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# Sensitivity of Pension Fund's Balance Sheet: a non-linear risk factor approach\*

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## Abstract

In this paper, we study the funding situation of a representative pension fund when it is exposed to extreme shocks of financial markets. We measure the exposure of both asset and liability sides of the fund's balance sheet and especially when the benefit obligations use a market based discount rate. By assigning different market indexes to the main items of the fund's balance sheet, we are able to compute the expected funded status conditionally on extreme shocks to different financial markets. We also take into account the links between the corresponding indexes thanks to the CVine Risk Factor (CVRF) model combining factors and copulas. In particular we are able to measure the exposure of the funding status of the fund to extreme shocks to different risk factors (equity, bond, real estate etc...). We find that the fund is particularly exposed to large shocks to the equity risk factor, even if diversification benefits can exist because such shocks simultaneously induce a drop in the asset values and a decrease in the discounted value of the liabilities. However, the first decrease is larger than the second one, thus the funding situation is declined.

*Keywords:* Regular vine copula, Factorial model, Pension Funds, Stress testing

*JEL classification:* G23; G32; J32

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# 1 Introduction

Pension provision institutions manage assets that are used to provide retired workers a flow of income which they need to maintain their preretirement standard of living. They control relatively large pool of capital and represent the largest institutional investors in many developed countries. Exposed to various market risks, the assets of a pension plan may affect the retirement benefit. Falling returns can cause serious under-funding situation especially with demographic challenges faced by the pension industry. Difficult market conditions have made risk management practices even more important for institutional investors and especially for Defined-Benefit (DB) pension plans. Market risk of pension funds refers to the sensitivity of his portfolio to the overall market price movements such as interest rates, inflation, equities, currency and property.

Regulatory reforms of pension provisions that have recently taken place intend to better meet the objective of benefit security.<sup>1</sup> The regulatory system is moving toward a marked-to-market valuation of the balance sheet and a risk-based supervisory framework (e.g. Boon et al. (2014), Pugh and Yermo (2008)).

Examining the sensitivity of the funding situation of the pension fund to market risks is therefore important to assess its security.

A pension plan consists of two primary elements: benefit obligations and plan assets that are employed to finance retiree benefits. Treated as a portfolio, the asset side is clearly exposed to market risks. The liability side is also sensitive to these risks. More precisely, according to the recent regulatory recommendations, the marked-to-market value of the benefit obligations may be affected by market movements. Indeed, to assess the asset liability matching, a pension plan is expected to evaluate its funding ratio (assets divided by liabilities) or, equivalently, its funded status (assets minus liabilities) by discounting the future benefit payments of its liabilities with a market based discount rate. For example, in the US Public case, the discount rate is the expected return of

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<sup>1</sup>In Europe, the European Commission (EC) has published a proposal for the IORP II Directive (new rules on Institutions for Occupational Retirement Provision (IORPs)). The European Insurance and Occupational Pensions Authority (EIOPA) is continuing its work on the “Holistic Balance Sheet” (HBS). As a part of the work, the Quantitative Impact Study (QIS) on IORPs is used to “provide stakeholders with information on the impact of EIOPA’s advice on the review of the IORP Directive”.

assets.

In this paper, we aim at measuring the sensitivities of the assets and liabilities of a pension fund to the extreme shocks to financial markets.

To do that, in the lines of Kahlert and Wagner (2015)<sup>2</sup>, we assign a market index to each of the items of the fund's balance sheet. For example, the equity item is assigned to a representative stock index like MSCI World Index.

Moreover, we take into account the complex inter-dependencies between the different financial markets by using a CVine Risk Factor model (CVRF) as proposed by Bruneau et al. (2015). The model combines copula dependencies with a factorial structure and allows to extract different market risk factors (equity, bond, sovereign, currency, commodity...) from representative market indexes. We can therefore assess the funding situation of a plan under extreme circumstances observed on a financial market. More precisely, we compute the expected funded status conditionally to extreme shocks to the different market risk factors identified in the CVRF model. This calculation can be viewed as a generalization of the traditional sensitivity analyses performed from a linear multibeta relationship in the lines of Ross (1976).

Our analysis enters in the recent literature about risk management in pension industry and more specifically about stress testing issues. Since the two major financial crisis (2000-2003, 2007-2009) risk management tools have been applied to the pension industry with concepts like Value-at-Risk, fat tail events and risk budgeting (Franzen , 2010). Concerning the stress tests, they have been only recently introduced in the pension industry. EIOPA has launched its first stress test for IORPs in May 2015.

Compared to the existing approaches, our contribution is twofold. First, the methodology we propose allows to measure tail dependencies and to account for the indirect effects of a shock to a specific market transmitted by the other markets. For example, due to the links between the stock and the bond markets, a shock to the stock market not only impacts the equity of the fund but also its fixed income assets and the market based discount rate which is used to calculate the present value of the liabilities.

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<sup>2</sup>Kahlert and Wagner (2015) work with a detailed breakdown of asset and liability positions in a bank's balance sheet with each position assigned to a market index.

Second, our model allows us to measure the impact of any discount rate on the evaluation of the funding status of a pension fund. Such a measure can be useful to assess the consequences of choosing one or another marked based discount factor on the risk management behavior of pension funds. Indeed, as shown by Andonov et al. (2014), imposing a discount rate related to the asset returns as in the case of U.S. public funds encourages the pension plan to increase the investment in riskier assets in order to maintain high discount rates.

