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Intertemporal equilibrium with heterogeneous agents, endogenous dividends, and borrowing constraints*

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Abstract

We build dynamic general equilibrium models with heterogeneous producers and financial market imperfections. First, we prove the existence of equilibrium. Second, we investigate the role of financial market imperfection in growth and land prices. Third, we introduce land dividends, then define and study land bubbles as well as individual land bubbles.

Keywords: Infinite horizon, general equilibrium, financial market imperfection, land bubbles.

JEL Classifications: C62, D53, D9, E44, G10.

1 Introduction

This paper analyzes the impact of the financial market on the economic activities and study land bubbles with endogenous dividends. We do so by using infinite-horizon deterministic general equilibrium models with finitely heterogeneous agents and an imperfect financial market. There are three assets: a consumption good, land used to produce consumption good, and a financial asset with zero supply.

There are a finite number m of agents who differs on initial resource, borrowing limit, preference, and production functions.[1] At each date, each agent is endowed a positive amount of consumption good. Agents can use land to produce consumption good by using their own technology.[2] Agents can also invest by buying the financial

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[1]In our model, agents can be interpreted as producers.

[2]We do not take into account the physical capital in this paper. It would be a potential research in the future.

asset. In the spirit of Geanakoplos and Zame (2002), Kiyotaki and Moore (1997), we assume that each agent i can borrow from the financial asset but they must hold land which is used as collateral. The repayment of agent i does not exceed a fraction $f_i \leq 1$ of the income coming from her land.[3]

Before explore equilibrium analysis, we prove the existence of equilibrium. We cannot directly use the method of Becker, Bosi, Le Van, and Seegmuller (2015) and Le Van and Pham (2015) because the financial asset in our model is a short-lived asset with zero supply while the financial asset in Le Van and Pham (2015) is a long-lived asset bringing strictly positive exogenous dividend. The challenge is to prove that the financial asset volume held by each agent is bounded. To overcome this difficulty, we introduce an intermediate economy with a nominal asset whose structure is different from that of the financial asset in the original economy. In this intermediate economy, we can bound the volume of the financial asset, and so can prove the existence of equilibrium by adapting the method of Becker, Bosi, Le Van, and Seegmuller (2015) and Le Van and Pham (2015). Last, we construct an equilibrium of the original economy from an equilibrium of the intermediate economy. Our method can be used for a large class of general equilibrium models.

We introduce the new concept *the dividends of land*, denoted by (d_t) which are endogenously determined the following asset pricing equation

$$\frac{q_t}{p_t} = \gamma_{t+1}\left(\frac{q_{t+1}}{p_{t+1}} + d_{t+1}\right) \tag{1}$$

where q_t, p_t is the price of land and consumption good at date t, and γ_{t+1} is the endogenous discount factor of the economy from date t to date $(t+1)$.

In our framework, land plays 3 different roles: (1) it can be sold, (2) it produces fruits (consumption good), and (3) it can be used as collateral to borrow. The land dividend represents two latter roles of land. By the way we can interpret that land can be resold and bring dividend at each date. The land dividend is proved to be greater than the lowest marginal productivity but less than the highest marginal productivity. Note that when the production function of any agent is identical and linear, we recover the Lucas tree with exogenous dividend.

We then analyze the impact of the financial market on economic activities. The production activity here is to produce consumption good by using land. Since any agent can produce, there is a competition on the financial market. Note that in our framework lenders and borrowers are endogenously identified. We point out several points.

First, we show that if any agent produces and the financial system is good enough (in the sense that any agent can totally access to the financial market, i.e. $f_i = 1$ for any i) then the marginal productivities of agents are the same and equal to the dividend of land. However, there may be some cases where some agents do not produce. This happens under two conditions: (1) the productivity of these agents are very low, and (2) agents having high productivity can access totally to the credit market.

[3]This income consists the value of land and consumption good produced by using land.

Second, we prove that if the marginal productivity of some agent is strictly greater than the land dividend, this agent will borrow until her borrowing constraint is binding.

Last, we give an analysis at the steady state in a simple case. In the long run the most patient agent may not hold the entire land stock. This is different from Becker and Mitra (2012) where they prove the Ramsey's conjecture that in the long run the most patient agent holds the entire capital stock. The reason for this difference is that each agent in our model is a producer while in Becker and Mitra (2012) there is a unique firm and all consumers rent capital to this firm.

The last part of the paper focuses on land bubbles. In the standard literature, given an asset, the asset bubble exists if the equilibrium price of this asset is strictly greater than its fundamental value. However these two variables are endogenous. The equilibrium price is determined by the market clearing condition. The more critical concept is the fundamental value of an asset. Considering an asset bringing exogenous dividends, Kocherlakota (1992), Santos and Woodford (1997) define the fundamental value of this asset is the sum (over time) of discounted values of dividends. Different from this literature, land in our paper is used by agents to produce consumption good and the sequence of land dividends (d_t) is endogenously determined by the asset pricing equation (1). The fundamental value of land is then defined by the sum of discounted value of land dividends. As discussed above, land dividends represent the two roles of land: land is used to produce consumption good and as collateral to borrow, hence the fundamental value of land represents the value of these two roles. We say that land bubbles exist if the equilibrium price of land is strictly greater than its fundamental value.

Our definition of the land bubble contributes to the literature studying bubbles of an asset with endogenous dividends. Let us mention two approaches among others.

The first one introduced by Miao and Wang (2012, 2015) where they consider bubbles on the firm value with endogenous dividends. They decompose the value of a firm $V(K)$ having K units of capital at the beginning into two parts: $V(K) = QK+B$ where Q represents Tobin's marginal Q, Q is endogenous. They interpret QK as the fundamental value of the firm and B as bubbles of the firm value.

The second one is the concept *physical capital bubble* introduced by Becker, Bosi, Le Van, and Seegmuller (2015). They define the fundamental value of physical capital is the sum of discounted value of capital returns (after having been depreciated), then they say that physical capital bubbles exist if the equilibrium price of physical capital is strictly greater than its fundamental value. Like physical capital, land in our framework is also used to produce consumption. However Becker, Bosi, Le Van, and Seegmuller (2015) assume that there is a unique representative firm while any agent in our framework can be viewed as a firm.

We investigate whether land bubbles exist. To answer this question, we start by studying transversality conditions (TVCs). We prove two different kinds of TVCs. The first one determined with respect to individual discount factors (or individual expected interest rates) represents the optimality of agents. By contrast the second one determined with respect to discount factors of the economy (or interest rates of the economy) shows the behavior of borrowing constraints.

We then demonstrate that land bubbles are ruled out if borrowing constraints of any agent are not binding from some date. In other words, bubbles only arise if borrowing constraints of some agent are binding infinitely many dates. The intuition is that when borrowing constraints are not binding, the discount factors of any agents coincide with the discount factor of the economy. In this case, no-bubble condition is equivalent to no Ponzi scheme. Since TVCs are satisfied, so is no-bubble condition.

Our next finding is to show that when the borrowing limit of any agent equals 1 (the maximum level), endowments are uniformly bounded from above, and technologies are stationary, there is no land bubble. Our result suggests that rational land bubble can not appear in a bounded economy (an economy where any fundamental variable is uniformly bounded from above and away from zero) with a good financial system (in the sense that $f_i = 1$ for any i). We also give some examples of bubble where $f_i = 0$, endowments may tend to infinity, and/or technologies are not stationary. The two key points of these examples are that (1) there is a fluctuation on borrowing constraints of agents, and (2) land dividends are low with respect to endowments.

Last we introduce the concept *individual bubble* and show the connection between bubble and individual bubbles. Consider an agent, say i. We take her marginal productivity as individual land dividend, then use her individual discount factor to define the individual fundamental value with respect to this agent. We say that individual bubbles with respect to (w.r.t.) agent i (for short we write $i-$bubbles) exist if the market price of land is strictly greater than the individual fundamental value of land w.r.t. agent i. We find out three points. First, if $i-$bubbles exist for some agent i then land bubbles exist, the converse is not true. Second, if the individual fundamental value of each agent equals the fundamental value then there is no land bubble. Third, there always exists an agent i such that i-bubbles are ruled out. Our finding suggests that the way we choose the discount factors to evaluate the fundamental value plays a very important role on asset bubbles.

The paper is organized as follows. Section 2 presents our framework and provides some preliminary equilibrium properties. Section 3 investigates the role of the financial market. Section 4 studies land bubbles. Section 5 concludes.

All formal proofs are gathered in Appendices 6 and 7.

2 The framework

This is an infinite horizon general equilibrium model without uncertainty. The time is discrete and run from date 0 (initial date) to infinity. There are a finite number of agents. Let us denote I the set of agents.

Consumption good. There is a single consumption good. At each period $t = 0, 1, 2, \ldots, \infty$, the price of consumption good is denoted by p_t and agent i consumes $c_{i,t}$ units of consumption good. Each agent i is endowed $e_{i,t}$ units of consumption good.

Land. The total supply of exogenous land is L and its price at date t is denoted by q_t. At date t, if agent i buys $l_{i,t+1}$ units of land with price q_t then: (1) on the one

hand, agent i uses this land to produce $F_i(l_{i,t+1})$ units of consumption good, (2) on the other hand, agent i can resell land with price q_{t+1}.

The financial market only opens from the initial date. R_t is the gross return at date t. If agent i buys $a_{i,t}$ units of financial asset at date $t-1$ then agent i receives $R_t a_{i,t}$ at date t. In this framework, agents can also get credit from financial market. However, if agents want to borrow, they are required to hold land as collateral. We will discuss borrowing constraints below.

Each household i takes the sequence of prices $(p, q, R) := (p_t, q_t, R_t)_{t=0}^{\infty}$ as given and chooses sequences of consumption, land, and asset volume $(c_i, l_i, a_i) := (c_{i,t}, l_{i,t+1}, a_{i,t+1})_{t=0}^{+\infty}$ in order to maximizes her intertemporal utility

$$(P_i(p, q, R)) : \quad \max \Big[\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t}) \Big] \tag{2}$$

subject to, for each t,

$$l_{i,t+1} \geq 0 \tag{3}$$

$$p_t c_{i,t} + q_t l_{i,t+1} + p_t a_{i,t+1} \leq p_t e_{i,t} + q_t l_{i,t} + p_t F_i(l_{i,t}) + R_t a_{i,t} \tag{4}$$

$$R_{t+1} a_{i,t+1} \geq -f_i \big[q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1}) \big] \tag{5}$$

where $l_{i,0} > 0$ are given. We assume that there is no debt before the opening of the financial market, that is $a_{i,0} = 0$.

Borrowing constraint (5) means that agent i can borrow an amount but the repayment of this amount does not exceed a fraction of the incoming coming from her land. This fraction is f_i which is set by law and less than 1 in order to ensure that the incoming coming from land of each agent is greater than its debt. f_i is the borrowing limit of agent i.

Condition (5) can also be viewed as a collateral constraint: agents can borrow but they must have land as collateral. By the way, our model is also related to the literature on general equilibrium with collateral constraints (Geanakoplos and Zame, 2002; Kiyotaki and Moore, 1997). Our framework is also related to land-price dynamics (Liu, Wang, and Zha, 2013).

Remark 1. *In Kiyotaki and Moore (1997), they considered two types of agents: farmer and gatherer whose time preferences are $\beta < \beta'$. Farmer has a linear production function and gatherer has a decreasing return to scale production function. They assumed that $\frac{R_t}{p_t}$ be exogenous and equal to the $\frac{1}{\beta'}$. They also assumed that $f_i = 1$ for every i.*

Remark 2. *Our model is related to the one in Farhi and Tirole (2012). The difference is that we consider dynamic firms in an infinite-horizon GE model while they consider firms living for 3 periods in an OLG model.*

The economy, denoted by $\mathcal{E}$, is characterized by a list

$$\mathcal{E} := (u_i, \beta_i, e_i, f_i, l_{i,0}, F_i).$$

Definition 1. *A list* $\Big(\bar{p}_t, \bar{q}_t, \bar{R}_t, (\bar{c}_{i,t}, \bar{l}_{i,t+1}, \bar{a}_{i,t+1})_{i=1}^m\Big)_{t=0}^{+\infty}$ *is an equilibrium of the economy* $\mathcal{E}$ *if the following conditions are satisfied:*

(i) Price positivity: $\bar{p}_t, \bar{q}_t, \bar{R}_{t+1} > 0$ *for* $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$,

$$\begin{aligned} \textit{good}: \quad & \sum_{i=1}^m \bar{c}_{i,t} = \sum_{i=1}^m (e_{i,t} + F_i(\bar{l}_{i,t})) \\ \textit{land}: \quad & \sum_{i=1}^m \bar{l}_{i,t} = L \\ \textit{financial asset}: \quad & \sum_{i=1}^m \bar{a}_{i,t} = 0. \end{aligned}$$

(iii) Agents' optimality: for each i, $(\bar{c}_{i,t}, \bar{l}_{i,t+1}, \bar{a}_{i,t+1})_{t=0}^{\infty}$ *is a solution of the problem* $(P_i(\bar{p}, \bar{q}, \bar{R}))$.

The financial asset in our framework is a short-lived asset with zero supply, which is different from the long-lived asset bringing exogenous positive dividends in Kocherlakota (1992), Santos and Woodford (1997), Le Van and Pham (2015).

2.1 The existence of equilibrium

In what follows, if we do not explicitly mention, we work under the following assumptions.

Assumption 1. *For each* i, *the function* F_i *is concave, continuously differentiable,* $F_i' > 0$ *and* $F_i(0) = 0$.

Assumption 2. $l_{i,0} > 0$ *for any* i, *and* $e_{i,t} > 0$ *for any* i *and for any* t.

Assumption 3. $f_i > 0$ *for any* i.

