaining

$$K^*(t) = \langle k^*(t), \mathbb{1}\rangle = e^{At}\left\langle k_0, e^{Gt}\mathbb{1}\right\rangle - \int_0^t e^{(t-s)A}\left\langle \eta\,\langle k^*(s), \mathbb{1}\rangle\,\mathbb{1}, e^{(t-s)G}\mathbb{1}\right\rangle \mathrm{d}s, \tag{B.8}$$

where K^* is the aggregate capital along the optimal trajectory. We used two properties. The first is intuitive: the scalar product can "enter inside" the integral because both the scalar product and the integral are linear. The second is that e^{tG} is "self-adjoint': for any $f, g \in L^2(\mathbb{T})$ one has $\langle f, e^{tG}g\rangle = \langle e^{tG}f, g\rangle$. Moreover using that $e^{Gt}\mathbb{1} = \mathbb{1}$ (an explanation of this fact is given in the proof of Theorem 3.3) the expression above becomes:

$$K^*(t) = e^{tA}K(0) - \int_0^t e^{(t-s)A}K^*(s)\left[2\pi\frac{\rho - A(1-\sigma)}{2\pi\sigma}\right] \mathrm{d}s = e^{tA}K(0) - \int_0^t e^{(t-s)A}\frac{\rho - A(1-\sigma)}{\sigma}K^*(s)\,\mathrm{d}s. \tag{B.9}$$

Note that $\langle \mathbb{1}, \mathbb{1} \rangle = \int_0^{2\pi} 1 \, \mathrm{d}\theta = 2\pi$. (B.9) is a a standard one-dimensional ordinary differential equation in $K^*(t)$ in integral form. One can prove by inspection that (B.9) has a unique solution $K^*(t) = K(0)e^{\beta t}$ with $\beta = (A - \rho)/\sigma$. This finishes the proof. $\square$

Proof of Theorem 3.3. We need to write $k_D(t, \theta)$ using Fourier series. This is one of the advantages of studying the problem on $\mathbb{T}$: only functions on $\mathbb{T}$ (or 2π-periodic functions on $\mathbb{R}$) can be written using Fourier series.[10] For $n \in \mathbb{Z}$, we call e_n the function $e_n \colon \mathbb{T} \to \mathbb{R}$, given by

$$e_n(\theta) := \begin{cases} \frac{\cos(n\theta)}{\sqrt{\pi}}, & \text{if } n \geq 1, \\ \frac{\sin(-n\theta)}{\sqrt{\pi}} = \frac{\sin(n\theta)}{\sqrt{\pi}}, & \text{if } n \leq -1, \\ \frac{1}{\sqrt{2\pi}}, & \text{if } n = 0. \end{cases} \tag{B.10}$$

They are the sines and cosines needed to write the Fourier expansion.

For a fixed $n \in \mathbb{Z}$ e_n is very regular and $e_n(0) = e_n(2\pi)$ so they are in the domain of G and $Ge_n = \frac{\mathrm{d}^2 e_n(\theta)}{\mathrm{d}\theta^2} = -n^2 e_n$. Hence, for fixed θ and calling $\phi(t, \theta) := e^{tG} e_n(\theta)$ we have that $\frac{\mathrm{d}\phi(t,\theta)}{\mathrm{d}t} = e^{tG} G e_n(\theta) = -n^2 e^{tG} e_n(\theta) = -n^2 \phi(t, \theta)$ so $\phi(t, \theta) = e^{-n^2 t} \phi(0, \theta)$. That is $e^{tG} e_n(\theta) = e^{-n^2 t} e_n(\theta)$. This implies in particular that $e^{tG} \mathbb{1} = \mathbb{1}$.

The Fourier coefficients of a 2π-periodic function f are given by $\int_0^{2\pi} e_n(s) f(s) \, \mathrm{d}s$ and this is exactly equal to $\langle f, e_n \rangle$. Hence to determinate the Fourier coefficients, we use (B.3) and take the scalar product with e_n for all n:

$$\begin{aligned} \langle k_D(t), e_n \rangle = e^{-\beta t} \langle k(t), e_n \rangle &= e^{-\beta t} \left\langle e^{At} e^{Gt} k_0, e_n \right\rangle - e^{-\beta t} \int_0^t \left\langle e^{(t-s)A} e^{(t-s)G} \eta \langle k(s), \mathbb{1} \rangle \mathbb{1}, e_n \right\rangle \mathrm{d}s \\ &= e^{-\beta t} \left\langle k_0, e^{At} e^{Gt} e_n \right\rangle - e^{-\beta t} \int_0^t \eta \langle k(s), \mathbb{1} \rangle \left\langle \mathbb{1}, e^{(t-s)A} e^{(t-s)G} e_n \right\rangle \mathrm{d}s. \end{aligned} \tag{B.11}$$

For $n \neq 0$, $\langle e_n, \mathbb{1} \rangle = 0$ since it is the integral on $[0, 2\pi]$ of a constant times $\sin(n\theta)$ or $\cos(n\theta)$. Using the last remark and that $e^{At} e^{Gt} e_n = e^{(-n^2 + A)t} e_n$:

$$\langle k_D(t), e_n \rangle = e^{-\beta t} \left\langle k_0, e^{(A - n^2)t} e_n \right\rangle = e^{(A - n^2 - \beta)t} \langle k_0, e_n \rangle, \qquad n \neq 0.$$

If $n = 0$ we have $\langle k_D(t), e_0 \rangle = \frac{K(0)}{\sqrt{2\pi}}$ (it follows immediately from Proposition 3.2).

