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Egalitarianism under population change: age structure does matter*

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Abstract

We study the compatibility of the optimal population size concepts produced by different social welfare functions and egalitarianism meant as "equal consumption for all individuals of all generations". Social welfare functions are parameterized by an altruism parameter generating the Benthamite and Millian criteria as polar cases. The economy considered is in continuous time and is populated by homogeneous cohorts with a given life span. Production functions are linear in labor, (costly) procreation is the unique way to transfer resources forward in time. First, we show that egalitarianism is optimal whatever the degree of altruism in "perpetual youth" model, that's when lifetime span is finite but age structure does not matter: in this case egalitarianism does not discriminate between the social welfare functions considered. Then we show that, when life span is finite but age structure matters, egalitarianism does not arise systematically as an optimal outcome. In particular, in a growing economy, that is when population growth is optimal in the long-run, this egalitarian rule can only hold when the welfare function is Benthamite. When altruism is impure, egalitarianism is impossible in the context of a growing economy. Either in the Benthamite or impure altruism cases, procreation is never optimal for small enough life spans, leading to finite time extinction and maximal consumption for all existing individuals.

Key words: Egalitarianism, age structure, total utilitarianism, impure altruism, endogenous growth

JEL numbers: D63, D64, C61, O40

1 Introduction

Population growth, and notably the consequences of overpopulation on the living standards of present and future generations, are on the top of research agendas of many demographers, philosophers and economists. In particular, the role of population size in the genesis of inequality has become central in the so-called population ethics. Dasgupta (2005) is an excellent survey of research in this area. A considerable part of the related contributions has been devoted to study the extent to which the classical forms of utilitarianism can make the job of ranking populations of different sizes according to the kind of equality meant. Throughout our paper, we study equality in terms of welfare as measured by utility from consumption. This is certainly a benchmark (see the basic model in this area in Dasgupta, 2005) but consumption can be taken, as always, in a very broad sense. A central contribution in the area of population ethics is Parfit (1984). According to Parfit, total utilitarianism (that is the Benthamite social welfare function) may lead to prefer a situation with a very large population size while the standards of living are quite low compared to a situation with a smaller population and better standards of living (as measured by consumption per capita for example). Parfit calls this outcome a *repugnant conclusion*.[1] Actually, Edgeworth (1925) was the first to claim that total utilitarianism leads to a bigger population size and lower standard of living. So this discussion has also always been important in normative economic theory as well. An interesting connected theoretical question is the notion of optimal population size, which is admittedly another old question in economic theory (see for example Dasgupta, 1969). Typically, in all the papers that have been written to study the robustness of Edgeworth claim (see a short survey below), population size is chosen so as to maximize the considered social welfare functions. *In fine*, the key question is: is the optimal population size concept produced by this or that social welfare function compatible with standard and less standard egalitarian principles? This is indeed the question we treat in this paper in a novel framework, which will be described later.

First if all, let us mention that population ethics is currently a very active research area with many open questions and debates. Two literature streams have emerged. One, a sort of natural continuation of the Beckerian endogenous fertility model (see for example, Barro and Becker, 1989), is concerned by the construction of Pareto efficiency principles in overlapping-generations models involving quite naturally external effects within dynasties running from parents to children and vice versa. A subtle representative of this type of literature is Golosov et al. (2007) which presents several efficiency concepts depending on the way unborn are treated.[2] The second stream is much more directly connected to the literature initiated by Parfit (1984). In particular, this stream does not rely on the dynastic model and is not concerned with the externalities inherent to its structure. Representatives of this approach are either axiomatic (Blackorby et al., 2005, or Asheim and Zuber, 2012) or non-axiomatic (Nerlove et al., 1982, or Palivos and Yip, 1993).

In this paper, we also depart from the dynastic approach and take the latter avenue with a special emphasis on populations' age structures. More specifically, we revisit some old population ethics questions within the modern framework of endogenous growth, having in mind that growth, by relaxing resource constraints, might ease avoiding the paradoxical outcomes

[1] Dasgupta (2005) discusses to which extent the term "repugnant" is appropriate.
[2] Another excellent reference is Conde Ruiz et al. (2010).

outlined by Parfit, and even might pave the way to reach more egalitarian allocations across individuals facing different time horizons at given date. Actually, the robustness of Edgeworth's claim when societies experience long periods (say infinite time periods) of economic growth has been already discussed in two previous papers, namely Razin and Yuen (1995) and Palivos and Yip (1993). We shall rely on the same class of parameterized social welfare functions used by these authors. The parametrization consists in weighting the utility of individuals at any given date t by the term $N^{\gamma}(t)$ where $N(t)$ is the size of the population at t and $0 \leq \gamma \leq 1$. When γ increases, the time discount rate goes down, inducing a larger weight for individuals of future generations in the social welfare functions. In this precise sense, γ measures a kind of degree of altruism towards individuals to be born in future dates as outlined by Palivos and Yip for example. To fix the terminology, we shall refer to γ as the degree of altruism. This terminology is chosen for convenience.[3] When $\gamma = 1$ (Resp. $\gamma = 0$) one gets the standard Benthamite (Resp. Millian) social welfare function. We may treat γ as a continuous parameter and interpret the cases where $0 < \gamma < 1$ as cases where altruism is impure or imperfect.

Using the same class of social welfare functions, Palivos and Yip (1993) showed that Edgeworth's claim cannot hold for the realistic parameterizations of their model. Precisely, they established their results in the framework of endogenous growth driven by an AK production function. The determination of the optimum relies on the following trade-off: on one hand, the utility function depends explicitly on population growth rate; on the other, population growth has the standard linear dilution effect on physical capital accumulation. Palivos and Yip proved that in such a framework the Benthamite criterion leads to a smaller population size and a higher growth rate of the economy provided the intertemporal elasticity of substitution is lower than one (consistently with empirical evidence), that is when the utility function is negative. Indeed, a similar result could be generated in the setting of Razin and Yuen (1995) when allowing for negative utility functions.[4] It goes without saying that the value of not-living is essential in the outcomes:[5] in the class of models surveyed just above, this value is typically zero, so that negative utility functions imply that living gives inferior value than not living.

Our paper goes much beyond the technical point mentioned just above. Essentially it aims at investigating the compatibility between utilitarianism and egalitarianism in an economy where human resources, and therefore population size, is the engine of growth. Specifically, our set-up has the following three distinctive features:

1. First, we shall consider a minimal model in the sense that we do not consider neither capital accumulation (as in Palivos and Yip, 1993) nor natural resources (as in Makdissi, 2001): we consider one productive input, population (that's labor), and the production function is AN with N the population size. By taking this avenue, population growth and economic growth will coincide in contrast to the previous related AK literature (and in particular to Razin and Yuen, 1995). More importantly our model is clearly at odds with the typical *genesis problem* as presented by Dasgupta (2005) in his survey:

[3] We could have fixed the terminology referring more to the role of γ in intertemporal discounting to show better the distance with respect to dynastic models.

[4] See also Boucekkine and Fabbri (2013).

[5] Dasgupta (2005) has already underlined the crucial nature of this point.

not only we have constant returns to scale (*vs.* decreasing returns in Dasgupta), but apparently we don't have any type of investment to transfer resources to the future. As one will see, our model does actually entail a form of forward resource transfer simply through having children: having children is costly (investment) but they are the workers of tomorrow, and therefore they are the exclusive wealth producers in the future (forward income transfer). Because birth costs are supposed linear in our AN model, one would expect to have the same outcomes as in a standard AK model. In particular, detrended consumption would be constant. Since demographic and economic growth coincide in our model, one would infer that constant per capita consumption is a possible outcome. Indeed it is the latter important observation that led us to select this minimal model for the study of the compatibility between total utilitarianism and egalitarianism. Accordingly, one can choose "equal consumption per capital for all individuals and all generations" as the **natural egalitarian principle** in our framework.

2. Second, we bring into the analysis human life span and age structures of populations. Concerning life spans, we shall assume that all individuals of all cohorts live a fixed amount of time, say T. The value of T will be shown to be crucial for the outcomes of the analysis. As outlined above, procreating is the unique way to transfer resources forward in time. Durability of these resources, captured by the life span T, is therefore likely to be key for the design of the optimal procreation plan. We shall assume that life span is exogenous in our model. Admittedly, a large part of the life spans of all species is the result of a complex evolutionary process (see the provocative paper of Galor and Moav, 2007). Also it has been clearly established that for many species life span correlates with mass, genome size, and growth rate, and that these correlations occur at differing taxonomic levels (see for example Goldwasser, 2001). Of course, part of the contemporaneous increase of humans' life span is, in contrast, driven by health spending and medical progress. We shall abstract from the latter aspect.

3. Third, we shall prove that above all consideration, the age structure is key in the main outcome of the paper, that is on the set of social welfare functions supporting egalitarianism as an optimal outcome. In comparison with "perpetual youth" AK models with finite lifetime but irrelevant age structure (detailed later), our AN model does display transitional dynamics because of the finite lifetime assumption comes with an explicit age structure (just like in the AK vintage capital model studied in Boucekkine et al., 2005, and Fabbri and Gozzi, 2008). The deep reason of these different behaviors is that as the finite life span setting we use allows to take into account the whole age-distribution structure of the population, the evolution of the system is much more complex: indeed the engine of the transitional dynamics of detrended variables is the rearrangement of the shares among the cohorts. This clearly distinguishes the approach we use from the models with "radioactive" decay of the population (and in particular from the zero-decay case, corresponding to $T = +\infty$) where all the individual are identical. As a consequence, the property that at the optimum one gets "equal consumption per capita for all individuals and all generations" is quite challenging. This makes our problem either technical and theoretically fundamentally nontrivial.

Resorting to AN production functions has also the invaluable advantage to allow for (nontrivial) analytical solutions to the optimal dynamics in certain parametric conditions. In particular, we shall provide optimal dynamics in closed-form in the Benthamite case and for a class of intermediate parameterizations of the social welfare function (impure altruism), the Millian case being trivial (and degenerate in a sense to be given later). It is important to notice here that considering finite lifetimes changes substantially the mathematical nature of the optimal control problem under study. Because the induced state equations are no longer ordinary differential equations but delay differential equations, the problem is infinitely dimensioned. Problems with these characteristics are tackled in Boucekkine et al. (2005), Fabbri and Gozzi (2008) and recently by Boucekkine, Fabbri and Gozzi (2010). We shall follow the dynamic programming approach used in the two latter papers since, in this case, differently from the maximum principle approach, it allows to find explicitly the value function and the optimal policy function. It is important to observe that the optimal control problem studied in this paper is structurally different (especially due to the presence of the additional multiplicative term $N(t)^{\gamma}$ in the objective function) from the ones treated in the three quoted papers so, to be solved, it needs a nontrivial methodological progress. See Remark A.1.

For the reader convenience, the main technical details are reported in the appendix.

Main findings

Several findings will be highlighted along the way. At the minute, we enhance two of them.

1. A major contribution of the paper is the striking implication of age structures induced by finite life spans for the optimal consumption pattern across cohorts. Indeed, we study under which conditions the successive cohorts will be given the same consumption per capita. We show that our egalitarian rule "equal consumption per capital for all individuals and all generations" is optimal whatever the degree of altruism in "perpetual youth" model, that's when lifetime span is finite but the age structure does not matter and, as a particular case, in the zero decay-infinite life spans case, so it does not discriminate between the social welfare functions considered. However, when finite life spans are combined with active age structures, egalitarianism does not arise systematically as an optimal outcome and it depends on the degree of altruism (see below). In particular, in the finite life span case, when altruism is impure, egalitarianism is impossible in the context of a growing economy.

