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Growth and financial liberalization under capital collateral constraints: The striking case of the stochastic AK model with CARA preferences

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Growth and financial liberalization under capital collateral constraints: The striking case of the stochastic AK model with CARA preferences[*]

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Abstract

We consider a small-open, collateral-constrained AK economy. We show that the combination of CARA preferences and uncertainty on capital inflows in such an economy generates long-term (expected) growth while the deterministic counterpart does not. In this framework, long-term growth is entirely driven by precautionary savings. In particular, we show that the asymptotic growth rate of the expected capital stock is an increasing function of both the risk parameter and the Arrow-Prat absolute risk aversion parameter. The model also predicts that economies that are more financially integrated through international borrowing experience lower consumption growth volatility relative to output growth volatility.

Keywords: Financial liberalization, growth, CARA preferences, collateral constraints, precautionary savings

JEL Classification: F34, F43, O40

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1 Introduction

Admittedly, a strong argument in favor of financial liberalization is international risk-sharing. As outlined by Obstfeld (1994), portfolio diversification thanks to open international financial assets markets pave the way to significant welfare gains through (expected) consumption growth. Obviously, the argument is particularly strong in the absence of imperfections in international financial markets. Recently, Boucekkine et al. (2012), relying on Boucekkine and Pintus (2012), show that financial liberalization is not always welfare-improving when the national economies are subject to capital collateral constraints; for financial openness to be welfare-increasing, the corresponding autarkic growth rates should be large enough, which is somehow consistent with the empirical "threshold" literature (see for example, Kose et al., 2011) according to which financial liberation is beneficial only to the extent that the "fundamentals" of the countries under scrutiny are good enough. Boucekkine et al. (2012) use a deterministic AK model, this note examines a stochastic extension of the Boucekkine et al.'s model.[1] We do not specifically examine Obstfeld's diversification argument but a simpler stochastic extension where uncertainty lies on the magnitude of international financial flows. Would this additional channel alter the main conclusions of Boucekkine et al. (2012)? We more specifically examine the implications for growth.

It is known since Weil (1990) and his concept of certainty equivalent return on saving, that the presence of risk generates conflicting intertemporal substitution and intertemporal income effects, the total outcome depending on risk aversion. An earlier application of this apparatus to stochastic AK models is due to Steger (2005). However, the model considered by Steger is a closed economy and the source of uncertainty is total factor productivity. In this note, we study a small open economy subject to capital collateral constraints, and uncertainty lies on the size of international capital inflows, implying a much trickier stochastic process affecting the economy. As a consequence, the model is much more involved from the analytical point of view.[2] Indeed, in contrast to Steger (2005), using constant relative risk aver-

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[1]Boucekkine et al. (2012) also allow for the absence of commitment, the creditors lending up to some fraction of past values of the collateral. We abstract from this refinement here for sake of simplicity, the stochastic extension is already quite demanding from the analytical point of view.

[2]Another highly interesting AK stochastic growth small open economy model has been developed by Epaulard and Pommeret (2005). This contribution does not account for collateral constraints and is essentially quantitative.