Assumption 4. *For each* i, *the function* u_i *is continuously differentiable, concave,* $u_i' > 0$, $u_i'(0) = \infty$.[4]

Assumption 5. *For each* i

$$\sum_{t=0}^{\infty} \beta_i^t u_i(W_t) < \infty, \tag{6}$$

where $W_t := \sum_{i=1}^m \Big(e_{i,t} + F_i(L)\Big)$.

[4]In the proof of the existence of equilibrium, we do not require $u_i'(0) = \infty$. This condition is to ensure that $c_{i,t} > 0$ for any t, which is used in next Sections.

Proposition 1. *Under the above assumptions, there exists an equilibrium.*

We cannot directly use the method of Becker, Bosi, Le Van, and Seegmuller (2015) and Le Van and Pham (2015) because the financial asset in our model is a short-lived asset with zero supply. The difficulty is to prove that the asset volume $a_{i,t}$ is bounded. To overcome this difficulty, we introduce an intermediate economy with a nominal asset whose structure is different from that of the financial asset in the original economy. In this intermediate economy, we can bound the volume of the financial asset, and so can prove the existence of equilibrium by adapting the method of Becker, Bosi, Le Van, and Seegmuller (2015) and Le Van and Pham (2015): (1) we prove the existence of equilibrium for each $T-$ truncated economy $\mathcal{E}^T$; (2) we show that this sequence of equilibria converges for the product topology to an equilibrium of our economy $\mathcal{E}$.

Last, we construct an equilibrium for the original economy from an equilibrium of the intermediate economy.

Let us introduce the intermediate economy $\tilde{\mathcal{E}}$ as follows. We only change the structure of the financial asset. We consider a nominal asset b with the sequence of returns $(r_t)_{t\geq 1}$. In this economy, each household i takes the sequence of prices $(p,q,r) = (p_t,q_t,r_t)_{t=0}^{\infty}$ as given and chooses sequences of consumption, land, and asset volume $(c_i,l_i,b_i) := (c_{i,t},l_{i,t+1},b_{i,t+1})_{t=0}^{+\infty}$ in order to maximizes her intertemporal utility $\sum\limits_{t=0}^{+\infty}\beta_i^t u_i(c_{i,t})$ subject to sequences of budget and borrowing constraints. Her maximization problem is

$$(\tilde{P}_i(p,q,r)): \quad \max\Big[\sum_{t=0}^{+\infty}\beta_i^t u_i(c_{i,t})\Big] \tag{7}$$

subject to, for each t,

$$l_{i,t+1} \geq 0 \tag{8}$$

$$p_t c_{i,t} + q_t l_{i,t+1} + b_{i,t+1} \leq p_t e_{i,t} + q_t l_{i,t} + p_t F_i(l_{i,t}) + r_t b_{i,t} \tag{9}$$

$$r_{t+1} b_{i,t+1} \geq -f_i\big[q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1})\big] \tag{10}$$

where $l_{i,0} \geq 0$ is given and there is no debt before the opening of the financial market, that is $b_{i,0} = 0$.

Since $l_{i,t} \leq L$ for any i,t, we can use (9) and the induction argument, to prove that the asset volume $b_{i,t}$ is bounded from above if p_t, q_t, r_t are bounded from above.

Definition 2. *A list* $\Big(\bar{p}_t,\bar{q}_t,\bar{r}_t,(\bar{c}_{i,t},\bar{l}_{i,t+1},\bar{b}_{i,t+1})_{i=1}^m\Big)_{t=0}^{+\infty}$ *is an equilibrium of the economy* $\tilde{\mathcal{E}}$ *if the following conditions are satisfied:*

(i) $\bar{p}_t,\bar{q}_t,\bar{r}_{t+1} \in (0,+\infty)$ *for any* $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$,

$$\begin{aligned} \textit{good}: &\quad \sum_{i=1}^{m} \bar{c}_{i,t} = \sum_{i=1}^{m} (e_{i,t} + F_i(\bar{l}_{i,t})) \\ \textit{land}: &\quad \sum_{i=1}^{m} \bar{l}_{i,t} = L \\ \textit{financial asset}: &\quad \sum_{i=1}^{m} \bar{b}_{i,t} = 0. \end{aligned}$$

(iii) Agents' optimality: for each i, $(\bar{c}_{i,t}, \bar{l}_{i,t+1}, \bar{b}_{i,t+1})_{t=0}^{\infty}$ is a solution of the problem $(\tilde{P}_i(\bar{p}, \bar{q}, \bar{r}))$.

By changing variables ($a_{i,t} := b_{i,t}/p_{t-1}, R_t := r_t p_{t-1}$), it is easy to prove the following result.

Lemma 1. *If $\left(\bar{p}_t, \bar{q}_t, \bar{r}_t, (\bar{c}_{i,t}, \bar{l}_{i,t+1}, \bar{b}_{i,t+1})_{i=1}^{m}\right)_{t=0}^{+\infty}$ is an equilibrium for the economy $\tilde{\mathcal{E}}$ then $\left(\bar{p}_t, \bar{q}_t, \bar{R}_t, (\bar{c}_{i,t}, \bar{l}_{i,t+1}, \bar{a}_{i,t+1})_{i=1}^{m}\right)_{t=0}^{+\infty}$, where $\bar{a}_{i,t} := \bar{b}_{i,t}/\bar{p}_{t-1}, \bar{R}_t := \bar{r}_t \bar{p}_{t-1}$, is an equilibrium for the economy $\mathcal{E}$.*

See Appendix 7 for a proof of the existence of equilibrium for the economy $\tilde{\mathcal{E}}$. Note that in this proof, we allow for non-stationary production functions. However, in this paper (except Section 4.3.1), we assume that the technology is stationary for the sake of simplicity.

2.2 Preliminary properties of equilibrium

Let $\left(p, q, R, (c_i, l_i, a_i)_{i=1}^{m}\right)$ be an equilibrium.

We write all first order conditions (henceforth FOCs) for the economy $\mathcal{E}$. Denote by $\lambda_{i,t}$ the multiplier with respect to the budget constraint of agent i and by $\eta_{i,t+1}$, $\mu_{i,t+1} \geq 0$ the multipliers with respect to constraints (3), (5). We have

$$\beta_i^t u_i'(c_{i,t}) = \lambda_{i,t} p_t \tag{11}$$

$$\lambda_{i,t} p_t = (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} \tag{12}$$

$$\lambda_{i,t} q_t = (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})) + \eta_{i,t+1} \tag{13}$$

$$\eta_{i,t+1} l_{i,t+1} = 0 \tag{14}$$

$$\mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i\big[q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1})\big]\Big) = 0 \tag{15}$$

Remark 3. *Since we allow for $F_i'(0) < \infty$, there may be some agent who do not use land to produce.*

We define $\gamma_{i,t+1}$ the individual discount factor of agent i from date t to date $t+1$ and $Q_{i,t}$ the discount factor of agent i from initial date to date t

$$\gamma_{i,t+1} := \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}, \quad Q_{i,0} := 1, \quad Q_{i,t} := \gamma_{i,1} \cdots \gamma_{i,t} = \frac{\beta_i^t u_i'(c_{i,t})}{u_i'(c_{i,0})}. \tag{16}$$

The following transversality condition is one of fundamental results of our paper.

Proposition 2 (Transversality condition)**.** *We have*

$$\lim_{t\to\infty} Q_{i,t}(\frac{q_t}{p_t}l_{i,t+1} + a_{i,t+1}) = 0 \tag{17}$$

$$\infty > \sum_{t=0}^{\infty} Q_{i,t}c_{i,t} = \big(F_i(l_{i,0}) + \frac{q_0}{p_0}l_{i,0}\big) + \sum_{t=0}^{\infty} Q_{i,t}e_{i,t}$$
$$+ \sum_{t=1}^{\infty} Q_{i,t}\Big(1 + f_i\frac{\mu_{i,t}}{\lambda_{i,t}}\Big)\Big(F_i(l_{i,t}) - l_{i,t}F_i'(l_{i,t})\Big). \tag{18}$$

Interpretation. We see that $\sum_{t=1}^{\infty} Q_{i,t}\Big(1 + f_i\frac{\mu_{i,t}}{\lambda_{i,t}}\Big)\Big(F_i(l_{i,t}) - l_{i,t}F_i'(l_{i,t})\Big)$ is the total gain of the productive process of agent i, which is divided in two parts:

- gain from the pure production process: $\sum_{t=1}^{\infty} Q_{i,t}\Big(F_i(l_{i,t}) - l_{i,t}F_i'(l_{i,t})\Big)$.

- gain from the financial market (due to the financial market's imperfection): $\sum_{t=1}^{\infty} Q_{i,t}f_i\frac{\mu_{i,t}}{\lambda_{i,t}}\Big(F_i(l_{i,t}) - l_{i,t}F_i'(l_{i,t})\Big)$.

Lemma 2. *Assume that $f_i > 0$ for any i. Then, we have*

$$\frac{p_{t+1}}{R_{t+1}} = \max_{i\in\{1,\dots,m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}. \tag{19}$$

We define the discount factor γ_{t+1} of the economy from date t to $t+1$, and the discount factor Q_t of the economy from initial date to date t as follows

$$\gamma_{t+1} := \max_{i\in\{1,\dots,m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}, \quad Q_0 := 1, \quad Q_t := \gamma_1 \dots \gamma_t, \forall t \geq 1. \tag{20}$$

Note that $\gamma_{i,t} \leq \gamma_t$ for any t and for any i.

We rewrite borrowing constraint (5) as

$$Q_{t+1}\frac{R_{t+1}}{p_{t+1}}a_{i,t+1} \geq -f_i Q_{t+1}\big[\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1})\big]. \tag{21}$$

According to definition of (Q_t) and (19), we see that $Q_t = \frac{R_{t+1}}{p_{t+1}}Q_{t+1}$. Therefore, (5) is equivalent to

$$Q_t a_{i,t+1} \geq -f_i Q_{t+1}\big[\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1})\big] \tag{22}$$

Proposition 3 (Fluctuation of borrowing constraints)**.** *1. For each i, there are only 2 cases*

(a) there does not exist $\lim_{t\to\infty}\Big(Q_t a_{i,t+1} + f_i Q_{t+1}(\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1}))\Big)$.

(b) $\lim_{t\to\infty}\Big(Q_t a_{i,t+1} + f_i Q_{t+1}(\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1}))\Big) = 0$.

2. We have, for each i,

$$\liminf_{t\to\infty}\Big(Q_t a_{i,t+1} + f_i Q_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} l_{i,t+1} + F_i(l_{i,t+1})\big)\Big) = 0. \tag{23}$$

Remark 4. *There are two kinds of transversality conditions. The first one is (17) which is determined by the individual discount factor $\frac{\beta_i^t u_i'(c_{i,t})}{u_i'(c_{i,0})}$. The second one is (23) based on the economy's discount factor Q_t.*

Remark 5. *All results in this Section still hold for non-stationary production functions.*

3 The role of the financial market

For each $t \geq 0$, we define

$$\underline{d}_{t+1} := \min_{i\in\{1,\dots,m\}} F_i'(l_{i,t+1}), \quad \bar{d}_{t+1} := \max_{i\in\{1,\dots,m\}} F_i'(l_{i,t+1}). \tag{24}$$

We have the following result

Lemma 3. *We have*

$$\gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + \underline{d}_{t+1}\big) \leq \frac{q_t}{p_t} \leq \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + \bar{d}_{t+1}\big). \tag{25}$$

According to Lemma 3, we introduce the land dividends as follows.

Definition 3. *(**Dividends of land**)*
We define dividends of land $(d_t)_t$ by

$$\frac{q_t}{p_t} = \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + d_{t+1}\big). \tag{26}$$

Note that dividends of land are endogenously determined. In our framework, the land's structure is the following: if agents buy land at date t, they can: (1) resell land at date $t+1$, (2) use land as collateral in order to borrow from financial market and (3) receive an amount of consumption good from their production process. Definition 3 states that two latter roles of land can be represented by dividends of land. By the way, we interpret that land can be resold and give dividends at each date. Condition (26) can be viewed as an asset pricing equation or non-arbitrage condition.

According to (25), we see that land dividend d_t is greater than the lowest marginal productivity $\underline{d}_t$ but less than the highest marginal productivity $\bar{d}_t$.

In what follows we will show other properties of land's dividends.

Proposition 4. *We have $d_{t+1} = \bar{d}_{t+1}$ if $f_i = 1$ for any i or (5) is not binding for any i.*

We can interpret $f_i = 1$ as a total access of agent i to credit market. Proposition 4 indicates that the land's dividend equals the highest marginal productivity if any one can totally access to credit market or borrowing constraints of any agent are not binding.

This following result shows that if every agent buys land, then the land's dividend equals the lowest marginal productivity.

Proposition 5. *Consider the date t. Assume that $l_{i,t+1} > 0$ for every i. We then have,*

$$\frac{q_t}{p_t} = \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + \underline{d}_{t+1}\big). \tag{27}$$

And then, we have $d_{t+1} = \underline{d}_{t+1}$.

We point out some consequences of Propositions 4 and 5.

Corollary 1. *Assume that $F_i'(0) = +\infty$ for every i. We have $d_t = \underline{d}_t$ for any t.*

Corollary 2. *(Equal marginal productivities)*
Assume that $\mu_{i,t} = f_i\mu_{i,t}$ and $l_{i,t} > 0$. We have $F_i'(l_{i,t}) = d_t$ for such i.

Corollary 3. *Assume that $f_i = 1$ and $F_i'(0) = \infty$ for any i. We have $d_t = F_i'(l_{i,t})$ for every i, t.*

3.1 Who buys land? Who needs credits?

In this section, we examine which agents produce and which agents need to borrow.

Let us start by the following result.

Proposition 6. *If $l_{i,t+1} > 0$ then $F_i'(l_{i,t+1}) \geq d_{t+1}$.*
If $F_i'(l_{i,t+1}) > d_{t+1}$ then borrowing constraint (5) of agent i is binding.