2. Second, the analysis illustrates the crucial role of the degree of altruism in the shape of the optimal allocation rules for given finite life span, and the framework allows for striking clear-cut analytical results for optimal dynamics while the vast majority of the papers in the topic only are working on balanced growth paths. In particular, we deliver explicit optimal extinction results depending on the level of altruism, the life span and the technological parameters of the model.[6] For example in the Benthamite case, we identify two threshold values for individuals' lifetime, say $T_0 < T_1$: below T_0, finite time extinction is optimal; above T_1, balanced growth paths (at positive rates) are optimal. In between, asymptotic extinction is optimal. That's to say, if life span is large enough, Parfit's *repugnant conclusion* for total utilitarianism does not hold: even

[6]For a more positive theory of extinction, see de la Croix and Dottori (2008).

more, all individuals of all generations will receive the same consumption, and therefore will enjoy the same welfare. In a growing economy, when economic growth depends on human resources (which is a reasonable assumption), total utilitarianism need not be *repugnant*. On the other hand, our analysis implies that the Benthamite criterion is not necessarily pro-natalist: in particular, if life spans are small enough, this criterion would legitimate finite time extinction.

The paper is organized as follows. Section 2 describes the optimal population model, gives some technical details on the maximal admissible growth and the boundedness of the associated value function, and displays some preliminary results on extinction. Section 3 studies a benchmark AK model without active age structure, based on the popular "perpetual youth" assumption. Section 4 derives the optimal dynamics corresponding to the Benthamite and impure (or imperfect) altruism cases. Section 5 concludes. The Appendices A and B are devoted to collect most of the proofs.

2 The problem

2.1 The model

Let us consider a population in which every cohort has a fixed finite life span equal to T. Assume for simplicity that all the individuals remain perfectly active (i.e. they have the same productivity and the same procreation ability) along their life time. At every moment t we denote by $N(t)$ the size of population at t and by $n(t)$ the size of the cohort born at time t.

The dynamics of the population size $N(t)$ is then driven by the following delay differential equation (in integral form):

$$N(t) = \int_{t-T}^{t} n(s)\,\mathrm{d}s, \qquad t \geq 0, \tag{1}$$

and

$$n(t) \geq 0, \qquad t \geq 0. \tag{2}$$

The past history of $n(r) = n_0(r) \geq 0$ for $r \in [-T, 0)$ is known at time 0: it is in fact the initial datum of the problem[7]. This features the main mathematical implication of assuming finite lives. Pointwise initial conditions, say $n(0)$ or $N(0)$, are no longer sufficient to determine a path for the state variable, $N(t)$. Instead, an initial function is needed. The problem becomes infinitely dimensioned, and the standard techniques do not immediately apply. Summarizing, (1) becomes:

$$N(t) = \int_{t-T}^{t} n(s)\,\mathrm{d}s,\ n(r) = n_0(r) \geq 0 \ \text{ for } r \in [-T, 0),\ N(0) = \int_{-T}^{0} n_0(r)\,\mathrm{d}r. \tag{3}$$

Note that the constraint (2) together with the positivity of n_0 ensure the positivity of $N(t)$ for all $t \geq 0$. Note also that, if $N(\bar{t}) = 0$ for a certain $\bar{t} \geq 0$ then we must have $N(t) = 0$ for every $t \geq \bar{t}$, as we expect.

[7]To ensure treatability of the problem we also assume that $n_0(\cdot) \in L^2(-T, 0)$, for a definition of such a functional space see the initial lines of Appendix A.

We consider a closed economy, with a unique consumption good, characterized by a labor-intensive aggregate production function exhibiting constant returns to scale, that is

$$Y(t) = aN(t). \tag{4}$$

Note that by equation (1) we are assuming that individuals born at any date t start working immediately after birth. Delaying participation into the labor market is not an issue but adding another time delay into the model will complicate unnecessarily the (already extremely tricky) computations. Note also that there is no capital accumulation in our model. The linearity of the production technology is necessary to generate long-term growth, it is also adopted in the related bulk of papers surveyed in the introduction. If decreasing returns were introduced, that is $Y(t) = aN^{\alpha}$ with $\alpha < 1$, growth will vanish, and we cannot in such a case connect life span with economic and demographic growth.

Output is partly consumed, and partly devoted to raising the newly born cohort, say rearing costs. In this benchmark we assume that the latter costs are linear in the size of the cohort, which leads to the following resource constraint:

$$Y(t) = N(t)c(t) + bn(t) \tag{5}$$

where $b > 0$. Again we could have assumed that rearing costs are distributed over time but consistently with our assumption of immediate participation in the labor market, we assume that these costs are paid once for all at time of birth. On the other hand, the linearity of the costs is needed for the optimal control problem considered above to admit closed-form solutions. As it will be clear along the way, this assumption is much more innocuous than the AN production function considered.

Let us describe now accurately the optimal control problem handled. The controls of the problem are $n(\cdot)$ and $c(\cdot)$ but, using (4) and (5), one obtains

$$aN(t) = c(t)N(t) + bn(t). \tag{6}$$

so we have only to choose $n(t)$ (or, equivalently, $c(t)$) for all $t \geq 0$ and the other will be given by (6). We choose to work with the control $n(\cdot)$ to ease proving and writing down the results. Then $c(\cdot)$ will be given by

$$c(t) = \frac{aN(t) - bn(t)}{N(t)}. \tag{7}$$

From such equation it is clear that $c(t)$ is well defined only when $N(t) > 0$, it does not make sense after extinction arise. We come back to this issue later.

Concerning the constraints, since we want both per-capita consumption and the size of new cohorts to remain positive, using (6) we require, in term of $n(\cdot)$:

$$0 \leq n(t) \leq \frac{a}{b}N(t), \qquad \forall t \geq 0 \tag{8}$$

or, in terms of $c(\cdot)$,

$$0 \leq c(t) \leq a, \; N(t) \geq 0, \qquad \forall t \geq 0. \tag{9}$$

So we consider the controls $n(\cdot)$ in the set[8]

$$\mathcal{U}_{n_0} := \left\{ n(\cdot) \in L^1_{\mathrm{loc}}(0,+\infty;\mathbb{R}^+) \; : \; \text{eq. (8) holds for all } t \geq 0 \right\}. \tag{10}$$

We shall consider the following social welfare functional to be maximized within the latter set of controls:

$$J(n_0(\cdot); n(\cdot)) := \int_0^{+\infty} e^{-\rho t} u(c(t)) N^{\gamma}(t)\, \mathrm{d}t,$$

or equivalently

$$J(n_0(\cdot); n(\cdot)) := \int_0^{+\infty} e^{-\rho t} u\left(\frac{aN(t) - bn(t)}{N(t)}\right) N^{\gamma}(t)\, \mathrm{d}t, \tag{11}$$

where $\rho > 0$ is the time discount factor, $u\colon (0,+\infty) \to (0,+\infty)$ is a continuous, strictly increasing and concave function, and $\gamma \in [0,1]$. We denote the social welfare functional by $J(n_0(\cdot); n(\cdot))$ to underline its dependency, beyond the control strategy $n(\cdot)$, also on the initial datum $n_0(\cdot)$. γ is interpreted as the degree of altruism of the social planner towards individuals of future generations. As explained in the introduction, this interpretation is consistent with the fact that the term $N^{\gamma}(t)$ is a determinant of the discount rate at which the welfare of future generations is discounted. At this early stage, it is important to observe that the social welfare function considered does not account for the age-structure of population (in contrast to the state equation): we implicitly assume that the benevolent planner gives the same consumption $c(t)$ for all individuals living at date t whatever their age. So a kind of instantaneous egalitarianism is already included in the specification of the problem. We will see that it does not guarantee "equal consumption per capita for all individuals and all generations".[9] Last but not least, notice also that we only consider positive utility functions. Indeed, our modeling implicitly implies that the value of not living is zero. As explained in the introduction, a (strictly) negative utility function therefore implies that not living is superior to living, implying that the the optimal cohort is zero. As a result, for any initial conditions and any lifetime, T, the planner would choose extinction at finite time. This argument is formalized in the discussion paper version of the paper.[10]

Once the social welfare functional is defined we can go ahead defining the value function of our problem as

$$V(n_0(\cdot)) := \sup_{n(\cdot) \in \mathcal{U}_{n_0}} J(n_0(\cdot); n(\cdot))$$

We now give the definition of optimal control strategy adapted to our case.

Definition 2.1 *An admissible control strategy* $\bar{n}(\cdot) \in \mathcal{U}_{n_0}$ *is optimal for the initial datum* $n_0(\cdot)$ *if* $J(n_0(\cdot); \bar{n}(\cdot))$ *is finite and* $J(n_0(\cdot); \bar{n}(\cdot)) = V(n_0(\cdot))$.

[8] The space $L^1_{\mathrm{loc}}(0,+\infty;\mathbb{R}^+)$ in the definition of $\mathcal{U}_{n_0}$ is defined as $L^1_{\mathrm{loc}}(0,+\infty;\mathbb{R}^+) := \left\{ f\colon [0,+\infty) \to \mathbb{R}^+ \; : \; f \text{ measurable and } \int_0^S |f(x)|\,\mathrm{d}x < +\infty,\ \forall S > 0 \right\}$.

[9] A more general formulation would incorporate the age structure of the population not only in the social welfare function but also in the production function through a given age profile of productivity for example. We abstract away from this potential extension here.

[10] see Proposition 2.3 of an earlier version at: `http://halshs.archives-ouvertes.fr/docs/00/53/60/73/PDF/DTGREQAM2010_40.pdf`. Also, utility functions with no sign restriction are explicitly handled in this paper, see Section 3.2.2.

Note that, since the utility is positive, the value function is positive, too. Moreover, to avoid unnecessary technical complications we restrict ourselves to study cases where the value function is finite.

Maximization of the social welfare function specified above in the control set given by (10) is not only mathematically nontrivial given the infinite dimension of the problem and its possible non-convexity. More importantly, it is economically nontrivial at least when $\gamma > 0$, that's when the social welfare function is not Millian. The involved trade-off is rather simple. Procreation is costly but it is beneficial for society for two reasons in our setting: it allows to secure more production now and in the future, and the social planner is intertemporally altruistic in the sense given in the introduction and below the definition of the social welfare in this section. The second effect vanishes if $\gamma = 0$. The trade-off is clearly nontrivial when $\gamma > 0$. It is even less trivial when we add the finite life span characteristic. And it's definitely tricky if one has to guess whether the optimal solution of these trade-offs leads to egalitarian allocations over time. Only in very few cases an analogy can be established between our framework and the standard AK models.[11]

Let's end the presentation of our age-structured optimal population problem with a remark on the Millian case. As outlined just above, this case looks already trivial. It's actually a degenerate case from the point of view of optimal control, and we will disregard it from now on. Indeed, when $\gamma = 0$, the objective function only depends on consumption per capita, leading to choose maximal consumption, $c^\star(t) = a$, which implies zero optimal procreation given the budget constraint. This in turn implies extinction at finite time $\bar{t}$ ($N(\bar{t}) = 0$). This may seem the natural optimal control solution. Unfortunately, without further constraint (for example on the shape of the social welfare function), this is not true. Indeed, let us write the objective function under finite time extinction explicitly as

$$J(n_0(\cdot); n(\cdot)) = \int_0^{\bar{t}} e^{-\rho t} u\left(c(t)\right) N^\gamma(t)\,\mathrm{d}t + \int_{\bar{t}}^{+\infty} e^{-\rho t} u\left(c(t)\right) N^\gamma(t)\,\mathrm{d}t.$$

If $\gamma > 0$, then finite time extinction (coupled with bounded control, $c(t)$) implies that the second term of the decomposition just above is zero. When $\gamma = 0$, this is not the case, which causes the optimal control problem to not admit a solution strictly speaking since it's possible to manipulate the value of the objective function after extinction (without violating the budget constraint since $n(t) = N(t) = 0$ after finite time extinction). One easy way to force the natural solution to the Millian case presented above to be an optimal control is to require the social welfare function to be continuous: with this additional restriction, $c^\star(t) = a$ and $n^\star(t) = 0$, $\forall t \geq 0$, will be the optimal control. Instead of adding these technical restrictions to force a well-defined solution to an otherwise economically trivial case, we prefer to omit the latter.

2.2 Maximal admissible growth

We begin our analysis by giving a sufficient condition ensuring the boundedness of the value function of the problem. The arguments used are quite intuitive so we mostly sketch the proofs.

[11] If we interpret n as investment and N as the capital stock, then one can find analogies with standard AK models only under some very special parameterizations, as we will explain along the way.

