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Stability of rejections and Stable Many-to-Many Matchings

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Abstract

For models of many-to-many matchings, stable outcomes exist if agents on both sides have path-independent choice functions. We show that stable outcomes exists if the agents on one side have outcast choice functions and the agents on another have path independent choice functions. All known results on existence of stable outcomes in many-to-many matchings follows from this result. Many-to-many matchings with contracts have stable outcomes under the same conditions. In order to prove our existence theorem, we introduce a new class of telescopic choice functions. We also consider non-symmetric blocking situations and prove that in such a case path-independence can be weakened.

1 Introduction

A market is two-sided if there are two sets of agents, and if an agent from one side of the market can not be matched with an agent from the same side. Gale and Shapley [10] proposed in the so-called one-to-one two-sided markets that a matching (of men and women) could be regarded as stable only if it left no pair of agents on opposite sides of the market who were not matched to each other but would both prefer to be. They showed that a special property of two-sided (as opposed to one or three-sided) markets is that stable matchings always exist (at least when agents' preferences are uncomplicated).

A natural application of two-sided matching models is to labor markets. Shapley and Shubik [22] showed that the properties of stable matchings are robust to generalizations of the model which allow both matching and wage determination to be considered together. Kelso and Crawford [14] showed how far these results can be generalized when firms have substitutable preferences over sets of workers. Gul and Stachetii [11] showed that the Kelso-Crawford gross-substitutability is of importance for stability. Danilov et al [6, 7] showed that gross-substitutability is one of instances of discrete convexity, and that the discrete convexity is the key for existence of equilibria with indivisibles was established in [8]. There are non isomorphic classes of discrete convexity determined by non isomorphic totally unimodular sets of vectors (see [4]). Each of these classes can be thought as a new non-trivial instance for gross-substitutability ([16]).

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In many-to-many two-sided markets, that is if an agent from one side of the market can be matched with a set of agents from the other side, matchings could be regarded as stable either as pair-wise as above, or as not blocking by one side or by both sides (see, for example, Roth [20]).

Roth [20] showed existence of stable many-to-many matchings without money transfers under the assumption of substitutability agents' preferences. This assumptions is equivalent to path-independence introduced by Plott [19] indeed (see Section 5).

We consider a class of telescopic choice functions and show that stable matching exist if agents on one side have idempotent 1-telescopic choice functions and on another side idempotent 2-telescopic functions. These conditions can be consider as necessary and sufficient for existence of stable matchings, but here we do not go to details here.

Under these conditions stable outcomes for models many-to-many matchings with contracts also exist and, thus, generalize existence results (see, for example [9, 12, 13, 20] and the literature cited there).

We also consider a non-symmetry in blocking facilities and show that the telescopic condition can be weakened.

2 Many-to-many matching

We consider a standard model many-to-many matchings with contracts: there are finite sets D and H of workers (doctors, teachers) and firms (hospitals, schools), and a finite set C of contracts. Each contract $c \in \mathcal{C}$ specifies a relationship between a worker, $d(c)$, and a firm, $h(c)$. The usual matching model is obtained by setting $\mathcal{C}$ to be isomorphic to $D \times H$, that is a case in which each worker and each firm deals in a single contract.

For a set $C \subset \mathcal{C}$ of contracts, we denote by C_d the subset contracts of C such that, for any $a \in C_d$ it holds $d = d(a)$. Similarly, C^h denotes the subset of contracts such that, for any $a \in C^h$ it holds $h = h(a)$.

Without loss of generality, we assume that each worker has the same amounts of contracts (a finite set) with each firm and vice versa. Let K be a set of contracts between a worker and firm. Then one can consider a contract as a tripe (d, h, k) in the box $D \times H \times K$, and, hence, $\mathcal{C}_d$ is isomorphic to $H \times K$, and $\mathcal{C}^h$ is isomorphic to $D \times K$.