The rest of the paper is organized as follows. Section 2 introduces the way to represent the balance sheet of a pension fund. Section 3 gives the methodology. Section 4 reports results. Section 5 gives concluding remarks.

## 2 A hypothetical pension fund

In this section we describe the characteristics of the hypothetical pension fund.

### 2.1 Balance sheet and risks

In the following, we briefly describe the balance sheet of a typical pension plan. It is inspired by the balance sheets of the Top 100 American pension funds. The asset side of the balance sheet is viewed as a portfolio composed of appropriate market indexes which are assigned to each of the marked-to-market positions, namely, Fixed Income, Equity and Alternatives.

According to recent regulatory rules, the discounted rate used to value liabilities is associated with a market based yield curve and, more precisely, with an AA corporate bond yield curve <sup>3</sup>.

According to the composition of the balance sheet, we observe that the pension fund inevitably takes risks on the two sides. Investment risk is one of the major risks that pension funds are facing. It changes along with the investment strategies. Longevity risk is another important risk for pension funds since future benefits are sensitive to this risk.

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<sup>3</sup>International accounting standards SFAS 87.44 and IAS19.78 recommend that pension obligations be valued by referring to an AA corporate bond yield curve.

Table 1: Balance sheet

Pension Fund	
Assets	Liabilities
Fixed Income	Present value of Future Benefit Obligation
Equity	
Alternatives	Surplus
Cash	
Others	

The table indicates the principal elements in the balance sheet of a pension plan.

The liability side is also exposed to risks such as inflation increase risk, salary increase risk and retirement age risk. In this study, we focus on the market risks and assume stable future benefit payments<sup>4</sup>. However, even if they are stable, these future payments have to be discounted and their present value depends on the discount rate. The liability side is exposed to the market risks if an AA corporate bond curve is used for discounting.

On the asset side, each market-related item is assigned to a representative market. Pension funds may have different types of fixed income securities in their portfolios. In our study, we retain two of them: inflation linked bonds and government bonds. Accordingly, we choose the *Barclays World Inflation Linked Bonds* and *Citi World Government Bond Index* as respective representative indexes. We associate the equity position with the *MSCI World Index*. Alternative investments could contain hedge funds, private equity and real estate. Accordingly, we retain the *HFRX Global Hedge Fund EUR Index*, *LPX50 Index* and *S&P Global REIT<sup>5</sup> Index* as representative indexes.

As previously indicated, the discount rate used for the evaluation of the liabilities is based on the yield of corporate bonds and, more precisely, refers to the *iBoxx EUR Corporates AA 10+ Index*. Instead of using the index prices, we use the average yield to maturity of the index to derive the discount rate. We assume that the average maturity is  $M_{AA} = 17$  years for bonds with maturities of more than ten years (10+). Following Kahlert and Wagner (2015), we convert the yield to maturity  $R_{AA,t}$  into a discount factor

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<sup>4</sup>A pension fund operates in a more complex way with some risk-mitigating instruments like technical provisions, sponsor support and pension protection arrangements. In order to simplify the analysis, these back-up instruments are momentarily excluded from our study.

<sup>5</sup>Real Estate Investment Trust

as follows:

$$DF_{AA,t} = \frac{1}{(1 + R_{AA,t})^{M_{AA}}} \quad (1)$$

In the following, the discount factor series as defined in (1) is considered as a representative index accounting for the risk of AA corporate bonds<sup>6</sup>.

An overview of the representative indexes that we assign to the different asset classes is given in Table 2.

Table 2: Asset classes and Indexes

Asset Classes	Index names
Inf. Linked Bonds	Barclays World Inflation Linked Bonds Index
Gov. Bonds	Citi World Government Bond Index
Equity	MSCI World Index
DF Corp. AA 10+	Calculated from iBoxx EUR Corporates AA 10+ Index
Hedge Fund	HFRX Global Hedge Fund EUR Index
Private Equity	LPX50 Index
Real Estate	S&P Global REIT Index

The table provides an overview of indexes assigned to different asset classes in the balance sheet of a pension fund. Note that the Discount Factor (DF) of an AA rated corporate bond index are calculated from the corresponding index's average yield.

After assigning the market indexes to the different items of the balance sheet, we focus on the funded status (FS) which is defined as:

$$FS = PA - PL \quad (2)$$

where  $PA$  is the asset value and  $PL$  is the present value of the Pension Benefit Obligation (PBO). Another health indicator for a pension fund is the funding ratio (FR):

$$FR = \frac{PA}{PL} \quad (3)$$

By definition, both indicators are clearly affected by the market risks through the market value of the fund's assets and the market related discount rate which is used to

<sup>6</sup>Note that  $DF_{AA,t}$  is the price of a representative zero-coupon bond of maturity  $M_{AA}$  at date  $t$ .

calculate the present value of the PBO.

## 2.2 Pension fund characteristics

Inspired by the 100 largest American pension funds, our hypothetical pension fund has an asset allocation as described in the following table.

Table 3: Asset allocation

Inflation linked Bonds	10%
Government Bonds	35%
Equity	35%
Hedge funds	5%
Private Equity	5%
Real Estate	5%
Cash	5%

The table gives the asset allocation of a hypothetical pension fund

Moreover, we assume that the pension plan has an average maturity (denoted as  $ML$ ) of 20 years. For the sake of simplicity, we assume that the pension plan has an initial balanced status (i.e. Funding ratio = 1). Then, by considering large shocks to the different representative market indexes, we investigate the changes in the balance sheet situation as explained in the next sections.