Consider the state equation (3) with the constraint (9). Given an initial datum $n_0(\cdot) \geq 0$ (and then $N(0) = \int_{-T}^{0} n_0(r)\, \mathrm{d}r$), we consider the admissible consumption path defined as $c_{MAX} \equiv 0$. This gives the maximal population size allowed, associated with the control $n_{MAX}(t) = \frac{a}{b} N(t)$ by equation (6): it is the control/trajectory in which all the resources are assigned to raising the newly born cohorts with nothing left to consumption. Call the trajectory related to such a control $N_{MAX}(\cdot)$. By definition $N_{MAX}(\cdot)$ is a solution to the following delay differential equation (written in integral form):

$$N_{MAX}(t) = \int_{(t-T)\wedge 0}^{0} n_0(s)\, \mathrm{d}s + \frac{a}{b} \int_{(t-T)\vee 0}^{t} N_{MAX}(s)\, \mathrm{d}s. \tag{12}$$

The characteristic equation of such a delay differential equation is[12]

$$z = \frac{a}{b} \left(1 - e^{-zT}\right). \tag{13}$$

It can be readily shown[13] that, if $\frac{a}{b}T > 1$, the characteristic equation has a unique strictly positive root ξ. This root belongs to $(0, \frac{a}{b})$ and it is also the root with maximal real part. If $\frac{a}{b}T \leq 1$, then all the roots of the characteristic equation have non-positive real part and the root with maximal real part is 0. In that case, we define $\xi = 0$. We have that (see for example Diekmann et al., 1995, page 34), for all $\epsilon > 0$,

$$\lim_{t\to\infty} \frac{N_{MAX}(t)}{e^{(\xi+\epsilon)t}} = 0, \tag{14}$$

and that the dynamics of $N_{MAX}(t)$ is asymptotically driven by the exponential term corresponding to the root of the characteristic equation with the largest real part. As it will be shown later, this result drives the optimal economy to extinction when individuals' lifetime is low enough. At the minute, notice that since $N_{MAX}(\cdot)$ is the trajectory obtained when all the resources are diverted from consumption, it is the trajectory with the largest population size. More formally, one can write:

Lemma 2.1 *Consider a control $\hat{n}(\cdot) \in \mathcal{U}_{n_0}$ and the related trajectory $\hat{N}(\cdot)$ given by (1). We have that*

$$\hat{N}(t) \leq N_{MAX}(t), \qquad \textit{for all } t \geq 0.$$

The previous lemma, coupled with property (14), implies the following sufficient condition for the value function of the problem to be bounded:

Proposition 2.1 *The following hypothesis*

$$\rho > \gamma\xi \tag{15}$$

is sufficient to ensure that the value function

$$V(n_0(\cdot)) := \sup_{\hat{n}(\cdot)\in\mathcal{U}_{n_0}} \int_0^{+\infty} e^{-\rho t} u(\hat{c}(t)) \hat{N}^{\gamma}(t)\, \mathrm{d}t$$

[12] As for any linear dynamic equation (in integral or differential form), the characteristic equation is obtained by looking at exponential solutions, say e^{zt}, of the equation.

[13] All these statements about the roots of (13) follow as particular cases of Theorem 3.2 p. 312 and Theorem 3.12 p. 315 of Diekmann et al. (1995) once we observe that z is a root of (13) if and only if $w = zT$ is a root of $w = aT - aTe^{-w}$.

is finite (again we denoted with $\hat{N}(\cdot)$ the trajectory related to the control $\hat{n}(\cdot)$).

The proofs of the two results above are given in Appendix B.

Once we know that the value function is finite we can prove another crucial property of it: the γ-homogeneity.

Proposition 2.2 *Assume that (15) is satisfied. Then, for every $\gamma \in [0, 1]$ the value function is positively homogeneous of degree γ i.e., for every positive $n_0(\cdot) \in L^2(-T, 0)$ and $\lambda_0 > 0$ we have*

$$V(\lambda_0 n_0(\cdot)) := \lambda_0^\gamma V(n_0(\cdot)).$$

The proof is in Appendix B. This property helps understanding some of the nontrivial results obtained along the paper.

We are now ready to provide the first important result of the paper highlighting the case of asymptotic extinction.

2.3 A preliminary extinction result

We provide now a general extinction property inherent to our model. Recall that when $\frac{a}{b}T \leq 1$, all the roots of the characteristic equation of the dynamic equation describing maximal population, that is equation (12), have non-positive real part, which may imply that maximal population goes to zero asymptotically (asymptotic extinction). The next proposition shows that this is actually the case for any admissible control in the case where $\frac{a}{b}T < 1$.

Proposition 2.3 *If $\frac{a}{b}T < 1$ then for every admissible control $n(\cdot)$ the associated state trajectory $N(\cdot)$ satisfies*

$$\lim_{t \to +\infty} N(t) = 0$$

i.e. it "drives the system to extinction".

The proof is in the Appendix B. The value of individuals' lifetime is therefore crucial for the optimal (and non-optimal) population dynamics. This is not really surprising: if people do not live long enough to bring in more resources than it costs to raise them, then one might think that eventually the population falls to zero. The proposition identifies indeed a threshold value independent of the welfare function (and so independent in particular of the strength of intertemporal altruism given by the parameter γ) such that, if individuals' lifetime is below this threshold, the population will vanish asymptotically. While partly mechanical, the result has some interesting and nontrivial aspects. First of all, one would claim that in a situation where an individual costs more than what she brings to the economy, the optimal population size could well be zero at finite time. Our result is only about asymptotic extinction. As we shall show later, whether finite time extinction could be optimal, that's ethically legitimate, requires additional conditions, notably on the degree of altruism. Even under $\frac{a}{b}T > 1$, finite time or asymptotic extinction could be optimal depending on other parameters of the model.

Second, the result is interesting in that it identifies an explicit and interpretable threshold value, equal to $\frac{b}{a}$, for individuals' lifetime: the larger the productivity of these individuals, the lower this threshold is, and the larger the rearing costs, the larger the threshold is.[14]

[14]If $T = \frac{b}{a}$, not all the admissible trajectories drive the system to extinction: indeed if we have for example

An originally non-sustainable economy can be made sustainable by two types of exogenous impulses: technological shocks (via a or b) or demographic shocks (via T).[15]

3 Benchmark

In order to disentangle accurately the implications of age structures, we develop now the properties of a benchmark model without **active** age structures. Let's be precise on what we mean by active age structures. In our model, one could write the law of motion of population size as:

$$\dot{N}(t) = n(t) - \mu(t)\, N(t),$$

where $\mu(t) = \frac{n(t-T)}{N(t)}$ is the endogenous population "destruction" or "replacement" rate implied by our model. The endogeneity of the latter rate features the demographic replacement dynamics, which do depend on the initial age structure of the economy. In this sense, the age structure is active in our model, provided T is finite. If T is infinite, the destruction rate is nil ($\mu(t) = 0$), and the age structure is inactive. This could be a benchmark case. A definitely much more interesting benchmark is the case where $\mu(t)$ is exogenous (without necessary being equal to zero). In such a case, the initial age structure becomes also irrelevant. Such a benchmark can be interpreted as a Blanchard-Yaari-like perpetual youth model. Instead of considering that people have necessarily an infinite lifetime as in the previous case ($\mu = 0$), assume that people die at a constant flow probability of $\mu > 0$, whatever their age, therefore making the latter irrelevant. In this "perpetual youth" setting, life expectancy is typically measured by the inverse of the mortality rate, so our lifetime measure T is captured by $\frac{1}{\mu}$. The infinite lifetime case can be recovered as a limit case (when μ tends to zero). We shall therefore analyze hereafter this definitely much more interesting benchmark.

One can reformulate this benchmark model in a more straightforward way. Denote by $m(t)N(t) = n(t)$ the flow of newborn where $m(t)$ is the fertility rate, and by $c(t) = a - bm(t)$ the per-capita consumption at time t. Also, for sake of closed-form solutions, choose the isoelastic function, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, with $\sigma \in (0, 1)$.[16] Then the problem becomes the maximization of the functional

$$\int_0^{+\infty} e^{-\rho t} \frac{(a - bm(t))^{1-\sigma}}{1-\sigma} N^{\gamma}(t)\, \mathrm{d}t \tag{16}$$

subject to the state equation:

$$\dot{N}(t) = (m(t) - \mu)\, N(t), \qquad N(0) = N_0 \tag{17}$$

the constant initial datum $N(t) = 1$ for all $t < 0$ or $n(t) = a/b$ for all $t < 0$, the (admissible) maximal control $N_{MAX}(t)$ allows to maintain the population constant equal to 1 for every t.

[15]This is largely consistent with unified growth (positive) theory – see Galor and Weil (1999), Galor and Moav (2002), and Boucekkine, de la Croix and Licandro (2002).

[16]The values considered for σ guarantee the positivity of the utility function. It might be argued following Palivos and Yip (1993) that such values imply unrealistic figures for the intertemporal elasticity of substitution, which require $\sigma > 1$. We show in the discussion paper version of the paper that our main results still hold qualitatively on the utility function: $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, whose positivity is compatible, under certain scale conditions, with the more realistic $\sigma > 1$. See Section 3.2.2 at http://halshs.archives-ouvertes.fr/docs/00/53/60/73/PDF/DTGREQAM2010_40.pdf.

varying $m(\cdot)$ among controls that satisfy the constraint $m(t) \in \left[0, \frac{a}{b}\right]$ for every $t \geq 0$. The benchmark model can be completely solved; we summarize its behavior in the following proposition.

Proposition 3.1 *The condition*

$$\rho > \left(\frac{a}{b} - \mu\right)\gamma \tag{18}$$

is necessary and sufficient to ensure the boundedness of the functional in the benchmark model. Moreover

(i) If $\frac{a}{b}\gamma > (\rho + \gamma\mu)(1 - \sigma)$ then the optimal control is constant over time and it is given by $m^*(t) = \theta_\mu$ for every $t \geq 0$, where,

$$\theta_\mu := \frac{1}{\gamma\sigma}\left(\frac{a}{b}\gamma - (\rho + \gamma\mu)(1 - \sigma)\right) > 0;$$

so the optimal per-capita consumption does not depend on time and, for every $t \geq 0$, is given by $c^*(t) = a - b\theta_\mu$, and the evolution of the optimal population size can be written explicitly as $N^*(t) = N_0 e^{(\theta_\mu - \mu)t}$.

(ii) If $\frac{a}{b}\gamma \leq (\rho + \gamma\mu)(1 - \sigma)$ then the optimal control is constantly equal to zero: $m^*(t) = 0$ for every $t \geq 0$, the optimal per-capita consumption is constantly equal to a and the optimal trajectory of the population size is $N^*(t) \equiv N_0 e^{-\mu t}$.

In the next remark we collect a series of corollaries of the previous result that will be useful to compare the behavior of the benchmark with that of the main age-structured model.

Remark 3.1 The optimization program above has the following properties (among others):

(i) If $\frac{a}{b}\ \mu^{-1} < 1$, then (asymptotic) extinction is optimal whatever γ.

(ii) If $\gamma = 0$ (Millian case), $m(t) = 0$ for every $t \geq 0$ is optimal.

(iii) Whatever is the value of the altruism parameter γ and the optimal fertility rate θ_μ, consumption per capita is the same for all individuals of all cohorts.

The remark highlights several properties of the "perpetual youth" model. Similarly to the main model of Section 2, a large enough lifetime span (here measured by μ^{-1}) is necessary to avoid extinction (property (i)). Notice that in contrast to the model of Section 2, with given life span T, the "perpetual youth" model does only deliver asymptotic extinction: by the demographic law of motion (17), population only vanishes asymptotically. Regarding the altruism parameter γ: low enough γ (and in particular, the Millian case, $\gamma = 0$, depicted in property (ii)) lead to zero optimal fertility rate. More importantly, consumption per capita is constant across individuals and cohorts, this holds even in the optimal non-zero procreation case in item (iii) of the remark. Indeed, since consumption per capita at time t is given by $a - bm(t)$, and as optimal $m(\cdot)$ is time-invariant, we get the egalitarian outcome by construction in this second benchmark. The main conclusion from the exercise is that in the case where lifetime span is finite but not associated with an active age structure, egalitarianism does not discriminate between the social welfare functions considered. The rest of the paper will be devoted to the analysis of the model of Section 2 where not only lifetime spans are finite but associated with an active age structure.