To evaluate contracts each worker $d \in D$ has a choice function $f_d : 2^{H\times K} \to 2^{H\times K}$, $f_d(H') \subset H'$, $H' \subset H \times K$, and each firm has a choice function $f_h : 2^{D\times K} \to 2^{D\times K}$.

We consider a class of *idempotent* choice functions $f : 2^X \to 2^X$ (here X denotes universal set of elements under choice), that is, for any $Y \subseteq X$, it holds that $f(f(Y)) = f(Y)$.

2.1 Stable matchings

Stability of outcomes in the many-to-many matchings is defined as follows.

Definition 1 *For a profile of choice functions $f_d : 2^{H\times K} \to 2^{H\times K}$, $d \in D$, and, $f_h : 2^{D\times K} \to 2^{D\times K}$, $h \in H$, a set of contracts $A \subset H \times D \times K$ is* stable *if it is individually rational and unblocked, that is*

IR *For all* $d \in D$ *and* $h \in H$, *it holds that* $f_d(A_d) = A_d$, *and* $f_h(A^h) = A^h$, *where* $A_d = A \cap (H \times d \times K)$, $A^h = A \cap h \times D \times K$.

UB *There does not exists a worker* d *and a set of firms* $H' \subset H$, *and non-empty set of contracts* $(d \times A') \subset D \times H' \times K$ *such that*

$A' \subseteq f_d(A' \cup A_d)$ *and, for any* $(d, h', a') \in (d \times A')$,

$d \notin A^{h'}$ *and* $(d, a') \in f_{h'}(A^{h'} \cup (d, a'))$.

And there does not exists a firm h *and a set of workers* $D' \subset D$, *and a set of contracts* $(h \times A') \subset D' \times H \times K$ *such that*

$D' \subseteq f_h(A' \cup A^h)$ *and, for any* $(d', h, a') \in A' \times h$,

$h \notin A_{d'}$ *and* $(h, a') \in f_{d'}(A_{d'} \cup (h, a'))$.

Stability in many-to-many matchings without contracts is due to the specification of the above definition to a case with a singleton K:

Definition 2 *For a profile of choice functions* $f_d : 2^H \to 2^H$, $d \in D$, *and,* $f_h : 2^D \to 2^D$, $h \in H$, *a set of matchings* $A \subset H \times D$ *is* stable *if it is individually rational and unblocked, that is*

IR *For all* $d \in D$ *and* $h \in H$, *it holds that* $f_d(A_d) = A_d$, *and* $f_h(A^h) = A^h$, *where* $A_d = A \cap (H \times d)$, $A^h = A \cap (h \times D)$.

UB *There does not exists a worker* d *and a set of firms* $H' \subset H$ *such that*

$H' \subseteq f_d(H' \cup A_d)$ *and, for any* $h' \in H'$, $d \notin A^{h'}$ *and* $d \in f_{h'}(A^{h'} \cup d)$;

and there does not exists a firm h *and a set of workers* $D' \subset D$ *such that*

$D' \subseteq f_h(D' \cup A^h)$ *and, for any* $d' \in D'$, $h \notin A_{d'}$ *and* $h \in f_{d'}(A_{d'} \cup h)$.

3 Stability of rejections and stable matchings

Let us introduce a tower of classes of choice functions.

For a set A and partition $A = A_1 \coprod A_2 \coprod \ldots \coprod A_k$ (some A_i's are allowed to be empty sets), we let to denote $C_1 := f(A_1)$, $C_2 := f(C_1 \cup A_2) = f(f(A_1) \cup A_2)$, ..., $C_i := f(C_{i-1} \cup A_i), \ldots$.