## 3 Methodology

We use the methodology developed by Bruneau et al. (2015) to analyze the co-movements of the different financial markets considered in the study. The CVRF (Canonical Vine Risk Factor) model we use can be viewed as a non-linear version of a risk factor model in a vine-copula framework. Before implementing stress tests on the balance sheet with this model, we recall the main underlying principles of the methodology.



### 3.1 The CVRF model

The CVRF model allows to characterize the joint distribution of  $n$  random variables as constrained by a factorial structure.

According to Sklar's theorem (Sklar, 1959) the  $n$ -dimensional cumulative distribution function (cdf)  $F$  of any vector of  $n$  random variables  $\mathbf{X} = (X_1, \dots, X_n)$  with marginals  $F_1(\cdot), \dots, F_n(\cdot)$  can be written as:

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)), \quad (4)$$

where  $C(F_1(x_1), \dots, F_n(x_n)) = F(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n))$  is some appropriate  $n$ -dimensional copula,  $F_i(X_i) = U_i$  are uniformly distributed variables and the  $F_i^{-1}$ s denote the quantile functions of the marginals.

Moreover, for any absolutely continuous  $F$  with strictly increasing continuous marginal cdf  $F_i$ , the corresponding density function  $f$  can be obtained by differentiation from (4) as:

$$f(x_1, \dots, x_n) = c_{1:n}(F_1(x_1), \dots, F_n(x_n)) \cdot f_1(x_1) \cdots f_n(x_n), \quad (5)$$

which is the product of the density  $c_{1:n}(\cdot)$  of the  $n$ -dimensional copula and the marginal densities  $f_i(\cdot)$ . Accordingly, the modeling of the marginal distributions is separated to the modeling of dependence.

Furthermore, the  $n$ -dimensional density  $c_{1:n}$  can be decomposed as a product of bi-variate copulas. The decomposition is not unique. To help organize the possible factorization, Bedford et Cooke (2001, 2002) have introduced a graphical model denoted the regular vine. Aas et al. (2009) have introduced two useful types of vines in the field of risk management, namely the C-vine (canonical vine) and the D-vine copulas. The C-vine structure is retained in the CVRF model as it gives a natural way to model risk factors since the variables can be related to each other in a hierarchical way.

Finally, the factorial structure is introduced in the model by constraining the density of some of the bivariate copulas to be equal to 1 in the C-vine factorization of the density

$c_{1:n}$ . Indeed, for any a set of conditioning variables  $v$ , two variables  $X, Y$  are conditionally independent given  $v$  if and only if:

$$c_{xy|v}(F_{x|v}(x|v), F_{y|v}(y|v)) = 1. \quad (6)$$

For example, if the links between two asset returns  $X_1$  and  $X_2$  are summarized by a third one  $X_3$  which plays the role of a common factor- for example the return of a relevant market index-, we expect that  $X_1$  and  $X_2$  are independent conditionally on  $X_3$  and we impose  $c_{x_1x_2|x_3}(F_{x_1|x_3}(x_1|x_3), F_{x_2|x_3}(x_2|x_3))$  to be equal to 1. Thus, the cdf  $F_{x_1|x_3}$  and  $F_{x_2|x_3}$  characterize the idiosyncratic part of  $X_1$  and  $X_2$  respectively.

To estimate the model, we proceed in two steps. First, we transform the random variables into uniform ones by inverting the empirical cdf of the marginal distributions as in Meucci (2007). In a second stage, we fit a Canonical Vine (C-Vine) copula structure to the uniform variables.

## 3.2 Return simulations

Following Bruneau et al. (2015), we implement three types of simulations that allow us to perform stress tests.

### 3.2.1 General Simulation

To run the simulations, we proceed as follows. First, by using the estimated parameters of the different bivariate copulas and the algorithm 2 described in Aas et al. (2009), we simulate  $N$  samples from an  $n$ -dimensional canonical vine for the next period, i.e.  $\hat{u}_{1:T+1}$ . Then, the inverse empirical distribution functions ( $F_i^{-1}$ ) produce a sample of returns, i.e.  $\hat{r}_{i,T+1} = F_i^{-1}(\hat{u}_{i,T+1})$ , for  $i = 1, 2, \dots, n$ .

In the following, we focus on the simulations of the uniforms from the vine structure. The process transformation from the uniforms to returns remains the same.

### 3.2.2 Simulation with extreme shocks

With the tool we can implement simulations in accordance to an extreme behavior of one specific index. Indeed, by using the algorithm proposed by Bedford et Czado (2013), we draw samples from a extreme zone for the stressed variable  $\hat{u}_{i_0}$ ,  $i_0 \in \{1, \dots, n\}$  instead of drawing all the  $\hat{u}_i; i = 1, \dots, n$  between 0 and 1. Since the dependence structure is supposed to be unaffected by the shock, a stress situation for one asset impacts not only the variables which are directly related to this asset but also the other variables in an indirect way, by affecting the key variable at the root node of the C-Vine (which is related to all variables). This means that a sharp decrease in the return of one particular asset can cause the distress of the whole portfolio if the other assets are positively related with the stressed one. In the case where some assets are negatively related to the stressed asset, the portfolio benefits from the diversification effects and is less affected by the shock than a single asset.