The first statement indicates that if an agent buys land, its marginal productivity must be greater than land dividends.

The second one shows that if an agent has marginal productivity which is strictly greater than land dividend, she will borrow until her borrowing constraints are binding. In other word this agent needs credit.

The following result shows that agents having low productivity do not buy land to produce.

Proposition 7. *Consider the agent i. Assume that there exists an agent j such that $f_j = 1$ and $F_i'(0) < F_j'(L)$. We have $l_{i,t} = 0$ for every t.*

We note that Proposition 7 holds whatever the form of the utility function and the size of the discount rate β_i. We now point out some implications of Proposition 7.

This result is in line with Proposition 1 in Le Van and Pham (2015) where they proved that no one invests in the productive sector if the productivity of this sector is too low.

We can interpret $f_j = 1$ as a total access of agent j to credit market. In this case, any agent, say i, having low productivity so that $F_i'(0) < F_j'(L)$ never produce.

If we interpret agents as countries, our economy becomes the world economy with free trade. Each country i is endowed $l_{i,0}$ units of land. Our result indicates that when the trade is totally free and the international financial market is good enough (in the sense that $f_i = 1$ for any i), countries with low productivity never produce and land in these countries will be held by other countries with high productivity.

However, when there exist financial (or political) frictions characterized by $f_j < 1$, the analysis becomes more complex.

3.2 A steady state analysis

In this section, we assume that no agent holds endowment, that is $e_{i,t} = 0$ for every i, t. For simplicity, we also assume that there are two agents, say i and j, with different rates of time preference $\beta_i < \beta_j$.

We will give steady state analysis.

Lemma 4. *Assume that $e_{i,t} = 0$ for any i, t and $F_i(l_i) = A_i l_i^{\alpha}$, where $\alpha \in (0, 1)$, for any i. Assume that there are two agents i and j with $\beta_i < \beta_j$. Then there is a unique steady state (up to scalar of prices) determined by*

$$\frac{R}{p} = \frac{1}{\beta_j} \tag{28}$$

$$\left(\frac{q}{p}\right)^{\frac{1}{1-\alpha}} L = \left(\frac{\alpha A_i}{\frac{1}{\beta_i + f_i(\beta_j - \beta_i)} - 1}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\alpha A_j}{\frac{1}{\beta_j} - 1}\right)^{\frac{1}{1-\alpha}} \tag{29}$$

$$l_i = \left(\frac{\alpha A_i}{\frac{1}{\beta_i + f_i(\beta_j - \beta_i)} - 1} \frac{p}{q}\right)^{\frac{1}{1-\alpha}}, \quad l_j = L - l_i \tag{30}$$

$$R a_i + f_i\big[q l_i + p F_i(l_i)\big] = 0, \quad a_i + a_j = 0. \tag{31}$$

Who will own land in the long-run?

With Cobb-Douglas production functions, we see that $l_i > 0$ and $l_j > 0$. Each agent holds a strictly positive amount of land to produce them-self. This is different from the result in Becker and Mitra (2012) where the most patient agent holds the entire stock of capital in the long-run.

There are two reasons for this difference.

First, in Becker and Mitra (2012) there is a unique representative firm and consumers cannot produce while agents in our framework can produce by using their technologies and they can be viewed as heterogeneous credit-constrained firms.

Second, capital returns in Becker and Mitra (2012) are determined by the marginal productivity of the representative firm while in our framework land dividends which can be interpreted as land returns are determined by "non-arbitrage" condition (26).

The role of borrowing limit

Corollary 4. *The price of land $\frac{q}{p}$ increases in f_i.*

The intuition is that when f_i increases, agent i can borrow more, and then land demand will increase which do increase the price of land.

Corollary 5. *The output in the long run $Y := F_i(l_i) + F_j(l_j)$ is increasing in borrowing limit f_i*

The reason is intuitive: the higher level of f_i, the more quantity agent with higher productivity can borrow, and therefore the more output is produced.

4 Land bubbles

We know that $Q_{t+1} = \gamma_{t+1} Q_t$. Combining with (26), we get

$$Q_t \frac{q_t}{p_t} = Q_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + d_{t+1}\big). \tag{32}$$

We have

$$\begin{aligned} \frac{q_0}{p_0} &= \gamma_1(\frac{q_1}{p_1} + d_1) = Q_1 d_1 + Q_1 \frac{q_1}{p_1} \\ &= Q_1 d_1 + Q_1 \gamma_2 (\frac{q_2}{p_2} + d_2) = Q_1 d_1 + Q_2 d_2 + Q_2 \frac{q_2}{p_2} \\ &= \ldots = \sum_{t=1}^{T} Q_t d_t + Q_T \frac{q_T}{p_T}, \forall T \geq 1. \end{aligned} \tag{33}$$

Definition 4. *The fundamental value of the land is defined by*

$$FV_0 := \sum_{t=1}^{\infty} Q_t d_t \tag{34}$$

Interpretation. As discussed above, land dividends represent the two roles of land: land is used to produce consumption good and as collateral to borrow, hence the fundamental value of land represents the value of these two roles.

Definition 5. *(**bubble**)*
We say that land bubbles exist if the market price (in term of consumption good) of the land is greater than its fundamental value, i.e., $\dfrac{q_0}{p_0} > FV_0$.

As in Montrucchio (2004), Le Van, Pham, and Vailakis (2014) we have

Proposition 8. *The three following statement are equivalent*

(i) Land bubbles exist.

(ii) $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} > 0.$

(iii) $\sum_{t=1}^{\infty} \frac{p_t d_t}{q_t} < +\infty.$

Since $d_t \geq \min_i F_i'(l_{i,t}) \geq \min_i F_i'(L) > 0$ for every t, we have

Corollary 6. *If land bubble exists, we have* $\sum_{t=1}^{\infty} \frac{p_t}{q_t} < +\infty$.

We see here that the existence of land bubbles implies that the land price tends to infinity. However this fact only holds for the stationary technology. In Section 4.3.1, we will discuss more about this issue.

Interest rates and bubbles

According to (19), we see that $Q_t = \frac{p_1}{R_1}\frac{p_2}{R_2}\cdots\frac{p_t}{R_t}$. According to (33), we have $\sum_{t=1}^{\infty} Q_t d_t \leq q_0/p_0 < \infty$. We also have $d_t \geq \min_i F_i'(l_{i,t}) \geq \min_i F_i'(L) > 0$ for every t, hence we get $\sum_{t=1}^{\infty} Q_t < \infty$.

We write $\gamma_t = \frac{p_t}{R_t} = \frac{1}{1+\rho_t}$ where ρ_t is interpreted as the real interest rate of the economy at date t. Note that these interest rates may be negative. The condition $\sum_{t=1}^{\infty} Q_t < \infty$ is rewritten as follows

$$\sum_{t=0}^{\infty} \frac{1}{\prod_{s=1}^{t}(1+\rho_s)} < \infty. \tag{35}$$

We can interpret that real interest rates are not "too low". We also see that there exists an infinite sequence of date $(t_n)_n$ such that $\rho_{t_n} > 0$ for any n.

According to Proposition 8, land bubbles exist if and only if

$$\lim_{t\to\infty} \frac{1}{\prod_{s=1}^{t}(1+\rho_s)} \frac{q_t}{p_t} > 0. \tag{36}$$

This condition implies that there exists the following limit

$$\lim_{t\to\infty} \frac{\frac{q_{t+1}}{p_{t+1}}}{\frac{q_t}{p_t}} \frac{1}{1+\rho_{t+1}} = 1. \tag{37}$$

We obtain that if land bubbles exist, in the long-run the rate of growth of land prices is equal to the gross interest rate.

4.1 No-bubble results

Proposition 9. *Assume that for each i, $Q_t/Q_{i,t}$ is uniformly bounded from above. Then there is no bubble.*

We write $\gamma_{i,t} = \frac{1}{1+\rho_{i,t}}$, where ρ_t is interpreted as the the real expected interest rate of agent i at date t. Note that these interest rates may be negative. According to Proposition 9, if bubble exists there exists an agent i such that her expected interest rates are high w.r.t. those of the economy in the sense that

$$\prod_{s=1}^{T} \frac{1+\rho_{i,t}}{1+\rho_t} \xrightarrow{T\to\infty} \infty.$$

We point out some consequences of Proposition 9

Corollary 7. *Assume that there exists $T > 0$ such that $\mu_{i,t+1} = 0$ for any $t \geq T$ and for any i. Then there is no land bubble.*

The intuition of this result is that when $\mu_{i,t+1} = 0$ for any $t \geq T$ and for any i, the discount factors of any agents coincide with the discount factors of the economy. In this case, no-bubble condition is equivalent to no Ponzi scheme. Since TVCs are satisfied, so is no-bubble condition.

Proposition 7 implies that if borrowing constraints of all agents are not binding, then there is no bubble. In other words, we have

Corollary 8. *If land bubbles exist, there exists an agent i and an infinite sequence of dates $(t_n)_n$ such that borrowing constraints of agent i are binding at each date t_n, i.e, for any t_n*

$$R_{t_n} a_{i,t_n} = -f_i\big[q_{t_n} l_{i,t_n} + p_{t_n} F_i(l_{i,t_n})\big]. \tag{38}$$

This result is mentioned in Kocherlakota (1992) where he considers the borrowing constraints: $x_{i,t} \geq x$ where $x_{i,t}$ is the asset quantity held by agent i at date t and $x \leq 0$ is an exogenous bound. He claims that $\liminf_{t\to\infty}(x_{i,t} - x) = 0$ and interprets this means that borrowing constraints of agent i are frequently binding.

We define the aggregate output of the economy at date t by $Y_t := \sum_{i=1}^{m}(e_{i,t} + F_i(l_{i,t}))$. We then define the present value of the aggregate output by

$$\sum_{t=0}^{\infty} Q_t Y_t. \tag{39}$$

We prepare our main result in this section by following claims whose proofs are presented in Appendix 6.

Lemma 5. *If $\sup_{i,t} e_{i,t} < \infty$ and technologies are stationary, the present value of the aggregate output is finite.*

Lemma 6. *Assume that* $\sup_{i,t} e_{i,t} < \infty$ *and technologies are stationary. Given an equilibrium, we have that* $Q_t\big(\frac{q_t}{p_t} l_{i,t+1} + a_{i,t+1}\big)$ *is uniformly bounded from below and from above as well.*

Lemma 7. *Assume that* $\sup_{i,t} e_{i,t} < \infty$ *and technologies are stationary. Given an equilibrium, there exist the following limits.*

$$\lim_{t\to\infty} Q_t\big(\frac{q_t}{p_t} l_{i,t+1} + a_{i,t+1}\big) = \lim_{t\to\infty} \big(Q_t \frac{q_t}{p_t} l_{i,t} + Q_{t-1} a_{i,t}\big) \tag{40}$$

for each i.

Lemma 8. *Assume that* $\sup_{i,t} e_{i,t} < \infty$ *and technologies are stationary. Given an equilibrium, if there exists* T *such that*

$$f_i\big(\frac{q_t}{p_t} + \frac{F_i(l_{i,t})}{l_{i,t}}\big) l_{i,t} \geq \big(\frac{q_t}{p_t} + d_t\big) l_{i,t} \tag{41}$$

for every $t \geq T$. *Then* $\lim_{t\to\infty} Q_t\big(a_{i,t+1} + \frac{q_t}{p_t} l_{i,t+1}\big) \leq 0$.

By using these above results, we obtain the main result of this section.

Proposition 10. *Assume that* $\sup_{i,t} e_{i,t} < \infty$, *technologies are stationary, and* $f_i = 1$ *for every* i. *Then, land bubbles are ruled out at equilibrium.*

Our result indicates that there is no land bubble at equilibrium when the financial system is good enough (in the sense that $f_i = 1$ for any i), exogenous endowments are bounded from above, and the technology is stationary. We also notice that our result still holds for any technology with the form $A_{i,t} F_i$ where $A_{i,t}$ is bounded from above and away from zero for any i.

Proposition 10 is in line with the result in Kocherlakota (1992), Santos and Woodford (1997), Huang and Werner (2000), Le Van, Pham, and Vailakis (2014), where they prove that bubbles are ruled out if the present value of the aggregate endowment is finite. Indeed, the asset in Kocherlakota (1992) is a particular case of the land in our model for the case where $F_{i,t}(X) = \xi_t X$ for any X. Proposition 10 also shows that land bubbles are ruled out in model in Kiyotaki and Moore (1997).

Proposition 10 suggests that land bubbles only appear when technologies are non-stationary or agents cannot easily access to financial market, i.e., $f_i < 1$. We will present some examples of bubbles in Section 4.3.1, where these conditions are violated.

4.2 Bubbles vs $i-$bubbles

According to (13), we have

$$\frac{q_t}{p_t} = \frac{\lambda_{i,t+1} p_{i,t+1}}{\lambda_{i,t} p_t}\big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big) + \frac{f_i \mu_{i,t+1} p_{t+1}}{\lambda_{i,t} p_t}\big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big) + \frac{\eta_{i,t+1}}{\lambda_{i,t} p_t} \tag{42}$$

We then use (11) and (12) to get that

$$\frac{q_t}{p_t} = \gamma_{i,t+1}\big(\frac{q_{t+1}}{p_{t+1}} + d_{i,t+1}\big) \tag{43}$$

where

$$d_{i,t+1} := \Big(F_i'(l_{i,t+1}) + \frac{\eta_{i,t+1}}{\lambda_{i,t+1}p_{t+1}}\Big) + f_i\big(\frac{\gamma_{t+1}}{\gamma_{i,t+1}} - 1\big)\big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big) \tag{44}$$

We call $d_{i,t+1}$ *the individual dividend of agent i at date $t+1$.* $d_{i,t+1}$ is divided into two terms. The first term $X_{i,t+1} := F_i'(l_{i,t+1}) + \dfrac{\eta_{i,t+1}}{\lambda_{i,t+1}p_{t+1}}$ represents the return from the production process. Note that $X_{i,t+1}l_{i,t+1} = F_i'(l_{i,t+1})l_{i,t+1}$. The second term $f_i\big(\frac{\gamma_{t+1}}{\gamma_{i,t+1}} - 1\big)\big(\dfrac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big)$ can be interpreted as the collateral return. Note that the collateral return is equal to zero if $f_i = 0$ or the discount factors of agent i and of the economy are identical.