sion (CRRA) utility functions renders the problem intractable. We therefore resort to constant absolute risk aversion (CARA) utility functions, which do produce closed-form solutions. Of course, CARA utility functions have some very well known specific implications due to the fact that the implied risk premium is wealth-independent. This said, the full resolution of the model with CARA utility functions shows truly striking and nontrivial results on the role of uncertainty in the context of our capital collateral constrained economy. First of all, in the deterministic counterpart, that's when considering the Boucekkine et al. model with CARA instead of CRRA preferences, capital grows linearly from $t = 0$, consumption being an affine function of time. In other words, the long-term growth of both capital and consumption is zero while it's strictly positive (under mild conditions) in the CRRA case. This is not surprising at all since CARA utility functions display intertemporal elasticities of substitution strictly decreasing in the level of consumption (while the latter are constant in the CRRA case). Despite constant returns to capital, the incentives to invest drop as consumption rises, which kills exponential growth. Second, when uncertainty on financial inflows is introduced, the asymptotic growth rate of the expected value of capital turns out to be a strictly increasing function of the risk (as captured by a parameter showing up multiplicatively in the non-deterministic part of the law of motion of financial inflows), the same property holding for the expected value of consumption. The larger this risk and the larger the Arrow-Prat absolute risk aversion parameter, the bigger the latter asymptotic growth rates. In other words, **risk-taking is the engine of long-term growth in this model**: the model predicts a clear-cut positive relationship between growth and risk, long-term growth droping to zero when the risk parameter goes to zero. In addition, the model predicts that economies that are more financially liberalized experience lower volatility of consumption growth relative to output growth volatility. Of course, this striking set of results entirely relies on the CARA specification but we believe it examplifies in a nice way the working of risk-induced intertemporal substitution effects (in the absence of wealth effects) under capital constraints, that's the joint role of the latter and precautionary savings.

The paper is organized as follows. Section 2 describes the stochastic model. Section 3 solves the model and provides with the main results. Section 4 concludes.

2 The model

The economy considered is the one described in Boucekkine and Pintus (2012). It's a small open economy endowed with an AK production technology: $Y = A\,K$, $A > 0$. The output good is tradeable, and we assume that the capital input is not (capital immobility). The world interest rate is $r > 0$, and the level of net foreign debt is denoted D. We shall place ourselves in the case of net debtors, so $D \geq 0$. Last but not least, international borrowing is subject to capital collateral constraints in the spirit of Cohen and Sachs (1986). In this note, we only focus on the case with investment commitment. More specifically, we assume that at each time $t \geq 0$ we have

$$D(t) = \lambda K(t) \tag{1}$$

for some $\lambda \in [0,1)$. λ is the credit multiplier, it is a measure of financial markets imperfection: the larger λ, the lower these imperfections. Boucekkine and Pintus (2012) also study the case of no-commitment, that's when borrowing depends on realized investment (typically past investment). As shown in Boucekkine et al. (2012), the mathematical treatment required in the latter case is quite heavy (this amounts to the optimal control of functional differential equations). Since we add stochastics, we prefer to build on the more standard deterministic counterpart, that's the one relying on investment commitment.

We now introduce the stochastic structure. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a real standard Brownian motion $W \colon [0, +\infty) \times \Omega \to \mathbb{R}$. Denote by $\mathcal{F}_t$ the filtration generated by W. The state equations describing the evolution of our economy are given by the following system

$$\begin{cases} \mathrm{d}K(t) = (-\delta K(t) + I(t))\,\mathrm{d}t \\ \mathrm{d}D(t) = (rD(t) - AK(t) + I(t) + C(t))\,\mathrm{d}t - \sqrt{\gamma D(t)}\,\mathrm{d}W(t) \\ K(0) = K_0 > 0, \qquad D(0) = D_0 \geq 0, \end{cases} \tag{2}$$

with $\gamma \geq 0$ and $\delta \geq 0$ is the capital depreciation rate (the sign in front of the term $\sqrt{\gamma D(t)}\,\mathrm{d}W(t)$ is irrelevant, we take the minus sign to have the positive sign in our main state equation given here below). The two equations are standard. Notice that uncertainty only affects the law of motion of foreign debt: in contrast to Steger (2005) for example, uncertainty is not technological (laying on parameter A) but totally related to the working of international financial markers, which expand or shrink randomly. The considered stochastic law of motion considered is therefore more involved than

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Steger's, and it is as important as the CARA preference specification in the genesis of our results. Last but not least, it's important to single out the role of parameter γ, this is the risk parameter outlined in the introduction. This parameter is a quite straigthforward measurement of the magnitude of the random financial inflows: the larger it is, the larger this magnitude is likely to be. Notice that when $\gamma = 0$, the model degenerates into the deterministic counterpart.