Definition 3 *A choice function* f *is* k-telescopic *if, for any set* $A \subset X$ *and partition* $A = A_1 \coprod A_2 \coprod \ldots \coprod A_k$, *and any* $B \subset A \setminus C_k$ *it holds true*

$$f(B \cup C_k) \cap B \neq B. \tag{1}$$

A class of $|X|$*-telescopic choice functions, we call* T-choice *functions.*

For example, for $k = 1$, the 1-telescopic property is

$$f(f(A) \cup B) \cap B \neq B$$

for any A and $B \subset A \setminus f(A)$.

We call this property *1-T*. Obviously, the class of 1-T choice functions is bigger than the class of 2T-choice functions etc.

The condition (1) is a kind of formalization a requirement of stability of rejections: elements of $A \setminus C_k$ constitute the set of rejected alternatives due to the choice procedure determined by the partition, and (1) says that no subset of such elements can be in the choice of the union the final choice set and such a subset. Notice, that, for a singleton B , it holds $B \notin f(B \cup C_k)$.

Our main result is the following theorem.

Theorem 4 *Suppose each worker $d \in D$ has an idempotent 1-T choice function, and each firm $h \in H$ has an idempotent T-choice function. Then the stable outcomes exist.*

Proof. We consider a slightly modified algorithm due to Roth [20] (see also [12]). In the beginning, for simplicity of notations, we consider a case of singleton K.

Specifically, we consider the *worker-active* algorithm:

at each next step the workers who have no contracts apply according to the following rule: such a worker applies for firms which are her/his best choice among the list of those firms which not rejected this worker at previous steps. Firms in their term make the following choices: a firm chooses among the set of workers which were chosen at previous step union the workers who apply to this firm at this step.

Because, for each worker, on each step the available set for the choice does not increase, the algorithm terminates.

Let A^f_d, $d \in D$, and A^f_h, $h \in H$, be outputs of the algorithm. Then the individual rationality is obviously holds true. Let us check UD- property.

Let d be a worker potentially blocking the algorithm outputs. Then there exists a set $H' \subset H \setminus A^f_d$ such that it hold

$$H' \subset f_d(A^f_d \cup H')$$

and, for any $h \in f_d(A^f_d \cup H')$,

$$d \in f_h(A^f_h \cup d).$$

Denote A'_d the set of firms which did not rejected the worker d in their choice at all steps of the algorithm. Then $A^f_d = f(A'_d)$. Because of *1-T* property, $H' \not\subset A'_d$. In fact, if $H' \subset A'_d$, then

$$f_d(H' \cup A^f_d) \cap H' \neq H'.$$

Let $h \in H' \setminus A'_d$. Then h rejects d at some step of the algorithm. Hence by the T property $d \notin f_h(A^f_h \cup d)$. This shows that d can not block the algorithm output.

Suppose a firm h can block the outcome with a set D' of workers. Then, due to the above arguments, for any $d \in D'$, $h \notin A'_d$. Hence each of $d \in D'$ had been rejected by h at some step of the algorithm. Due to T property, we get

$$f(D' \cap A_h^f) \cap D' \neq D'.$$

In fact, let $A_1, \dots A_i, \dots, A_k$ be the sets of workers who apply for h at some step of algorithm. Due to the definition, for any $i \neq j$, $A_i \cap A_j = \emptyset$. Hence these sets form a partition, let us denote $A := A_1 \coprod \dots \coprod A_k$. Then due to the design of the algorithm, we have $A_h^f = C_k$, and $D' \subset A \setminus C_k$. This and T property imply the above claim.

Thus, h can not block the algorithm outcome.

For arbitrary K, the algorithm has to be modified as follows, the workers set D is replaced by $D \times K$. □

Conditions *1-T* and *T* are not less not more what we need to ensure a stable matching due to the worker-active algorithm in general. We can swap the role of workers and firms and consider the firm-active algorithm. Then the stable outcomes exist if firms are endowed with *1-T* choice functions and workers are endowed with *T*-functions.

We are now going to compare these classes of *1-T* choice functions and *T*-choice functions with known classes of choice functions.

