**Intertemporal equilibrium with production: bubbles and efficiency**

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# Intertemporal equilibrium with production: bubbles and efficiency[*]

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### Abstract

We consider a general equilibrium model with heterogeneous agents, borrowing constraints, and exogenous labor supply. First, the existence of intertemporal equilibrium is proved even if the aggregate capitals are not uniformly bounded above and the production functions are not time invariant. Second, we call by physical capital bubble a situation in which the fundamental value of physical capital is lower than its market price. We show that there is a physical capital bubble if and only if the sum (over time) of capital returns is finite. We also point out that there is no causal relationship between physical capital bubble and the fact that the present value of output is finite. Last, with linear technologies, every intertemporal equilibrium is efficient in sense of Malinvaud (1953). Moreover, there is a room for both efficiency and bubble.

**Keywords**: Intertemporal equilibrium, physical capital bubble, efficiency, infinite horizon.

**JEL Classifications**: C62, D31, D91, G10

# 1  Introduction

We consider a dynamic general equilibrium model with heterogeneous agents, exogenous labor supply, and non-stationary technologies. Heterogeneous agents decide to invest and consume. If they invest in physical capital, this asset will not only give them return in term of consumption good at the next period but also give back a fraction of the same asset (after being depreciated). We assume that agents cannot borrow.

Our first contribution is to prove the existence of intertemporal general equilibrium. To do so, we firstly prove the existence of equilibrium for each $T-$truncated economy. Hence, we have a sequence of equilibria which depend on $T$. We then prove that this sequence has a limit (for the product topology) which is an equilibrium for the infinite horizon economy. The added value of our proof is that we allow non-stationary and linear production functions. Moreover, the capital stocks are not necessarily uniformly bounded. This general setup allows us to point out some relevant results on bubble and efficiency of equilibria.

In our framework, the physical capital is viewed as a long-lived asset which is depreciated and gives dividends at each date. Therefore, we can define the fundamental value of the physical capital. We then say that a physical capital bubble (for short, *bubble*) occurs at equilibrium if the market price of the physical asset is greater than its fundamental value. Our second contribution is to prove that physical capital bubble exists if and only if the sum (over time) of real capital returns is finite, which we call, for short, *low returns.*

We prove that, if the technology is stationary, there is no physical capital bubbles. By the way, the no-bubble result in Becker, Bosi, Le Van and Seegmuller (2014) can be viewed as a particular case of our result. Indeed, in Becker, Bosi, Le Van and Seegmuller (2014), thanks to specific conditions of the stationary technology, the aggregate capital stock is uniformly bounded, and then real return of the physical capital is uniformly bounded away from zero. Therefore, the sum of capital returns equals infinity. According to our result, the physical capital bubble is ruled out.

However, when we allow for non-stationary production functions, there may be a bubble at equilibrium. To see the point, take linear production functions whose productivity at date $t$ is denoted by $a_t$. At equilibrium, real return of physical capital at date $t$ must be $a_t$. As mentioned above, there is a bubble if and only if $\sum_{t=0}^{\infty} a_t < \infty$. We can now see clearly that there is a bubble if productivities decrease with sufficiently high speed. Note that, the physical capital bubble may exist in the model with a unique agent. This is different from the standard model with pure financial asset as in Kocherlakota (1992, 2008), Santos and Woodford (1997), Huang and Werner (2000), Le Van and Vailakis (2012).

We also point out that there is no relationship between physical capital bubble and the fact that the present value of output is finite.

Our third contribution is about the efficiency of intertemporal equilibrium. An intertemporal equilibrium is called efficient if its aggregate capital path is efficient in sense of Malinvaud (1953). We prove that with linear production functions, every intertemporal equilibrium is efficient. However, as we mentioned above, this

efficient intertemporal equilibrium may have bubble if productivities decrease with sufficiently high speed. Therefore, we have both efficient and bubble at equilibrium with such technologies. Note that our result does not require any conditions about the convergence or boundedness of the capital path as in previous literature.

Our paper is related to several strands of research.

(1) *On bubbles*: In infinite horizon general equilibrium models with incomplete financial markets and without capital accumulation (Tirole , 1982; Kocherlakota , 1992; Santos and Woodford , 1997; Huang and Werner , 2000; Le Van and Vailakis , 2012; Kocherlakota , 2008), one considered bubbles of a long-lived financial asset which gives exogenous dividends at each date. A well-known sufficient condition for no-bubble is that the present value of aggregate endowment is finite. Note that the present value of aggregate endowments is endogenously determined.

We can also define bubble for a long-lived asset which does not give dividends. For such an asset, its fundamental value is zero. The standard definition of bubble is the following: We say that there is a bubble if the market price of this asset is strictly positive. There is a large litterature on this kind of bubbles. Some of them are Tirole (1985), Ventura (2012), Farhi and Tirole (2012).[1]

Different from these kinds of bubbles, we study bubbles of the physical capital. There are two structural differences between the physical capital and the financial asset in their framework: (i) the physical capital is depreciated at each date, (ii) the dividend of this asset at each date is endogenous, which is determined by the marginal productivity of the production function.

(2) *On the efficiency of a capital path.* Malinvaud (1953) introduced the concept of efficiency of a capital path and gave a sufficient condition for the efficiency: $\lim_{t\to\infty} P_t K_t = 0$, where $(P_t)$ is a sequence of competitive prices, $(K_t)$ is the capital path.[2] Following Malinvaud, Cass (1972) considered capital path which is uniformly bounded from below. Under the concavity of a stationary production function and some mild conditions, he proved that a capital path is inefficient if and only if the sum (over time) of future values of a unit of physical capital is finite. Cass and Yaari (1971) gave a necessary and sufficient condition for a consumption plan $(C)$ to be efficient: the inferior limit of differences between the present value of any consumption plan and the plan $(C)$ is negative.

Our paper is also related to Becker and Mitra (2012) where they proved that a Ramsey equilibrium is efficient if the most patient household is not credit constrained from some date. However, their result is based on the fact that the consumption of each household is uniformly bounded from below. In our paper, we do not need this condition. Instead, the efficient capital path in our model may converge to zero. Mitra and Ray (2012) studied the efficiency of a capital path with nonconvex production technologies and examined whether the Phelps-Koopmans theorem is valid. However, their results are no longer valid without the convergence or the boundedness of capital paths.[3]

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[1] A survey on bubbles in models with asymmetric information, overlapping generation, heterogeneous-beliefs can be found in Brunnermeier and Oehmke (2012).

[2] See Malinvaud (1953), Lemma 5, page 248.

[3] Another concept of efficiency is *constrained efficiency*. Constrained inefficiency occurs when

The remainder of the paper is organized as follows. Section 2 describes the model. In section 3, existence of equilibrium is proved. Section 4 studies physical capital bubble. Section 5 explores our results on the efficiency of equilibria. Conclusion will be presented in Section 6. Technical details are gathered in Appendix.