Using the collateral constraint specification (1), one can reduce the number of state equation to a single (stochastic) one in the capital stock:

$$\begin{cases} \mathrm{d}K(t) = \left(\dfrac{A - \delta - r\lambda}{1 - \lambda} K(t) - \dfrac{1}{1 - \lambda} C(t) \right) \mathrm{d}t + \dfrac{\sqrt{\gamma\lambda}\sqrt{K(t)}}{1 - \lambda} \mathrm{d}W(t) \\ K(0) = K_0 > 0. \end{cases}$$

$$(3)$$

We now set the preferences of the representative consumers. As argued in the introduction section, we set CARA preferences in order to get closed-form solutions to the associated stochastic optimal growth model. More precisely, given two positive constants θ and η we consider the functional

$$J(C(\cdot)) := \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(-\theta e^{-\eta C(t)} \right) dt \right] \tag{4}$$

to be maximized varying the control $C(\cdot)$ under the state equation (3). One can directly see that the Arrow-Prat absolute risk aversion associated to the considered CARA utility function is given by $\eta > 0$.

Define the set of the admissible controls as follows:

$$\mathcal{U}_{K_0} := \left\{ C(\cdot) \colon [0, +\infty) \times \Omega \to \mathbb{R} \; : \; \begin{array}{c} C(\cdot) \text{ is } \mathcal{F}^t - \text{progressively measurable} \\ \text{and } K(\cdot) \text{ remains positive} \end{array} \right\}.$$

Denote with

$$V(K_0) = \sup_{C(\cdot) \in \mathcal{U}_{K_0}} J(C(\cdot)). \tag{5}$$

the value function of the problem. In the next section, we present the main outcomes of the problem, we focus on the relationship between growth and uncertainty, or in other words, between growth and risk-taking.

3   Main results

The first theorem stated just below characterize the optimal solution to the stochastic optimal control problem by identifying the corresponding value function in closed-form.[3]

Theorem 3.1. *Assume that*

$$A - \delta - r\lambda > 0 \tag{6}$$

and that

$$\frac{1}{\eta(1-\lambda)} - \frac{\rho\left(\frac{1}{2}\frac{\gamma\lambda}{1-\lambda} + \frac{1}{\eta}\right)}{A - \delta - r\lambda} > \frac{\gamma\lambda}{2(1-\lambda)^2} \tag{7}$$

then the value function (5) can be written explicitly and it is given by

$$V(K) = -\beta e^{-\alpha K} \tag{8}$$

where the expression of α is

$$\alpha = \frac{A - \delta - r\lambda}{\frac{1}{2}\frac{\gamma\lambda}{1-\lambda} + \frac{1}{\eta}} > 0 \tag{9}$$

and, denoted with $\mu := -\dfrac{A-\delta-r\lambda}{\eta(1-\lambda)\left(\frac{1}{2}\frac{\gamma\lambda}{1-\lambda}+\frac{1}{\eta}\right)} < 0$, the expression of β is

$$\beta = -\frac{\theta}{\mu}\exp\left(\frac{\rho}{\mu} + 1\right) > 0. \tag{10}$$

As the utility function is CARA, an exponential value function is worth a try. We identify in closed-form such a value function. Condition (6) is needed for the value function to be increasing in the capital stock (basically it states as usually that capital productivity, A, should be larger than the marginal cost, $\delta + r\lambda$). The much trickier condition (7) is needed to ensure the positivity of the corresponding optimal capital trajectory. Notice that the positivity of capital is required for the volatility term $\sqrt{K(t)}$ to make sense. One can check that the value function formula degenerates to the right known formulas in the case of closed and non-stochastic economies (that is when $\gamma = 0 = \lambda$). More importantly, the previous theorem is a necessary step to characterize the optimal trajectories for consumption and capital as follows.

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[3]A technical appendix including the complete proofs of all the results stated is available upon request.