### 3.2.3 Simulation with extreme conditional shocks

In addition, we can characterize shocks as extreme values drawn from conditional distributions. Thus the shocks are interpreted as shocks to specific risk sources. First of all, some definitions of risk sources need to be clarified. The unconditional distribution of an index-factor  $f$  summarizes a set of different risk sources, whereas the conditional distribution of factor  $f_i$  given another factor  $f_j$  can be interpreted as a combination of the remaining risk sources when the risk associated with  $f_j$  has been removed. By considering the difference between the expected returns of relevant unconditional and conditional distributions, we can isolate the effect of a shock to a specific risk source. Simulations of such shocks require implementing a specific algorithm given in Bruneau et al. (2015).

## 3.3 Conditional expected funded status

By applying shocks to one market, we can get the response of all other markets. With the hypothetical allocation described in section 2.2, the conditional expected asset value of the pension plan's portfolio can be estimated.

Since the present value of the pension benefits depends on the market related discount rate (through the bond index), the liability of the pension plan and more precisely the value of the PBO is also affected by a shock to one of the financial markets.

We denote  $S$  an extreme circumstance. This event can be, for example, an extreme negative shock to a given market index as discussed more precisely later in this section. The expected value of any asset  $i$  at time  $t + 1$  given  $S$  is calculated as follows:

$$E(P_{t+1}^i|S) = P_t^i (1 + E(r_{t+1}^i|S)). \quad (7)$$

where  $P_t^i$  is the value of asset  $i$  at time  $t$  and  $E(r_{t+1}^i|S)$  is the expected return of this asset over  $[t, t + 1]$  given  $S$ .

Consequently, as the values of the asset side  $PA_t$  and  $PA_{t+1}$  at dates  $t$  and  $t + 1$  are related according to:

$$PA_{t+1} = PA_t \times \sum_i \omega_i (1 + r_{t+1}^i). \quad (8)$$

where  $\omega_i$  is the weight of asset  $i$  in the initial fund portfolio, the conditional expected value of the asset side at time  $t + 1$  given  $S$ ,  $E(PA_{t+1}|S)$ , is simply given by:

$$E(PA_{t+1}|S) = PA_t \times \sum_i \omega_i (1 + E(r_{t+1}^i|S)). \quad (9)$$

We therefore get the conditional expected return of the assets between  $t$  and  $t + 1$ , given  $S$  as:

$$\Delta_{PA} = \frac{E(PA_{t+1}|S)}{PA_t} = \sum_i \omega_i (1 + E(r_{t+1}^i|S)). \quad (10)$$

On the other side, the discounted liabilities at time  $t$  is defined as:

$$PL_t = \frac{PBO}{(1 + DR_t)^{ML}} \quad (11)$$

where  $PBO$  is the Pension Benefit Obligation of the pension plan which is assumed to

be constant in the short term,  $DR_t$  is the discount rate at time  $t$  and  $ML$  is the average maturity of the pension plan.

As we have chosen the iBoxx EUR Corporates AA 10+ index to specify the discount rate,  $DR_t = R_{AA,t}$ . Accordingly the values of liabilities at dates  $t$  and  $t + 1$  are related as follows:

$$PL_{t+1} = PL_t \times \left( \frac{1 + R_{AA,t}}{1 + R_{AA,t+1}} \right)^{ML} \quad (12)$$

Moreover, according to the equation (1), the return rate of the discount factor between  $t$  and  $t + 1$  is defined as:

$$r_{t+1}^{DF_{AA}} = \left( \frac{1 + R_{AA,t}}{1 + R_{AA,t+1}} \right)^{M_{AA}} - 1 \quad (13)$$

From the equations (12) and (13), we deduce the present value of the liabilities at  $t + 1$ :

$$PL_{t+1} = PL_t \times \left( 1 + r_{t+1}^{DF_{AA}} \right)^{\frac{ML}{M_{AA}}} \quad (14)$$

and the corresponding expected value given  $S$ :

$$E(PL_{t+1}|S) = PL_t \times E \left( \left( 1 + r_{t+1}^{DF_{AA}} \right)^{\frac{ML}{M_{AA}}} | S \right) \quad (15)$$

from which we get the conditional expected return of the liabilities between  $t$  and  $t + 1$ , given  $S$ :

$$\Delta_{PL} = \frac{E(PL_{t+1}|S)}{PL_t} = E \left( \left( 1 + r_{t+1}^{DF_{AA}} \right)^{\frac{ML}{M_{AA}}} | S \right) \quad (16)$$

The equations (8) and (14) give the conditional expected funded status given  $S$ :

$$\begin{aligned} E(FS_{t+1}|S) &= E(PA_{t+1}|S) - E(PL_{t+1}|S) \\ &= PA_t \times \sum_i \omega_i (1 + E(r_{t+1}^i|S)) - PL_t \times E \left( \left( 1 + r_{t+1}^{DF_{AA}} \right)^{\frac{ML}{M_{AA}}} | S \right) \end{aligned} \quad (17)$$

Finally the conditional expected funding ratio can be deduced from the equation (8) and (14).

$$E\left(\frac{PA_{t+1}}{PL_{t+1}}|S\right) = \frac{PA_t}{PL_t} \times E\left(\frac{\sum_i \omega_i (1 + r_{t+1}^i)}{(1 + r_{t+1}^{DFAA})^{\frac{ML}{M_{AA}}}}|S\right) \quad (18)$$

### 3.4 Stressful events

For the sake of simplicity, the stressful events are identified as  $S$  in the previous discussions. However, we can distinguish two types of extreme events as in Bruneau et al. (2015), which are large shocks to market indexes involving different underlying risks and shocks to the risk factors of the CVRF model we retain. Table 4 gives the correspondence between the indexes and the risk factors in the retained CVRF model.