The asset pricing equation (43) shows the way that agent i evaluates the price of land. With the individual discount factor $\gamma_{i,t+1}$, once agent i buys land she will be able to resell land with price q_{t+1} and expect to receive $d_{i,t+1}$ units of consumption good as dividends. Since the individual discount factor $\gamma_{i,t+1}$ is less than that of economy γ_{t+1}, the individual dividend $d_{i,t+1}$ expected by agent i is greater than the dividend d_{t+1} of the economy.

According to (43) and by using the same argument in (33), we obtain

$$\frac{q_0}{p_0} = \sum_{t=1}^{T} Q_{i,t}d_{i,t} + Q_{i,T}\frac{q_T}{p_T}, \forall T \geq 1. \tag{45}$$

Definition 6. *(**individual-bubble**) $FV_i := \sum_{t=1}^{\infty} Q_{i,t}d_{i,t}$ is called the i-fundamental value of land. We say that i-land bubbles exist if $q_0/p_0 > \sum_{t=1}^{\infty} Q_{i,t}d_{i,t}$.*

The concept $i-$bubble is related to bubbles of durable goods and collateralized assets in Araujo, Pascoa, and Torres-Martinez (2011). Given an equilibrium, Araujo, Pascoa, and Torres-Martinez (2011) provide asset pricing conditions (Corollary 1, page 263) based on the existence of what they call *deflators* and *non-pecuniary returns* which are not necessarily unique. They then defined bubbles associated to each deflators and non-pecuniary returns. In our framework, for each equilibrium, we give closed formulas for two types of deflators. They are (γ_t) and $(\gamma_{i,t})$ which we call discount factors and individual discount factors respectively.

By using the same argument in Proposition 8, we obtain the following result.

Proposition 11. *The following statements are equivalent*

(i) i-land bubbles exist.

(ii) $\lim_{t\to\infty} Q_{i,t}\dfrac{q_t}{p_t} > 0.$

(iii) $\sum_{t=1}^{\infty} \frac{p_t d_{i,t}}{q_t} < +\infty$.

We now state our results showing the connection between two concepts: bubble and $i-$bubble.

Proposition 12. *There exists an agent i such that $i-$bubble is ruled out.*

Proposition 13.

1. *If i-land bubbles exist for some agent i then land bubbles exist.*
2. *$FV_0 \leq FV_i$ for any i. Moreover, if $FV_0 = FV_i$ for any i then $FV_0 = FV_i = q_0/p_0$ for any i, i.e. there is neither bubble nor $i-$bubble.*

Note that in Section 4.3.1 we present an example where $i-$bubble does not exist for any i while bubble may arise.

We next introduce the following concept.

Definition 7 (Strong bubble). *We say that strong bubble exists if the asset price is strictly greater than every individual value of land, i.e. $q_0 > \max_i FV_i$.*

It is easy to see that if strong land bubbles exist then $i-$bubble exists for any i. However this is impossible because of Proposition 12. In other words, we have the following result.

Proposition 14. *Strong land bubbles are ruled out.*

Our concept strong bubble is related to the concept *speculative bubble* in Werner (2014) where he considers an asset bringing exogenous dividends in a model with ambiguity. Werner (2014) defines the asset's fundamental value under agent's i beliefs is the sum of discounted expected future dividends under her beliefs. He then says that speculative bubble exists if the asset price is strictly higher than every agent's fundamental value.

The readers may ask why strong bubbles are ruled out while speculative bubbles in Werner (2014) may exist. It is hard to compare these two results since the two concepts of bubbles are defined in two different frameworks (with and without ambiguity). Another reason may be the linearity of utility functions: In our paper, we impose Inada condition for utility functions to ensure that $c_{i,t}$ is strictly positive at equilibrium while Werner (2014) works with linear utility function. If we consider linear utility functions, the ratio $\frac{p_{i,t+1}\lambda_{i,t+1}}{p_{i,t}\lambda_{i,t}}$ may be higher than $\frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} = \beta_i$. As a result, it is easy to have a situation where the asset price q_0/p_0 may be greater than $\sum_{t=1}^{\infty} \beta_i^t d_{i,t}$ which can be interpreted as the fundamental value with respect to agent i. Hence strong bubbles may exist. To sum up, in the theoretical point of view, the existence of bubbles depends on the way we define the asset's fundamental value which is an ambiguous concept.

4.3 Land bubbles when the economy has no financial sector

We focus on the case where there is no financial market. In this section, we allow for non-stationary production functions. We rewrite the problem of each agent. The household i takes the sequence of prices $(p,q) = (p_t, q_t)_{t=0}^{\infty}$ as given and chooses sequences of consumption and land $(c_i, l_i) := (c_{i,t}, l_{i,t+1})_{t=0}^{+\infty}$ in order to maximize her intertemporal utility

$$(P_i(p,q)): \quad \max \Big[\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t}) \Big] \tag{46}$$

$$\text{subject to, for each } t, : \quad l_{i,t+1} \geq 0 \tag{47}$$

$$p_t c_{i,t} + q_t l_{i,t+1} \leq p_t e_{i,t} + q_t l_{i,t} + p_t F_{i,t}(l_{i,t}), \tag{48}$$

where $l_{i,0} > 0$ is given.

Definition 8. *A list* $\Big(\bar{p}_t, \bar{q}_t, (\bar{c}_{i,t}, \bar{l}_{i,t+1})_{i=1}^m\Big)_{t=0}^{+\infty}$ *is an equilibrium of the economy without financial market if the following conditions are satisfied:*

(i) Price positivity: $\bar{p}_t, \bar{q}_t > 0$ *for* $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$,

$$\textit{good}: \quad \sum_{i=1}^m \bar{c}_{i,t} = \sum_{i=1}^m (e_{i,t} + F_{i,t}(\bar{l}_{i,t}))$$

$$\textit{land}: \quad \sum_{i=1}^m \bar{l}_{i,t} = L.$$

(iii) Agents' optimality: for each i, $(\bar{c}_{i,t}, \bar{l}_{i,t+1})_{t=0}^{\infty}$ *is a solution of the problem* $(P_i(\bar{p}, \bar{q}))$.

If we consider the linear technology $F_{i,t}(x) = \xi_t x$ for every i, the structure of land becomes the same structure of asset in Kocherlakota (1992), Santos and Woodford (1997), Huang and Werner (2000). If $F_i = 0$ for every i, the land becomes pure bubble asset as in Tirole (1985). In this case, the fundamental value of this asset is zero.

Let $\Big(p, q, (c_i, l_i)_{i=1}^m\Big)$ be an equilibrium.

Firstly, we write all FOCs for the economy $\mathcal{E}$. Denote by $\lambda_{i,t}$ the multiplier with respect to the budget constraint of agent i and by $\mu_{i,t+1}$ the multiplier with respect to the borrowing constraint $l_{i,t+1} \geq 0$

$$\beta_i^t u_i'(c_{i,t}) = \lambda_{i,t} p_t \tag{49}$$

$$\lambda_{i,t} q_t = \lambda_{i,t+1}(q_{t+1} + p_{t+1} F_{i,t+1}'(l_{i,t+1})) + \mu_{i,t+1} \tag{50}$$

$$\mu_{i,t+1} l_{i,t+1} = 0. \tag{51}$$

We also define dividends of land as follows.

$$\frac{q_t}{p_t} = \gamma_{t+1}\Big(\frac{q_{t+1}}{p_{t+1}} + d_{t+1}\Big). \tag{52}$$

where γ_{t+1} is the discount factor of the economy from date t to date $t+1$: $\gamma_{t+1} := \max_{i\in\{1,\dots,m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}$. We define the discount factor of the economy from initial date to date t as follows: $Q_0 := 1, Q_t := \prod_{s=1}^{t} \gamma_s, \forall t \geq 1$. Then the fundamental value of the land is defined by $FV_0 := \sum_{t=1}^{\infty} Q_t d_t$. We say that land bubbles exist $q_0/p_0 > FV_0$.

4.3.1 Examples of land bubbles

We now construct equilibria with bubbles. We assume that there are two agents A and B. with non-stationary technologies $F_{A,t}(L) = A_t L, F_{B,t}(L) = B_t L$, and utility functions $u_A(x) = ln(x) = u_B(x)$.

Let $e_{A,2t} = e_{B,2t+1} = 0$ for any t and the land supply $L = 1$.

We need the following conditions to ensure FOCs and to identify what are the discount factors of the economy (γ_t).

$$\beta_A(\frac{\beta_B e_{B,2t}}{1+\beta_B} + A_{2t})(\frac{\beta_A e_{A,2t+1}}{1+\beta_A} + A_{2t+1}) < \beta_B \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t+1}}{1+\beta_A} \tag{53}$$

$$\beta_B(\frac{\beta_A e_{A,2t-1}}{1+\beta_A} + B_{2t-1})(\frac{\beta_B e_{B,2t}}{1+\beta} + B_{2t}) < \beta_A \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t-1}}{1+\beta_A} \tag{54}$$

$$\beta_A(\frac{\beta_B e_{B,2t}}{1+\beta_B} + A_{2t})(\frac{\beta_A e_{A,2t+1}}{1+\beta_A} + B_{2t+1}) \leq \beta_B \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t+1}}{1+\beta_A} \tag{55}$$

$$\beta_B(\frac{\beta_A e_{A,2t-1}}{1+\beta_A} + B_{2t-1})(\frac{\beta_B e_{B,2t}}{1+\beta} + A_{2t}) \leq \beta_A \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t-1}}{1+\beta_A}. \tag{56}$$

Note that these conditions are satisfied if $\beta_A = \beta_B = \beta$, and $A_{2t}, B_{2t} < \frac{(1-\beta)e_{B,2t}}{1+\beta}$ and $A_{2t+1}, B_{2t+1} < \frac{(1-\beta)e_{A,2t+1}}{1+\beta}$ for any t.

We will construct an equilibrium $\left(p_t, q_t, (c_{i,t}, l_{i,t+1})_{i\in I}\right)_t$ as follows. Let us normalize by setting $p_t = 1$ for any t.

Allocations: for each $t \geq 0$,

$$l_{A,2t} = L, l_{A,2t+1} = 0, \quad l_{B,2t} = 0, l_{B,2t+1} = L \tag{57}$$

$$c_{A,2t} = q_{2t}L + A_{2t}L, \quad c_{B,2t} + q_{2t}L = e_{B,2t} \tag{58}$$

$$c_{A,2t+1} + q_{2t+1}L = e_{A,2t+1}, \quad c_{B,2t+1} = q_{2t+1}L + B_{2t+1}L. \tag{59}$$

The land prices are given by

$$q_{2t} = \frac{\beta_B}{1+\beta_B}e_{B,2t}, \quad q_{2t+1} = \frac{\beta_A}{1+\beta_A}e_{A,2t+1}. \tag{60}$$

In Appendix 6.3.1, we check that this is an equilibrium. In this example, we have

$$\gamma_{2t+1} = \frac{\beta_B u_B'(c_{B,2t+1})}{u_B'(c_{B,2t})}, \quad d_{2t+1} = B_{2t+1} \tag{61}$$

$$\gamma_{2t} = \frac{\beta_A u_A'(c_{A,2t})}{u_A'(c_{A,2t-1})}, \quad d_{2t} = A_{2t}. \tag{62}$$

According to Proposition 8, land bubbles exist if and only if $\sum_{t=0}^{\infty} \frac{d_t}{q_t} < \infty$ which is equivalent to

$$\sum_{t=0}^{\infty} \frac{A_{2t}}{e_{B,2t}} + \sum_{t=0}^{\infty} \frac{B_{2t+1}}{e_{A,2t+1}} < \infty. \tag{63}$$

It means that land dividends are low with respect to endowments. The intuition of the existence of bubble in our example is the following: agent B does not have endowment at the date $2t+1$ and has logarithm utility (which satisfies Inada condition). This agent needs to smooth her consumption, and hence she accepts to buy land with price $q_{2t+1} \geq \frac{\beta_A}{1+\beta_A} e_{A,2t+1}$ which does not depend on the productivity of agents. The lower productivity, the lower dividend, the lower the fundamental value of land. As a consequence, when dividends tend to zero, the land price will be higher than the land's fundamental value.

Remark 6 (bubble vs $i-$bubble). *For each $i = A, B$ we can verify that $\lim_{t\to\infty} \beta_i^t u_i'(c_{i,t}) q_t = 0$. It means that $i-$bubble does not exist for any i. However bubble may arise.*

Corollary 9. *Consider our example. Assume that $A_t = B_t = A$ for any t. Then land bubbles exist if and only if*

$$\sum_{t=0}^{\infty} \frac{1}{e_{B,2t}} + \sum_{t=0}^{\infty} \frac{1}{e_{A,2t+1}} < \infty. \tag{64}$$

Corollary 9 illustrates Proposition 10. Here with stationary production function and $f_i = 0$ for any i, land bubbles may appear if endowments tend to infinity. In this example, we see that land bubbles exist if and only if $\sum_{t=1}^{\infty} 1/q_t < \infty$. By the way, this result also illustrates Corollary 6.