Theorem 3.2. *Under the hypotheses of Theorem 3.1 the optimal consumption can be expressed in feedback form. In other words it can be written is as a function ϕ of the state variable K:*

$$C = \phi(K) := \frac{\alpha}{\eta}K - \frac{1}{\eta} + \frac{\rho(1-\lambda)}{\alpha}. \tag{11}$$

The optimal trajectory is the unique solution of the following SDE

$$\begin{cases} \mathrm{d}K(t) = \left(\frac{1}{2}\frac{\gamma\lambda\alpha}{(1-\lambda)^2}K(t) + \left(\frac{1}{\eta(1-\lambda)} - \frac{\rho}{\alpha} \right) \right) \mathrm{d}t + \frac{\sqrt{\gamma\lambda}}{1-\lambda}\sqrt{K(t)}\,\mathrm{d}W(t) \\ K(0) = K_0 > 0. \end{cases} \tag{12}$$

Remark 3.3. *Observe that in particular Theorem 3.2 states that the control originated in the feedback (11) is admissible, and then in particular along the corresponding trajectory, $K > 0$ a.s. and that (12) has a unique solution.*

The previous theorem is already clear enough to visualize the main implications of the model in terms of expected asymptotic growth rate. Let us focus on equation (12). Start with the deterministic case $\gamma = 0$. In such case, equation (12) degenerates into:

$$\mathrm{d}K(t) = \left(\frac{1}{\eta(1-\lambda)} - \frac{\rho}{\alpha} \right) \mathrm{d}t.$$

Because of the feedback form (11), both capital and consumption are linear in t, yielding zero growth in long-run as announced in the introduction. Once again, this is not surprising since CARA utility functions display intertemporal elasticities of substitution strictly decreasing in the level of consumption. This said, one can easily visualize the additional mechanisms arising from the small open economy and collateral constraint characteristics. Since α (under $\gamma = 0$) is a decreasing function of both r and λ, the time slope of capital, that's $\frac{1}{\eta(1-\lambda)} - \frac{\rho}{\alpha}$, is an unambiguously decreasing function of the world interest rate, r, and is an increasing function of the credit multiplier, λ for admissible parameterizations of the model, that is when the product $r\rho$ is second order. When uncertainty is added, an exponential deterministic term emerges, that is

$$\mathrm{d}K(t) = \frac{1}{2}\frac{\gamma\lambda\alpha}{(1-\lambda)^2}K(t).$$

It's readily shown (using equation (9) which expresses α as a function of γ) that this term is strictly increasing in parameter γ, therefore featuring the announced striking role of risk-taking under risk aversion as an engine of growth in this model. Notice also that this new growth term is an increasing function of the absolute aversion parameter, η. In this AK model, what drives growth is not constant returns to capital but access to a volatile financial market to finance investment. This property is of course due to the CARA preferences which kills the wealth-dependence of the associated risk premium, and gives first-order importance to the risk-induced intertemporal substitution effects, that's to precautionary savings. The latter depresses consumption and stimulates investment, and therefore growth. The fact that in our model growth depends on capital accumulation, which itself depends on collateral-constrained borrowing is another crucial ingredient which pushes the optimal solution towards massive investment (via precautionary savings): growth builds on investment, investment is needed to borrow more, and borrowing more is made possible when the magnitude of the random capital inflows is bigger. The combination of the CARA preferences and the latter characteristics of the model causes precautionary savings to be the exclusive engine of long-term growth, which is indeed a striking outcome. It is possible to derive a more explicit characterization of (expected) long-term growth using the same argument as in Wiersema (2008), Section 5.7 page 112.

Proposition 3.4. *The expected value of $K(t)$ can be computed explicitly and we have*

$$\mathbb{E}[K(t)] = \left[K(0) + \frac{2(1-\lambda)^2}{\gamma\lambda\alpha}\left(\frac{1}{\eta(1-\lambda)} - \frac{\rho}{\alpha}\right)\right] e^{\frac{1}{2}\frac{\gamma\lambda}{(1-\lambda)^2}\alpha t} - \frac{2(1-\lambda)^2}{\gamma\lambda\alpha}\left(\frac{1}{\eta(1-\lambda)} - \frac{\rho}{\alpha}\right)$$

So in particular the asymptotic growth rate of the expected value of the capital is

$$\frac{1}{2}\frac{\gamma\lambda}{(1-\lambda)^2}\alpha.$$

One can recover again the positive relationship between expected long-run growth and the risk parameters γ and η. Straightforwardly using equation (9) as above for disclosing α as a function of γ and η, one can obtain that the asymptotic growth rate of expected capital is increasing in both parameters, which reinforces again the massive role of precautionary savings in this model.