## 2 Model

We consider an infinite horizon general equilibrium model with heterogeneous agents. Time runs from $t = 0$ to infinity.

**Consumption good**: There is a single consumption good. At each period $t = 0, 1, 2, \ldots, \infty$, the price of consumption good is denoted by $p_t$, and agent $i$ consumes $c_{i,t}$ units of consumption good.

**Physical capital**: at time $t$, if agent $i$ decides to buy $k_{i,t+1} \geq 0$ units of new capital, then at period $t+1$, after being depreciated, agent $i$ will receive $(1-\delta)k_{i,t+1}$ units of old capital and a return on capital $k_{i,t+1}$ at the rate $r_{t+1}$. Here, $\delta$ is the capital depreciation rate.

**Firm**: For each period $t$, there is a representative firm whose production function is $F_t$ which may be non-stationary. This firm takes prices $(p_t, r_t)$ as given, and maximizes its profit.

$$(P(r_t)): \quad \pi_t(p_t, r_t) := \max_{K_t \geq 0} \Big[ p_t F_t(K_t) - r_t K_t \Big]$$

We write $\pi_t$ instead of $\pi_t(p_t, r_t)$ if there is no confusion.

**Households**: There are $m$ heterogeneous households. Each household $i$ takes the sequence of prices and capital returns $(p, r) = (p_t, r_t)_{t=0}^{\infty}$ as given and maximizes her intertemporal utility by choosing the sequences of consumption and capital subject to the sequences of budget constraints and borrowing constraints. The problem of agent $i$ is the following

$$(P_i(p, r)): \quad \max_{(c_{i,t}, k_{i,t+1})_{t=0}^{+\infty}} \Big[ \sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t}) \Big] \tag{1}$$

$$\text{subject to} : \quad \forall t \geq 0, \quad k_{i,t+1} \geq 0 \text{ and} \tag{2}$$

$$p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) \leq r_t k_{i,t} + \theta^i \pi_t(p_t, r_t), \tag{3}$$

where $(\theta^i)_{i=1}^m$ is the share of profit, which is exogenous, $\theta^i \geq 0$ for all $i$ and $\sum\limits_{i=1}^{m} \theta^i = 1$. $\beta_i \in (0, 1)$ is the time preference of the agent $i$, $u_i$ is her utility function. Here, $k_{i,0}$ is given.

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there exists a welfare improving feasible redistribution subject to constraints (these constraints depends on models). About the constrained efficiency in general equilibrium models with financial asset, see Kehoe and Levine (1993), Alvarez and Jermann (2000), Bloise and Reichlin (2011). About the constrained efficiency in the neoclassical growth model, see Davila, Hong, Krusell and Rios-Rull (2012).

**Remark 1.** *If we interpret that the household $i$ has $\theta_i$ units of time and she uses all this time to work, we can say that the salary of household $i$ is*

$$\theta_i \pi_t = \theta_i (p_t F_t(K_t) - r_t K_t).$$

*By the way, our model can be interpreted as a model with exogenous labor supply as in Becker and Mitra (2012).*

We need the following assumptions:

**Assumption (H1):** For each $i$, the utility function $u_i$ of agent $i$ is strictly increasing, strictly concave, continuously differentiable, and $u_i(0) = 0, u_i'(0) = \infty$.

**Assumption (H2):** $F_t(\cdot)$ is continuously differentiable, strictly increasing, concave, the input is essential ($F_t(0) = 0$) and $F_t(\infty) = \infty$.

**Assumption (H3):** $\delta \in (0, 1)$ and $k_{i,0} > 0$ for every $i$.[4]

**Definition 1.** *A sequence of prices and quantities $\left( \bar{p}_t, \bar{r}_t, (\bar{c}_{i,t}, \bar{k}_{i,t+1})_{i=1}^m, \bar{K}_t \right)_{t=0}^{+\infty}$ is an equilibrium of the economy $\mathcal{E} = \left( (u_i, \beta_i, k_{i,0}, \theta_i)_{i=1}^m, (F_t)_{t=0}^\infty \right)$ if the following conditions holds.*

(i) *Price positivity: $\bar{p}_t, \bar{r}_t > 0$ for $t \geq 0$.*

(ii) *All markets clear: at each $t \geq 0$,*

$$\text{consumption good} \ : \ \sum_{i=1}^m [\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}] = F_t(\bar{K}_t) \qquad (4)$$

$$\text{physical capital} \ : \ \bar{K}_t = \sum_{i=1}^m \bar{k}_{i,t}. \qquad (5)$$

(iii) *Optimal consumption plans: for each $i$, $(\bar{c}_{i,t}, \bar{k}_{i,t+1})_{t=0}^\infty$ is a solution to problem $(P_i(\bar{p}, \bar{r}))$.*

(iv) *Optimal production plan: for each $t \geq 0$, $(\bar{K}_t)$ is a solution to problem $(P(\bar{r}_t))$.*

## 3 The existence of equilibrium

First, we prove the existence of equilibrium for each $T-$ truncated economy $\mathcal{E}^T$. Second, we show that this sequence of equilibria converges for the product topology to an equilibrium of our economy $\mathcal{E}$.

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[4] Becker, Bosi, Le Van and Seegmuller (3) weekens H3 by assuming $\sum_{i=1}^m k_{i,0} > 0$ because they assume that every agent has 1 unit of labor.

## 3.1 Existence of equilibrium in $\mathcal{E}^T$

We define a $T-$ truncated economy $\mathcal{E}^T$ as the econonomy obtained from $\mathcal{E}$ by imposing that there are no activities from period $T + 1$ to infinity, i.e., $c_{i,t} = k_{i,t} = 0$ for every $i = 1, \ldots, m$, and for every $t \geq T + 1$.

In the economy $\mathcal{E}^T$, agent $i$ takes the sequence of prices $(p, r) = (p_t, r_t)_{t=0}^T$ as given and maximizes her intertemporal utility by choosing consumption and investment levels.

$$(P_i(p,r)): \qquad \max_{(c_{i,t}, k_{i,t+1})_{t=0}^T} \Big[ \sum_{t=0}^T \beta_i^t u_i(c_{i,t}) \Big]$$

$$\text{subject to:} \quad k_{i,t+1} \geq 0,$$

$$\text{(budget constraints)} \quad p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) \leq r_t k_{i,t} + \theta^i \pi_t,$$

where $k_{i,T} = 0$.

We then define the bounded economy $\mathcal{E}_b^T$ as obtained from $\mathcal{E}^T$ by assuming all variables are bounded in the following compact sets:

$$
\begin{aligned}
(c_{i,t})_{t=0}^T &\in \mathcal{C}_i := [0, B_c]^{T+1} \\
(k_{i,t})_{t=1}^{T+1} &\in \mathcal{K}_i := [0, B_k]^T \\
K := (K_t)_{t=1}^{T+1} &\in \mathcal{K} := [0, B]^T,
\end{aligned}
$$

where $B_c > \max_t F_t(B) + (1-\delta)B$, $B > m B_k$.