Corollary 10. *Consider our example. Assume that $e_{A,2t+1} = e_{B,2t} = e > 0$ for any t. Then land bubbles exist if and only if*

$$\sum_{t=0}^{\infty} (A_{2t} + B_{2t+1}) < \infty. \tag{65}$$

This result is consistent with the one in Le Van, Pham, and Vailakis (2014) where they give an example of financial asset bubble under condition that the sum of exogenous dividends is less than 1. Our result is also related to Bosi, Le Van and Pham (2014) where they show that physical capital bubbles arise if the sum of capital returns is finite.

Land bubbles and prices

Corollary 6 points out that with stationary technologies, if land bubbles exist then the land price must tend to infinity. However, in our example with non-stationary

technologies, the land prices are given by $q_{2t} = q_{2t+1} = \frac{\beta}{1+\beta} e_{t+1}$. We see that the land prices can be increasing or decreasing over time and bubbles exist. Our result is in line with Weil (1990) where he gives an example of bubble with decreasing asset prices. His model is a particular case of our model (when we take $A_t = B_t = 0$ for any $t \geq T$, i.e. land will not give fruits from some date).

Pure bubbles

We consider $A_t = B_t = 0$. In this case, the fundamental value of land is zero and an equilibrium is bubbly if the prices of land are strictly positive, i.e., $q_t > 0$ for any t. This is called pure bubble (Tirole, 1985). We see that in our example there exists an equilibrium with pure bubbles.

5 Conclusion

In our general equilibrium models with many agents, we discussed about the role of financial market on the economy. Agents with high productivity would borrow until their borrowing constraints bind and produce while agents with low productivity lend and do not produce. In the long-run the most patient may not hold the entire of land stock.

We defined the endogenous land dividends and studies land bubbles. Under standard assumptions, land bubbles are ruled out if the borrowing limit of any agent equals 1 and the production functions are stationary. When no one can borrow land bubbles may arise. Note that land bubbles may exist whenever land prices increase or decrease.

We also point out that the existence of bubbles of an asset depends not only on the structure of this asset but also the way we choose the discount factors to evaluate the fundamental value of the asset.

6 Appendix: formal proofs

6.1 Proofs for Section 2.2

Proof of Lemma 2. According to (12), we have $\lambda_{i,t} p_t \geq \lambda_{i,t+1} R_{t+1}$ for every i.

Since $f_i > 0$ for any i, it is easy to see that there exists an agent i whose borrowing constraint (5) is not binding. Thus $\mu_{i,t+1} = 0$ which implies that $\lambda_{i,t} p_t = \lambda_{i,t+1} R_{t+1}$. As a result, we get (19). □

Proof of Proposition 2. Denote $x_i := (l_i, a_i) = (l_{i,t}, a_{i,t})_t$. We say x_i is feasible if for every t we have $l_{i,t} \geq 0$ and

$$R_t a_{i,t} \geq -f_i(q_t l_{i,t} + p_t F_i(l_{i,t})) \tag{66}$$

$$q_t l_{i,t+1} + p_t a_{i,t+1} \leq p_t e_{i,t} + q_t l_{i,t} + p_t F_i(l_{i,t}) + R_t a_{i,t}. \tag{67}$$

We claim that: if x_i is feasible then $(x_{i,0}, \ldots, x_{i,t}, \lambda x_{i,t+1}, \lambda x_{i,t+2}, \ldots)$ is also feasible for each $t \geq 1$ and $\lambda \in [0, 1)$.

We have to prove that:

$$q_t\lambda l_{i,t+1} + p_t\lambda a_{i,t+1} \leq p_t e_{i,t} + q_t l_{i,t} + p_t F_i(l_{i,t}) + R_t a_{i,t} \tag{68}$$

and

$$\lambda q_s l_{i,s+1} + \lambda p_s a_{i,s+1} \leq p_s e_{i,s} + \lambda q_s l_{i,s} + p_s F_i(\lambda l_{i,s}) + \lambda R_s a_{i,t} \tag{69}$$

$$\lambda R_s a_{i,s+1} + f_i(\lambda q_{s+1} l_{i,s+1} + p_{s+1} F_i(\lambda l_{i,s+1})) \geq 0 \tag{70}$$

for each $s \geq t$.

(69) and (70) are proved by using the fact that $F_i(\lambda x) > \lambda F_i(x)$ for every $\lambda \in [0,1)$. (68) is clear if $q_t l_{i,t+1} + p_t a_{i,t+1} < 0$. If $q_t l_{i,t+1} + p_t a_{i,t+1} \geq 0$, we have

$$q_t\lambda l_{i,t+1} + p_t\lambda a_{i,t+1} \leq q_t l_{i,t+1} + p_t a_{i,t+1} \leq p_t e_{i,t} + q_t l_{i,t} + p_t F_i(l_{i,t}) + R_t a_{i,t}. \tag{71}$$

By using the same argument in Theorem 2.1 in Kamihigashi (2002),[5] we obtain that $\limsup_{t\to\infty} \lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1}) \leq 0$.

We now have

$$\lambda_{i,t}(p_t c_{i,t} + q_t l_{i,t+1} + p_t a_{i,t+1}) = \lambda_{i,t}(p_t e_{i,t} + q_t l_{i,t} + p_t F_i(l_{i,t}) + R_t a_{i,t}) \tag{72}$$

$$\lambda_{i,t} p_t a_{i,t+1} = (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} a_{i,t+1} \tag{73}$$

$$\lambda_{i,t} q_t l_{i,t+1} = (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})) l_{i,t+1} \tag{74}$$

$$\mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i(q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1}))\Big) = 0 \tag{75}$$

According to (73) and (74), we have

$$\begin{aligned}
&\lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1}) && (76)\\
&= (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})) l_{i,t+1} + (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} a_{i,t+1} \\
&= \lambda_{i,t+1} R_{t+1} a_{i,t+1} + \lambda_{i,t+1}\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1} \\
&+ \mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1}\Big) && (77)
\end{aligned}$$

Therefore, combining with (75), we get

$$\begin{aligned}
&\lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1}) - \lambda_{i,t+1} R_{t+1} a_{i,t+1} - \lambda_{i,t+1}\big(q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1})\big) \\
&= \lambda_{i,t+1}\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1} + \mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1}\Big) \\
&\quad - \lambda_{i,t+1}\big(q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1})\big) && (78)\\
&= -\lambda_{i,t+1} p_{t+1}(F_i(l_{i,t+1}) - l_{i,t+1} F_i'(l_{i,t+1})) + \mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1}\Big) \\
&- \mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i\big(q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1})\big)\Big) && (79)\\
&= -\lambda_{i,t+1} p_{t+1}(F_i(l_{i,t+1}) - l_{i,t+1} F_i'(l_{i,t+1})) - f_i \mu_{i,t+1} p_{t+1}(F_i(l_{i,t+}) - l_{i,t+1} F_i'(l_{i,t+1})) \\
& && (80)
\end{aligned}$$

[5] In Kamihigashi (2002), he worked with positive allocations. Our model is different from that in Kamihigashi (2002) because $a_{i,t}$ may be negative.

By summing (72) from $t = 0$ to T, and then using (78), we obtain that

$$\sum_{t=0}^{T} \lambda_{i,t} p_t c_{i,t} + \lambda_{i,T}(q_t l_{i,T+1} + p_T a_{i,T+1}) = \sum_{t=0}^{T} \lambda_{i,t} p_t e_{i,t} + \lambda_{i,0}(q_0 l_{i,0} + p_t F_i(l_{i,0}) + R_0 a_{i,0}) + \sum_{t=1}^{T} p_t(\lambda_{i,t} + f_i \mu_{i,t})(F_i(l_{i,t}) - l_{i,t} F_i'(l_{i,t})). \tag{81}$$

Under Assumption (5), the utility of agent i is finite, we have that

$$\sum_{t=0}^{\infty} \lambda_{i,t} p_t c_{i,t} = \sum_{t=0}^{\infty} \beta_i^t u_i'(c_{i,t}) c_{i,t} \leq \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) < \infty. \tag{82}$$

Combining with $\limsup_{t\to\infty} \lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1}) \leq 0$, we obtain that there exists the sum

$$\sum_{t=0}^{\infty} \lambda_{i,t} p_t e_{i,t} + \sum_{t=1}^{\infty} p_t(\lambda_{i,t} + f_i \mu_{i,t})(F_i(l_{i,t}) - l_{i,t} F_i'(l_{i,t})) < \infty.$$

We now use (81) to get that $\lim_{t\to\infty} \lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1})$ exists and it is non positive.

We again use (72) and note that $q_t l_{i,t} + p_t F_i(l_{i,t}) + R_t a_{i,t} \geq 0$ (because of borrowing constraint) to obtain that $\liminf_{t\to\infty} \lambda_{i,t}(p_t c_{i,t} + q_t l_{i,t+1} + p_t a_{i,t+1}) \geq 0$.

(82) implies that $\lim_{t\to\infty} \lambda_{i,t} p_t c_{i,t} = 0$. As a result, we get $\liminf_{t\to\infty} \lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1}) \geq 0$. Therefore, we have $\lim_{t\to\infty} \lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1}) = 0$ and then

$$\infty > \sum_{t=0}^{\infty} \lambda_{i,t} p_t c_{i,t} = \sum_{t=0}^{\infty} \lambda_{i,t} p_t e_{i,t} + \sum_{t=1}^{\infty} p_t(\lambda_{i,t} + f_i \mu_{i,t})(F_i(l_{i,t}) - l_{i,t} F_i'(l_{i,t})) + \lambda_{i,0}(q_0 l_{i,0} + p_t F_i(l_{i,0}) + R_0 a_{i,0}). \tag{83}$$

By combining with the fact that $Q_{i,t} = \dfrac{\beta_i^t u_i'(c_{i,t})}{u_i'(c_{i,0})} = \dfrac{\lambda_{i,t} p_t}{\lambda_{i,0} p_0}$, we obtain (17) and (18). □

Proof of Proposition 3. Assume that there exists $\lim_{t\to\infty} \Big(Q_t a_{i,t+1} + f_i Q_{t+1} \big[\frac{q_{t+1}}{p_{t+1}} l_{i,t+1} + F_i(l_{i,t+1})\big]\Big) > 0$. Hence, there exists a date $T \geq 1$ such that borrowing constraint (5) is not binding for every $t \geq T$. Therefore, $\lambda_{i,t} p_t = \lambda_{i,t+1} R_{t+1}$ for every $t \geq T$. As a consequence, there exists a constant $C_i \in (0, \infty)$ such that $Q_t = C_i \lambda_{i,t} p_t$ for every $t \geq T$. According to transversality condition (17), we get $\lim_{t\to\infty} Q_t\big(a_{i,t+1} + \frac{q_t}{p_t} l_{i,t+1}\big) = 0$.

According to (18), we have $\lim_{t\to\infty} Q_t c_{i,t} = \lim_{t\to\infty} Q_t e_{i,t} = 0$. Therefore, by using (72), we get

$$\lim_{t\to\infty} Q_t\big(\frac{R_t}{p_t} a_{i,t} + \frac{q_t}{p_t} l_{i,t} + F_i(l_{i,t})\big) = 0.$$

Since $f_i \in [0, 1]$ and $Q_t \frac{R_t}{p_t} = Q_{t-1}$, we obtain the statement (b).

(23) is proved by using the same argument. □

6.2 Proofs for Section 3

Proof of Lemma 3. According to (13), we obtain $\frac{q_t}{p_t} \geq \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + \underline{d}_{t+1}\big)$.

We prove the second inequality. We see that there exists an agent, say i, such that $l_{i,t+1} > 0$. Thus, $\eta_{i,t+1} = 0$. Therefore, we have

$$\begin{aligned}\lambda_{i,t}q_t &= (\lambda_{i,t+1} + f_i\mu_{i,t+1})(q_{t+1} + p_{t+1}F_i'(l_{i,t+1})) \\ &\leq (\lambda_{i,t+1} + \mu_{i,t+1})(q_{t+1} + p_{t+1}F_i'(l_{i,t+1})) \leq \frac{\lambda_{i,t}p_t}{R_{t+1}}(q_{t+1} + p_{t+1}\bar{d}_{t+1}). \qquad (84)\end{aligned}$$

By combining with Lemma 2, we get the second inequality in (25).

□

Proof of Propostion 4. According to (25), we obtain $d_{t+1} \leq \bar{d}_{t+1}$.

Since $f_i = 1$ for any i or (5) is not binding for any i, we always have $\mu_{i,t+1} = f_i\mu_{i,t+1}$ for every i. So, we get

$$\frac{q_t}{p_t} = \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big) + \frac{\eta_{i,t+1}}{\lambda_{i,t}p_t} \geq \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big)$$

for any i. Therefore $d_{t+1} \geq \bar{d}_{t+1}$. As a result, we have $d_{t+1} = \bar{d}_{t+1}$. □

Proof of Proposition 5. Since $l_{i,t+1} > 0$ at equilibrium, and then $\eta_{i,t+1} = 0$. As a consequence, we obtain, for every i, t,

$$\lambda_{i,t}p_t = (\lambda_{i,t+1} + \mu_{i,t+1})R_{t+1} \qquad (85)$$

$$\lambda_{i,t}q_t = (\lambda_{i,t+1} + f_i\mu_{i,t+1})(q_{t+1} + p_{t+1}F_i'(l_{i,t+1})). \qquad (86)$$

We see that, for every i, t,

$$\begin{aligned}q_t = \frac{\lambda_{i,t+1} + f_i\mu_{i,t+1}}{\lambda_{i,t}}(q_{t+1} + p_{t+1}F_i'(l_{i,t+1})) &\leq \frac{\lambda_{i,t+1} + \mu_{i,t+1}}{\lambda_{i,t}}(q_{t+1} + p_{t+1}F_i'(l_{i,t+1})) \\ &= \gamma_{t+1}(q_{t+1} + p_{t+1}F_i'(l_{i,t+1}))\end{aligned}$$

Therefore, we obtain that $\frac{q_t}{p_t} \leq \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + \underline{d}_{t+1})\big)$.