Namely, we have

Proposition 5 *The following conditions are equivalent*

a) choice function f is idempotent and satisfies 1-T *;*

b) for any $B \subset A$ such that $f(A) \subseteq B$ there holds true

$$f(B) = f(A).$$

Proof. Suppose a) holds true, but b) is violated for some pairs $B \subset A$. Choose among the set of such pairs a pair with A being of minimal possible cardinality. Let $B \subset A$ be such a pair. Suppose $f(B)$ is not a subset of $f(A)$. Then, because of the cardinality of B is smaller than that of A, and $(f(A) \cup (f(B) \setminus f(A)) \subset B$ and $f(B) \subseteq (f(A) \cup (f(B) \setminus f(A))$, we have

$$f(f(A) \cup (f(B) \setminus f(A)) = f(B).$$

The latter contradicts *1-T*.

If $f(B) \subseteq f(A)$, then form idempotence and *1-T*, we get the equality $f(A) = f(B)$.

Suppose b) holds true. Then f is idempotent and *1-T* holds true. In fact, $f(A) \subseteq f(A) \cup C \subset A$ for any $C \subset A \setminus f(A)$, and hence $f(f(A) \cup C) = f(A)$ that is nothing else but the *1-T* property. □

Remark. The item b) is known under many names: the Nash axiom, or Postulate 5 in Chernoff (1951), or Outcast axiom **O** (see [1, 2, 3, 17]).)

Theorem 6 *The following conditions are equivalent* *i) choice function f is idempotent and satisfies 2-*T *;*

ii) for any A and B, it holds true

$$f(A \cup B) = f(f(A) \cup B).$$

Before proving this theorem, let us recall that ii) is known under the name *path-independence* [5, 15, 17, 19].

Let us recall the following two well-known characterizations for path-independence (see, for example, [5, 15, 18])

H&O path-independence is equivalent to fulfilling Outcast axiom **O** and Heredity axiom **H**, the latter says that for any $A \subset B$ there holds $f(A) \cap B \supseteq f(B)$;

Mult for a choice function f defined on a set X, there exist a set of linear orders on X such that $f(A)$ is the union of best elements of these orders restricted to A.

Specifically, for a set X, a bijection $\ell : X \to \{1, \dots, |X|\}$ is a *linear order*, $i <_\ell j$ if $\ell(i) < \ell(j)$.

For a set $Y \subset X$ and a linear order l, we denote $l|_Y$ the restriction of l to Y, and $\operatorname{Arg\,max}_Y l$ is an element of $y \in Y$ having minimal value $l(y)$. Let $\mathbf{l} := \{l_1, \dots, l_s\}$ be a collection of linear orders on X. Then such a collection defines a choice function defined by

$$f_{\mathbf{l}}(Y) = \cup_{i=1,\dots,s} \operatorname{Arg} \max_Y l_i.$$

This function is path independent and, moreover, any path independent function has such a form (see, for example, [1, 5, 15]).

Proof Theorem 6. We use characterization **H&O** for proving the implication $i) \Rightarrow ii)$. Because of Theorem 5, it remains to show that *2-T* implies **H**.

Suppose to the contrary that **H** is violated for a pair $A \subset B$. This means that $A \cap f(B)$ is not a subset of $f(A)$. Let us denote $C := (A \cap f(B)) \setminus f(A)$, C is nonempty. Consider the following partition

$$B = A \coprod B \setminus A.$$

Denote by $D := f(f(A) \cup (B \setminus A))$. Then, since C is non-empty, $E := f(B) \setminus D$ is also non-empty.

Because $f(B) \subset D \cup E \subset B$, due to **O**, we have

$$f(D \cup E) = f(B) \Rightarrow E = f(B) \cap E.$$

That is not the case due to *2-T*, and, hence, the implication holds true.