4 Egalitarianism and optimal population dynamics under finite lives and active age structures

We provide now with the complete analysis of the Benthamite case ($\gamma = 1$) and the case of impure altruism ($0 < \gamma < 1$).

4.1 The Benthamite case, $\gamma = 1$

The mathematics needed to characterize the optimal dynamics is complex, relying on advanced dynamic programming techniques in infinite-dimensioned Hilbert spaces. Technical details are given in Appendix A. The same technique is used to handle the impure altruism case studied in the next section. Here, since $\gamma = 1$, the objective function simplifies into

$$\int_0^{+\infty} e^{-\rho t} \frac{(aN(t) - bn(t))^{1-\sigma}}{1-\sigma} N^{\sigma}(t)\,\mathrm{d}t. \tag{19}$$

For the value-function to be bounded, we can use the general sufficient condition (15): when $\gamma = 1$, it amounts to

$$\rho > \xi. \tag{20}$$

Recall that we have $\xi = 0$ when (13) does not have any strictly positive roots, i.e. when $\frac{a}{b}T \leq 1$. Moreover, if we define

$$\beta := \frac{a}{b}(1 - e^{-\rho T}), \tag{21}$$

then equation (20) implies

$$\rho > \beta > \xi \quad \textit{and} \quad \frac{\rho}{1 - e^{-\rho T}} > \frac{a}{b}. \tag{22}$$

The following theorem states a sufficient parametric condition ensuring the existence of an optimal control and characterizes it.

Theorem 4.1 *Consider the functional (19) with $\sigma \in (0, 1)$. Assume that (20) holds and let β given by (21). Then there exists a unique optimal control $n^*(\cdot)$.*

- *If*

$$\beta \leq \rho(1-\sigma) \quad \Longleftrightarrow \quad \frac{\rho}{1 - e^{-\rho T}} \geq \frac{a}{b} \cdot \frac{1}{1-\sigma}, \tag{23}$$

then the optimal control is $n^*(\cdot) \equiv 0$ and we have extinction at time T.

- *If*

$$\beta > \rho(1-\sigma) \quad \Longleftrightarrow \quad \frac{\rho}{1 - e^{-\rho T}} < \frac{a}{b} \cdot \frac{1}{1-\sigma}, \tag{24}$$

then we call

$$\theta := \frac{a}{b} \cdot \frac{\beta - \rho(1-\sigma)}{\beta\sigma} = \frac{a}{b}\left[\frac{1}{\sigma} - \frac{\rho(1-\sigma)}{\beta\sigma}\right] = \frac{1}{\sigma}\frac{a}{b} + \frac{\rho}{1 - e^{-\rho T}}\left(1 - \frac{1}{\sigma}\right) \tag{25}$$

and we have $\theta \in (0\ ,\frac{a}{b})$*. The optimal control* $n^*(\cdot)$ *and the related trajectory* $N^*(\cdot)$ *satisfy*

$$n^*(t) = \theta N^*(t). \tag{26}$$

Along the optimal trajectory the per-capita consumption is constant and its value is

$$c^*(t) = \frac{aN^*(t) - bn^*(t)}{N^*(t)} = a - b\theta \in (0, a) \qquad \textit{for all } t \geq 0. \tag{27}$$

Moreover the optimal control $n^*(\cdot)$ *is the unique solution of the following delay differential equation*

$$\begin{cases} \dot{n}(t) = \theta\left(n(t) - n(t-T)\right), & \textit{for } t \geq 0 \\ n(0) = \theta N_0 & \\ n(s) = n_0(s), & \textit{for all } s \in [-T, 0). \end{cases} \tag{28}$$

The proof is in Appendix A. Some comments on the optimal control identified are in order.

1. First of all, one has to notice that the condition for growth in Theorem 4.1, $\beta > \rho(1-\sigma)$ leads exactly to the growth condition uncovered in the benchmark with infinite lifetime (that's when $\mu = 0$), as one can see by making T going to infinity and putting $\gamma = 1$. The same can be claimed on the optimal constant fertility rate, θ, which now depends on individuals' lifetime: it is an increasing function of life spans and it converges to the constant fertility rate identified in the benchmark case if $\gamma = 1$ and $\mu = 0$ when T goes to infinity. Though the results obtained need not be algebraically identical to those one extracts from the general "perpetual youth" benchmark (for given μ), when $\gamma = 1$, still they are largely analogous. In both cases, the longer individuals' lives, the larger the fertility rate since individuals' are active for a longer time in our model. This anti-demographic transition mechanism can be counter-balanced if one introduces fixed labor time and costly pensions. This extension goes beyond the objectives of this paper.

2. Second, and related to the previous comparison point, the Benthamite case with finite lives displays qualitatively the same growth regime as in the benchmark: both consumption per capita and the fertility rate are constant over time. Such an outcome is far from obvious: in our model with active age structures, an endogenous or state-dependent depreciation term $\mu(t)$ shows up in the law of motion of population size, which either does not exist when people live forever, or is constant in the "perpetual youth" model; it is far unclear that the state independence property outlined in the benchmarks for fertility and consumption optimal dynamics can survive to this depreciation term. Theorem 4.1 shows that it does. However, we shall show in Section 4.2 that in contrast to the two benchmarks, the constancy of the optimal fertility rate and the corresponding intergenerational egalitarian consumption rule do not hold under impure altruism. This is consistent with Proposition 2.2 establishing the γ-homogeneity of the value function of our program. In particular, the value function is linear in the

Benthamite case: in such a case, the optimal policy in feedback form is also linear, leading to the optimality of the egalitarian solution in this case. When $\gamma < 1$, the value function is nonlinear, and as we shall see in the next section, the optimal policy in feedback form need to be nonlinear.

3. Third, one has to notice that finite time extinction is a possible optimal outcome in the Benthamite case (and not only the result of the anti-natalist Millian criteria as it is usually claimed in the related literature). Actually, finite time optimal extinction occurs when parameter β is low enough. By definition, this parameter measures a kind of adjusted productivity of the individual: productivity, a, is adjusted for the fact that individuals live a finite life (through the term $1 - e^{-\rho T}$), and also for the rearing costs they have to pay along their lifetime. If this adjusted productivity parameter is too small, the economy goes to extinction at finite time. And this possibility is favored by larger time discount rates and intertemporal elasticities of substitution (under $\sigma < 1$). Longer lives, better productivity and lower rearing costs can allow to escape from this scenario, although even in such cases, the economy is not sure to avoid extinction asymptotically (see Proposition 4.1 below). In particular, it is readily shown that condition (24), ruling out finite time extinction, is fulfilled if and only if $T > T_0$, where $T_0 = -(1/\rho)\ln(1 - \rho(1-\sigma)b/a)$ is the threshold value induced by (24), which depends straightforwardly on the parameters of the model.

4. Finally it is worth pointing out that there is a major difference between the finite life Benthamite case and the benchmark seen in Section 3: while the latter does not exhibit any transitional dynamics, the former does. Equation (28) gives the optimal dynamics of cohort's size $n(t)$. This linear delay differential equation is similar to the one analyzed by Boucekkine et al. (2005) and Fabbri and Gozzi (2008). The dynamics depend on the initial function, $n_0(t)$, and on the parameters θ and T in a way that will be described below. They are generally oscillatory reflecting replacement echoes as in the traditional vintage capital theory (see Boucekkine et al., 1997). In our model, the mechanism of generation replacement induced by finite life spans is the engine of these oscillatory transitions.

We now dig deeper in the dynamic properties and asymptotics of optimal trajectories. The following proposition summarizes the key points.

Proposition 4.1 *Consider the functional (19) with $\sigma \in (0,1)$. Assume that (20) and (24) hold, so $\theta \in (0\ ,\frac{a}{b})$. Then*

- *If $\theta T < 1$ then $n^*(t)$ (and then $N^*(t)$) goes to 0 exponentially.*
- *If $\theta T > 1$ then the characteristic equation of (28)*

$$z = \theta\left(1 - e^{-zT}\right), \tag{29}$$

has a unique strictly positive solution h belonging to $(0,\theta)$ while all the other roots have negative real part; h is an increasing function of T. Moreover the population and cohort *sizes both converge to an exponential solution at rate* h[17]*:*

$$\lim_{t\to\infty} \frac{n^*(t)}{e^{ht}} = \frac{\theta}{1-T(\theta-h)} \int_{-T}^{0} \left(1 - e^{(-s-T)h}\right) n_0(s)\, \mathrm{d}s > 0$$

and

$$\lim_{t\to\infty} \frac{N^*(t)}{e^{ht}} = \frac{1-e^{-hT}}{h} \frac{\theta}{1-T(\theta-h)} \int_{-T}^{0} \left(1 - e^{(-s-T)h}\right) n_0(s)\, \mathrm{d}s > 0$$

Finally, convergence is generally non-monotonic.

The proof is in Appendix B. The proposition above highlights the dynamic and asymptotic properties of the optimal control when finite time extinction is ruled out, that it is when $T > T_0$.

Indeed the proposition adds another threshold value $T_1 > T_0$ on individuals' lifetime: we have extinction in finite time when $T < T_0$, asymptotic extinction when individuals' lifetime is between T_0 and T_1, population and economic growth when $T > T_1$. Notice that the emergence of asymptotic extinction is consistent with Proposition 2.2[18]. It is very interesting to note that even in the case of asymptotic extinction, the Benthamite criterion does assure egalitarianism in the sense of "equal consumption per capital for all individuals and all generations": though population size goes to zero as time increases indefinitely, consumption per capita is constant along the transition by equation (27) of Theorem 4.1. Egalitarianism is therefore ensured in this optimal asymptotic extinction configuration.

The existence of such second threshold T_1 would be trivial if θ be independent of T. Since θ do depend on T the argument can be made precise observing that the function $T \mapsto T\theta(T)$ is always strictly increasing in T. This allows to formulate the following important result.

Corollary 4.1 *Under the conditions of Theorem 4.1, there exist two threshold values for individuals' lifetime,* T_0 *and* T_1*,* $0 < T_0 < T_1$ *such that:*

1. *for* $T \leq T_0$*, finite-time extinction is optimal,*
2. *for* $T_0 < T < T_1$*, asymptotic extinction is optimal,*
3. *for* $T > T_1$*, economic and demographic growth (at positive rate) is optimal.*

Proposition 4.1 brings indeed further important results. If individuals' lifetime is large enough (i.e. above the threshold T_1), then both the cohort size and population size will grow asymptotically at a strictly positive rate. In other words, these two variables will go to traditional balanced growth paths (BGPs). Proposition 4.1 shows that the longer the lifetime, the higher the BGP growth rate, which is a quite natural outcome of our setting. Moreover, consistently with standard endogenous growth theory, the levels of the BGPs depend notably on the initial conditions, here the initial function $n_0(t)$. Proposition 4.1 derives explicitly these long-run levels and their dependence on the initial datum is explicitly given.

We now examine a case of impure altruism.

[17]Observe that $\left(1 - e^{(-s-T)h}\right)$ is always positive for $s \in [-T, 0]$ and the constant $\frac{1}{1-T(\theta-h)}$ can be easily proved to be positive too.