We also see that

$$\frac{q_t}{p_t} \geq \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + \underline{d}_{t+1}\big).$$

As a result, we get (27). □

Proof of Proposition 6. According to FOCs, we get

$$q_t = \frac{\lambda_{i,t+1} + f_i\mu_{i,t+1}}{\lambda_{i,t+1} + \mu_{i,t+1}}\frac{\lambda_{i,t+1} + \mu_{i,t+1}}{\lambda_{i,t}}(q_{t+1} + p_{t+1}F_i'(l_{i,t+1})) + \frac{\eta_{i,t+1}}{\lambda_{i,t}}.$$

Therefore, we obtain

$$1 = \frac{\lambda_{i,t+1} + f_i\mu_{i,t+1}}{\lambda_{i,t+1} + \mu_{i,t+1}} \frac{q_{t+1} + p_{t+1}F_i'(l_{i,t+1})}{q_{t+1} + p_{t+1}d_{t+1}} + \frac{\eta_{i,t+1}}{\lambda_{i,t}q_t} \tag{87}$$

If $l_{i,t+1} > 0$, we obtain $\eta_{i,t+1} = 0$. By combining with $f_i \leq 1$, we get $d_{t+1} \leq F_i'(l_{i,t+1})$.

If (5) is not binding, we have $\mu_{i,t+1} = 0$ which implies that $d_{t+1} \geq F_i'(l_{i,t+1})$, contradiction. □

Proof of Proposition 7. Since $f_j = 1$, (13) implies that

$$\frac{q_t}{q_{t+1} + p_{t+1}F_j'(l_{j,t+1})} \geq \frac{\lambda_{j,t+1} + \mu_{j,t+1}}{\lambda_{j,t}} = \frac{p_t}{R_{t+1}}.$$

Assume that $l_{i,t+1} > 0$, we have $\eta_{i,t+1} = 0$ which implies that

$$\frac{q_t}{q_{t+1} + p_{t+1}F_i'(l_{i,t+1})} = \frac{\lambda_{i,t+1} + f_i\mu_{i,t+1}}{\lambda_{i,t}} \leq \frac{p_t}{R_{t+1}} \leq \frac{q_t}{q_{t+1} + p_{t+1}F_j'(l_{j,t+1})}.$$

Therefore, $F_j'(L) \leq F_j'(l_{j,t+1}) \leq F_i'(l_{i,t+1}) < F_i'(0)$, contradiction! □

Proof of Lemma 4. Let $\big((c_i, l_i, a_i), (c_j, l_j, a_j), p, q, R\big)$ be a steady state equilibrium. We rewrite the system (11, 12, 13, 14, 15)

$$\beta_i^t u_i'(c_{i,t}) = \lambda_{i,t}p_t \tag{88}$$

$$1 = \frac{R_{t+1}}{p_{t+1}}\Big(\frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} + \frac{\mu_{i,t+1}p_{t+1}}{\lambda_{i,t}p_t}\Big) \tag{89}$$

$$\frac{q_t}{p_t} = \Big(\frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} + f_i\frac{\mu_{i,t+1}p_{t+1}}{\lambda_{i,t}p_t}\Big)\Big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\Big) + \frac{\eta_{i,t+1}}{\lambda_{i,t}p_t} \tag{90}$$

$$\eta_{i,t+1}l_{i,t+1} = 0 \tag{91}$$

$$\mu_{i,t+1}\Big(R_{t+1}a_{i,t+1} + f_i\big[q_{t+1}l_{i,t+1} + p_{t+1}F_i(l_{i,t+1})\big]\Big) = 0 \tag{92}$$

Let denote $x_{i,t} := \dfrac{\mu_{i,t+1}p_{t+1}}{\lambda_{i,t}p_t}$, $\sigma_{i,t} := \dfrac{\eta_{i,t+1}}{\lambda_{i,t}p_t}$.

At steady state, we have

$$1 = \frac{R}{p}(\beta_i + x_i) \tag{93}$$

$$\frac{q}{p} = (\beta_i + f_i x_i)(\frac{q}{p} + F_i'(l_i)) + \sigma_i. \tag{94}$$

Since $\beta_i < \beta_j$, we have $x_i > x_j$, which implies that $x_i > 0$. Therefore, we have

$$Ra_i + f_i\big[ql_i + pF_i(l_i)\big] = 0.$$

Hence, $a_i < 0$ and then $a_j > 0$ which implies that $x_j = 0$. The impatient agent borrow from the patient agent.

We consider the case where $F_i(l_i) = A_i l_i^\alpha, F_j(l_i) = A_j l_j^\alpha$. Then $F_h'(l_h) = \alpha A_h l_h^{\alpha-1}$. In this case $l_i, l_j > 0$, hence $\sigma_i = \sigma_j = 0$.

We see that $a_i < 0$, which implies that $a_j > 0$. Hence, $x_j = 0$. The asset price is $\dfrac{R}{p} = \dfrac{1}{\beta_j} = 1 + r$, r is the real interest rate. We have $\dfrac{q}{p}\Big(\frac{1}{\beta_j} - 1\Big) = F_i'(l_i) = \alpha A_j l_j^{\alpha-1}$, therefore

$$l_j = \Big(\frac{\alpha A_j}{\frac{1}{\beta_j} - 1}\frac{p}{q}\Big)^{\frac{1}{1-\alpha}} \tag{95}$$

Since $\beta_i + x_i = \beta_j + x_j$, we get $x_i = \beta_j - \beta_i$. As a consequence, we can compute

$$l_i = \Big(\frac{\alpha A_i}{\frac{1}{\beta_i + f_i(\beta_j - \beta_i)} - 1}\frac{p}{q}\Big)^{\frac{1}{1-\alpha}} \tag{96}$$

Using $l_i + l_j = L$, we can compute the price of land

$$\big(\frac{q}{p}\big)^{\frac{1}{1-\alpha}} L = \Big(\frac{\alpha A_i}{\frac{1}{\beta_i + f_i(\beta_j - \beta_i)} - 1}\Big)^{\frac{1}{1-\alpha}} + \Big(\frac{\alpha A_j}{\frac{1}{\beta_j} - 1}\Big)^{\frac{1}{1-\alpha}} \tag{97}$$

□

6.3 Proofs for Section 4

Proof of Proposition 8. According to (33), it is easy to see that (i) is equivalent to (ii). We now prove that (ii) and (iii) are equivalent.

According to (32), we get that

$$\frac{q_0}{p_0} = Q_T \frac{q_T}{p_T} \prod_{t=1}^{T} (1 + \frac{p_t d_t}{q_t}). \tag{98}$$

Since $\frac{q_0}{p_0} > 0$, we see that $\lim\limits_{t \to +\infty} Q_t \frac{q_t}{p_t} > 0$ if and only if $\lim\limits_{t \to \infty} \prod\limits_{t=1}^{T} (1 + \frac{p_t d_t}{q_t}) < \infty$. It is easy to prove that this condition is equivalent to $\sum_{t=1}^{\infty} \dfrac{p_t d_t}{q_t} < +\infty$. □

Proof of Proposition 9. Assume that $Q_t / Q_{i,t}$ is uniformly bounded from above. According to Lemma 2, we have $\lim\limits_{t \to \infty} Q_{i,t}(\frac{q_t}{p_t} l_{i,t+1} + a_{i,t+1}) = 0$, therefore

$$\lim_{t \to \infty} Q_t(\frac{q_t}{p_t} l_{i,t+1} + a_{i,t+1}) = \lim_{t \to \infty} \frac{Q_t}{Q_{i,t}} Q_{i,t}(\frac{q_t}{p_t} l_{i,t+1} + a_{i,t+1}) = 0$$

for any i. Note that $\sum\limits_i l_{i,t+1} = L$ and $\sum\limits_i a_{i,t+1} = 0$ for any t, we obtain that $\lim\limits_{t \to \infty} Q_t \frac{q_t}{p_t} = 0$ □

Proof of Corollary 7. Since $\mu_{i,t+1} = 0$ for every $t \geq T$, we have $\lambda_{i,t} p_t = \lambda_{i,t+1} R_{t+1}$ for every $t \geq T$. As a consequence, $\gamma_{i,t} = \gamma_t$ for any $t \geq T+1$. This implies that $Q_t/Q_{i,t}$ is uniformly bounded from above. According to Proposition 9, there is no bubble. □

Proof of Lemma 5. According to (33), we have $\sum_{t=0}^{\infty} Q_t d_t < \infty$. However we have $d_t > F_i'(L) > \min_i F_i'(L) > 0$ for every t. Therefore, we obtain $\sum_{t=0}^{\infty} Q_t < \infty$. Since $\sup_{i,t} e_{i,t} < \infty$ and $F_i(l_{i,t}) \leq F_i(L)$ for every i, t, we obtain that $\sum_{t=0}^{\infty} Q_t Y_t < \infty$. □

Proof of Lemma 6. We will claim that $\sup_{i,t} Q_t a_{i,t+1} < \infty$. Indeed, (5) is rewritten as

$$Q_{t+1} \frac{R_{t+1}}{p_{t+1}} a_{i,t+1} + f_i Q_{t+1} \Big(\frac{q_{t+1}}{p_{t+1}} l_{i,t+1} + F_i(l_{i,t+1}) \Big) \geq 0. \tag{99}$$

Since $Q_t = \dfrac{R_{t+1}}{p_{t+1}} Q_{t+1}$, (5) is equivalent to

$$Q_t a_{i,t+1} \geq -f_i Q_{t+1} \Big(\frac{q_{t+1}}{p_{t+1}} l_{i,t+1} + F_i(l_{i,t+1}) \Big). \tag{100}$$

It is easy to see that $0 \leq Q_t \frac{q_t}{p_t} l_{i,t+1} \leq \dfrac{q_0}{p_0} L < \infty$. Therefore, we have

$$f_i Q_{t+1} \Big(\frac{q_{t+1}}{p_{t+1}} l_{i,t+1} + F_i(l_{i,t+1}) \Big) \leq f_i \frac{q_0}{p_0} L + f_i Q_{t+1} F_i(L). \tag{101}$$

As a consequence, we obtain

$$Q_t a_{i,t+1} \geq -f_i \frac{q_0}{p_0} L - f_i Q_{t+1} F_i(L). \tag{102}$$

According to the proof of Lemma 5 , we see that $\lim_{t \to \infty} Q_t = 0$, and hence we get that $\inf_{i,t} Q_t a_{i,t+1} > -\infty$. Since $\sum_{i=1}^{m} Q_t a_{i,t+1} = 0$, have $-\infty < \inf_{i,t} Q_t a_{i,t+1} \leq \sup_{i,t} Q_t a_{i,t+1} < \infty$. □

Proof of Lemma 7. We rewrite the budget constraint of agent i at date t

$$Q_t c_{i,t} + Q_t \frac{q_t}{p_t} l_{i,t+1} + Q_t a_{i,t+1} = Q_t (e_{i,t} + F_i(l_{i,t})) + Q_t \frac{q_t}{p_t} l_{i,t} + Q_t \frac{R_t}{p_t} a_{i,t}. \tag{103}$$

According to (19) and (26), we get

$$Q_t \frac{q_t}{p_t} = Q_{t+1} \big(\frac{q_{t+1}}{p_{t+1}} + d_{t+1} \big), \quad Q_t = \frac{R_{t+1}}{p_{t+1}} Q_{t+1}.$$

Therefore, we have

$$\sum_{t=0}^{T} Q_t c_{i,t} + \sum_{t=1}^{T} Q_t d_t l_{i,t} + Q_T(\frac{q_T}{p_T} l_{i,T+1} + a_{i,T+1}) = \sum_{t=0}^{T} Q_t(e_{i,t} + F_i(l_{i,t})) + \frac{q_0}{p_0} l_{i,0} + \frac{R_0}{p_0} a_{i,0}.$$

By combining this with Lemmas 5 and 6, we obtain that

$$\sup_{T\to\infty} \Big(\sum_{t=0}^{T} Q_t c_{i,t} + \sum_{t=1}^{T} Q_t d_t l_{i,t} \Big) < \infty.$$

This implies that there exists the sum $\sum_{t=0}^{\infty}(Q_t c_{i,t} + Q_t d_t l_{i,t})$, and so does $\lim_{t\to\infty} Q_t\big(\frac{q_t}{p_t} l_{i,t+1} + a_{i,t+1}\big)$.

Note that $\lim_{t\to\infty} Q_t c_{i,t} = \lim_{t\to\infty} Q_t(e_{i,t} + F_i(l_{i,t})) = 0$. Then, by using (103), we get (40). □

Proof of Lemma 8. If $\lim_{t\to\infty} Q_t\big(a_{i,t+1} + \frac{q_t}{p_t} l_{i,t+1}\big) > 0$, there exists $T_1 \geq T$ such that $Q_t\big(a_{i,t+1} + \frac{q_t}{p_t} l_{i,t+1}\big) > 0$ for every $t \geq T_1$. Hence, we get

$$\begin{aligned} &Q_{t+1}\frac{R_{t+1}}{p_{t+1}} a_{i,t+1} + f_i Q_{t+1}\big[\frac{q_{t+1}}{p_{t+1}} l_{i,t+1} + F_i(l_{i,t+1})\big] \\ &\geq Q_{t+1}\frac{R_{t+1}}{p_{t+1}} a_{i,t+1} + Q_{t+1}\big[\frac{q_{t+1}}{p_{t+1}} + d_{t+1}\big] l_{i,t+1} = Q_{t+1}\frac{R_{t+1}}{p_{t+1}} a_{i,t+1} + Q_t \frac{q_t}{p_t} l_{i,t+1} > 0 \end{aligned}$$

for every $t \geq T_1$. This implies that $\mu_{i,t+1} = 0$ for every $t \geq T_1$

Therefore, $\lambda_{i,t} p_t = \lambda_{i,t+1} R_{t+1}$ for every $t \geq T_1$. As a consequence, there exists a constant $C_i > 0$ such that $Q_t = C_i \lambda_{i,t} p_t$ for every $t \geq T_1$. According to transversality condition (17), we get $\lim_{t\to\infty} Q_t\big(a_{i,t+1} + \frac{q_t}{p_t} l_{i,t+1}\big) = 0$, contradiction! □

Proof of Propostion 10. If $l_{i,t} = 0$ then condition (41) is satisfied.

