**No-arbitrage and equilibrium in finite dimension:
a general result**

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No-arbitrage and Equilibrium in Finite Dimension: A General Result

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Abstract

We consider an exchange economy with a finite number of assets and a finite number of agents. The utility functions of the agents are concave, strictly increasing and their suprema equal infity. We use weak no-arbitrage prices a la Dana and Le Van [5]. Our main result is: an equilibrium exists if, and only if, their exists a weak no-arbitrage price common to all the agents.

Keywords: asset market equilibrium, individually rational attainable allocations, individually rational utility set, no-arbitrage prices, weak no-arbitrage prices, no-arbitrage condition

JEL Classification: C62, D50, D81,D84,G1

1 Introduction

The literature on the existence of an equilibrium on financial asset markets is very huge. Because short-sales are allowed, the consumption set is not bounded any more from below. As a consequence, unbounded and mutually compatible arbitrage opportunities can arise. In such cases, prices at which all arbitrage opportunities can be exhausted may fail to exist, and thus, equilibrium may fail to exist. The literature focuses on conditions which ensure the compactness of the individually rational feasible allocations set or of the individually rational utility set. These conditions are known as *no-arbitrage* conditions. We can classify them in three categories:

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- Conditions on prices, like Green [10], Grandmont [8], [9], Hammond [15] and Werner [27],

- Conditions on net trade, like Hart [16], Page [22], Nielsen [21], Page and Wooders [25], Allouch et ali [1], Page, Wooders and Monteiro [24],

- Conditions on utility set, like Brown and Werner [4], Dana, Le Van, Magnien [6].

A natural question arises: under which conditions there is an equivalence between these conditions? In [1], Allouch, Le Van and Page prove the equivalence between Hart's condition and No Unbounded Arbitrage of Page with the assumption that the utility functions have no half-line, i.e. there exists no trading direction in which the agent's utility is constant. These conditions imply existence of a general equilibrium. But the converse is not always true, i.e., the existence of equilibrium does not ensure these no-arbitrage conditions are satisfied. We can find in Ha-Huy and Le Van [12] an example of economy where these conditions fail but an equilibrium exists.

Observe in the papers we cite above, no-arbitrage prices are the ones at which arbitrage opportunities are exhausted. Dana and Le Van in [5] introduce weak no-arbitrage price, a no-arbitrage price weaker than the one in Werner [27], or in [1]. Their no-arbitrage prices are, up to a scalar, the marginal utilities of the consumptions. Following [5], we use in this paper these weak no-arbitrage prices.

Our main result is just as follows. Suppose the utility functions of the agents are concave, strictly increasing with suprema equal to infinity, then an equilibrium exists **if, and only if**, their exists a weak no-arbitrage price common to all these agents. This result is quite new since we only assume concavity and increasingness of the utility functions. In many papers, additional assumptions are required to get this result, for instance, no half-line, strict concavity, closedness of the gradients. We emphasize that the proof of this result does not not pass by the proof of the compactness of the individually rational utility set as in the papers of the existing literature on the existence of equilibrium on assets markets.

The paper is organized as follows. In Section 2, we present the model with the definitions of equilibrium, individually rational attainable allocations set, individually rational utility set, useful vectors and useless vectors. In Section 3, we review some no-arbitrage conditions in the literature. In particular we define weak no-arbitrage prices. Section 4 presents our main result. We explain there the different steps of the proof of our result. Most of of the proofs are gathered in Appendix 1 and Appendix 2.

2 The model

We have an exchange economy $\mathcal{E}$ with m agents. Each agent is characterized by a consumption set $X^i = \mathbb{R}^S$, an endowment e^i and a utility function $U^i : \mathbb{R}^S \to \mathbb{R}$. We suppose that $\sup_{x \in \mathbb{R}^S} U(x) = +\infty$.

For the sake of simplicity, we suppose that utility functions are concave, strictly increasing.

We first define an equilibrium of this economy.

Definition 1 *An equilibrium is a list $\big((x^{*i})_{i=1,\ldots,m}, p^*)\big)$ such that $x^{*i} \in X^i$ for every i and $p^* \in \mathbb{R}^S_+ \setminus \{0\}$ and*
(a) For any i, $U^i(x) > U^i(x^{*i}) \Rightarrow p^* \cdot x > p^* \cdot e^i$
(b) $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m e^i$.