[18]Here the threshold is indeed larger than the one identified in Proposition 2.2, see Corollary 4.1

4.2 A case of impure altruism

We now study whether the findings of the previous section are altered when the intertemporal altruism parameter γ varies in $(0\,,1)$. In this section we study the intermediate case $\gamma = 1-\sigma$ since it is a good and "cheap" way to address such a crucial question. Indeed from the mathematical point of view, and in contrast to the case $\gamma = 1$ handled above (and to the case $\gamma \neq 1-\sigma$), the case $\gamma = (1-\sigma)$ leads to the same infinitely dimensioned optimal control problem solved out explicitly by Fabbri and Gozzi (2008) using dynamic programming.[19] Moreover, by varying σ in $(0\,,1)$, one can extract some insightful lessons on the outcomes of our optimal control problem for any γ in $(0\,,1)$.[20]

The answer we find to our crucial question is that, for $\gamma \in (0\,,1)$, the optimal dynamics show some similarities with the Benthamite case concerning notably the optimal extinction properties and the oscillatory dynamics exhibited by population and cohort's sizes but they are also quite different in some aspects like the optimal consumption and fertility rate dynamics.[21]

As in the previous sections, we consider the optimal control problem of maximizing the objective function (11), that in the case $\gamma = 1-\sigma$ simplifies to

$$\int_0^{+\infty} e^{-\rho t} \frac{(aN(t) - bn(t))^{1-\sigma}}{1-\sigma} \, \mathrm{d}t, \tag{30}$$

over $n(\cdot) \in \mathcal{U}_{n_0}$. Also, as discussed in Section 2.2, we call ξ the unique strictly positive root of equation

$$z = \frac{a}{b}\left(1 - e^{-zT}\right),$$

if it exists, otherwise we pose $\xi = 0$. From Section 2.2, we know that $\xi > 0$ if individuals' lifetime is large enough: $T > \frac{b}{a}$. The condition (15) needed for the boundedness of the value function becomes:

$$\rho > \xi(1-\sigma). \tag{31}$$

It is then possible to characterize the optimal control of our problem as follows:

Theorem 4.2 *Consider the optimal control problem driven by (3), with constraint (8) and functional (30). If (31) and the following condition (needed to rule out corner solutions)*

$$\frac{\rho - \xi(1-\sigma)}{\sigma} \leq \frac{a}{b} \tag{32}$$

[19] Indeed, these authors identified a closed-form solution to the Hamilton-Jacobi-Bellman equation induced by the optimal growth model with AK technology and "one-hoss-shay" depreciation, i.e. all machines of any vintage are operated during a fixed time T. The objective function (with obvious notations) is $\int_0^{+\infty} e^{-\rho t} \frac{(ak(t)-i(t))^{1-\sigma}}{1-\sigma} \, \mathrm{d}t$ under the state equation $k(t) = \int_{t-T}^{t} i(\tau)\, \mathrm{d}\tau$, which is formally identically to our problem if and only if $\gamma = 1-\sigma$.

[20] In contrast to the infinite life case, we have found no way to identify an explicit solution to the Hamilton-Jacobi-Bellman equation under finite lives and active age structures for any value of the altruism parameter, γ.

[21] This fact can be also assessed (with some hard mathematical work) in the case $\gamma \neq 1-\sigma$ studying the qualitative properties of the optimal dynamics through the dynamic programming approach.

are satisfied, then, along the unique optimal trajectory, $n^*(\cdot)$ and the related optimal trajectory $N^*(\cdot)$ satisfy

$$n^*(t) = \frac{a}{b} N^*(t) - \Lambda e^{gt} \tag{33}$$

where

$$g := \frac{\xi - \rho}{\sigma} \tag{34}$$

and

$$\Lambda := \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{b\xi} \right) \left(\int_{-T}^{0} \left(1 - e^{\xi r}\right) n_0(r)\, \mathrm{d}r \right).$$

Moreover $n^*(\cdot)$ is characterized as the unique solution of the following delay differential equation:

$$\begin{cases} \dot{n}(t) = \frac{a}{b}\left(n(t) - n(t-T)\right) - g\Lambda e^{gt}, & t \geq 0 \\ n(0) = \frac{a}{b}\left(N_0 - \Lambda\right) \\ n(r) = n_0(r), \qquad r \in [-T, 0). \end{cases}$$

The proof is in Appendix B, it is a simple adaptation of previous work of Fabbri and Gozzi (2008). The closed-form solution identified allows indeed for a much finer characterization of this impure altruism case. For example, one can show in detail how close this case is to the Benthamite configuration studied in Section 4.1. Indeed, condition (32) rules out finite time extinction as an optimal outcome: if it is not verified, we get, as in Section 4.1, a case of optimal finite time extinction. Since the root ξ is an increasing function of the life span T,[22] one can also interpret condition (32) as putting a first threshold value for T below which finite extinction is optimal. Above this first threshold, either sustainable positively growing or asymptotically vanishing populations (and economies) are optimal. In particular, note that when $T < \frac{b}{a}$, $\xi = 0$ and therefore $g < 0$: in this case we necessarily have asymptotic extinction. Sustainable growth is not guaranteed even if $T > \frac{b}{a}$ because even if in this case the root $\xi > 0$ is not necessarily bigger than ρ for g to be necessarily positive. Just like in the Benthamite case, there exist a second threshold value of life span above which positive growth is optimal.

Two important comments should be made here. First of all, one can see that the properties extracted in the theorem above are not applicable to the limit case $\gamma = 1$ because this amounts to study the limit case $\sigma = 0$: in the latter case, magnitudes, like the growth rate g given in equation (34), are not defined. In contrast, the theorem can be used to study possible dynamics of optimal controls when γ is close to zero, or when σ is close to one (but not equal to 1 of course). Theorem 4.2 shows that is when γ is close to zero (but not equal to zero), that when the social welfare function tends to the Millian case, finite-time extinction is not the unique optimal outcome: population may even grow at a rate close to $g = \xi - \rho$ which might well be positive if the lifetime T is large enough (see a finer characterization below). In this sense, the impure altruism cases considered mimic to a large extent the properties identified for the Benthamite configuration.

[22] A formal statement and proof of this claim can be found in Proposition 4.2 of the earlier version of the paper quoted before, and available at: `http://halshs.archives-ouvertes.fr/docs/00/53/60/73/PDF/DTGREQAM2010_40.pdf`.

Much more importantly, Theorem 4.2 highlights a crucial specificity of the latter case. A major difference comes from the fact that the fertility rate can be hardly constant when altruism is imperfect given the γ-homogeneity property demonstrated in Proposition 2.2. Indeed, as specified in the point 2 after Theorem 4.1, the γ-homogeneity of the value function in the impure altruism case $\gamma < 1$, implies that the value function and the feedback optimal policy are nonlinear making impossible the constancy of the fertility rate arising e.g. in (26).

As a consequence, per capita consumption cannot be constant (when finite time extinction is ruled out). Recall that, in the Benthamite case, optimal consumption per capita and the fertility rate are constant and independent of the initial procreation profile when growth is optimal: our egalitarianism principle is ensured. This reflects the specificity of the latter case: when intertemporal altruism is maximal, the social planner abstracts from the initial conditions when fixing optimal consumption level and the fertility rate. Under intermediate altruism, the planner takes into account the initial data, and the optimal dynamics of the latter variables do adjust to this data: optimal consumption per capita (and fertility rates) cannot be constant in general, and therefore our egalitarianism principle is not generally compatible with impure altruism under finite lifetimes. Notice finally that the fact that optimal fertility rate and per capita consumption are non-constant in the impure altruism case goes at odds with the counterpart properties in the benchmark without active age structure studied in Section 3. For example, when $\gamma = 1 - \sigma$ and $\mu = 0$, the latter variables are constant (no transition dynamics). That's to say, age structures do significantly matter! A direct way to get this crucial aspect is to visualize the role of population's age distribution, which is irrelevant in the benchmark. Here observe that the younger the population, the higher the value of Λ. Therefore, the optimal decision $n(t) = aN(t)/b - \Lambda e^{gt}$ means that for a younger population a higher per-capita consumption and a lower fertility are optimal in the short-run. Still, as it will be shown below, the latter magnitudes are independent in the long run of the initial age-distribution of the population and the age-share profile converges to the "exponential" one.

Finally the transition dynamics in the impure altruism case can be described in detail.

Proposition 4.2 *Under the hypotheses of Theorem 4.2 the following limits exist*

$$\lim_{t\to\infty} \frac{n^*(t)}{e^{gt}} =: n_L$$

and

$$\lim_{t\to\infty} \frac{N^*(t)}{e^{gt}} =: N_L.$$

Moreover, if $g \neq 0$ *we have:*

$$n_L = \frac{\Lambda}{\frac{a}{bg}(1-e^{-gT}) - 1}$$

and

$$N_L = \frac{b}{a}(n_L + \Lambda) = \frac{\Lambda(1-e^{-gT})}{\frac{a}{b}(1-e^{-gT}) - g} = n_L \cdot \frac{1-e^{-gT}}{g}.$$

In particular, if $\rho > \xi$ *in the long run* $N(t)$ *and* $n(t)$ *go to zero exponentially; if* $\rho < \xi$*, they grow exponentially with rate* g *defined in (34). If* $\rho = \xi$ *they stabilize respectively to* n_L *and*

N_L. Moreover

$$\lim_{t\to\infty} c^*(t) = \lim_{t\to\infty} \frac{aN^*(t) - bn^*(t)}{N^*(t)} = a - \frac{bg}{1 - e^{-gT}}.$$

Finally $c(t)$, detrended $n(t)$ and detrended $N(t)$ exhibit oscillatory convergence to their respective asymptotic values.

The proposition shows that, as in the Benthamite case and despite the extra non-autonomous term, the economy will converge to a balanced growth path at rate g given in equation (34). As before, the long-run levels corresponding to total population and cohort sizes depend on the initial procreation profile via the parameter Λ. It should be noted here that despite the latter feature, both per capita consumption and the fertility rate are independent of the parameter Λ in the long-run. Therefore, and though the two latter variables do show up transition dynamics, they converge to magnitudes which are independent of the initial conditions, contrary to the traditional AK model.

5 Conclusion

In this paper, we have studied how egalitarianism in consumption within and across generations could be compatible with the optimal population size concepts produced by different social welfare functions. First, we have shown that egalitarianism does not discriminate between the social welfare functions considered in the benchmark case where age structure is irrelevant. In contrast, egalitarianism ceases to be systematically optimal as we move to active age structures. In particular, the final outcome depends on the degree of altruism: in a growing economy, that is when population growth is optimal, this egalitarian rule can only hold when the welfare function is Benthamite. When altruism is impure, egalitarianism is impossible in the context of a growing economy. That's to say, if life span is large enough, Parfit's *repugnant conclusion* for total utilitarianism does not hold: even more, all individuals of all generations will receive the same consumption, and therefore will enjoy the same welfare. Therefore, in a growing economy, when economic growth depends on human resources (which is a reasonable view), total utilitarianism need not be *repugnant.* On the other hand, our analysis implies that the Benthamite criterion is not necessarily pronatalist: in particular, if life spans are small enough, this criterion would legitimate finite time extinction.