Each market index is representative of one asset class but it can depend on several risk factors. Firstly, the inflation linked bond index is supposed to be exposed to the sole real interest rate risk; in that case the risk factor and the index coincide. Secondly, the world government bond index is jointly exposed to the real rate and inflation risks and therefore covers the two corresponding risk factors. Finally, the MSCI World index is the reference index for the stock markets and is generally considered as a good representation of the risk aversion; it is supposed to be related to three risk factors, the two previous ones and the equity risk factor.

Table 4: Asset classes and risk exposure

Related Asset Classes	Potential Risk Factors		
World Inflation Linked Bonds	Real Rates		
Citigroup WGBI All Maturities	Real Rates	Inflation	
MSCI World	Real Rates	Inflation	Equity Market Risk Aversion

Thus, to examine how the funding situation of the pension fund responds to the two

types of extreme shocks, we perform two types of simulations, as described in sections 3.2.2 and 3.2.3, respectively.

Let us examine the case where the pension fund is exposed to large shocks to the different risk factors. We denote  $D$  one of the variables  $r_{t+1}^i$ ,  $PA_{t+1}$ ,  $PL_{t+1}$  and  $FS_{t+1}$ . Following the idea of Bruneau et al. (2015), we measure the exposures of  $D$  to the different risk factors under extreme circumstances by running simulations under the condition that one relevant conditional distribution function is at the 1% worst case.

More precisely, by denoting  $R_1$ ,  $R_2$  and  $R_3$  the returns of the three main indexes of table 4, (Inflation linked bond, World government bond and MSCI World indexes respectively), we define the sensitivity of  $D$  to the Real Rate (RR) risk by the difference:

$$\gamma_{RR}(D) = E(D|F_1(R_1) = 1\%) - E(D) \quad (19)$$

where  $E(D|F_1(R_1) = 1\%)$  is the conditional expected value of  $D$  given that the cumulative distribution of the Inflation linked bond index is equal to 1% and  $E(D)$  is the unconditional expectation of  $D$  considered as a reference value.

The exposure of  $D$  to Inflation risk (INF), is defined as:

$$\gamma_{INF}(D) = E(D|F_{2|1}(R_2) = 1\%) - E(D) \quad (20)$$

where  $E(D|F_{2|1}(R_2) = 1\%)$  is the conditional expected value of  $D$  given that the conditional cumulative distribution  $F_{2|1}(R_2)$  is equal to 1%.

The idea is that when  $F_{2|1}(R_2)$  is equal to 1%,  $R_2$  takes an extreme value, conditionally on a given value of  $R_1$ , or equivalently for a given level of the real rate risk. In other words, this extreme event only involves the other source of risk which affects  $R_2$ , namely the inflation risk.

Similarly, the sensitivity to equity risk is computed as :

$$\gamma_M(D) = E(D|F_{3|1,2}(R_3) = 1\%) - E(D) \quad (21)$$

with  $E(D|F_{3|1,2}(R_3) = 1\%)$  denoting the expected value of  $D$  given that the conditional cumulative distribution  $F_{3|1,2}(R_3)$  is equal to 1%.

Finally the sensitivities to the remaining risk sources related to the alternatives (equity, private equity and real estate) are measured as:

$$\gamma_{SR_j}(D) = E(D|F_{j|1,2,3}(R_j) = 1\%) - E(D) \quad (22)$$

where  $E(D|F_{j|1,2,3}(R_j) = 1\%)$  is the conditional expected value of  $D$  given that the conditional cumulative distribution  $F_{j|1,2,3}(R_j)$  is equal to 1% where  $R_j$ , ( $j = 4, 5, 6$ ), denote the returns of the representative indexes assigned to the alternatives.

## 4 Empirical study

### 4.1 Data

All the data are from Bloomberg and Thomas Reuters. We work with total return indexes if they exist. Descriptive statistics of weekly arithmetic returns are reported in Table 5. We can observe that equity, private equity and real estate are more volatile than the other asset classes. The value of the Jarque-Bera test statistic along with the negative skewness and the positive excess kurtosis indicate that the non-normality of the weekly returns is simultaneously driven by asymmetry (a long tail to the left<sup>7</sup>) and fat tails.

### 4.2 Dependence

Table 6 shows the empirical dependencies between the different indexes as measured with the CVRF model. Compared with a standard correlation matrix, the dependence matrix under the C-vine structure has to be read as follows: the first column reports the unconditional dependencies between the inflation linked bonds and other assets. The second column gives the conditional dependencies between the returns of government bonds and other assets conditionally on the return of the inflation linked bonds index.

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<sup>7</sup>except for the DF Corp. AA 10+ which has a positive skewness.



Table 5: Descriptive statistics for index returns

	Inf. Linked Bonds	Gov. Bonds	Equity	DF Corp. AA 10+	Hedge Fund	Private Equity	Real Estate
Min	-4.51%	-1.24%	-19.17%	-6.82%	-6.13%	-27.18%	-16.83%
Max	3.12%	1.55%	11.38%	10.95%	2.30%	13.66%	19.01%
Mean	0.10%	0.07%	0.17%	0.11%	0.01%	0.19%	0.24%
Median	0.12%	0.10%	0.42%	0.20%	0.13%	0.48%	0.42%
Std. Dev.	0.71%	0.39%	2.29%	1.66%	0.67%	3.31%	3.14%
Skewness	-0.46	-0.06	-1.08	0.33	-2.52	-1.42	-0.46
Kurtosis	3.65	0.40	10.40	4.74	15.53	11.96	7.74
JB Test	358***	4.55*	2845***	578***	6715***	6715***	1534***

Descriptive statistics for weekly returns of indexes. Sample period: July 4, 2003 to December 26, 2014. Sample with 600 observations. The Jarque-Bera test has critical values for different level of  $\alpha$ : 4.37 ( $\alpha = 0.1, *$ ), 5.88 ( $\alpha = 0.05, **$ ), 10.56 ( $\alpha = 0.01, ***$ ).