An interesting corollary follows from our previous results, in line with the precautionary savings effect previously underlined. Direct inspection of

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(11) shows that the ratio of consumption's variance over capital's variance depends positively on α/η and it is easy to see that this property about the levels extends to the growth rates of consumption and capital or, for that matter, output. Because α is, under our assumptions, a decreasing function of λ, it follows that economies that are more financially integrated through international borrowing experience lower consumption growth volatility relative to output growth volatility, in line with empirical evidence (e.g. Bekaert et al., 2006).

Corollary 3.5. *Under the assumptions of Theorem 3.2, the ratio of consumption growth volatility to output growth volatility depends negatively on parameters λ, γ, η and r. Therefore, economies that are more financially integrated through international borrowing experience lower volatility of consumption growth relative to output growth volatility.*

4 Conclusion

We have shown how the combination of CARA preferences and uncertainty on capital inflows in a capital collateral constrained economy generates long-term growth while the deterministic counterpart does not. In addition, the model predicts that deeper international financial integration leads to lower consumption volatility relative to output growth volatility. We have outlined the preeminent role of risk-taking and precautionary savings in generating this striking set of results. We do think that the mechanism singled out is useful though its preeminence is certainly specific to the set-up used. Intuition suggests that extending the analysis so as to include international risk-sharing should strengthen our results about the level and volatility of output growth. This is left for future research.

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Appendices

A   Proofs (*Not intended for publication*)

Proof of Theorems 3.1 and 3.2. We want to solve the problem using the dynamic programming. So, as a first step we introduce the Hamilton-Jacobi-Bellman (HJB) equation related to the problem (3)-(4) and we try to solve it.

The HJB of the system is defined as follows

$$\rho v(K) = Dv(K)\frac{A - \delta - r\lambda}{1 - \lambda}K + \frac{1}{2}\frac{\gamma\lambda}{(1 - \lambda)^2}KD^2v(K)$$
$$+ \sup_C \left(-\theta e^{-\eta C} - \frac{1}{1 - \lambda}Dv(K)C\right) \quad (13)$$

that is

$$\rho v(K) = Dv(K)\frac{A - \delta - r\lambda}{1 - \lambda}K + \frac{1}{2}\frac{\gamma\lambda}{(1 - \lambda)^2}KD^2v(K)$$
$$+ \frac{1}{\eta}\frac{1}{1 - \lambda}Dv(K)\left(-1 + \ln\left(\frac{1}{\theta\eta(1 - \lambda)}Dv(K)\right)\right). \quad (14)$$

We look for a solution of the form

$$w(K) = -\beta e^{-\alpha K} \quad (15)$$

in this case one has
$$Dw(K) = \alpha\beta e^{-\alpha K}$$

and
$$D^2w(K) = -\alpha^2\beta e^{-\alpha K}.$$

So $w(K)$ of the described form can be a solution of the (14) if and only if

$$-\rho\beta e^{-\alpha K} = \alpha\beta e^{-\alpha K}\frac{A - \delta - r\lambda}{1 - \lambda}K - \frac{1}{2}\frac{\gamma\lambda}{(1 - \lambda)^2}K\alpha^2\beta e^{-\alpha K}$$
$$+ \frac{1}{\eta}\frac{1}{1 - \lambda}\alpha\beta e^{-\alpha K}\left(-1 + \ln\left(\frac{\alpha\beta}{\theta\eta(1 - \lambda)}\right) - \alpha K\right)$$

i.e.