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Appendices

A The case $\gamma = 1$: the infinite-dimensional setting and the proof of Theorem 4.1

We denote by $L^2(-T,0)$ the quotient space of all functions f from $[-T,0)$ to $\mathbb{R}$ that are Lebesgue measurable and such that $\int_{-T}^0 |f(x)|^2 \, dx < +\infty$ with respect to the relation of equality almost everywhere. It is an Hilbert space when endowed with the scalar product $\langle f, g\rangle_{L^2} = \int_{-T}^0 f(x)g(x)\, dx$. We consider the Hilbert space $M^2 := \mathbb{R} \times L^2(-T,0)$ (with the scalar product $\langle (x_0,x_1),(z_0,z_1)\rangle_{M^2} := x_0 z_0 + \langle x_1, z_1\rangle_{L^2}$). Following Bensoussan et al. (2007) (see Chapter II-4 and in particular Theorem 5.1), given an admissible control $n(\cdot)$ and the related trajectory $N(\cdot)$, if we define $x(t) = (x_0(t), x_1(t)) \in M^2$ for all $t \geq 0$ as

$$\begin{cases} x_0(t) := N(t) \\ x_1(t)(r) := -n(t-T-r), \qquad \text{for all } r \in [-T,0), \end{cases} \tag{35}$$

we have that $x(t)$ satisfies the following evolution equation in M^2:

$$\dot{x}(t) = A^* x(t) + B^* n(t). \tag{36}$$

where A^* is the adjoint of the generator of a C_0-semigroup[23] A defined as[24]

$$\begin{cases} D(A) \stackrel{def}{=} \{(\psi_0,\psi_1) \in M^2 \ : \ \psi_1 \in W^{1,2}(-T,0), \ \psi_0 = \psi_1(0)\} \\ A\colon D(A) \to M^2, \qquad A(\psi_0,\psi_1) \stackrel{def}{=} (0, \frac{\mathrm{d}}{\mathrm{d}s}\psi_1) \end{cases} \tag{37}$$

and B^* is the adjoint of $B\colon D(A) \to \mathbb{R}$ defined as[25] $B(\psi_0,\psi_1) := (\psi_1(0) - \psi_1(-T))$. Moreover, using the new variable $x \in M^2$ defined in (35) we can rewrite the welfare functional as

$$J(n(\cdot)) = \int_0^{+\infty} e^{-\rho t} \frac{(a x_0(t) - b n(t))^{1-\sigma}}{1-\sigma} x_0^\sigma(t)\, \mathrm{d}t. \tag{38}$$

Our optimal control problem of maximizing the welfare functional (19) over the set $\mathcal{U}_{n_0}$ defined in (10) with the state equation (3) can be equivalently rewritten as the problem of maximizing the functional above with the state equation (36) over the same set $\mathcal{U}_{n_0}$ (if we read x_0 instead of N in the definition (10)). The value function V depends now on the new variable x that can be expressed in term of the datum n_0 using (35) for $t=0$. The associated Hamilton-Jacobi-Bellman equation for the unknown v is[26]:

$$\rho v(x) = \langle x, ADv\rangle_{M^2} + \sup_{n \in \left[0, \frac{a}{b}x_0\right]} \left(nBDv(x) + \frac{(a x_0 - bn)^{1-\sigma}}{1-\sigma} x_0^\sigma \right). \tag{39}$$

As far as

$$BDv > a^{-\sigma} b \tag{40}$$

the supremum appearing in (39) is a maximum and the unique maximum point is strictly positive (since $x_0 > 0$) and is

$$n_{max} := \frac{a}{b} \left(1 - \left(\frac{BDv(x)}{a^{-\sigma} b} \right)^{-1/\sigma} \right) x_0 \tag{41}$$

so (39) can be rewritten as

$$\rho v(x) = \langle x, ADv\rangle_{M^2} + \frac{a}{b} x_0 BDv(x) + \frac{\sigma}{1-\sigma} x_0 \left(\frac{1}{b} BDv(x) \right)^{1-\frac{1}{\sigma}}. \tag{42}$$

[23] See e.g. Pazy (1983) for a standard reference to the argument.

[24] $W^{1,2}(-T,0)$ is the set $\{f \in L^2(-T,0) \ : \ \partial f \in L^2(-T,0)\}$ where ∂f is the distributional derivative of f.

[25] Recall that the elements of $W^{1,2}(-T,0)$ have a representative that is continuous on the whole interval $[-T,0]$ (so taking as ψ_1 such a representative) B is well defined.

[26] Dv is the Gateaux derivative.

When

$$BDv \leq a^{-\sigma} b \tag{43}$$

then the supremum appearing in (39) is a maximum and the unique maximum point is $n_{max} = 0$. In this case (39) can be rewritten as

$$\rho v(x) = \langle x, ADv \rangle_{M^2} + \frac{a^{1-\sigma} x_0}{1-\sigma} \tag{44}$$

We expect that the value function of the problem is a (the) solution of the HJB equation. Since the value function is 1-homogeneous (see Proposition 2.2), we look for a linear solution of the HJB equation. We have the following result:

Proposition A.1 *Suppose that (20) (and then (22)) holds and consider $\sigma \in (0,1)$. If*

$$\beta > \rho(1-\sigma) \tag{45}$$

then the function

$$v(x) := \alpha_1 \left(x_0 + \int_{-T}^{0} x_1(r) e^{\rho r} \, \mathrm{d}r \right) \tag{46}$$

where

$$\alpha_1 = a^{1-\sigma} \frac{1}{\beta} \left(\frac{1-\sigma}{\sigma} \cdot \frac{\rho - \beta}{\beta} \right)^{-\sigma}$$

is a solution of (42) in all the points s.t. $x_0 > 0$.

On the other side, if

$$\beta \leq \rho(1-\sigma) \tag{47}$$

then the function

$$v(x) := \alpha_2 \left(x_0 + \int_{-T}^{0} x_1(r) e^{\rho r} \, \mathrm{d}r \right) \tag{48}$$

where

$$\alpha_2 = \frac{a^{1-\sigma}}{\rho(1-\sigma)}$$

is a solution of (44) in all the points s.t. $x_0 > 0$.

Proof. Let $i = 1, 2$. We first observe that the function v is C^1 (since it is linear). Setting $\phi(r) = e^{\rho r}$, $r \in [-T, 0]$ we see that its first derivative is constant and is

$$Dv(x) = \alpha_i (1, \phi) \qquad \text{for all } x \in M^2.$$

Looking at (37) we also see that such derivative belongs to $D(A)$ so that all the terms in (39) make sense. We have $ADv(x) = (0, \alpha_i \rho \phi)$ and $BDv(x) = \alpha_i (1 - e^{-\rho T})$. Then, thanks to (45) (resp. (47)) we have that (40) (resp. (43)) is satisfied and (39) can be written in the form (42) (resp. (44)). To verify the statement we have only to check it directly: the left hand side of (42) (resp. (44)) is equal to $\rho \alpha_i \left(x_0 + \langle x_1, \phi \rangle_{L^2} \right)$. The right hand side is, for $i = 1$,

$$\langle x_1, \alpha_1 \rho \phi \rangle_{L^2} + \frac{a}{b} x_0 \alpha_1 (1 - e^{-\rho T}) + \frac{\sigma}{1-\sigma} x_0 \left(\frac{1}{b} \alpha_1 (1 - e^{-\rho T}) \right)^{1 - \frac{1}{\sigma}}$$
$$= \langle x_1, \alpha_1 \rho \phi \rangle_{L^2} + x_0 \alpha_1 \beta + \frac{\sigma}{1-\sigma} x_0 \left(\frac{\alpha_1 \beta}{a} \right)^{1 - \frac{1}{\sigma}} = \langle x_1, \alpha_1 \rho \phi \rangle_{L^2} + x_0 \frac{\alpha_1 \beta}{a} \left[1 + \frac{\sigma}{1-\sigma} \left(\frac{\alpha_1 \beta}{a} \right)^{-\frac{1}{\sigma}} \right].$$

Since the expression in square brackets is equal to $a\rho/\beta$ thanks to the definition of α_1, we have the claim for $i = 1$. For $i = 2$ the right hand side of (44) is (using the expression of α_2 above)

$$\langle x_1, \alpha_2 \rho \phi \rangle_{L^2} + \frac{a^{1-\sigma}}{1-\sigma} x_0 = \langle x_1, \alpha_2 \rho \phi \rangle_{L^2} + \alpha_2 \rho x_0$$

and this proves the claim for $i = 2$. □

Once we have a solution of the Hamilton-Jacobi-Bellman equation we can try to prove that it is the value function and so use it to find a solution of our optimal control problem in feedback form.

Theorem A.1 *Suppose that (20) (and then (22)) holds and $\sigma \in (0,1)$. Consider the optimal control problem having state equation (36), functional (38) and constraints $x_0(t) \geq 0$ and $n(t) \in \left[0, \frac{a}{b} x_0(t)\right]$ for all $t \geq 0$ with an initial datum of the form $x(0) := (x_0(0), x_1(0)(\cdot)) := \left(\int_{-T}^0 n_0(s)\,\mathrm{d}s, -n_0(-T-\cdot)\right)$ for some positive $n_0 \colon [-T,0) \to \mathbb{R}$.*

If (45) holds then the function v defined in (46) computed at $x(0)$ is the value function V of the problem and there exists a unique optimal control/trajectory starting from that initial datum. The optimal control $n^*(\cdot)$ and the related trajectory $x^*(\cdot)$ satisfy the following feedback relation:

$$n^*(t) = \frac{a}{b}\left(1 - (\alpha_1\beta)^{-\frac{1}{\sigma}}\right) x_0^*(t) = \theta x_0^*(t) \tag{49}$$

where θ is given by (25). If (47) is satisfied then the function v defined in (48), computed at $x(0)$, is the value function V computed at $x(0)$ and there exist a unique optimal control/trajectory starting from $x(0)$. In this case the optimal control $n^*(\cdot)$ is identically zero.

Proof. We give the proof in the case when (45) holds as the proof in the other case (when (47) holds) is completely analogous and indeed simpler). For brevity during the proof we denote $\int_{-T}^0 n_0(s)\,\mathrm{d}s$ by N_0.

Step 1: existence and admissibility of the feedback control (49).
First of all we observe that if we substitute the feedback function (49) in the state equation (36) we get

$$\begin{cases} \dot{x}^*(t) = A^* x(t) + B^* \theta x_0^*(t) \\ x^*(0) = (x_0(0), x_1(0)(\cdot)) = (N(0), -n_0(-T-\cdot)) \end{cases}$$

Using Theorem 4.3 page 254 of Bensoussan et al. (2007) we can see that such an equation has a unique solution $x^*(t) = (x_0^*(t), x_1^*(t)(\cdot))$ in

$$\mathcal{V} := \left\{ x \in C(0,+\infty; M^2) \;:\; \frac{\mathrm{d}x(t)}{\mathrm{d}t} \in L^2_{loc}(0,+\infty; M^2) \right\}.$$

and that can be characterized as follows:

- $x_0^*(t)$ is the unique (continuous) solution of the following DDE on $[0,+\infty)$:

$$\begin{cases} \dot{x}_0(t) = \theta\left(x_0(t) - x_0(t-T)\right), & \text{for } t \geq 0 \\ x_0(0) = N_0 & \\ x_0(s) = \frac{1}{\theta} n_0(s), & \text{for all } s \in [-T,0). \end{cases} \tag{50}$$

- $x_1^*(t)$ is given by $x_1^*(t)(s) = -n^*(t-T-s)$ for any $t \geq 0$ and $s \in [-T,0]$ where n^* is the unique solution of the following DDE:

$$\begin{cases} \dot{n}(t) = \theta\left(n(t) - n(t-T)\right), & \text{for } t \geq 0 \\ n(0) = \theta N_0 & \\ n(s) = n_0(s), & \text{for all } s \in [-T,0). \end{cases} \tag{51}$$

  From (50) and (51) we clearly have that $n^*(t) = \theta x_0^*(t)$ for any $t \geq 0$. The solution n^* of (51) is our candidate-optimal control.

Observe that, since $N_0 = \int_{-T}^0 n_0(s)\,\mathrm{d}s$, by integrating (50) and using a simple recursive argument we get, for $t \in [0,+\infty)$,

$$x_0^*(t) = \theta \int_{t-T}^t x_0^*(s)\,\mathrm{d}s.$$

Now we use this fact to prove that, if $N_0 > 0$ (this means that $n_0(s) > 0$ in a set of points of (Lebesgue) positive measure in the interval $[-T,0)$) then $x_0(t)$ can never be negative. Indeed recall that $x_0^*(t)$ is continuous and $x_0^*(0) = N_0 > 0$ and denote, by contradiction, by $\bar{t} > 0$ the first point in $[0,+\infty)$ in which x_0^* is equal to zero. We have

$$0 = x_0^*(\bar{t}) = \theta \int_{\bar{t}-T}^{\bar{t}} x_0^*(s)\,\mathrm{d}s > 0$$

the last inequality follows from the fact that, by hypothesis, $x_0^*(t) > 0$ for all $t \in [0, \bar{t})$ so when we integrate we obtain a strictly positive value. This gives a contradiction and then $x_0^*(t)$ never become zero (and then never negative).

We prove now that the feedback described in (49) gives an evolution of the system that satisfies the constraints $n^*(t) \in \left[0, \frac{a}{b} x_0^*(t)\right]$ for all $t \geq 0$; in other words we prove now that the control written in feedback form is admissible. Indeed, since $n^*(t) = \theta x_0^*(t)$ and since we have just seen that x_0^* is always positive, we have the positivity of $n^*(t)$ for any $t \geq 0$. On the other hand, since $n^*(t) = \theta x_0^*(t)$ and, from (24) and (25), $\theta \leq \frac{a}{b}$, we have that $n^*(t) \leq \frac{a}{b} x_0^*(t)$ for any $t \geq 0$.