**Proposition 1.** *Under Assumptions $(H1) - (H3)$, there exists an equilibrium for $\mathcal{E}_b^T$.*

*Proof.* See Appendix. $\qquad \qquad \square$

**Proposition 2.** *An equilibrium of the economy $\mathcal{E}_b^T$ is also an equilibrium of the unbounded economy $\mathcal{E}^T$.*

*Proof.* Similar to the one in Becker, Bosi, Le Van and Seegmuller (2014). $\qquad \square$

## 3.2 Existence of equilibrium in $\mathcal{E}$

The following result proves that the feasible aggregate capital and the feasible consumption are bounded from above for the product topology.

**Lemma 1.** *Feasible individual and aggregate capitals and feasible consumptions are in a compact set for the product topology. Moreover, they are uniformly bounded if there exists $t_0$ and an increasing, concave function $G$ such that: (i) for every $t \geq t_0$ we have $F_t(K) \leq G(K)$ for every $K$, (ii) there exists $x > 0$ such that $G(y) + (1-\delta)y \leq y$ for every $y \geq x$.*

*Proof.* Denote

$$
\begin{aligned}
D_0 := D_0(F_0, \delta, K_0) \quad &:= \quad F_0(K_0) + (1-\delta)K_0, \\
D_t := D_t((F_s)_{s=0}^t, \delta, K_0) \quad &:= \quad F_t(D_{t-1}((F_s)_{s=0}^{t-1}, \delta, K_0)) \\
&\quad\ + (1-\delta)D_{t-1}((F_s)_{s=0}^{t-1}, \delta, K_0), \ \forall t \geq 0.
\end{aligned}
$$

Then $\sum_{i=1}^m c_{i,t} + K_{t+1} \leq D_t$ for every $t \geq 0$. Since $D_t$ is exogenous, capital and consumption stocks are in a compact set for the product topology.

We now assume $t_0$ and the function $G$ (as in Lemma 1) exist. We are going to prove that $0 \leq K_t \leq \max\{D_0, ..., D_{t_0-1}, x\} =: M$. Indeed, $K_t \leq D_{t-1} \leq M$ for every $t \leq t_0$. For $t \geq t_0$, we have

$$
K_{t+1} = \sum_{i=1}^m k_{i,t+1} \quad \leq \quad G(K_t) + (1-\delta)K_t.
$$

Then $K_{t_0+1} \leq G(K_{t_0}) + (1-\delta)K_{t_0} \leq G(M) + (1-\delta)M \leq M$. Iterating the argument, we obtain $K_t \leq M$ for each $t \geq 0$.

Feasible consumptions are bounded because $\sum_{i=1}^m c_{i,t} \leq G(K_t) + (1-\delta)K_t$. $\qquad\square$

**Assumption (H4)**: For each $i$, the utility of agent $i$ is finite

$$
\sum_{t=0}^\infty \beta_i^t u_i(D_t) < \infty.
$$

**Theorem 1.** *Under Assumptions (H1)-(H4), there exists an equilibrium.*

*Proof.* We have shown that for each $T \geq 1$, there exists an equilibrium for the economy $\mathcal{E}^T$. We denote by $(\bar{p}^T, \bar{r}^T, (\bar{c}_i^T, \bar{k}_i^T)_{i=1}^m, \bar{K}^T)$ an equilibrium of $T-$ truncated economy $\mathcal{E}^T$. We can normalize by setting $\bar{p}_t^T + \bar{r}_t^T = 1$ for every $t \leq T$. We see that

$$
0 < \bar{c}_{i,t}^T, \bar{K}_t^T \quad \leq \quad D_t.
$$

Thus, consumption and capital stocks are in a compact set for the product topology. Therefore, without loss of generality, we can assume that

$$
(\bar{p}^T, \bar{r}^T, (\bar{c}_i^T, \bar{k}_i^T)_{i=1}^m, \bar{K}^T) \xrightarrow{T \to \infty} (\bar{p}, \bar{r}, (\bar{c}_i, \bar{k}_i)_{i=1}^m, \bar{K})
$$

for the product topology.

We are going to prove that: (i) all markets clear, (ii) at each date $t$, $\bar{K}_t$ is a solution to the firm's maximization problem, (iii) $\bar{r}_t > 0$ for each $t \geq 0$, (iv) $(\bar{c}_i, \bar{k}_i)$ is a solution to the maximization problem of agent $i$ for each $i = 1, \ldots, m$, (v) $\bar{p}_t > 0$ for each $t$. Consequently, we obtain that $(\bar{p}, \bar{r}, (\bar{c}_i, \bar{k}_i)_{i=1}^m, \bar{K})$ is an equilibrium for the economy $\mathcal{E}$.

(i) By taking the limit of market clearing conditions for the truncated economy, we obtain the market clearing conditions for the economy $\mathcal{E}$.

(ii) Take $K \geq 0$ arbitrary. We have $\bar{p}_t^T F_t(K) - \bar{r}_t^T K \leq \bar{p}_t^T F_t(\bar{K}_t^T) - \bar{r}_t^T \bar{K}_t^T$. Let $T$ tend to infinity, we obtain that $\bar{p}_t F_t(K) - \bar{r}_t K \leq \bar{p}_t F_t(\bar{K}_t) - \bar{r}_t \bar{K}_t$. Therefore, the optimality of $\bar{K}_t$ is proved.

(iii) If $\bar{r}_t = 0$ then $\bar{p}_t = 1$ (since $\bar{r}_t^T + \bar{p}_t^T = 1$). The optimality of $\bar{K}_t$ implies that $\bar{K}_t = \infty$. This is a contradiction, because we have $\bar{K}_t = \lim_{T \to \infty} \bar{K}_t^T \leq D_t < \infty$. As a result, we have $\bar{r}_t > 0$.

(iv) We start by giving some notations. For each $i$ and $t$, we define $B_i^T(\bar{p}, \bar{r})$ and $C_i^T(\bar{p}, \bar{r})$ as follows

$$B_i^T(\bar{p}, \bar{r}) := \Big\{ (c_{i,t}, k_{i,t+1})_{t=0}^T \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : \text{(a) } k_{i,T+1} = 0, \text{(b) } \forall t = 0, \ldots, T,$$

$$k_{i,t+1} > 0, \quad \bar{p}_t[c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}] < \bar{r}_t k_{i,t} + \theta^i \pi_t(\bar{p}_t, \bar{r}_t) \Big\},$$

$$C_i^T(\bar{p}, \bar{r}) := \Big\{ (c_{i,t}, k_{i,t+1})_{t=0}^T \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : \text{(a) } k_{i,T+1} = 0, \text{(b) } \forall t = 0, \ldots, T,$$

$$k_{i,t+1} \geq 0, \quad \bar{p}_t[c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}] \leq \bar{r}_t k_{i,t} + \theta^i \pi_t(\bar{p}_t, \bar{r}_t) \Big\}.$$

Since $\bar{r}_t > 0$ for every $t$, it is easy to prove that $B_i^T(\bar{p}, \bar{r}) \neq \emptyset$.