$$-\frac{\rho}{\alpha} = \frac{A - \delta - r\lambda}{1 - \lambda}K - \frac{1}{2}\frac{\gamma\lambda}{(1 - \lambda)^2}K\alpha$$
$$+ \frac{1}{\eta}\frac{1}{1 - \lambda}\left(-1 + \ln\left(\frac{\alpha\beta}{\theta\eta(1 - \lambda)}\right) - \alpha K\right).$$

In order to satisfy for all $K > 0$ such an equation we need that

$$\begin{cases} -\frac{\rho}{\alpha} = -\frac{1}{\eta}\frac{1}{1-\lambda} + \frac{1}{\eta}\frac{1}{1-\lambda}\ln\left(\frac{\alpha\beta}{\theta\eta(1-\lambda)}\right) \\ 0 = \frac{A-\delta-r\lambda}{1-\lambda} - \frac{1}{2}\frac{\gamma\lambda}{(1-\lambda)^2}\alpha - \frac{1}{\eta}\frac{1}{1-\lambda}\alpha. \end{cases}$$

From the second we have

$$\alpha = \frac{A - \delta - r\lambda}{\frac{1}{2}\frac{\gamma\lambda}{(1-\lambda)} + \frac{1}{\eta}} \quad (16)$$

and from the first

$$\beta = \frac{\theta\eta(1 - \lambda)}{\alpha}e^{\frac{-\rho\eta(1-\lambda)}{\alpha}+1}. \quad (17)$$

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So far we have proved that the function $w(\cdot)$ defined in (15) is in fact a solution of the HJB equation (13). We do not know yet if it is the value function of the optimal control problem (4)-(3).

We cannot directly use standard existence and uniqueness results for the solution of the HJB equation (that ensure that the unique solution of the HJB is in fact the valued function of the problem) as those presented for instance by Yong and Zhou (1999) (see in particular Section 4.3.3, page 182) for two reasons: (i) we have a non-Lipschitz coefficient in the drift part of the SDE, (ii) we have a state constraint since we want to be sure that K remains positive. So we give here an independent and self-contained proof.

We will argue in three steps: Step (I): we will prove that the feedback associated to the solution of the HJB we have (i.e. our candidate-value function (8)) is admissible i.e. the related trajectory of the capital remains positive; Step (II): we will show that the feedback associated to (8) is optimal i.e. the related control is optimal; Step (III): we will show that (8) is in fact the value function of the problem. Steps (I) and (II) will give us the claim of Theorem 3.2 while Step (III) is the content of Theorem 3.1.

Step (I):

The feedback associated to (8) is defined as follows:

$$
\left\{
\begin{array}{l}
\phi \colon \mathbb{R} \to \mathbb{R} \\
\phi(K) := \arg\max_C \left(-\theta e^{-\eta C} - \frac{1}{1-\lambda} Dw(K)C \right) = \frac{\alpha}{\eta} K - \frac{1}{\eta} \ln \left(\frac{\alpha\beta}{\eta\theta(1-\lambda)} \right) \\
\hspace{7cm} = \frac{\alpha}{\eta} K - \frac{1}{\eta} + \frac{\rho(1-\lambda)}{\alpha}
\end{array}
\right.
\tag{18}
$$

(in the last step we used the expression of α given in (16)) that is indeed the feedback function described in (11). The related trajectory is the solution of the following SDE (that is the same of (12)

$$
\left\{
\begin{array}{l}
\mathrm{d}K(t) = \left(\frac{1}{2} \frac{\lambda\gamma\alpha}{(1-\lambda)^2} K(t) + \left(\frac{1}{\eta(1-\lambda)} - \frac{\rho}{\alpha} \right) \right) \mathrm{d}t + \frac{\sqrt{\gamma\lambda}}{(1-\lambda)} \sqrt{K(t)}\, \mathrm{d}W(t) \\
K(0) = K_0 > 0.
\end{array}
\right.
\tag{19}
$$

The equation above has solution only for some choice of the parameters $A, \delta, \gamma, r, \lambda, \rho, \eta$. The problem is understanding in which cases the trajectory $K(\cdot)$ remains a.s. positive so the term $\sqrt{K(t)}$ makes sense. This is the