The implication $ii) \Rightarrow i)$ follows from the characterization **Mult**. □

Because of Theorems 5 and 6, we can reformulate Theorem 4 as follows:

Theorem 7 *Suppose that each worker $d \in D$ has an outcast choice function, and each firm $h \in H$ has a path independent choice function. Then the stable outcomes exist.*

Remark. One can consider conditions in Theorem 4 (Theorem 7) as necessary and sufficient for existence of stable outcome in the *worker-active* algorithm. Namely, if for some workers or firms the conditions are weakened, then one can find a model with such weakened conditions without stable outcomes. We come to details in another paper.

4 Non-symmetric stability

Typically, role of workers and firms is not symmetric: a firm offers vacancies (contracts) and workers apply for some of them. A firm makes choices from sets of applicants. Hence, workers look alike active players: they are looking for job openings and make applications. Firms are not active: if a firm needs for workers, it offers a list of vacancies (contracts). Since worker are active and can observe all lists of vacancies and firms can not observe all workers and their possible applications, we suppose that a workers can improve her/his current situation by offering contracts to a set of firms, while a firm is limited in at most a single extra contract. Roth and Sotomayor [21] called a similar situation as stability on the worker side.

Because of our assumptions, we can consider singleton K.

Definition 8 *For a profile of choice functions $f_d : 2^H \to 2^H$, $d \in D$, and, $f_h : 2^D \to 2^D$, $h \in H$, a set of matchings $A \subset H \times D$ is* W-stable *if it is individually rational, that is*

IR *For all $d \in D$ and $h \in H$, it holds that $f_d(A_d) = A_d$, and $f_h(A^h) = A^h$;*

unblocked on the side of workers:

UB *There does not exist a worker d and a set of firms $H' \subset H$ which form a blocking pair;*

and cannot be improved by any pair of a firm and a worker:

NI *There do not exist a firm h and a worker $d \notin A^h$ such that $d \in f_h(A^h \cup d)$.*

We weaken *2-T* property as follows. Namely, a choice function $f : 2^X \to 2^X$ is *2-TS*-choice function if, for any tuple (y, Y) such that $y \notin f(Y \cup y)$, and any $Z \subseteq X \setminus Y$ it holds the *2-T singleton* property:

$$y \notin f(f(Z \cup f(Y)) \cup y).$$

We have

Theorem 9 *Suppose that a choice function f_d, $d \in D$, is an* 1-T *idempotent choice functions and a choice function f_h, $h \in H$ is a* 2-TS *and* 1-T *idempotent choice function. Then the W-stable outcomes exist.*

Proof. We apply the same algorithm as in proving Theorem 4.

Let us show that under Theorem assumptions any final outcome of such an algorithm is W-stable.

From the contrary, suppose there is a worker d who blocks A^f_d with a set of firms H'. Then by the same reasons as in the proof of Theorem 4 , d was rejected by each $h' \in H'$ at some non terminal step of the algorithm.

But the inclusion $d \in f_{h'}(A^f_{h'} \cup d)$ contradicts *2-TS*, since for each step t' after rejecting d, it holds $d \notin f_{h'}(A^{t'}_{h'} \cup d)$.

Since, for any $h \in H$, f_h satisfies **O**, the inclusion $d \in f_h(A^f_h \cup d)$ holds true iff d has been rejected at some step of the algorithm. The same reasons as above shows that this can not be the case. $\square$

4.1 2-T singleton choice functions

Here is an example of *2-TS* outcast choice function which does not satisfy the substitutability axiom.

Example 1. Let $X = \{1, 2, 3\}$, and the choice function specified as follows, $f(123) = 123$, and

$$f(1) = f(12) = 1,$$
$$f(2) = 2,$$
$$f(3) = f(23) = f(13) = 3.$$

Because $1 \in f(123)$, and $1 \notin f(13)$, f does not satisfy the substitutability axiom **H**, and does satisfy the outcast axiom **O**

However, f satisfies *2-TS* axiom: $2 \notin f(1 \cup 2)$ implies $2 \notin f(f(13) \cup 2)$; $2 \notin f(3 \cup 2)$ implies $2 \notin f(f(13) \cup 2)$; $1 \notin f(3 \cup 1)$ implies $1 \notin f(f(23) \cup 1)$. Therefore axiom *2-TS* is valid for this choice function. $\square$

Here is an example of O-choice function which do not satisfy the axiom *2-TS*.