Let $(c_i, k_i)$ be a feasible allocation of the problem $P_i(\bar{p}, \bar{r})$. We have to prove that $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

We define $(c'_{i,t}, k'_{i,t+1})_{t=0}^T$ as follows: $c'_{i,t} = c_{i,t}$ for every $t \leq T$, $= 0$ if $t > T$; $k'_{i,t+1} = k_{i,t+1}$ for every $t \leq T - 1$, $= 0$ if $t \geq 0$. We see that $(c'_{i,t}, k'_{i,t+1})_{t=0}^T$ belongs to $C_i^T(\bar{p}, \bar{r})$. Since $B_i^T(\bar{p}, \bar{r}) \neq \emptyset$, there exists a sequence $\Big( (c_{i,t}^n, k_{i,t+1}^n)_{t=0}^T \Big)_{n=0}^{\infty} \in B_i^T(\bar{p}, \bar{r})$ with $k_{i,T+1}^n = 0$, and this sequence converges to $(c'_{i,t}, k'_{i,t+1})_{t=0}^T$ when $n$ tends to infinity. We have

$$\bar{p}_t(c_{i,t}^n + k_{i,t+1}^n - (1-\delta)k_{i,t}^n) < \bar{r}_t k_{i,t}^n + \theta^i \pi_t(\bar{p}_t, \bar{r}_t)$$

We can chose $s_0 > T$, high enough, such that: for every $s \geq s_0$, we have

$$\bar{p}_t^s(c_{i,t}^n + k_{i,t+1}^n - (1-\delta)k_{i,t}^n) < \bar{r}_t^s k_{i,t}^n + \theta^i \pi_t(\bar{p}_t^s, \bar{r}_t^s).$$

It means that $(c_{i,t}^n, k_{i,t+1}^n)_{t=0}^T \in C_i^T(\bar{p}^s, \bar{r}^s)$. Therefore, we get $\sum_{t=0}^T \beta_i^t u_i(c_{i,t}^n) \leq \sum_{t=0}^s \beta_i^t u_i(\bar{c}_{i,t}^s)$. Let $s$ tend to infinity, we obtain $\sum_{t=0}^T \beta_i^t u_i(c_{i,t}^n) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

Let $n$ tends to infinity, we have $\sum_{t=0}^T \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$ for every $T$.

Let $T$ tend to infinity, we obtain $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

(v) $p_t$ is strictly positive thanks to the strict increasingness of the utility functions.

$\square$

# 4 Physical capital bubble

## 4.1 Definition of physical capital bubble

Let $\left(p_t, r_t, (c_{i,t}, k_{i,t})_{i=1}^m, K_t\right)_{t=0}^{+\infty}$ be an equilibrium.

**Lemma 2.** *For each $t$, we have*

$$1 = (1 - \delta + \rho_{t+1})\gamma_{t+1} \tag{6}$$

*where $\gamma_{t+1} := \max\limits_{i \in \{1,\ldots,m\}} \dfrac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}$ is the discount factor of the economy from date $t$ to date $t+1$, and $\rho_{t+1} = r_{t+1}/p_{t+1}$ is the return (in term of consumption good) of the physical capital at date $t+1$.*

*Proof.* Firstly, we write all FOCs for the economy $\mathcal{E}$. Denote by $\lambda_{i,t}$ the multiplier with respect to the budget constraint of agent $i$ and by $\mu_{t+1}$ the multiplier with respect to the borrowing constraint (i.e., $k_{i,t+1}^T \geq 0$) of agent $i$.

$$\beta_i^t u_i'(c_{i,t}) = \lambda_{i,t} p_t \tag{7}$$

$$\lambda_{i,t} p_t = \lambda_{i,t+1}(r_{t+1} + p_{t+1}(1 - \delta)) + \mu_{i,t+1} \tag{8}$$

$$\mu_{i,t+1} k_{i,t+1} = 0. \tag{9}$$

Therefore, we have $\dfrac{p_{t+1}}{r_{t+1} + p_{t+1}(1 - \delta)} \geq \dfrac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}$ for every $i$.

Since $K_t > 0$ at equilibrium, there exists $i$ such that $k_{i,t+1} > 0$. For such agent, we have $\mu_{i,t+1} = 0$. Thus, $\lambda_{i,t} p_t = \lambda_{i,t+1}(r_{t+1} + p_{t+1}(1 - \delta))$. Consequently, we get (6) □

**Definition 2.** *We define the discount factor of the economy from initial date to date $t$ as follows*

$$Q_0 := 1, \quad Q_t := \prod_{s=1}^t \gamma_s, \quad t \geq 1. \tag{10}$$

According to Lemma 2, we have $Q_t = (1 - \delta + \rho_{t+1})Q_{t+1}$ for every $t \geq 0$. In this framework, the physical capital can be viewed as a long-lived asset whose price (in term of consumption good) at initial date equals 1. If one buys one unit of the physical capital at date 0, he or she will anticipate as follows:

1. At date 1, one unit (from date 0) of this asset will give $(1-\delta)$ units of the physical capital and $\rho_1$ units of consumption good as its dividend. This argument is formalized by $1 = (1-\delta)Q_1 + \rho_1 Q_1$.

2. At date 2, $(1 - \delta)$ units of the physical capital will give $(1 - \delta)^2$ units of the physical capital and $(1 - \delta)\rho_2$ units of consumption good. This argument is formalized by $(1 - \delta)Q_1 = (1 - \delta)^2 Q_2 + (1 - \delta)\rho_2 Q_2$.

These argument are mathematically formalized as follows:

$$
\begin{aligned}
1 &= (1 - \delta + \rho_1)Q_1 = (1 - \delta)Q_1 + \rho_1 Q_1 \\
&= (1 - \delta)(1 - \delta + \rho_2)Q_2 + \rho_1 Q_1 = (1 - \delta)^2 Q_2 + (1 - \delta)\rho_2 Q_2 + \rho_1 Q_1 \\
&= \cdots \\
&= (1 - \delta)^T Q_T + \sum_{t=1}^{T}(1 - \delta)^{t-1}\rho_t Q_t.
\end{aligned}
\tag{11}
$$

Therefore, the fundamental value of capital at date 0 can be defined by

$$
FV_0 = \sum_{t=1}^{\infty}(1 - \delta)^{t-1}\rho_t Q_t.
\tag{12}
$$

**Definition 3.** *We say that there is a capital asset bubble if physical capital's price is greater that its fundamental value, i.e.,* $1 > \sum_{t=1}^{\infty}(1 - \delta)^{t-1}\rho_t Q_t.$

## 4.2 Comparison with litterature

In Araujo, Pascoa, Torres-Martinez (2011), for each equilibrium, they studied bubbles of durable goods and collateralized assets. Their asset pricing conditions (Corollary 1, page 263) are based on the existence of what they called **deflators** and **non-pecuniary returns** which are not unique. They then defined bubbles associated to each deflators and non-pecuniary returns. However, they did not mention how to determine these coefficients. In our framework, for each equilibrium, we have

$$
\lambda_{i,t}p_t = \lambda_{i,t+1}(r_{t+1} + p_{t+1}(1 - \delta)) + \mu_{i,t+1}.
\tag{13}
$$

Therefore, $\lambda_{i,t}$ and $\mu_{i,t+1}$ plays the same roles of deflators and non-pecuniary returns, respectively. By this way, we get

$$
1 = (1 - \delta + \rho_{t+1})\gamma_{i,t+1} + \alpha_{i,t+1}
\tag{14}
$$

where $\gamma_{i,t+1} := \dfrac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}$ and $\alpha_{i,t+1} := \dfrac{\mu_{i,t+1}}{\beta_i^t u_i'(c_{i,t})}$.