If $l_{i,t} > 0$, by combining with $f_i = 1$ and using Proposition 4, we have $d_t = F_i'(l_{i,t}) \leq \frac{F_i(l_{i,t})}{l_{i,t}}$. Therefore, condition (41) is satisfied. As a consequence, we have $\lim_{t\to\infty} Q_t\big(a_{i,t+1} + \frac{q_t}{p_t} l_{i,t+1}\big) \leq 0$ for any i. By summing over i, we obtain $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} L \leq 0$, which implies that bubbles are ruled out. □

Proof of Proposition 12. Assume that i-bubble exists, we have $\lim_{t\to\infty} Q_{i,t}\frac{q_t}{p_t} > 0$. Therefore, we get $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} > 0$. Since both these two limits are finite (less than q_0/p_0) we see that $\lim_{t\to\infty} Q_t/Q_{i,t} \in (0,\infty)$ for any i. According to Proposition 9 we have $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} = 0$, contradiction!

□

Proof of Proposition 13. Since $Q_t \geq Q_{i,t}$, it is easy to see that $FV_0 \leq FV_i$ for any i, and if i-land bubbles exist for some agent i then land bubbles exist.

We now assume that $FV_0 = FV_i$ for any i, which implies that $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} = \lim_{t\to\infty} Q_{i,t}\frac{q_t}{p_t}$. If land bubbles exist, we have $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} = \lim_{t\to\infty} Q_{i,t}\frac{q_t}{p_t} \in (0, q_0/p_0)$. Thus, we obtain $\lim_{t\to\infty} Q_{i,t}/Q_t = 1$. According to Proposition 9 we have $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} = 0 = \lim_{t\to\infty} Q_{i,t}\frac{q_t}{p_t}$, contradiction!

□

6.3.1 On the example of bubble

First, we give sufficient conditions for a sequence $\big(p_t, q_t, (c_{i,t}, l_{i,t+1})_{i\in I}\big)_t$ to be an equilibrium. The utility function may satisfy $u_i(0) = -\infty$.

Lemma 9. *If a sequence* $\big((c_{i,t}, l_{i,t+1}, \mu_{i,t})_{i\in I}, p_t, q_t\big)_t$ *satisfies*

(i) $\forall t,\ \forall i,\ c_{i,t} > 0, l_{i,t+1} \geqslant 0,\ \mu_{i,t} \geqslant 0.\ \forall t, p_t = 1, q_t > 0.$

(ii) First order conditions

$$q_t = \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}\big(q_{t+1} + F_{i,t}'(l_{i,t+1})\big) + \mu_{i,t+1} \tag{104}$$

$$\mu_{i,t+1} l_{i,t+1} = 0 \tag{105}$$

(iii) Transversality conditions: $\lim_{t\to\infty} \beta_i^t u_i'(c_{i,t}) q_t l_{i,t+1} = 0$ *for any* i.

(iv) $c_{i,t} + q_t l_{i,t+1} = e_{i,t} + q_t l_{i,t} + F_{i,t}(l_{i,t}),$

(vi) $\sum_{i\in I} l_{i,t} = L.$

then the sequence $\big(p_t, q_t, (c_{i,t}, l_{i,t+1})_{i\in I}\big)_t$ *is an equilibrium for the economy without financial market.*

Proof of Lemma 9. It is easy to see that market clearing conditions are satisfied. We now prove the optimality of the agents' plan. Let $(c_i', l_i') \geq 0$ be a plan satisfying all budget constraints and $l_{i,0}' = l_{i,0}$. We have

$$\begin{aligned}
&\sum_{t=0}^{T} \beta_i^t (u_i(c_{i,t}) - u_i(c_{i,t}')) \geq \sum_{t=0}^{T} \beta_i^t u_i'(c_{i,t})(c_{i,t} - c_{i,t}') = \\
&= \sum_{t=0}^{T} \beta_i^t u_i'(c_{i,t})\Big(q_t(l_{i,t} - l_{i,t}') + F_{i,t}(l_{i,t}) - F_{i,t}(l_{i,t}') - q_t(l_{i,t+1} - l_{i,t+1}')\Big) \\
&\geq \sum_{t=0}^{T} \beta_i^t u_i'(c_{i,t})(q_t + F_{i,t}'(l_{i,t}))(l_{i,t} - l_{i,t}') - \sum_{t=0}^{T} \beta_i^t u_i'(c_{i,t}) q_t (l_{i,t+1} - l_{i,t+1}') \\
&= \sum_{t=1}^{T} \Big(\beta_i^t u_i'(c_{i,t})(q_t + F_{i,t}'(l_{i,t})) - \beta_i^{t-1} u_i'(c_{i,t-1}) q_{t-1}\Big)(l_{i,t} - l_{i,t}') \\
&\quad - \beta_i^T u_i'(c_{i,T}) q_T (l_{i,T+1} - l_{i,T+1}').
\end{aligned}$$

According to (104), we obtain $\beta_i^t u_i'(c_{i,t})(q_t+F_{i,t}'(l_{i,t}))-\beta_i^{t-1}u_i'(c_{i,t-1})q_{t-1} = -\mu_{i,t}\beta_i^{t-1}u_i'(c_{i,t-1})$. By using this and the fact that $\mu_{i,t}l_{i,t} = 0$, we have

$$\sum_{t=0}^{T} \beta_i^t (u_i(c_{i,t}) - u_i(c_{i,t}')) \tag{106}$$

$$\geq \sum_{t=1}^{T} -\mu_{i,t}\beta_i^{t-1}u_i'(c_{i,t-1})(l_{i,t} - l_{i,t}') + \beta_i^T u_i'(c_{i,T})q_T(l_{i,T+1}' - l_{i,T+1}) \tag{107}$$

$$= \sum_{t=1}^{T} \beta_i^{t-1}u_i'(c_{i,t-1})\mu_{i,t}l_{i,t}' + \beta_i^T u_i'(c_{i,T})q_T(l_{i,T+1}' - l_{i,T+1}) \tag{108}$$

$$\geq -\beta_i^T u_i'(c_{i,T})q_T l_{i,T+1}. \tag{109}$$

Since there exists the sum $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) < \infty$ and $\lim_{t\to\infty} \beta_i^T u_i'(c_{i,T})q_T l_{i,T+1} = 0$, there exists the sum $\sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}') < \infty$. Therefore, we obtain $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \geq \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}')$. □

Check of the example in Section 4.3.1. We will check all conditions in Lemma 9. It is easy to see that the market clearing conditions are satisfied.

Let us check FOCs:

$$q_{2t} = \frac{\beta_B u_B'(c_{B,2t+1})}{u_B'(c_{B,2t})}\big(q_{2t+1} + B_{2t+1}\big) \geq \frac{\beta_A u_A'(c_{A,2t+1})}{u_A'(c_{A,2t})}(q_{2t+1} + A_{2t+1}) \tag{110}$$

$$q_{2t-1} = \frac{\beta_A u_A'(c_{A,2t})}{u_A'(c_{A,2t-1})}(q_{2t} + A_{2t}) \geq \frac{\beta_B u_B'(c_{B,2t})}{u_B'(c_{B,2t-1})}(q_{2t} + B_{2t}). \tag{111}$$

The equality in (110) is satisfied since

$$\frac{\beta_B u_B'(c_{B,2t+1})}{u_B'(c_{B,2t})}\big(q_{2t+1} + B_{2t+1}\big) = \frac{\beta_B(e_{B,2t} - q_{2t})}{q_{2t+1} + B_{2t+1}}\big(q_{2t+1} + B_{2t+1}\big) \tag{112}$$

$$= \beta_B(e_{B,2t} - q_{2t}) = q_{2t}. \tag{113}$$

We now prove the inequality in (110). We have

$$\frac{\beta_A u_A'(c_{A,2t+1})}{u_A'(c_{A,2t})}(q_{2t+1} + A_{2t+1}) = \frac{\beta_A(q_{2t} + A_{2t})}{e_{A,2t+1} - q_{2t+1}}(q_{2t+1} + A_{2t+1}) \tag{114}$$

$$= \frac{\beta_A(\frac{\beta_B}{1+\beta_B}e_{B,2t} + A_{2t})}{\frac{1}{1+\beta_A}e_{A,2t+1}}(\frac{\beta_A}{1+\beta_A}e_{A,2t+1} + A_{2t+1}) \tag{115}$$

As a consequence, the inequality in (110) is equivalent to

$$\beta_A(\frac{\beta_B e_{B,2t}}{1+\beta_B} + A_{2t})(\frac{\beta_A e_{A,2t+1}}{1+\beta_A} + A_{2t+1}) \leq \beta_B \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t+1}}{1+\beta_A} \tag{116}$$

which is the condition (53).

We have

$$\frac{\beta_B u_B'(c_{B,2t})}{u_B'(c_{B,2t-1})}(q_{2t}+B_{2t}) = \frac{\beta_B(q_{2t-1}+B_{2t-1})}{e_{B,2t}-q_{2t}}(q_{2t}+B_{2t}) \tag{117}$$

$$= \frac{\beta_B(\frac{\beta_A}{1+\beta_A}e_{A,2t-1}+B_{2t-1})}{\frac{1}{1+\beta_B}e_{B,2t}}(\frac{\beta_B}{1+\beta_B}e_{B,2t}+B_{2t}) \tag{118}$$

As a consequence, the inequality in (111) is equivalent to

$$\beta_B(\frac{\beta_A e_{A,2t-1}}{1+\beta_A}+B_{2t-1})(\frac{\beta_B e_{B,2t}}{1+\beta}+B_{2t}) \leq \beta_A \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t-1}}{1+\beta_A} \tag{119}$$

which is the condition (54).

We now check TVCs. We have

$$\beta_A^{2t} u_A'(c_{A,2t}) q_{2t} l_{A,2t+1} = 0 \tag{120}$$

$$\beta_A^{2t-1} u_A'(c_{A,2t-1}) q_{2t-1} l_{A,2t} = \frac{\beta_A^{2t-1}}{c_{A,2t-1}} q_{2t-1} = \beta_A^{2t} \to 0. \tag{121}$$

Similarly, we also have

$$\beta_B^{2t} u_B'(c_{B,2t}) q_{2t} l_{B,2t+1} = \beta_B^{2t+1} \to 0 \tag{122}$$

$$\beta_B^{2t-1} u_B'(c_{B,2t-1}) q_{2t-1} l_{B,2t} = 0. \tag{123}$$

We finally verify that, for each $t \geq 0$,

$$\frac{\beta_B u_B'(c_{B,2t+1})}{u_B'(c_{B,2t})} \geq \frac{\beta_A u_A'(c_{A,2t+1})}{u_A'(c_{A,2t})} \tag{124}$$

$$\frac{\beta_B u_B'(c_{B,2t})}{u_B'(c_{B,2t-1})} \leq \frac{\beta_A u_A'(c_{A,2t})}{u_A'(c_{A,2t-1})}. \tag{125}$$

Indeed, condition (124) is rewritten as

$$\frac{\beta_B(e_{B,2t}-q_{2t})}{q_{2t+1}+B_{2t+1}} \geq \frac{\beta_A(\frac{\beta_B}{1+\beta_B}e_{B,2t}+A_{2t})}{\frac{1}{1+\beta_A}e_{A,2t+1}}. \tag{126}$$

Since $q_{2t} = \frac{\beta_B}{1+\beta_B}e_{B,2t}, q_{2t+1} = \frac{\beta_A}{1+\beta_A}e_{A,2t+1}$, condition (124) is equivalent to

$$\beta_A(\frac{\beta_B e_{B,2t}}{1+\beta_B}+A_{2t})(\frac{\beta_A e_{A,2t+1}}{1+\beta_A}+B_{2t+1}) \leq \beta_B \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t+1}}{1+\beta_A}. \tag{127}$$

By the same argument, we see that condition (125) is equivalent to

$$\beta_B(\frac{\beta_A e_{A,2t-1}}{1+\beta_A}+B_{2t-1})(\frac{\beta_B e_{B,2t}}{1+\beta}+A_{2t}) \leq \beta_A \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t-1}}{1+\beta_A}. \tag{128}$$

□

7 Appendix: Existence of equilibrium for the intermediate economy $\tilde{\mathcal{E}}$

In this appendix, we present a proof of the existence of equilibrium for the economy $\tilde{\mathcal{E}}$. We allow for non-stationary technologies, i.e., the production functions $F_{i,t}$ depend on both i and t.

7.1 Existence of equilibrium for truncated economy

We define $T-$ truncated economy $\tilde{\mathcal{E}}^T$ as $\tilde{\mathcal{E}}$ but there are no activities from period $T+1$ to the infinity, i.e., $c_{i,t} = l_{i,t} = b_{i,t} = 0$ for every $i = 1, \ldots, m$, $t \geq T+1$.

We then define the bounded economy $\tilde{\mathcal{E}}_b^T$ as $\tilde{\mathcal{E}}^T$ but consumption level $(c_{i,t})_{t=0}^T$, land holding $(l_{i,t})_{t=1}^T$, and asset holding $(b_{i,t})_{t=1}^T$ are respectively bounded in the following sets:

$$\mathcal{C}_i \;:=\; [0, B_c]^{T+1}, \quad \mathcal{L}_i := [0, B_l]^T, \quad \mathcal{A}_i := \prod_{t=1}^{T} [-B_b, B_b]^T,$$

where $B_c > \max_{t \leq T} \sum_{i=1}^{m} \Big(e_{i,t} + F_{i,t}(B_l)\Big)$, $B_l > L$, and $B_b = m(B_c + B_l)$.