Example 2. Let $X = \{1, 2, 3\}$, and the O-choice function specified as follows, $f(123) = 123$, and

$$f(1) = f(13) = 1,$$
$$f(2) = f(12) = 2,$$
$$f(3) = f(23) = 3.$$

However, there hold $1 \notin f(12)$ and $f(f(f(12) \cup 3) \cup 1) = f(13) = 1$. But due to the *2-TS* axiom $1 \notin f(12)$ would imply $1 \notin f(f(f(12) \cup 3) \cup 1)$. $\square$

5 Roth's conditions

Here we demonstrate that Roth [20] considered a class of preferences isomorphic to the class of path-independent choice functions indeed (this is also the case for almost all literature on matching models without money, see for example [12, 9] and the literature cited there).

Specifically, Roth [20] considered a class of transitive binary relations on 2^X, such that, for each binary relation of the class and for any $A \subseteq X$, there is a unique subset

$A' \subset A$ which is preferred to any other subset of A (including A if $A \neq A'$). Such a unique subset defines a choice and a choice function, $f(A) = A'$, $A \in 2^X$. For a $\prec$, a transitive binary relation on 2^X, we denote $f_\prec$ such a defined choice function.

Lemma 10 *For a transitive binary relation $\prec$ on 2^X, the choice function $f_\prec$ satisfies the outcast axiom* **O**.

Proof. In fact, suppose $B \subset A$ and $f(A) \subset B$, but $f(B) \neq f(A)$. Then, by the definition, $f(A) \succ f(B)$ $(f(B) \subset A)$ and $f(B) \succ f(A)$ $(f(A) \subset B)$, this contradict transitivity of $\prec$. □

Let us note, that the reverse is not true. Namely, for an **O**-choice function f, the binary relation $<_f$ defined from

$$B <_f A \text{ if there exists } C \text{ such that } A = f(C),\ B \subseteq C,\ B \neq A$$

need not to be transitive. For example, this can be demonstrated by Example 2. In fact, from $1 = f(13)$ follows $3 <_f 1$, form $f(23) = 3$ follows $2 <_f 3$, and from $f(12) = 2$ follows $1 <_f 2$, so we get a cycle $1 <_f 2 <_f 3 <_f 1$. □

For an **O**-choice function f it holds true

$$B <_f A \quad \Leftrightarrow \quad A = f(A \cup B).$$

In fact, from $A \cup B \subset C$ and $f(C) = A$, it follows that $f(A \cup B) = A$.

Roth [20] called a transitive binary relation $\prec$ on 2^X a *substitutability preference* if the corresponding choice function $f_\prec$ satisfies the following condition: for all $x, z \in X$ and $Y \subset X$, if $x \notin f(Y \cup x)$, then $x \notin f(Y \cup x \cup z)$.

The latter condition is nothing else but the heredity axiom **H**.

Thus, we get that the class of substitutable preferences is isomorphic to a subclass of path-independent choice functions. But they are isomorphic indeed. This follows from the following

Proposition 11 *Let f be path-independent, then $<_f$ is transitive.*

Proof. In fact, let $A <_f B <_f C$. Then there is C' such that $B \subseteq C'$ and $C = f(C')$, and there is B' such that $A \subseteq B'$ and $B = f(B')$. Due to this and path-independence, we get

$$f(B' \cup C') = f(B \cup C') = f(C') = C.$$

Thus, $A \subset B' \cup C'$ and $C = f(B' \cup C')$, that is nothing but $A <_f C$. □

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