Denote $Q_{i,0} := 1$ and $Q_{i,t} := \prod_{s=1}^{t}\gamma_{i,s}$, for $t \geq 1$. In sense of Araujo, Pascoa, Torres-Martinez (2011), the fundamental value of capital with respect to deflators $(\lambda_{i,t})_t$ and non-pecuniary returns $(\alpha_{i,t+1})_t$ is defined by

$$
FV_{i,0} = \sum_{t=1}^{\infty}(1 - \delta)^{t-1}\rho_t Q_{i,t} + \sum_{t=1}^{\infty}(1 - \delta)^{t-1}\alpha_{i,t}Q_{i,t-1}.
\tag{15}
$$

However, this is not our way to define bubble. Instead, our definition of physical capital bubble is based on Lemma 2. If we use terminology of Araujo, Pascoa, Torres-Martinez (2011), our definition of bubble corresponds to bubble in sense of Araujo,

Pascoa, Torres-Martinez (2011) with deflators $\lambda_{i,t}$ where $i \in \arg\max_{i \in \{1,\ldots,m\}} \left( \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} \right)$, and non-pecuniary returns $\alpha_{i,t} = 0$.

In Tirole (1982), Kocherlakota (1992, 2008), Santos and Woodford (1997), Huang and Werner (2000), Le Van and Vailakis (2012), they considered a long-lived asset in general equilibrium models. The structure of such an asset is the following: If one buys 1 unit of this asset at date $t$ with price $q_t$, he or she can resell this asset at date $t+1$ with price $q_{t+1}$ and receive $\xi_{t+1}$ units of consumption good. The dividends ($\xi_t$) are exogenous. They define that there is a bubble if the market price (in term of consumption good), say $q_0$, at date 0 of the asset is greater than its fundamental value, i.e., $q_0 > \sum_{t=1}^{\infty} \Pi_t \xi_t$, where $\Pi_t$ is the discount factor of the economy from initial date to date $t$. We can also define bubble for a long-lived asset which does not give dividends (Tirole , 1985; Ventura , 2012; Farhi and Tirole , 2012). For such an asset, its fundamental value is zero. We say that there is a bubble if the market price of this asset is strictly positive.

The physical capital in our model is also a long-lived asset, which can be resold and gives dividends at each date. However, the difference is that the physical capital is depreciated at every period and its dividends, ($\rho_t$), are endogenous.

## 4.3  The nature of physical capital bubble

One unit of the physical capital at initial date will be depreciated to $(1-\delta)^t$ units of the same asset at date $t$. Hence, $B_t := (1-\delta)^t Q_t$ is the market value of one unit of capital at initial date. Denote $B := \lim_{t \to} B_t$. According to (11), the market price of capital equals its market value plus its fundamental value, that is

$$1 = B + FV_0. \tag{16}$$

We now state our main result in this section.

**Definition 4.** *At equilibrium, we say*

(i) *Capital returns are low if $\sum_{t=1}^{\infty} \rho_t < +\infty$.*

(ii) *The market value of capital is vanished if $\lim_{t \to \infty} (1-\delta)^t Q_t = 0$.*

**Proposition 3.** *The three following statement are equivalent*

(i) *There exists a physical capital bubble.*

(ii) *The market value of capital is not vanished*

(iii) *Capital returns are low.*

*Proof.* According to (16), it is easy to see that (i) is equivalent to (ii).

(ii) is equivalent to (iii): According to (6), we see that $Q_t = (1 - \delta + \rho_{t+1})Q_{t+1}$. Hence, we have

$$
\begin{aligned}
1 &= (1 - \delta + \rho_1)Q_1 = (1 - \delta + \rho_1)(1 - \delta + \rho_2)Q_2 \\
&= \ldots = Q_T \prod_{t=1}^{T}(1 - \delta + \rho_t) = Q_T(1 - \delta)^T \prod_{t=1}^{T}\left(1 + \frac{\rho_t}{1 - \delta}\right).
\end{aligned}
$$

Consequently, $\lim_{t \to \infty}(1 - \delta)^t Q_t > 0$ if and only if $\prod_{t=1}^{\infty}(1 + \frac{\rho_t}{1-\delta}) < +\infty$. This condition is equivalent to (iii). $\qquad\square$

Proposition 3 shows the nature of physical capital bubbles. To see the point, recall that the market value of capital at date $t$ is

$$
(1 - \delta)^t Q_t = \frac{(1 - \delta)^t}{(1 - \delta + \rho_1)\ldots(1 - \delta + \rho_t)}. \tag{17}
$$

$B$ is a function of capital returns, we write $B = B(\rho_1, \rho_2, \ldots)$ It is easy to see that $B$ is a decreasing function in each component. Moreover, we have $\lim_{\rho_1, \rho_2, \ldots \to 0} B = 1$. The lower level of capital returns, the lower present value of the physical capital. So, when capital returns are low, physical capital bubbles appear.

We point out some consequences of Proposition 3. We first consider the case where the technology is stationary.

**Corollary 1.** *Assume that $F_t = F$ for every $t$, $F$ is strictly increasing and concave. Then there is no bubble at equilibrium.*

*Proof.* Case 1: $F'(\infty) \geq \delta$. Therefore, we have $\rho_t \geq F'(K_t) \geq F'(\infty) \geq \delta$ for every $t$. As a result, $\sum_{t=1}^{\infty} \rho_t = \infty$ which implies that bubble is ruled out.

Case 2: $F'(\infty) < \delta$. Since $F$ is strictly increasing and strictly concave, aggregate capital stock is uniformly bounded, i.e., there exists $0 < K < \infty$ such that $K_t \leq K$. Consequently, $\rho_t = F'(K_t) > F'(K) > 0$ for every $t$. This implies that $\sum_{t=1}^{\infty} \rho_t = \infty$. According to Proposition 3, there is no bubble. $\qquad\square$

Corollary 1 is in line with no-bubble result in Becker, Bosi, Le Van and Seegmuller (2014). In Becker, Bosi, Le Van and Seegmuller (2014), they worked with an endogenous labor supply model and needed a specific condition of the production function, under which the capital stocks are uniformly bounded.[5] However, we do not require any specific condition on $F'(\infty)$, and we also allow for AK technology. By the way, their result can be viewed as a particular case of Proposition 3. Note that Becker, Bosi, Le Van and Seegmuller (2014) only gave a sufficient condition for no-bubble.