Functions on $\mathbb{T}$ and real 2π-periodic functions can be written using Fourier series. More precisely, for every function in $k \in L^2(\mathbb{T})$ one has $|k - \sum_{n \in \mathbb{Z}} e_n \langle (k, e_n \rangle|_{L^2(\mathbb{T})} = 0$ (see p.92 in Rudin, 1987). Furthermore, $|k|^2_{L^2(\mathbb{T})} = \sum_{n \in \mathbb{Z}} |\langle (k, e_n \rangle|^2$, so we can express $k_D(t)$ using its Fouries series:

$$k_D(t)(\theta) = \frac{K(0)}{2\pi} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{(A - n^2 - \beta)t} \langle k_0, e_n \rangle e_n(\theta). \tag{B.12}$$

Now consider $\varepsilon \in \left(0, \frac{A(1-\sigma)+\sigma-\rho}{\sigma}\right)$, whose existence is ensured by (13), so that $A - 1 - \beta + \varepsilon < 0$, and observe that, using the the expression of $k_D(t)$ as Fourier series,

$$\begin{aligned} &\sup_{\theta \in \mathbb{T}} \left| k_D(t)(\theta) - \frac{K(0)}{2\pi} \right| = \sup_{\theta \in \mathbb{T}} \left| \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{(A - n^2 - \beta)t} \langle k_0, e_n \rangle e_n(\theta) \right| \leq e^{-\varepsilon t} \left| \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{(A - n^2 - \beta + \varepsilon)t} |\langle k_0, e_n \rangle| \sup_{\theta \in \mathbb{T}} |e_n(\theta)| \right| \\ &= e^{-\varepsilon t} \left| \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{(A - n^2 - \beta + \varepsilon)t} |\langle k_0, e_n \rangle| \right| \leq e^{-\varepsilon t} \left(\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{2(A - n^2 - \beta + \varepsilon)t} \right)^{1/2} \left(\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} |\langle k_0, e_n \rangle|^2 \right)^{1/2} = e^{-\varepsilon t} \left(\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{2(A - n^2 - \beta + \varepsilon)t} \right)^{1/2} |k_0|_{L^2}. \end{aligned} \tag{B.13}$$

[10] If the Hilbert space is different from $L^2(\mathbb{T})$ and one has a optimal feedback rule similar to (B.3), one can use the same technique we use here in some cases. As a result, the optimal solution can be explicitly written as a sum of functions using a Hilbert basis. For example, if the Hilbert space is the space of square integrable functions on the sphere, one can use spherical harmonics.

In the second line of (B.13) we have used a version of the Cauchy-Schwartz inequality that ensures that given two sequences of real numbers $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ one has $\left|\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a_n b_n\right| \leq \left(\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a_n^2\right)^{1/2} \left(\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} b_n^2\right)^{1/2}$. To obtain the equality on the third line, we use that since $k_0 \in L^2(\mathbb{T})$, then $|k_0|^2_{L^2(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |\langle k_0, e_n\rangle|^2$.
If we consider $t \geq 1$ and calling $S := \sum_{n\in\mathbb{Z}} e^{(A-n^2-\beta+\varepsilon)} < \infty$, we can conclude from (B.13) that

$$\sup_{\theta\in\mathbb{T}} \left| k_D(t)(\theta) - \frac{K(0)}{2\pi} \right| \leq e^{-\varepsilon t} \left(S \, |k_0|_{L^2} \right) \xrightarrow{t\to\infty} 0,$$

and this concludes the proof. □

Remark B.1. *(B.12) allows to decompose $k_D(t)$ as a sum of terms with different exponential decays. The first term is $\frac{K(0)}{2\pi}$, which is the limit of k_D for $t \to \infty$. The second is $e^{(A-1-\beta)t} \langle k_0, e_1\rangle e_1(\theta)$, it tends to zero following the exponential factor $e^{(A-1-\beta)t}$. The third term is $e^{(A-2^2-\beta)t} \langle k_0, e_2\rangle e_2(\theta)$, etc.*

Hence we can approximate the behavior of k_D when $t \to \infty$ using a finite sum of these terms. Note indeed that such a behavior can be seen in the two pictures: in Picture 1 the first term of the series is more persistent than the others. In picture 2 $\rho = A(1-\sigma) + \sigma$ and the solution does not converge to a constant. Indeed, $\frac{K(0)}{2\pi}$ is dominated in the limit by $\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{(A-n^2-\beta)t} \langle k_0, e_n\rangle e_n(\theta)$.