More generally, in column  $i$ , one gets the conditional dependencies between returns of asset  $i$  and next assets ( $i + 1, i + 2, \dots$ ), given the returns of all the previous assets (from 1 to  $i - 1$ ).

Table 6: Kendall's tau under a C-vine Structure

	Inf. Linked Bonds	Gov. Bonds	Equity	DF Corp. AA 10+	Hedge Fund	Private Equity	Real Estate
Inf. Linked Bonds	1						
Gov. Bonds	0.60	1					
Equity	-0.20	-0.17	1				
DF Corp. AA 10+	0.49	0.37	0.06	1			
Hedge Fund	-0.07	-0.20	0.53	0.11	1		
Private Equity	-0.16	-0.15	0.59	0.14	0.02	1	
Real Estate	-0.05	-0.11	0.54	-0.04	-0.02	0.03	1

The table reports Kendall's tau between different asset classes under a C-vine Structure. Sample period: July 4, 2003 to December 26, 2014.

As regard to the risk levels, one can distinguish between two groups of assets: one quite risky group and another less risky one with inflation linked bonds, government bonds and AA rated corporate bonds. The “flight-to-quality” effect could explain the negative dependence between the two groups and could be exploited to get diversification benefits in managing the portfolio.

Furthermore, the positive dependency between the returns of the bond indexes and

the discount factor indicates that a negative shock to the bond indexes could have positive effects on the liability by reducing the discount factor (i.e. by increasing the discount rate). In this case, the under-funding risk could be reduced since such a stress to bonds could increase the equity and induce a drop in the liability.

In order to better understand the interest of the CVRF model, it is worth comparing the previous conditional dependence structure with an unconditional one as displayed in Table 7 which reports the unconditional Kendall’s tau coefficients for the sample period.

Table 7: Unconditional Kendall’s tau

	Inf. Linked Bonds	Gov. Bonds	Equity	DF Corp. AA 10+	Hedge Fund	Private Equity	Real Estate
Inf. Linked Bonds	1						
Gov. Bonds	0.61	1					
Equity	-0.21	-0.28	1				
DF Corp. AA 10+	0.48	0.59	-0.17	1			
Hedge Fund	-0.07	-0.17	0.56	-0.07	1		
Private Equity	-0.17	-0.24	0.62	-0.10	0.45	1	
Real Estate	-0.06	-0.12	0.55	-0.07	0.38	0.43	1

The table reports unconditional Kendall’s tau between different asset classes. Sample period: July 4, 2003 to December 26, 2014.

The first columns of the two matrices give nearly the same Kendall’s tau coefficients, in fact two estimates of the same coefficients. However it is worth comparing the fifth and sixth columns which provide the Kendall’s tau between the alternative assets. The unconditional relationships between these assets are quite strong as indicated by relatively high values of the Kendall’s tau (0.45, 0.38 and 0.43). But after conditioning on the previous assets, especially on equity, these conditional dependencies are close to nil (conditional Kendall’s tau equal to 0.02, -0.02 and 0.03 respectively). This probably means that the alternative assets are interconnected through the equity market. By contrast, the alternative assets remain strongly related to equity, even after removing the contribution of the bond indexes (conditional Kendall’s tau equal to 0.53, 0.59, 0.54). Such result indicates the existence of a “pure” equity risk contribution to the returns of the alternative assets.

The conditional dependencies as characterized from the C-vine structure obviously

play a central role in the stress test exercises as shown in the following section.

### 4.3 Sensitivity analyses

In this section, we measure the sensitivities of the funding status and the funding ratio to various benchmark risks as characterized in section 3.4.

First we focus on the sensitivities of the different market indexes assigned to the items of the asset side. They are reported in Table 8. According to the identification scheme retained for the risk factors, the inflation linked bonds index is only exposed to the real rate risk factor. In the case of an extreme shock (1% worst case) to this factor, the return of the inflation linked bond index decreases by 1.97% in one week. The government bond index return decreases by 0.79% and 0.6% respectively after an extreme shock (1% worst case) to the real rate risk factor and to the inflation risk factor. Due to its negative dependency with the first two risk factors, the equity index displays positive reactions when each of these factors are stressed: its return increases by 1.71% and 1.23% respectively. On the contrary, as expected, the equity has negative expected returns (-5.54%) when the equity risk factor is stressed. The private equity index and the real estate index are also exposed to the equity risk factor. The hedge fund appears to be the least sensitive asset to the different market risk factors in our sample.

Table 8: Sensitivities of assets to risk factors

	Real rate	Inflation	Equity	Corp. AA	Hedge fund	Private Equity	Real Estate
Inf. Linked Bonds	-1.97%						
Gov. Bonds	-0.79%	-0.60%					
Equity	1.71%	1.23%	-5.54%				
DF Corp. AA 10+	-1.82%	-1.88%	-0.24%	-2.64%	-0.75%	-0.34%	0.15%
Hedge Fund	0.10%	0.40%	-0.80%	-0.41%	-1.59%	-0.07%	0.03%
Private Equity	1.95%	1.54%	-6.47%	-0.58%	-0.35%	-4.51%	-0.20%
Real Estate	0.54%	1.23%	-6.46%	0.28%	0.16%	-0.25%	-5.20%

This table reports the sensibilities of all assets under a shock to one of the risk factors listed in the header row.