Step 2: optimality of the feedback control (49).

Fix the initial datum $x(0)$ as above and take any admissible (at $x(0)$) control/state couple $(\tilde{n}(\cdot), \tilde{x}(\cdot))$ (i.e. $\tilde{x}(\cdot)$ is the solution of (36 with control $\tilde{n}(\cdot)$ and initial datum $x(0)$). By a simple integration we have that

$$\tilde{x}_0(t) = \int_{-T+t}^{t} \tilde{n}(s)\,\mathrm{d}s = -\int_{-T}^{0} \tilde{x}_1(t)(s)\,\mathrm{d}s \tag{52}$$

and, thanks to (14) and (64),

$$|\tilde{x}_1(t)(s)| = |\tilde{n}(t-T-s)| \leq \frac{a}{b}|\tilde{x}_0(t-T-s)| = \frac{a}{b} S_{MAX} e^{\frac{\xi+\rho}{2}(t-T-s)} \tag{53}$$

for some constant $S_{MAX} > 0$.
We introduce the function:

$$v_0(t,x) \colon \mathbb{R} \times M^2 \to \mathbb{R}, \qquad v_0(t,x) := e^{-\rho t} v(x)$$

where v is the function defined in (46).
Observe that, fixed $\tau > 0$, we have, using (52),

$$\begin{aligned} |v_0(\tau, \tilde{x}(\tau))| &= \left|e^{-\rho t} v(\tilde{x}(\tau))\right| = \left| e^{-\rho t}\alpha_1 \left(\tilde{x}_0(t) + \int_{-T}^{0} \tilde{x}_1(t)(r) e^{\rho r}\,\mathrm{d}r \right)\right| \\ &\leq \left| e^{-\rho t}\alpha_1 \left(\int_{-T}^{0} (1-e^{\rho r}) \tilde{x}_1(t)(r)\,\mathrm{d}r \right)\right| \leq e^{-\rho t}\alpha_1 \left(\int_{-T}^{0} \frac{a}{b}(1-e^{\rho r})\frac{a}{b} S_{MAX} e^{\frac{\xi+\rho}{2}(t-T-r)}\,\mathrm{d}r \right) \end{aligned} \tag{54}$$

so, using (53) and (20),

$$|v_0(\tau, \tilde{x}(\tau))| \xrightarrow{\tau\to+\infty} 0. \tag{55}$$

Since we have proved in Proposition A.1 that, for any $x \in M^2$, $Dv(x) \in D(A)$ and that the application $x \mapsto Dv(x)$ is continuous with respect to the norm of $D(A)$ and since the solutions of (36) are defined in weak sense (see Theorem 4.3 page 254 of Bensoussan et al., 2007) we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} v_0(t, \tilde{x}(t)) &= -\rho v_0(t, \tilde{x}(t)) + \langle D v_0(t, \tilde{x}(t)), A^* \tilde{x}(t) + B^* \tilde{n}(t) \rangle_{D(A)\times D(A)'} = \\ &= -\rho e^{-\rho t} v(\tilde{x}(t)) + e^{-\rho t}\Big(\langle ADv(\tilde{x}(t)), \tilde{x}(t)\rangle_{M^2} + BDv(\tilde{x}(t))\tilde{n}(t) \Big) \end{aligned} \tag{56}$$

Given any $\tau > 0$ and integrating such an expression on $[0, \tau]$ we have:

$$v_0(\tau, \tilde{x}(\tau)) - v_0(0, \tilde{x}(0)) = \int_0^{\tau} e^{-\rho t}\Big(-\rho v(\tilde{x}(t)) + \langle ADv(\tilde{x}(t)), \tilde{x}(t)\rangle_{M^2} + BDv(\tilde{x}(t))\tilde{n}(t) \Big)\,\mathrm{d}t. \tag{57}$$

Thanks to (55), passing to the limit, we have

$$-v(x(0)) = -v_0(0, x(0)) = \int_0^{+\infty} e^{-\rho t}\Big(-\rho v(\tilde{x}(t)) + \langle ADv(\tilde{x}(t)), \tilde{x}(t)\rangle_{M^2} + \langle BDv(\tilde{x}(t)), \tilde{n}(t)\rangle_{\mathbb{R}} \Big)\,\mathrm{d}t. \tag{58}$$

Adding and subtracting $J(\tilde{n}(\cdot))$ (which is finite thanks to Proposition 2.1) to the above expression and rearranging the terms we have

$$v(x(0)) - J(\tilde{n}(\cdot)) = \int_0^{\infty} e^{-\rho t}\left(\rho v(\tilde{x}(t)) - \langle ADv(\tilde{x}(t)), \tilde{x}(t)\rangle_{M^2} - BDv(\tilde{x}(t))\tilde{n}(t) - \frac{(a\tilde{x}_0(t) - b\tilde{n}(t))^{1-\sigma}}{(1-\sigma)} \right)\mathrm{d}t$$

As a last step we use that v solves the HJB equation as proved in Proposition A.1, so the last equation is equivalent to

$$v(x(0)) - J(\tilde{n}(\cdot)) = \int_0^\infty e^{-\rho t} \Bigg(\sup_{n\in[0,a/b\tilde{x}_0(t)]} \left[BDv(\tilde{x}(t))n + \frac{(a\tilde{x}_0(t) - bn)^{1-\sigma}}{(1-\sigma)} \right) \mathrm{d}t \Bigg]$$

$$- \left[BDv(\tilde{x}(t))\tilde{n}(t) + \frac{(a\tilde{x}_0(t) - b\tilde{n}(t))^{1-\sigma}}{(1-\sigma)} \right] \Bigg) \mathrm{d}t. \quad (59)$$

Since the integrand of the last expression is always positive we have that $v(x(0)) - J(\tilde{n}(\cdot)) \geq 0$ and then (taking the supremum over $\tilde{n}(\cdot)$) $v(x(0)) \geq V(x(0))$. Moreover the original maximization problem is equivalent to the problem of finding a control $n(\cdot)$ that minimizes $v(x(0)) - J(n(\cdot))$ and the candidate-optimal control $n^*(\cdot)$ attains the minimum of this expression, in fact considering $\tilde{n}(\cdot) = n^*(\cdot)$ the integrand is zero for every time and then we have $v(x(0)) - J(n^*(\cdot)) = 0$. So n^* is optimal and $v(x(0)) = V(x(0))$ and this conclude the claim for the case (45). □

Proof of Theorem 4.1. Theorem 4.1 is nothing but Theorem A.1 once we write again $N^*(\cdot)$ instead of $x_0^*(\cdot)$. In particular (49) becomes (26). Finally, if we write $N^*(t)$ as $\int_{t-T}^t n(s)\,\mathrm{d}s$ and we take the derivative in (49) we obtain (28). □

Remark A.1 As recalled in the introduction the optimal control problem we have solved here is structurally different from the one treated in the previous literature. Indeed, differently from previous papers (see e.g. Fabbri and Gozzi (2008)), the objective functional here contains the term the multiplicative term $N(t)^\gamma$ and displays a different type of constraints (in particular here the consumption path is bounded which is not the case in the other papers). So, finding a solution to the HJB equation and proving that it is the value function, as we did just above, requires different arguments. In particular we note that two things that are different from what happens in other papers: first the value function is homogeneous not with exponent $1-\sigma$, but with exponent γ, so is the "new" term $N(t)^\gamma$ that drives its growth; second the optimal policy takes two different forms depending if the maximum point is in the interior or in the boundary of the control region and so here also cases with the so-called "corner solutions" are studied.

B Other proofs

Proof of Lemma 2.1. Observe first that, given any control $n(\cdot)$, (3) can be rewritten, for $t \in [0, T]$, as

$$N(t) = \int_{t-T}^0 n_0(s)ds + \int_0^t n(s)ds$$

so, thanks to (8), we have

$$N(t) \leq \int_{t-T}^0 n_0(s)ds + \int_0^t \frac{a}{b} N(s)ds.$$

On the other side N_{MAX} satisfies (12) i.e.

$$N_{MAX}(t) = \int_{(t-T)}^0 n_0(s)\,\mathrm{d}s + \frac{a}{b}\int_0^t N_{MAX}(s)\,\mathrm{d}s.$$

so, by applying the Gronwall inequality (see e.g. Henry (1981) page 6) we have

$$N(t) \leq N_{MAX}(t) \qquad \text{for any } t \in [0, T]$$

and, consequently,

$$n(t) \leq \frac{a}{b} N(t) \leq \frac{a}{b} N_{MAX}(t) \qquad \text{for any } t \in [0, T]. \quad (60)$$

For $t \in [T, 2T]$ we have, thanks to (60),

$$N(t) \leq \int_{t-T}^{0} n(s)ds + \int_{0}^{t} \frac{a}{b} N(s)ds \leq \frac{a}{b} \int_{t-T}^{0} N(s)ds + \frac{a}{b} \int_{0}^{t} N(s)ds$$
$$\leq \frac{a}{b} \int_{t-T}^{0} N_{MAX}(s)ds + \frac{a}{b} \int_{0}^{t} N(s)ds \quad (61)$$

on the other side N_{MAX} satisfies, for $t \in [T, 2T]$,

$$N_{MAX}(t) = \frac{a}{b} \int_{t-T}^{0} N_{MAX}(s)ds + \frac{a}{b} \int_{0}^{t} N_{MAX}(s)ds.$$

so, we can again apply the Gronwall inequality and we have

$$N(t) \leq N_{MAX}(t) \qquad \text{for any } t \in [T, 2T].$$

Iterating the argument we can obtain that

$$N(t) \leq N_{MAX}(t) \qquad \text{for any } t \geq 0. \quad (62)$$

and then

$$n(t) \leq \frac{a}{b} N(t) \leq \frac{a}{b} N_{MAX}(t) \qquad \text{for any } t \geq 0. \quad (63)$$

This concludes the proof. □

Proof of Proposition 2.1. Since we consider positive utility functions we always have $V \geq 0$ (and then V is always bounded from below). We now prove that V is bounded from above.

If $\gamma \in (0, 1]$ define $\eta := \frac{\xi + \frac{\rho}{\gamma}}{2}$, otherwise, if $\gamma = 0$, define $\eta = \xi + 1$. Observe that, thanks to (14), (15) and the continuity of N_{MAX} on $[0, +\infty)$, there exists a real (positive) constant S_{MAX} such that, for $t \geq 0$,

$$N_{MAX}(t) \leq S_{MAX} e^{\eta t}. \quad (64)$$

Then, given an initial datum n_0, for any control n we have, thanks to (62), (9) and the fact that u is increasing,

$$\int_{0}^{+\infty} e^{-\rho t} u(\hat{c}(t)) \hat{N}^{\gamma}(t)\, dt \leq \int_{0}^{+\infty} e^{-\rho t} u(a) N_{MAX}^{\gamma}(t)\, dt = u(a) \int_{0}^{+\infty} e^{-\rho t} N_{MAX}^{\gamma}(t)\, dt. \quad (65)$$

If $\gamma \in (0, 1]$, thanks to (64) last expression is less or equal to

$$u(a) \int_{0}^{+\infty} e^{-\rho t} S_{MAX}^{\gamma} e^{\eta \gamma t}\, dt = u(a) S_{MAX}^{\gamma} \int_{0}^{+\infty} e^{-\frac{\rho - \gamma \xi}{2} t}\, dt = u(a) S_{MAX}^{\gamma} \frac{2}{\rho - \gamma \xi} =: K_{n_0} < +\infty.$$

If $\gamma = 0$, (65) is less or equal to

$$u(a) \int_{0}^{+\infty} e^{-\rho t} S_{MAX}^{\gamma} e^{\eta \gamma t}(t)\, dt = u(a) \int_{0}^{+\infty} e^{-\rho t}\, dt = u(a) \frac{1}{\rho} =: K_{n_0} < +\infty.$$