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reason why we require hypothesis (7). Indeed it ensures that

$$\frac{2\left(\frac{1}{\eta(1-\lambda)} - \frac{\rho}{\alpha}\right)}{\frac{\gamma\lambda}{(1-\lambda)^2}} > 1$$

so (it is the same kind of condition one has e.g. for the Cox-Ingersoll-Ross interest rate model) we can apply for example Theorem 2.2 and Remark 2.2 page 79 by Mishura et al. (2008) and conclude that the trajectories of (19) remains positive with probability 1 and then it has a uinque solution that is a.s. continuous and positive.

This prove that the feedback defined in (18), with α and β specified in (16) and (17), is admissible.

Step (II):

We want to prove now that the feedback defined in (18) is optimal. In other words we denote with $K^*(t)$ the unique solution of (19), with

$$C^*(t) := \phi(K^*(t)) \qquad (20)$$

the control that drives the system along the trajectory $K^*(t)$ and we want to prove that C^* is an optimal control.

Denote with $\omega(t, K)$ the function

$$\omega(t, K) := e^{-\rho t} w(K)$$

where $w(\cdot)$ is defined in (15). Consider an admissible control $\tilde{C}(\cdot)$ at $K(0)$ and the related trajectory $\tilde{K}(\cdot)$. Choose $T > 0$. We have, using Ito formula (see Theorem 3.3 page 149 by Karatzas and Shreve, 1988),

$$\mathbb{E}\left[\int_0^T e^{-\rho t}\left(-\theta e^{-\eta\tilde{C}(t)}\right)\, \mathrm{d}t\right] - w(K(0)) + \mathbb{E}\left[\omega(T, \tilde{K}(T))\right]$$

$$= \mathbb{E}\left[\int_0^T e^{-\rho t}\left(-\theta e^{-\eta\tilde{C}(t)}\right)\, \mathrm{d}t\right] - \mathbb{E}\left[\omega(0, \tilde{K}(0)) - \omega(T, \tilde{K}(T))\right]$$

$$= \int_0^T e^{-\rho t}\left(-\theta e^{-\eta\tilde{C}(t)}\right)\, \mathrm{d}t + \mathbb{E}\left[\int_0^T \frac{\partial\omega}{\partial t}(t, \tilde{K}(t))\right.$$

$$+ D\omega(t, \tilde{K}(t))\left(\frac{A - \delta - r\lambda}{1 - \lambda}\tilde{K}(t) - \frac{1}{1 - \lambda}\tilde{C}(t)\right)$$

$$\left. + \frac{1}{2}D^2\omega(t, \tilde{K}(t))\left(\frac{\gamma\lambda}{(1 - \lambda)^2}\tilde{K}(t)\right)\, \mathrm{d}t\right]. \qquad (21)$$

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Since $w(\cdot)$ is a solution of (13) we have

$$
\begin{aligned}
\frac{\partial \omega}{\partial t}(t, \tilde{K}(t)) &= -\rho e^{-\rho t} w(\tilde{K}(t)) = -e^{-\rho t}\left(\rho w(\tilde{K}(t))\right) \\
&= -e^{-\rho t}\left[Dw(\tilde{K}(t))\frac{A-\delta-r\lambda}{1-\lambda}\tilde{K}(t) + \frac{1}{2}\frac{\gamma\lambda}{(1-\lambda)^2}\tilde{K}(t)D^2 w(\tilde{K}(t))\right. \\
&\qquad\qquad \left. + \sup_C\left(-\theta e^{-\eta C} - \frac{1}{1-\lambda}Dw(\tilde{K}(t))C\right)\right] \\
&= -\left(D\omega(t, \tilde{K}(t))\frac{A-\delta-r\lambda}{1-\lambda}\tilde{K}(t) + \frac{1}{2}\frac{\gamma\lambda}{(1-\lambda)^2}\tilde{K}(t)D^2\omega(t, \tilde{K}(t))\right. \\
&\qquad\qquad \left. + \sup_C\left(e^{-\rho t}\left(-\theta e^{-\eta C}\right) - \frac{1}{1-\lambda}D\omega(t, \tilde{K}(t))C\right)\right). \quad (22)
\end{aligned}
$$