By using the asset allocation<sup>8</sup> given in 2.2, we obtain the sensitivities of the asset

<sup>8</sup>Note that the return of cash remains zero in all cases.

value (PA) to the different risk factors. Instead of the expected asset value, we compute the ratio of the conditional expected asset value to the initial asset value (i.e  $\Delta_{PA}$ ) as it gives changes from initial value.

From the simulation without shock, we can compute the benchmark value of the  $\Delta_{PA}$  which is equal to 100.12% <sup>9</sup>. The first row of the Table 9 documents the changes of  $\Delta_{PA}$  in this benchmark value due to a shock to the different risk factors (see definition in section 3.4). As mentioned before, negative dependencies between assets can provide diversification effects. For example, the sensitivity of the portfolio to a shock to the real rate risk factor is lower than the ones of the bonds taken separately (0.25% for  $\Delta_{PA}$  against -1.97%, -0.79% and -1.82% for the different bonds). Even so, the asset side of the balance sheet is not completely immunized against risk factors. In the case of an extreme shock to the equity risk factor, the ratio  $\Delta_{PA}$  could lose 2.62% in one week.

Table 9: Sensitivities of funding status to risk factors

	Real rate	Inflation	Equity	Corp. AA	Hedge fund	Private Equity	Real Estate
$\Delta_{PA}$	0.25%	0.38%	-2.62%	-0.04%	-0.09%	-0.24%	-0.27%
$\Delta_{PL}$	-2.14%	-2.21%	-0.28%	-3.10%	-0.89%	-0.40%	0.18%
Funded Status	2.39	2.59	-2.35	3.06	0.80	0.16	-0.45
Funding Ratio	2.44%	2.68%	-2.34%	3.16%	0.81%	0.16%	-0.45%

This table documents the sensibilities of funding situation variables under a shock to one of the risk factors listed in the header row.

Following the same principle, we focus on the liability side and examine the reaction of the discount factor under market shocks. We calculate the benchmark value of the ratio ( $\Delta_{PL}$ ) (expected liability value to the initial liability value without shock, according to the equation (16)) and we find that it is equal to 100.14%. The second row of Table 9 reports the sensitivities of the  $\Delta_{PL}$  ratio to different risk factors as defined in section 3.4. We note that the response of this ratio (-0.28%) is negative after a (negative) shock to the equity risk factor. Normally, a decrease in the equity return is expected to involve an increase in the bond returns and in particular in the return of the DF Corp. AA 10+

<sup>9</sup>The unconditional ratio is calculated in a similar way as in the equation (10) except that the return of each asset is simulated in a normal case i.e. without shock (i.e.  $E(r_{t+1}^i)$ ).

index, which should induce an increase in  $\Delta_{PL}$  ; however in evaluating the impact of a shock to the stock risk factor, the important thing is the conditional dependence between the DF Corp. AA 10+ and the stock risk factor as obtained by conditioning on the two first bond indexes. This conditional dependence is positive as indicated in Table 6. This could mean that the sensitivity of DF Corp. AA 10+ index to the equity risk factor involves the sole credit risk channel.

More generally it is worth noting that the  $\Delta_{PL}$  ratio is more sensitive to the bond related risk factors (namely Real rate, inflation and Corporate bonds) than to the other risk factors.

Assuming that the pension fund is initially fully funded (i.e. Funded status = 0 or Funding ratio = 1), the initial asset value as well as the initial liability value can be given as 100 without loss of generality (i.e.  $PA_t = PL_t = 100$ ). Combining the equation (17) and the sensitivity definitions given in section 3.4, we can compute the funded status of the pension fund. We observe that it can benefit from a shock to the bond related risk factors while it can suffer from a shock to the equity risk factor.

The sensitivity of the funding ratio can be measured in a similar way according to the equation (18). We find a benchmark value of this ratio equal to 100.02%. The changes caused by extreme shocks to the different risk factors are given in the fourth row. As previously mentioned, we observe that a shock to the equity risk factor worsens the funding situation with a decrease in  $\Delta_{PA}$  (-2.62%) due to a drop in the asset values which is not compensated by the decrease in the discounted liability (-0.28% for  $\Delta_{PL}$ ).

## 5 Conclusion

When considering the asset-liability management of a pension fund, market risk is one of the most important risks faced by its balance sheet. Indeed the expected return of its assets which are directly exposed to market risk is a key factor of its future income. Moreover recent regulatory reforms tend to require pension funds to use a market based discount rate for discounting their liabilities. Consequently, both sides - asset and

liability- are exposed to market risk.

In this paper, we are interested in measuring the exposure of a pension fund to this risk. We use a CVine risk factor model (CVRF) which combines a factorial structure and copulas to perform stress tests to the balance sheet of an hypothetical pension fund. Each item of the pension fund assets is assigned to an appropriate index while the discount rate is linked to an AA-rated corporate bond index yield. The CVRF allows us to modelize the inter-dependencies between all indexes including the tail ones.

By running simulations under different extreme market circumstances, we evaluate the expected funding situation of the fund when it is exposed to extreme negative shocks to major market risk factors (real rate, inflation, equity and alternatives). We show that diversification effects may partly reduce the loss of the asset side during periods of market turbulence. We also find that large shocks to the equity market risk factor are most likely to severely affect the funding situation of the pension fund by inducing large asset losses.