Let us consider non-stationary linear technologies. The following result shows that the productivity decreases to zero with high speed, a bubble in physical capital will appear.

---

[5] They assumed that the production function $F(K, L)$ such that $\frac{\partial F}{\partial K}(\infty, m) = \frac{\partial F}{\partial L}(1, \infty) = 0$.

**Corollary 2.** *Assume that $F_t(K) = a_t K$ for each $t$. Then there is a bubble at equilibrium if and only if $\sum_{t=1}^{\infty} a_t < \infty$.*

*Proof.* This is a direct consequence of Proposition 3 . □

## 4.4 Physical capital bubble in an optimal growth model

We now consider the particular case of our framework, where there is a unique agent. This agent maximizes her utility $\sum_{t=0}^{+\infty} \beta_i^t u(c_t)$ by choosing sequences of consumption $(c_t)$ and capital $(k_t)$ subject to: for every $t \geq 0$, $k_{t+1} \geq 0$ and

$$c_t + k_{t+1} - (1-\delta)k_t \leq F_t(k_t). \tag{18}$$

It is easy to obtain that $1 = (1 - \delta + \tau_{t+1})\frac{\beta u'(c_{t+1})}{u'(c_t)}$. Then, we have

$$\Gamma_t = (1 - \delta + \tau_{t+1})\Gamma_{t+1}, \tag{19}$$

where $\tau_t := F'_t(k_t)$ and $\Gamma_t := \frac{1}{(1-\delta+\tau_1)\dots(1-\delta+\tau_t)}$.

We define bubble in the same way in Section 4.1. According to Proposition 3, bubble exists if and only if $\lim_{t\to\infty}(1-\delta)^t Q_t > 0$ which is equivalent to $\sum_{t=1}^{\infty} F'_t(k_t) < \infty$. We see that there does not exist physical capital bubble in the standard optimal growth model (i.e., when technology is stationary). However, when we take $F_t(K) = a_t K$ with $\sum_{t=1}^{\infty} a_t < \infty$, and then bubble exists. Thus, there exists bubble even there is a unique agent. This is different from the standard model with pure financial asset as in Kocherlakota (1992, 2008), Santos and Woodford (1997), Huang and Werner (2000), Le Van and Vailakis (2012)

**Remark 2.** *The transversality and the no-bubble condition are different. Indeed, the transversality condition is $\lim_{t\to\infty} \beta^t u'(c_t)k_{t+1} = 0$ which always holds at optimal. The no-bubble condition if $\lim_{t\to\infty}(1-\delta)^t Q_t = 0$ which may be not satisfied at optimal.*

*Note that $Q_t = \frac{\beta^t u'(c_t)}{u'(c_0)}$, and then the no-bubble condition can be rewritten as $\lim_{t\to\infty} \beta^t u'(c_t)(1-\delta)^t = 0$*

## 4.5 Physical capital bubble and the present value of output

A well known result on financial asset bubble is that if the present value of aggregate endowment is finite, there is no financial asset bubble (Santos and Woodford , 1997; Huang and Werner , 2000). In this section, we consider whether there is a relationship between the physical capital bubble and the present value of output. The present value of output is defined by

$$FV = \sum_{t=1}^{\infty} Q_t Y_t, \tag{20}$$

where $Y_t := F_t(K_t) + (1-\delta)K_t$ is the total output at date $t$.

We consider an example.

Let $u_i(c) = ln(c), \beta_i = \beta$ for every $i$, and $F_t(K) = a_t K$ for every $t$. A solution is given by the relationship $k_{i,t+1} = \beta(1 - \delta + a_t)k_{i,t}$.[6] Therefore,

$$k_{i,t} = \beta^t(1 - \delta + a_0)\ldots(1 - \delta + a_{t-1})k_{i,0}.$$

We then have

$$Y_t = (1 - \delta + a_t)K_t = (1 - \delta + a_t)\sum_{i=1}^{m} k_{i,t}$$

$$= \beta^t(1 - \delta + a_0)\ldots(1 - \delta + a_t)K_0 \tag{21}$$

$$Q_t = \frac{1}{(1 - \delta + a_1)\ldots(1 - \delta + a_t)}. \tag{22}$$

The present value of output is finite for every sequence $(a_t)_t$. Indeed,

$$FV = \sum_{t=0}^{\infty} Q_t Y_t = (1 - \delta + a_0)K_0 \sum_{t=1}^{\infty} \beta^t < \infty.$$

According to Corollary, when $\sum_{t=1}^{\infty} a_t = \infty$, there is no physical capital bubble and the present value of output is finite. When $\sum_{t=1}^{\infty} a_t < \infty$, there exists physical capital bubble and the present value of output is still finite.

Thus, there is no causal relationship between physical capital bubble and the fact that the present value of output is finite.

# 5    On the efficiency of equilibria

In this section, we study the efficiency of intertemporal equilibrium. Following Malinvaud (1953), we define the efficiency of a capital path as follows.

**Definition 5.** *Let $F_t$ be a production function, $\delta$ be the capital depreciation rate. A feasible path of capital is a positive sequence $(K_t)_{t=0}^{\infty}$ such that $0 \le K_{t+1} \le F_t(K_t) + (1 - \delta)K_t$ for every $t \ge 0$ and $K_0$ is given. A feasible path is efficient if there is no other feasible path $(K_t')$ such that*

$$F_t(K_t') + (1 - \delta)K_t' - K_{t+1}' \ge F_t(K_t) + (1 - \delta)K_t - K_{t+1}$$

*for every $t$ with strict inequality for some $t$.*

Here, aggregate feasible consumption at date $t$ is defined by $C_t := F_t(K_t) + (1 - \delta)K_t - K_{t+1}$.

**Definition 6.** *We say that an intertemporal equilibrium is efficient if its aggregate feasible capital path $(K_t)$ is efficient.*

---

[6]Indded, the Euler condition $c_{i,t+1} = \beta_i(1 - \delta + a_{t+1})c_{i,t}$ jointly with the budget constraint becomes $k_{i,t+2} - \beta_i(1 - \delta + a_{t+1})k_{i,t+1} = (1 - \delta + a_{t+1})[k_{i,t+1} - \beta_i(1 - \delta + a_t)k_{i,t}]$. Thus, a solution is given by $k_{i,t+1} = \beta(1 - \delta + a_t)k_{i,t}$.

Our main result in this section requires some intermediate steps. First, we have, as in Malinvaud (1953).