The economy $\tilde{\mathcal{E}}_b^T$ depends on bounds B_c, B_l, B_b. We write $\tilde{\mathcal{E}}_b^T(B_c, B_l, B_b)$.

Let denote

$$\mathcal{X}_b := \mathcal{C}_i \times \mathcal{L}_i \times \mathcal{A}_i, \quad \mathcal{X} := (\mathcal{X}_b)^{T+1}$$

We then define

$$\mathcal{P} := \{z_0 = (p, q, r) : r_0 = 0, q_T = 0; \textit{and} \tag{129}$$

$$0 \leq p_t, q_t, r_t; 2p_t + q_t + r_t = 1 \quad \forall t = 0, \ldots, T\} \tag{130}$$

$$\Phi := \mathcal{P} \times \mathcal{X}. \tag{131}$$

An element $z \in \Phi$ is in the form $z = (z_i)_{i=0}^m$ where $z_0 := (p, q, r), z_i := (c_i, l_i, b_i)$ for each $i = 1, \ldots, m$.

The following remark is useful.

Remark 7. *If $z \in \Phi$ is an equilibrium for the economy $\tilde{\mathcal{E}}$ then $c_{i,t} \in [0, B_c)$, $l_{i,t} \in [0, L]$. By using the fact that $2p_t + q_t + r_t = 1$, we get that $b_{i,t} \leq B_c + B_l$ for any i, t. Since $\sum_{i=1}^{m} b_{i,t} = 0$, we obtain that $b_{i,t} \in [-B_b, B_b]$ for any i and any t.*

Proposition 15. *Under our assumptions, there exists an equilibrium $(p, q, r, (c_i, l_i, b_i)_{i=1}^m)$, with $2p_t + q_t + r_t = 1$, for the economy $\tilde{\mathcal{E}}_b^T(B_c, B_l, B_b)$.*

Proof. We will prove this proposition as follows.

We define

$$\begin{aligned}C_i^T(p,q,r) := \big\{&(c_{i,t}, l_{i,t+1}, b_{i,t+1})_{t=0}^T \in \mathcal{X} : \text{(a) } l_{i,T+1}, b_{i,T+1} = 0,\\ &\text{(b) } p_0 c_{i,0} + q_0 l_{i,1} + b_{i,1} \le p_0 e_{i,0} + p_0 F_{i,0}(l_{i,0}) + q_0 l_{i,0}\\ &\text{(c) for each } 1 \le t \le T :\\ &\quad 0 \le r_t b_{i,t} + f_i\Big(q_t l_{i,t} + p_t F_{i,t}(l_{i,t})\Big)\\ &\quad p_t c_{i,t} + q_t l_{i,t+1} + b_{i,t+1} \le p_t e_{i,t} + p_t F_{i,t}(l_{i,t}) + q_t l_{i,t} + r_t b_{i,t}\big\}.\end{aligned}$$

We also define $B_i^T(p,q,r)$ as follows.

$$\begin{aligned}B_i^T(p,q,r) := \big\{&(c_{i,t}, l_{i,t+1}, b_{i,t+1})_{t=0}^T \in \mathcal{X} : \text{(a) } l_{i,T+1}, b_{i,T+1} = 0,\\ &\text{(b) } p_0 c_{i,0} + q_0 l_{i,1} + b_{i,1} < p_0 e_{i,0} + p_0 F_{i,0}(l_{i,0}) + q_0 l_{i,0}\\ &\text{(c) for each } 1 \le t \le T :\\ &\quad 0 < r_t b_{i,t} + f_i\Big(q_t l_{i,t} + p_t F_{i,t}(l_{i,t})\Big)\\ &\quad p_t c_{i,t} + q_t l_{i,t+1} + b_{i,t+1} < p_t e_{i,t} + p_t F_{i,t}(l_{i,t}) + q_t l_{i,t} + r_t b_{i,t}\big\}.\end{aligned}$$

Lemma 10. *$B_i^T(p,q,r) \neq \emptyset$ and $\bar{B}_i^T(p,q,r) = C_i^T(p,q,r)$.*

Proof. It can be easily proved by using the following remark: since $e_{i,0}, l_{i,0} > 0$ and $(p_0, q_0) \neq (0,0)$, we always have $p_0 e_{i,0} + p_0 F_{i,0}(l_{i,0}) + q_0 l_{i,0} > 0$. □

Lemma 11. *$B_i^T(p,q,r)$ is lower semi-continuous correspondence on $\mathcal{P}$. And $C_i^T(p,q,r)$ is continuous on $\mathcal{P}$ with compact convex values.*

Proof. It is clear since $B_i^T(p,q,r)$ is nonempty and has open graph. □

We now define correspondences.
First, we define φ_0 (for additional agent 0) : $\mathcal{X} \to 2^{\mathcal{P}}$ by

$$\begin{aligned}\varphi_0((z_i)_{i=1}^m) := \operatorname*{arg\,max}_{(p,q,r)\in\mathcal{P}} \Big\{ &\sum_{t=0}^T p_t \sum_{i=1}^m \big(c_{i,t} - e_{i,t} - F_{i,t}(l_{i,t})\big)\\ &+ \sum_{t=0}^{T-1} q_t \sum_{i=1}^m (l_{i,t+1} - l_{i,t}) + \sum_{t=1}^T r_t(-\sum_{i=1}^m b_{i,t})\Big\}.\end{aligned}$$

For each $i = 1, \ldots, m$, we define $\varphi_i : \mathcal{P} \to 2^{\mathcal{X}}$

$$\varphi_i((p,q,r)) \quad := \quad \operatorname*{arg\,max}_{(c_i,l_i,b_i)\in C_i(p,q,r)} \Big\{ \sum_{t=0}^T \beta_i^t u_i(c_{i,t}) \Big\}.$$

Lemma 12. *The correspondence φ_i is upper semi-continuous and non-empty, convex, compact valued for each $i = 0, 1, \ldots, m+1$.*

Proof. This is a direct consequence of the Maximum Theorem. □

According to the Kakutani Theorem, there exists $(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_{i=1}^m)$ such that

$$(\bar{p}, \bar{q}, \bar{r}) \in \varphi_0((\bar{c}_i, \bar{l}_i, \bar{b}_i)_{i=1}^m) \tag{132}$$
$$(\bar{c}_i, \bar{l}_i, \bar{b}_i) \in \varphi_i((\bar{p}, \bar{q}, \bar{r})). \tag{133}$$

Denote, for each $t \geq 1$,

$$\bar{X}_t := \sum_{i=1}^m (\bar{c}_{i,t} - e_{i,t} - F_{i,t}(\bar{l}_{i,t})) \tag{134}$$
$$\bar{Y}_t := \sum_{i=1}^m (\bar{l}_{i,t+1} - \bar{l}_{i,t}) \tag{135}$$
$$\bar{Z}_t := -\sum_{i=1}^m \bar{b}_{i,t}. \tag{136}$$

For every $(p, q, r) \in \mathcal{P}$, we have

$$\sum_{t=0}^T (p_t - \bar{p}_t)\bar{X}_t + \sum_{t=0}^{T-1} (q_t - \bar{q}_t)\bar{Y}_t + \sum_{t=1}^T (r_t - \bar{r}_t)\bar{Z}_t \leq 0. \tag{137}$$

By summing the budget constraints over i at date T, we get that

$$\bar{p}_T \bar{X}_T + \bar{r}_T \bar{Z}_T \leq 0. \tag{138}$$

As a consequence, we have, for every $(p_T, r_T) \geq 0$ with $2p_T + r_T = 1$,

$$p_T \bar{X}_T + r_t \bar{Z}_t \leq \bar{p}_T \bar{X}_T + \bar{r}_t \bar{Z}_t \leq 0. \tag{139}$$

Therefore, we have $\bar{X}_T, \bar{Z}_T \leq 0$ for any t, which means that

$$\sum_{i=1}^m \bar{c}_{i,T} \leq \sum_{i=1}^m (e_{i,T} + F_{i,t}(\bar{l}_{i,T})) \tag{140}$$
$$-\sum_{i=1}^m \bar{b}_{i,T} \leq 0. \tag{141}$$

By summing the budget constraints over i at date t, we get that

$$\bar{p}_t \bar{X}_t + \bar{q}_t \bar{Y}_t + \bar{r}_t \bar{Z}_t \leq \bar{Z}_{t+1}. \tag{142}$$

Since $\bar{Z}_T \leq 0$, we get,

$$\bar{p}_{T-1} \bar{X}_{T-1} + \bar{q}_{T-1} \bar{Y}_{T-1} + \bar{r}_{T-1} \bar{Z}_{T-1} \leq 0. \tag{143}$$

As a consequence, we have that, for any t,

$$p_{T-1} \bar{X}_{T-1} + p_{T-1} \bar{Y}_{T-1} + r_{T-1} \bar{Z}_{T-1} \leq 0. \tag{144}$$

This implies that $\bar{X}_{T-1}, \bar{Y}_{T-1}, \bar{Z}_{T-1} \leq 0$. Repeating this argument, we obtain that $\bar{X}_t, \bar{Y}_t, \bar{Z}_t \leq 0$ for any t, which means that

$$\sum_{i=1}^{m} \bar{c}_{i,t} \leq \sum_{i=1}^{m} (e_{i,t} + F_{i,t}(\bar{l}_{i,t})) \tag{145}$$

$$\sum_{i=1}^{m} (\bar{l}_{i,t+1} - \bar{l}_{i,t}) \leq 0 \tag{146}$$

$$-\sum_{i=1}^{m} \bar{b}_{i,t} \leq 0. \tag{147}$$

Lemma 13. *$\bar{p}_t, \bar{q}_t, \bar{r}_t > 0$ for $t = 0, \ldots, T$.*

Proof. We see that $\sum_{i=1}^{m} \bar{l}_{i,t} \leq L$ and $\sum_{i=1}^{m} \bar{b}_{i,t} \geq 0$. Hence, we have

$$\sum_{i=1}^{m} \bar{c}_{i,t} \leq \sum_{i=1}^{m} (e_{i,t} + F_{i,t}(L))$$

which allows us to prove that $\bar{p}_t > 0$. Indeed, if $\bar{p}_t = 0$ then $c_{i,t} = B_c > \sum_{i=1}^{m} (e_{i,t} + F_{i,t}(L))$, a contradiction.

If $\bar{q}_t = 0$, then $\bar{l}_{i,t} = B_l > L$ for any i, contradiction!

If $\bar{r}_t = 0$ then $\bar{b}_{i,t} = -B_a$ for any i, which implies that $\sum_{i=1}^{m} \bar{b}_{i,t} < 0$, contradiction! □

Lemma 14. *$\bar{X}_t = \bar{Z}_t = \bar{Y}_t = 0$.*

Proof. Using $\bar{p}_t \bar{X}_t + \bar{q}_t \bar{Y}_t + \bar{r}_t \bar{Z}_t \leq 0$. □

Lemma 15. *The optimality of (c_i, l_i, b_i).*

Proof. It is clear since $(\bar{c}_i, \bar{l}_i, \bar{b}_i) \in \varphi_i((\bar{p}, \bar{q}, \bar{r}))$. □

We have just prove that $(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_{i=1}^m)$ is an equilibrium for the economy $\tilde{\mathcal{E}}_b^T$. □

Proposition 16. *An equilibrium $(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_{i=1}^m)$, with $2p_t + q_t + r_t = 1$, of $\tilde{\mathcal{E}}_b^T$ is an equilibrium for $\tilde{\mathcal{E}}^T$.*

Proof. The proof is similar to Lemma 3 in Le Van and Pham (2015). □

7.2 Existence of equilibrium for the infinite horizon economy

Let us denote $W_t := \sum_{i=1}^{m}(e_{i,t} + F_{i,t}(L)$. We need the following assumption.

Assumption 6. *For each i*

$$\sum_{t=0}^{\infty} \beta_i^t u_i(W_t) < \infty. \tag{148}$$

Proposition 17. *Under Assumptions (1-4) and 7, there exists an equilibrium for the economy $\tilde{\mathcal{E}}$.*

Proof. We have shown that there exists an equilibrium, say $\big(p^T, q^T, r^T, (c_i^T, l_i^T, b_i^T)_i\big)$ for each truncated economy $\tilde{\mathcal{E}}^T$. Recall that $2p_t^T + q_t^T + r_t^T = 1$.

It is clear that $0 < c_{i,t}^T < W_t$ for each $t \geq 0$, and $l_{i,t}^T \in [0, L]$, $p_t^T, q_t^T, r_t^T \in [0, 1]$ and $2p_t^T + q_t^T + r_t^T = 1$.

We define a sequence (B_t) as follows.

$$B_1 = W_1, \quad B_{t+1} = B_t + W_{t+1} \quad \forall t. \tag{149}$$

It is easy to prove that $b_{i,t}^T < B_t$ for any t and any T. Since $\sum_{i=1}^{m} b_{i,t}^T = 0$ we have $b_{i,t}^T \in (-mB_t, mB_t)$ for any t and any T.

Therefore, without loss of generality, we can assume that

$$\big(p^T, q^T, r^T, (c_i^T, l_i^T, b_i^T)_i\big) \xrightarrow{T\to\infty} \big(p, q, r, (c_i, l_i, b_i)_i\big)$$

for the product topology.

By using the same argument in the proof of Theorem 1 in Le Van and Pham (2015) or the proof of Theorem 1 in Le Van, Pham, and Vailakis (2014), we obtain that $\big(p, q, r, (c_i, l_i, b_i)_i\big)$ is an equilibrium for the economy $\tilde{\mathcal{E}}$.

□

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