(B.12) can also be used for numerical simulation as we do in this paper.

Remark B.2. *Our method to find an explicit solution could work in principle in other geometric contexts but working in non-compact spaces or in spaces with a non-void boundary would complicate the state equation, the HJB equation and the search of an explicit solution of the HJB. The manifold used in this paper, the circle, is* compact *and* without boundary*, and the unique inherent constraint imposed on the state variable is $k(t,0) = k(t,2\pi)$. If we had worked on a segment $[a,b] \subset \mathbb{R}$, we would have to specify at every time the boundary conditions on $k(t,a)$ and $k(t,b)$. If we had worked on the straight line $\mathbb{R}$, we would have to specify at every time some form of limit of $k(t,x)$ for $x \to +\infty$ and $x \to -\infty$. In this case, if we rewrite the PDE as an evolution equation in the Hilbert space, new and typically "unbounded" terms arising from the boundary conditions will show up in the state equation (see e.g. Bensoussan et al., 2007, Section II-3.2 page 212), making much harder to find an explicit solution to the HJB equation. Moreover, some boundary conditions would enter in the definition of the domain $D(G)$ even in the simplest case when we impose a zero boundary condition (see e.g. Tanabe, 1997, Section 5.2 page 180). An explicit solution v satisfies $\nabla v(k) \in D(G)$ for all k in $L^2(\Omega)$, so the more complex the form of the domain, the harder the identification of an explicit solution. In particular, if we had worked on the segment $[a,b] \in \mathbb{R}$ and specified some boundary conditions $k(t,a)$ and $k(t,b)$ different from $k(t,a) = k(t,b)$, the function always equal to* 1 *would have not belonged to the domain of G. As a result, a solution of the form we have found could not work. The same would have hold if we had worked on $\mathbb{R}$.*

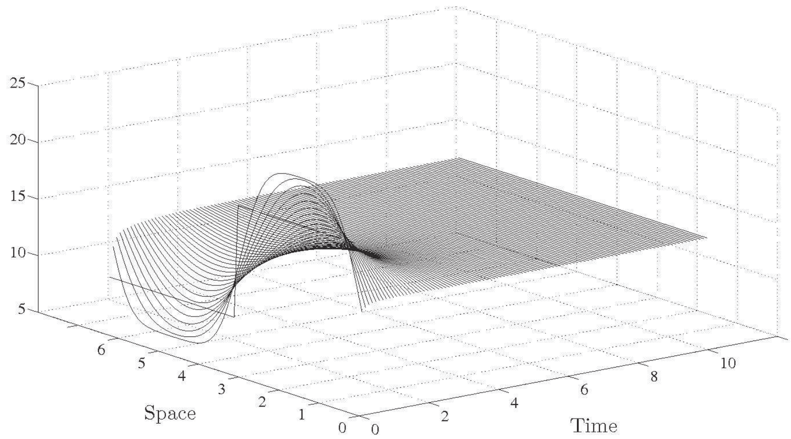

Figure 1: Spatial convergence of $k(t, x)$ when $\rho = 0.07$, $(t, x) \in [0, 8] \times [0, 2\pi]$.

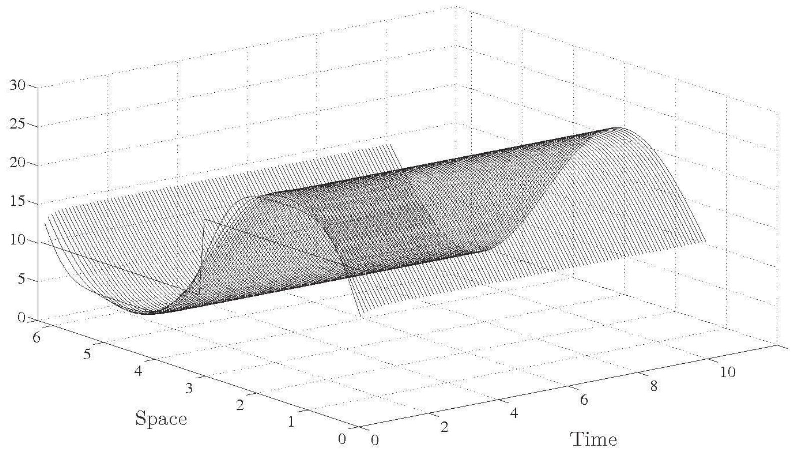

Figure 2: Case $\rho = A(1 - \sigma) + \sigma$, spatial divergence of $k(t, x)$, $(t, x) \in [0, 8] \times [0, 2\pi]$.