**Lemma 3.** *An equilibrium is efficient if* $\lim\limits_{t\to\infty} Q_t K_{t+1} = 0.$

*Proof.* Let $(K'_t, C'_t)$ be a feasible sequence. We have just to show that

$$\liminf_{T\to+\infty} \sum_{t=0}^{T} Q_t \left(C_t - C'_t\right) \geq 0. \tag{23}$$

It is enough to prove that feasibility and first-order conditions imply

$$\sum_{t=0}^{T} Q_t \left(C_t - C'_t\right) \geq -Q_T K_{T+1} \tag{24}$$

Let us prove inequality (24). We have

$$
\begin{aligned}
\Delta_T &\equiv \sum_{t=0}^{T} Q_t \left(C_t - C'_t\right) \\
&= \sum_{t=0}^{T} Q_t \left[F_t\left(K_t\right) - F_t\left(K'_t\right) + (1-\delta)\left(K_t - K'_t\right) - \left(K_{t+1} - K'_{t+1}\right)\right] \\
&\geq \sum_{t=0}^{T} Q_t \left[F'_t\left(K_t\right)\left(K_t - K'_t\right)\right] + (1-\delta)\left(K_t - K'_t\right)\right] - \sum_{t=0}^{T} Q_t \left(K_{t+1} - K'_{t+1}\right) \\
&= \sum_{t=0}^{T} Q_t \left(1 - \delta + \rho_t\right)\left(K_t - K'_t\right) - \sum_{t=0}^{T} Q_t \left(K_{t+1} - K'_{t+1}\right)
\end{aligned}
$$

By noticing that $K_0 = K'_0$ and $Q_{t+1}\left(1 - \delta + \rho_{t+1}\right) - Q_t = 0$, we then get:

$$
\begin{aligned}
\Delta_T &\geq \sum_{t=1}^{T} Q_t \left(1 - \delta + \rho_t\right)\left(K_t - K'_t\right) - \sum_{t=0}^{T} Q_t \left(K_{t+1} - K'_{t+1}\right) \\
&= \sum_{t=0}^{T-1} \left[Q_{t+1}\left(1 - \delta + \rho_{t+1}\right) - Q_t\right]\left(K_{t+1} - K'_{t+1}\right) - Q_T \left(K_{T+1} - K'_{T+1}\right) \\
&\geq \sum_{t=0}^{T-1} \left[Q_{t+1}\left(1 - \delta + \rho_{t+1}\right) - Q_t\right]\left(K_{t+1} - K'_{t+1}\right) - Q_T K_{T+1} \\
&= -Q_T K_{T+1}.
\end{aligned}
$$

$\square$

Since we impose borrowing constraint that is $k_{i,t} \geq 0$ for every $i$ and $t$, we can prove the transversality condition of each agent.

**Lemma 4.** *At any equilibrium, we have* $\lim\limits_{t\to\infty} \beta_i^t u'_i(c_{i,t})k_{i,t+1} = 0$ *for every $i$.*

*Proof.* See Theorem 2.1 in Kamihigashi (2002). □

The following result shows the impact of borrowing constraints on the efficiency of an intertemporal equilibrium.

**Lemma 5.** *Consider an equilibrium. If there exists a date such that, from this date on, the borrowing constraints of agents are not binding at this equilibrium, then it is efficient.*

*Proof.* Assume that there exists $t_0$ such that $k_{i,t} > 0$ for every $i$ and for every $t \geq t_0$. Then we have: for every $t \geq t_0$

$$\frac{Q_t}{Q_{t_0}} = \beta_i^{t-t_0} \frac{u_i'(c_{i,t})}{u_i'(c_{i,t_0})}.$$

According to Lemma 4, we have $\lim_{t\to\infty} \beta_i^t u_i'(c_{i,t}) k_{i,t+1} = 0$. Then $\lim_{t\to\infty} Q_t k_{i,t+1} = 0$ for every $i$. This implies that $\lim_{t\to\infty} Q_t K_{t+1} = 0$. Therefore, this equilibrium is efficient. □

We now state our main finding in this section.

**Proposition 4.** *Assume that the production functions are linear. Then every equilibrium path is efficient.*

*Proof.* Since production functions are linear, profit equals to zero. Recall that we have $c_{i,t} > 0$ for every $i$ and every $t$. This implies that $k_{i,t} > 0$ at equilibrium. According to Lemma 5, every equilibrium path is efficient. □

Our result is different from the one in Cass (1972), Becker and Mitra (2012), Mitra and Ray (2012) in two points: (i) we consider linear technologies (they consider strictly concave production functions), (ii) we do not need that the capital stocks are bounded as in their papers.

Corollary 2 and Proposition 4 indicate that there exists an equilibrium the capital path of which is efficient and a bubble may arise at this equilibrium. Note that, this is not a surprising result since the nature of bubbles is low returns while the nature of efficiency is the distribution of capital.

# 6  Conclusion

We build an infinite-horizon dynamic deterministic general equilibrium model with heterogeneous agents. We proved existence of equilibrium in this model, even if technologies are not stationary and aggregate capital stocks are not uniformly bounded.

At an equilibrium, we define that there is a bubble of physical capital if the physical capital's price is greater than its fundamental value. We pointed out that bubbles exist if and only if the sum (over time) of capital returns is finite. In particular case where the technology is stationary, there is no bubble.

With linear technologies, every intertemporal equilibrium is efficient. Interestingly, it is possible to have both bubble and efficient at equilibrium.

# 7  Appendix: Existence of equilibrium for the truncated economy

**Proof of Proposition 1.** Denote $\Delta := \{z_0 = (p, r) : 0 \le p_t, r_t \le 1, p_t + r_t = 1 \quad \forall t = 0, \ldots, T\}$,

$$
\begin{aligned}
B_i(p, r) := \{ \quad & (c_i, k_i) \in \mathcal{C}_i \times \mathcal{K}_i \text{ such that } : \forall t = 0, \ldots, T \\
& p_t(c_{i,t} + k_{i,t+1} - (1 - \delta)k_{i,t}) < r_t k_{i,t} + \theta^i \pi_t \},
\end{aligned}
$$

and

$$
\begin{aligned}
C_i(p, r) := \{ \quad & (c_i, k_i) \in \mathcal{C}_i \times \mathcal{K}_i \text{ such that } : \forall t = 0, \ldots, T \\
& p_t(c_{i,t} + k_{i,t+1} - (1 - \delta)k_{i,t}) \le r_t k_{i,t} + \theta^i \pi_t \},
\end{aligned}
$$

Denote by $\bar{B}_i(z_0)$ the closure of $B_i(z_0)$.

**Lemma 6.** *For every* $(p, r) \in \mathcal{P}$, *we have* $B_i(p, q) \ne \emptyset$ *and* $\bar{B}_i(p, q) = C_i(p, q)$.