Due to the fact that the number of risk factors is limited in this study, the diversification effects are rather under-estimated. This simplicity allows us to illustrate clearly the interaction between asset and liability and the resulting funding situation under stress scenarios. Future researches could extent this stress testing approach to a more complete pension fund balance sheet. In addition, considering different maturities of the future benefit payment could allow to examine more realistic implications on the liability side. We could also use this approach to assess the impact of the discount rate. For example we could compare the exposure of the funding situation to extreme shocks for different regulation requirements concerning the liability discount rate.

## **6 Index definitions**

### **Barclays World Government Inflation-Linked Bond (WGILB) Index**

Barclays Capital World Government Inflation-Linked Bond (WGILB) Index measures the performance of the major government inflation-linked bond markets. The index is designed to include only those markets in which a global government linker fund is likely and able to invest. Investability is therefore a key criterion for inclusion of markets in this index. The markets currently included in the index, in the order of inclusion, are the UK, Australia, Canada, Sweden, US, France, Italy, Japan and Germany.

### **Citi World Government Bond Index (WGBI)**

The World Government Bond Index (WGBI) measures the performance of fixed-rate, local currency, investment grade sovereign bonds. The WGBI is a widely used benchmark that currently comprises sovereign debt from over 20 countries, denominated in a variety of currencies, and has more than 25 years of history available. The WGBI provides a broad benchmark for the global sovereign fixed income market.

### **MSCI World Index**

The MSCI World Index captures large and mid cap equities across 23 Developed Markets (DM) countries (Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Hong Kong, Ireland, Israel, Italy, Japan, Netherlands, New Zealand, Norway, Portugal, Singapore, Spain, Sweden, Switzerland, the UK and the US). With 1,633 constituents, the index covers approximately 85% of the free float-adjusted equity market capitalization in each country.

## **HFRX Global Hedge Fund Index**

The HFRX Global Hedge Fund Index is designed to be representative of the overall composition of the hedge fund universe. It is comprised of all eligible hedge fund strategies; including but not limited to convertible arbitrage, distressed securities, equity hedge, equity market neutral, event driven, macro, merger arbitrage, and relative value arbitrage. The strategies are asset weighted based on the distribution of assets in the hedge fund industry.

## **LPX50 Index**

The LPX50 is a global index that consists of the 50 largest liquid LPE(Listed Private Equity) companies covered by LPX Group. It is a suitable benchmark index for private equity.

## **S&P Global REIT Index**

The S&P Global REIT (Real Estate Investment Trust) index serves as a comprehensive benchmark of publicly traded equity REITs listed in both developed and emerging markets. It consists of over 250 constituents from 19 developed and emerging markets.

## **iBoxx EUR Corporates AA 10+ Index**

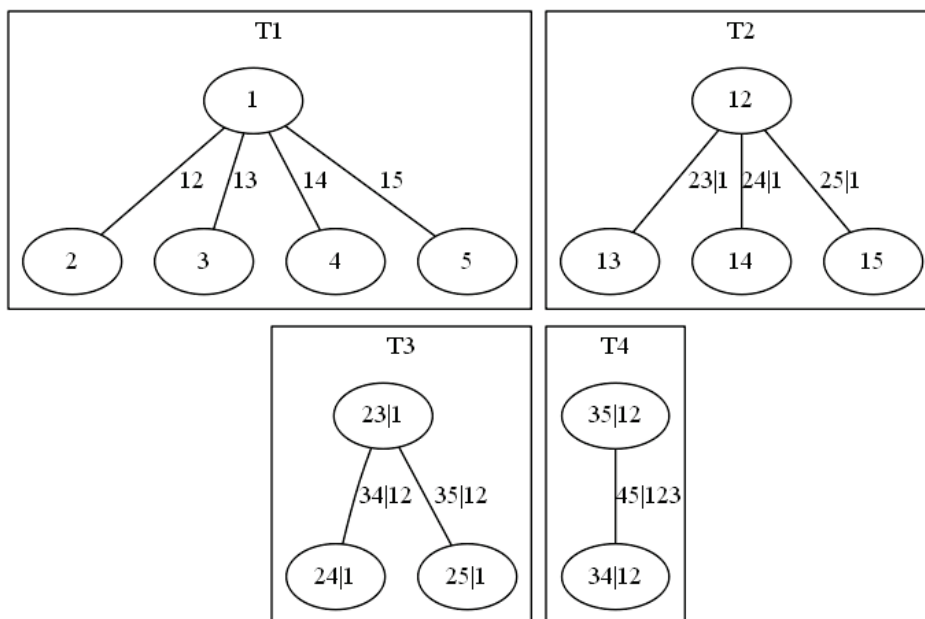
The iBoxx EUR Corporates AA 10+ Index measures the performance of AA-rated corporate bonds with a more than 10 years maturity.

## **7 An example of Canonical vine**

Figure 1 shows a canonical vine with five variables. From the figure, we observe that the variable 1 at the root node is a key variable that plays a leading role in governing interactions in the data set.



Figure 1: A five dimensional canonical vine tree



In the first tree, all nodes are associated with the  $X_1, \dots, X_5$  variables. For example, the edge 12 corresponds to the copula  $c(F_1(x_1), F_2(x_2))$ . In the second tree, the edge 23|1 denotes the copula  $c(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1))$ . The following trees are built according to the same rules.

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