Using last expression in (21) we get

$$
\begin{aligned}
\mathbb{E}&\left[\int_0^T e^{-\rho t}\left(-\theta e^{-\eta\tilde{C}(t)}\right)\,\mathrm{d}t - w(K(0)) + \omega(T, \tilde{K}(T))\right] \\
&= \mathbb{E}\left[\int_0^T e^{-\rho t}\left(\left(-\theta e^{-\eta\tilde{C}(t)}\right) - \frac{1}{1-\lambda}D\omega(t, \tilde{K}(t))\tilde{C}(t)\right)\right. \\
&\qquad\qquad \left. - \sup_C\left(e^{-\rho t}\left(-\theta e^{-\eta C}\right) - \frac{1}{1-\lambda}D\omega(t, \tilde{K}(t))C\right)\right)\,\mathrm{d}t\right] \leq 0 \quad (23)
\end{aligned}
$$

(the last inequality holds because the integrand is always ≤ 0). Observe now that along the admissible trajectories K remains positive then, by (15), $w(\tilde{K}(t)) \in [-\beta, 0]$ and then $\omega(T, \tilde{K}(T)) \xrightarrow{T\to+\infty} 0$. Moreover observe that

$$
\mathbb{E}\left[\int_0^T e^{-\rho t}\left(-\theta e^{-\eta\tilde{C}(t)}\right)\,\mathrm{d}t\right]
$$

is a decreasing function of T (the integrand is always negative) so it admits a limit (possibly equal to $-\infty$) for $T \to +\infty$. Since we are looking for an optimal solution we can restrict our attention to the set of controls $\tilde{C}(\cdot)$ s.t. such a limit is finite (the proof will show that the control induced by the feedback satisfies this condition and then such a set is non-void).

So we can pass to the limit in (23) and we find

$$J(C(\cdot)) - w(K(0)) = \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\left(-\theta e^{-\eta \tilde{C}(t)}\right) \, dt\right] - w(K(0))$$

$$= \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\left(\left(-\theta e^{-\eta \tilde{C}(t)}\right) - \frac{1}{1-\lambda}D\omega(t, \tilde{K}(t))\tilde{C}(t)\right)\right.$$

$$\left. - \sup_C\left(e^{-\rho t}\left(-\theta e^{-\eta C}\right) - \frac{1}{1-\lambda}D\omega(t, \tilde{K}(t))C\right)\right) dt\right] \le 0. \quad (24)$$

So for all admissible controls $\tilde{C}(\cdot)$ one has

$$J(C(\cdot)) \le w(K(0)), \quad (25)$$

Moreover, since $C^*(t)$ satisfied (20) for all $t \ge 0$, then along the trajectories K^* driven by C^* the integrand in the right hand side of (24) is always zero and then $J(C(\cdot)) - w(K(0)) = 0$ i.e.

$$J(C^*(\cdot)) = w(K(0)).$$

This fact together with (25), since $\tilde{C}(\cdot)$ is a generic admissible control, proves that

$$w(K(0)) = J(C^*(\cdot)) = \sup_{\tilde{C}(\cdot) \in \mathcal{U}_{K_0}} J(\tilde{C}(\cdot)) \quad (26)$$

and then (using the second equality of such an expression) the optimality of $C^*(\cdot)$.

Step (III): We prove that w defined in (15), that is the same of V defined in (8) is infact the value function of the problem.

This fact is straightforward since (26) proves infact that

$$w(K_0) = \sup_{\tilde{C}(\cdot) \in \mathcal{U}_{K_0}} J(\tilde{C}(\cdot))$$

and the right hand side is the definition of value function. $\qquad \square$

Bibliography for the appendix

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