*Proof.* We rewrite $B_i(p, r)$ as follows

$$
\begin{aligned}
B_i(p, r) := \{ (c_i, a_i) \in \mathcal{C}_i \times \mathcal{A}_i \text{ such that } : \forall t = 0, \ldots, T \\
0 < p_t((1 - \delta)k_{i,t} - c_{i,t} - k_{i,t+1}) + r_t k_{i,t} + \theta^i \pi_t \}.
\end{aligned}
$$

Since $(1 - \delta)k_{i,0} > 0$, we can choose $c_{i,0} \in (0, B_c)$ and $k_{i,1} \in (0, B_k)$ such that

$$
0 < p_0((1 - \delta)k_{i,0} - c_{i,0} - k_{i,1}) + r_0 k_{i,0} + \theta^i \pi_0.
$$

By induction, we see that $B_i(p, r)$ is not empty. $\qquad\square$

**Lemma 7.** $B_i(p, r)$ *is a lower semi-continuous correspondence on* $\mathcal{P} := \Delta^{T+1}$. *And* $C_i(p, r)$ *is upper semi-continuous on* $\mathcal{P}$ *with compact convex values.*

*Proof.* Clearly, since $B_i(p, r)$ is empty and has an open graph. $\qquad\square$

We define $\Phi := \Delta \times \prod_{i=1}^{m}(\mathcal{C}_i \times \mathcal{K}_i) \times \mathcal{K}$. An element $z \in \Phi$ is in the form $z = (z_i)_{i=0}^{m+1}$ where $z_0 := (p, r), z_i := (c_i, k_i)$ for each $i = 1, \ldots, m$, and $z_{m+1} = K$.

We now define correspondences. First, we define $\varphi_0$ (for additional agent 0)

$$
\varphi_0 : \prod_{i=1}^{m}(\mathcal{C}_i \times \mathcal{K}_i) \times \mathcal{K} \to 2^{\Delta}
$$

$$
\begin{aligned}
\varphi_0((z_i)_{i=1}^{m+1}) := \arg\max_{(p,r)\in\Delta} \Big\{ \sum_{t=0}^{T} p_t \Big( \sum_{i=1}^{m}[c_{i,t} + k_{i,t+1} - (1 - \delta)k_{i,t}] - F_t(K_t) \Big) \\
+ \sum_{t=0}^{T} r_t \Big( K_t - \sum_{i=1}^{m} k_{i,t} \Big) \Big\}.
\end{aligned}
$$

For each $i = 1, \ldots, m$, we define

$$\varphi_i : \qquad \Delta \to 2^{\mathcal{C}_i \times \mathcal{K}_i}$$

$$\varphi_i(p, r) \quad := \quad \underset{(c_i, k_i) \in C_i(p, r)}{\arg \max} \left\{ \sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}) \right\}.$$

For each $i = m + 1$, we define

$$\varphi_{m+1} : \qquad \Delta \to 2^{\mathcal{K}}$$

$$\varphi_i(p, r) \quad := \quad \underset{K \in \mathcal{K}}{\arg \max} \left\{ \sum_{t=0}^{T} p_t F_t(K_t) - r_t K_t \right\}.$$

**Lemma 8.** $\varphi_i$ *is upper semi-continuous convex-valued correspondence for each* $i = 0, 1, \ldots, m + 1$.

*Proof.* This is a direct consequence of the Maximum Theorem. $\qquad\square$

According to the Kakutani Theorem, there exists $(\bar{p}, \bar{r}, (\bar{c}_i, \bar{k}_i)_{i=1}^{m}, \bar{K})$ such that

$$(\bar{p}, \bar{r}) \in \varphi_0((\bar{c}_i, \bar{k}_i)_{i=1}^{m}, \bar{K}) \tag{25}$$

$$(\bar{c}_i, \bar{k}_i) \in \varphi_i(\bar{p}, \bar{r}) \tag{26}$$

$$\bar{K} \in \varphi_{m+1}(\bar{p}, \bar{r}). \tag{27}$$

Denote by $\bar{X}_t := \sum_{i=1}^{m} [\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1 - \delta)\bar{k}_{i,t}] - F_t(\bar{K}_t)$ and $\bar{Y}_t = \bar{K}_t - \sum_{i=1}^{m} \bar{k}_{i,t}$ the excess demands for goods and capital respectively. For every $(p, r) \in \Delta^{T+1}$, we have

$$\sum_{t=0}^{T} (p_t - \bar{p}_t)\bar{X}_t + \sum_{t=0}^{T} (r_t - \bar{r}_t)\bar{Y}_t \leq 0. \tag{28}$$

By summing the budget constraints, for each $t$, we get

$$\bar{p}_t \bar{X}_t + \bar{r}_t \bar{Y}_t \leq 0. \tag{29}$$

Hence, we have: for every $(p_t, r_t) \in \Delta$

$$p_t \bar{X}_t + q_t \bar{Y}_t \leq \bar{p}_t \bar{X}_t + \bar{r}_t \bar{Y}_t \leq 0. \tag{30}$$

Therefore, we have $\bar{X}_t, \bar{Y}_t \leq 0$, which implies that

$$\sum_{i=1}^{m} \bar{c}_{i,t} + \bar{k}_{i,t+1} \leq (1 - \delta) \sum_{i=1}^{m} \bar{k}_{i,t} + F_t(\bar{K}_t) \tag{31}$$

$$\bar{K}_t \leq \sum_{i=1}^{m} \bar{k}_{i,t}. \tag{32}$$

**Lemma 9.** $\bar{p}_t, \bar{r}_t > 0$ *for* $t = 0, \ldots, T$.

*Proof.* If $\bar{p}_t = 0$ then $\bar{c}_{i,t} = B_c > (1-\delta)B + F_t(B)$. Therefore, we get $\bar{c}_{i,t} + \bar{k}_{i,t+1} > (1-\delta)\sum_{i=1}^{m} \bar{k}_{i,t} + F_t(\bar{K}_t)$ which is a contradiction. Hence, $\bar{p}_t > 0$.

If $\bar{r}_t = 0$, then the optimality of $\bar{K}$ implies that $K_t = B$. However, we have $\bar{k}_{i,t} \leq B_k$ for every $i,t$. Consequently, $\sum_{i=1}^{m} \bar{k}_{i,t} \leq mB_k < B = K_t$, contradiction to (32). Therefore, we get $\bar{r}_t > 0$. $\qquad\square$

**Lemma 10.** $\sum_{i=1}^{m} \bar{k}_{i,t} = \bar{K}_t$ *and* $\sum_{i=1}^{m}[\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}] = F(\bar{K}_t)$

*Proof.* Since prices are strictly positive and the utility functions are strictly increasing, all the budget constraints are binding and, summing them across the individuals, we get

$$\bar{p}_t \bar{X}_t + \bar{r}_t \bar{Y}_t = 0. \tag{33}$$

We know that $\bar{X}_t, \bar{Y}_t \leq 0$ and $\bar{p}_t, \bar{r}_t > 0$. Then, $\bar{X}_t = \bar{Y}_t = 0$. The optimality of $(\bar{c}_i, \bar{k}_i)$ and $\bar{K}$ comes from (26) and (27). $\qquad\square$

$\qquad\square$

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