So, taking the sup among the set of admissible controls, we have

$$V(n_0) \leq K_{n_0} < +\infty.$$

This concludes the proof. □

Proof of Proposition 2.2. It is enough to note that, for every positive $n_0(\cdot) \in L^2(-T, 0)$ and every $\lambda_0 > 0$ we have, by the linearity of the state equation and of the constraints, $\mathcal{U}_{\lambda_0 n_0} = \lambda_0 \mathcal{U}_{n_0}$, so that

$$V(\lambda_0 n_0(\cdot)) = \sup_{n(\cdot) \in \mathcal{U}_{\lambda_0 n_0}} J(\lambda_0 n_0(\cdot); n(\cdot)) = \sup_{n(\cdot) \in \lambda_0 \mathcal{U}_{n_0}} J(\lambda_0 n_0(\cdot); n(\cdot)) = \sup_{n(\cdot) \in \mathcal{U}_{n_0}} J(\lambda_0 n_0(\cdot); \lambda_0 n(\cdot))$$

Now it is easy to check that

$$J(\lambda_0 n_0(\cdot); \lambda_0 n(\cdot)) = \lambda_0^{\gamma} J(n_0(\cdot); n(\cdot))$$

so the claim is proved. □

Proof of Proposition 2.3. Thanks to Lemma 2.1 it is enough to prove the statement for $N_{MAX}(t)$. Let us take $\bar{t} \in \arg\max_{s\in[T,2T]} N_{MAX}(s)$ (the argmax is non-void because N_{MAX} is continuous on $[0,+\infty)$). We have that $N_{MAX}(\bar{t}) = \frac{a}{b}\int_{\bar{t}-T}^{\bar{t}} N_{MAX}(s)\,\mathrm{d}s \le a/b(2T-\bar{t})\max_{s\in[0,T]} N_{MAX}(s) + a/b(\bar{t}-T)N_{MAX}(\bar{t})$ so $N_{MAX}(\bar{t}) \le \frac{a/b(2T-\bar{t})}{1-a/b(\bar{t}-T)}\max_{s\in[0,T]} N_{MAX}(s)$. Observe that, for all $\bar{t}\in[T,2T]$ we have that $\frac{a/b(2T-\bar{t})}{1-a/b(\bar{t}-T)} \in \left[0,\frac{a}{b}T\right]$, so $\max_{s\in[T,2T]} N_{MAX}(s) \le \frac{a}{b}T\max_{s\in[0,T]} N_{MAX}(s)$. In the same way we can prove that, for all positive integer n, $\max_{s\in[nT,(n+1)T]} N_{MAX}(s) \le \left(\frac{a}{b}T\right)^n \max_{s\in[0,T]} N_{MAX}(s)$. Since, by hypothesis, $\left(\frac{a}{b}T\right) < 1$ we have that $\lim_{t\to+\infty} N_{MAX}(t) = 0$ and then the claim. □

Proof of Proposition 3.1. We give the proof in the case $\gamma > 0$. The case $\gamma = 0$ is simpler.

Observe, as a first step, that (18) is sufficient to ensure the boundedness of the functional (16) along any possible admissible path. Indeed the constraint $m(t) \le a/b$ implies, for any possible control, $N(t) \le N_0 e^{\left(\frac{a}{b}-\mu\right)t}$. Conversely the constrain $m(t) \ge 0$ (and smaller than a/b) implies, for any $t \ge 0$, $(a-bm(t))^{1-\sigma} \le a^{1-\sigma}$, so that, for any possible admissible control $n(\cdot)$ and related trajectory of the state variable $N(\cdot)$, if (18) is verified, we have

$$\int_0^{+\infty} e^{-\rho t}\frac{(a-bm(t))^{1-\sigma}}{1-\sigma}N^\gamma(t)\,\mathrm{d}t \le \int_0^{+\infty} e^{-\rho t}\frac{a^{1-\sigma}}{1-\sigma}\left(N_0 e^{\left(\frac{a}{b}-\mu\right)t}\right)^\gamma \mathrm{d}t = \frac{a^{1-\sigma}N_0^\gamma}{1-\sigma}\left(\rho - \left(\frac{a}{b}-\mu\right)\gamma\right)^{-1} < +\infty.$$

Evaluating the utility along the optimal trajectory found in part (i) we will see that the condition (18) is indeed also necessary to ensure the boundedness of (16) along admissible paths.

Part (i): The Hamilton-Jacobi-Bellman equation associated to the problem is given by

$$\rho v(N) = \sup_{m\in[0,a/b]}\left((m-\mu)Nv'(N) + \frac{(a-bm)^{1-\sigma}}{1-\sigma}N^\gamma\right). \tag{66}$$

One can directly verify that the function $v(N) = \alpha N^\gamma$, where $\alpha := \frac{b}{\gamma}\left(\frac{\rho b - b\gamma\left(\frac{a}{b}-\mu\right)}{\gamma}\frac{1-\sigma}{\sigma}\right)^{-\sigma}$, is a solution of (66). So, using a standard verification argument (see for example Yong and Zhou (1999) Section 5.3), one proves that such v is indeed the value function of the problem and that the induced feedback map, given by

$$\begin{aligned} m = \phi(N) &:= \arg\max_{m\in[0,a/b]}\left((m-\mu)Nv'(N) + \frac{(a-bm)^{1-\sigma}}{1-\sigma}N^\gamma\right) \\ &= \arg\max_{m\in[0,a/b]}\left((m-\mu)N\alpha\gamma N^{\gamma-1} + \frac{(a-bm)^{1-\sigma}}{1-\sigma}N^\gamma\right) = \frac{1}{\gamma\sigma}\left(\frac{a}{b}\gamma - (\rho+\gamma\mu)(1-\sigma)\right) =: \theta_\mu, \end{aligned} \tag{67}$$

is the (unique) optimal feedback map of the problem, observe that it is independent of N. So the optimal control is given by $m^*(t) = \theta_\mu$ for every $t \ge 0$. The related trajectory, i.e. the unique solution of

$$\dot{N}^*(t) = (\phi(N^*(t)) - \mu)N^*(t) = (\theta_\mu - \mu)N^*(t), \qquad N^*(0) = N_0,$$

is then the (unique) optimal trajectory of the problem. It is given by $N^*(t) = N_0 e^{(\theta_\mu-\mu)t}$. The expression for $c^*(t)$ is then $c^*(t) = a - bm^*(t) = a - b\theta_\mu$ for every $t \ge 0$. As said, evaluating the utility along the trajectory $N^*(t)$ (driven by the control $m^*(\cdot)$ constantly equal to θ_μ), one can verify that the condition (18) is also necessary for the boundedness of the functional. The proof of Part (ii) is completely analogous. □

Proof of Remark 3.1. *Part (i)*: If $\frac{a}{b}\gamma \le (\rho+\gamma\mu)(1-\sigma)$ then we are in case (ii) of Proposition 3.1 then $m^*(t) \equiv 0$ and then, since $\mu > \frac{a}{b} > 0$ the asymptotic extinction is optimal. If $\frac{a}{b}\gamma > (\rho+\gamma\mu)(1-\sigma)$, using the expression of θ_μ given in Part (i) of Proposition 3.1, we get

$$\theta_\mu - \mu := \frac{1}{\sigma}\left(\frac{a}{b} - \mu(1-\sigma)\right) - \frac{\rho(1-\sigma)}{\gamma\sigma} - \mu = \frac{1}{\sigma}\left(\frac{a}{b} - \mu\right) - \frac{\rho(1-\sigma)}{\gamma\sigma}$$

If $\frac{a}{b} < \mu$ both the terms of the right hand side of previous equation are negative so $\theta_\mu - \mu < 0$ and then, since the optimal size of the population is $N^*(t) = N_0 e^{(\theta_\mu-\mu)t}$, the asymptotic extinction is optimal.

Part (ii): If $\gamma = 0$, since we always assume $\rho > 0$ and $\sigma \in (0,1)$, we always have $\frac{a}{b}\gamma \le (\rho+\gamma\mu)(1-\sigma)$ then the statement follows from case (ii) of Proposition 3.1.

Part (iii): It is a corollary of Proposition 3.1, indeed consumption per capita is the same for all individuals of all cohorts both if we are in case (i) or in case (ii) of Proposition 3.1. □

Proof of Proposition 4.1. Since $n^*(\cdot)$ solves (28) it can be written (see Diekmann et al., 1995, page 34) as a series

$$n^*(t) = \sum_{j=1}^{\infty} p_j(t) e^{\lambda_j t}$$

where $\{\lambda_j\}_{j=1}^{+\infty}$ are the roots of of the characteristic equation (29) and $\{p_j\}_{j=1}^{N}$ are $\mathbb{C}$-valued polynomial. If $\theta T > 1$, as already observed in Section 2.2 there exists a unique strictly positive root $\lambda_1 = h$. Moreover $h \in (0, \theta)$ and it is also the root with biggest real part (and it is simple). The polynomial p_1 associated to h is a constant (since h is simple) and can be computed explicitly (see for example Hale and Lunel (1993) Chapter 1, in particular equations (5.10) that gives the expansion of the fundamental solution and Theorem 6.1) obtaining that p_1 is constant and

$$p_1(t) \equiv \frac{\theta}{1 - T(\theta - h)} \int_{-T}^{0} \left(1 - e^{(-s-T)h}\right) n_0(s)\, \mathrm{d}s$$

this gives the limit for $n(t)^*/e^{ht}$. The limit for $N(t)^*/e^{ht}$ follows from the relation $N^*(t) = \int_{t-T}^{t} n^*(s)\, \mathrm{d}s$ and the previous result.

If $\theta T < 1$ each λ_j, for $j \geq 2$, has negative real part while $\lambda_1 = 0$ is the only real root. But again if we compute explicitly the polynomial p_1 (again a constant value) related to the root 0 we have

$$p_1(t) \equiv \frac{\theta N_0 + (-\theta) \int_{-T}^{0} n_0(r)\, \mathrm{d}r}{1 + \theta T} = \frac{\theta N_0 - \theta N_0}{1 + \theta T} = 0.$$

so only the contributions of the roots with negative real parts remain. This concludes the proof of the claims related to asymptotic behavior of detrended variables.

Let us prove now that h is an increasing function of T (recall that θ depends on T too). We use the implicit function theorem. Define

$$F(\lambda, T) = \theta(T)(1 - e^{-T\lambda}) - \lambda.$$

Given T such that $\theta(T)T > 1$ one has that $F(\lambda, T)$ is concave in λ, $F(0, T) = 0$ and $F(h, T) = 0$ (recall that $h \in (0, \theta(T))$). So it must be

$$\left.\frac{\partial}{\partial \lambda} F(\lambda, T)\right|_{\lambda = h} = \theta(T) T e^{-Th} - 1 < 0.$$

Moreover, since by the definition of θ in (25) we easily get $\theta'(T) > 0$, we have:

$$\frac{\partial F(h, T)}{\partial T} = \theta'(T)(1 - e^{-Th}) + \theta(T) h e^{-Th} > 0$$

Now, by the implicit function theorem we have

$$\frac{\mathrm{d}h}{\mathrm{d}T} = -\frac{\partial F}{\partial T} \left(\left.\frac{\partial F}{\partial \lambda}\right|_{\lambda = h} \right)^{-1} > 0$$

and this concludes the proof. □

Proof of Theorem 4.2. The statements follows from Lemma 2.3.3 and Theorem 2.3.4 of Fabbri and Gozzi (2008): here we have the control variable n instead of i and the state variable N instead of k. The state equation is the same. To rewrite the objective functional exactly in the form of the problem treated in Fabbri and Gozzi (2008) we only need to write

$$aN(t) - bn(t) = b\left(\frac{a}{b} N(t) - n(t)\right)$$

so the functional becomes

$$b^{1-\sigma} \int_0^{+\infty} e^{-\rho t} \frac{\left(\frac{a}{b} N(t) - n(t)\right)^{1-\sigma}}{1 - \sigma}\, \mathrm{d}t.$$

The constant $b^{1-\sigma}$ as it does not changes the optimal trajectories. Dropping it the functional is the same as the one of Fabbri and Gozzi (2008) where the constant a is substituted here by $\frac{a}{b}$. □

Proof of Proposition 4.2. Arguing as in the proof of Theorem 4.2 the statement is equivalent to that of Proposition 2.3.5 in Fabbri and Gozzi (2008). □