Definition 2 *A quasi-equilibrium is is a list $\big((x^{*i})_{i=1,\ldots,m}, p^*)\big)$ such that $x^{*i} \in X^i$ for every i and $p^* \in \mathbb{R}^S_+ \setminus \{0\}$ and*
(a) For any i, $U^i(x) > U^i(x^{*i}) \Rightarrow p^* \cdot x \geq p^* \cdot e^i$
(b) $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m e^i$.

Since short-sales are allowed, from Geistdorfer-Florenzano [7], actually any quasi-equilibrium is an equilibrium.

Definition 3   *1. The individually rational attainable allocations set $\mathcal{A}$ is defined by*

$$\mathcal{A} = \left\{ (x^i) \in (\mathbb{R}^S)^m \mid \sum_{i=1}^m x^i = \sum_{i=1}^m e^i \ \text{ and } \ U^i(x^i) \geq U^i(e^i) \text{ for all } i \right\}.$$

2. The individually rational utility set $\mathcal{U}$ is defined by

$$\mathcal{U} = \{ (v^1, v^2, \ldots, v^m) \in \mathbb{R}^m \mid \exists \, x \in \mathcal{A} \ \text{ such that } \ U^i(e^i) \leq v^i \leq U^i(x^i) \ \text{ for all } i \}.$$

Definition 4   *i) The vector w is called* useful vector *of agent i if for any $x \in \mathbb{R}^S$, for any $\lambda \geq 0$ we have $U^i(x + \lambda w) \geq U^i(x)$.*

ii) The vector w is called useless vector *of agent i if for any $x \in \mathbb{R}$, for any $\lambda \in \mathbb{R}$ we have $U^i(x + \lambda w) = U^i(x)$.*

iii) We say that $w \in \mathbb{R}^S$ is a half-line direction *for agent i if there exists $x \in \mathbb{R}^S$ such that $U^i(x + \lambda w) = U^i(x)$, $\forall \, \lambda \geq 0$.*

Denote by R^i the set of useful vectors. L_i the set of useless vectors. By the very definition, the set of useless vectors of agent i is the biggest linear subspace included in R^i:

$$L_i = R^i \cap (-R^i).$$

Observe that R^i has no empty interior since $\mathbb{R}^S_+ \subseteq R^i$.

3 Some no-arbitrage conditions in the literature

In this section, we will review some no-abitrage conditions in the literature.

3.1 Conditions on prices

1. We present the definition of *no-arbitrage prices* proposed by Werner [27].

Definition 5 *The vector $p \in \mathbb{R}^S$ is a* no-arbitrage price *for agent i if for any $w \in R^i \setminus L_i$ we have $p \cdot w > 0$, and for $w \in L_i$, $p \cdot w = 0$.*

Denote by S^i the set of no-arbitrage prices of agent i. It is a cone. The *no-arbitrage condition* is $\bigcap_i^m S^i \neq \emptyset$.

2 Allouch et al. [1] introduce a more general set of no-arbitrage prices

$$
\begin{aligned}
\tilde{S}^i &= \{p : p \cdot w > 0 \text{ if } w \in W^i \setminus L^i\} \\
&= L^{i\perp} \text{ if } W^i = L^i
\end{aligned}
$$

Their no-arbitrage condition is $\cap_i \tilde{S}^i \neq \emptyset$.

3. In [5], Dana and Le Van propose to use the marginal utilities as no-arbitrage price. They introduce *weak no-abitrage prices*.

Definition 6 *A vector p is a* weak no-arbitrage price *for agent i if their exists $\lambda > 0$ and $x^i \in \mathbb{R}^S$ such that $p = \lambda U^{i\prime}(x^i)$.*

Let P^i denote the set of weak no-arbitrage prices for the agent i. Their no-arbitrage condition is $\bigcap_i^m \text{int} P^i \neq \emptyset$

4. Ha-Huy and Le Van [12], following Dana and Le Van [5], use the marginal utilities of the agents as no-arbitrage prices, but for a model with a countably infinite number of states. Their no arbitrage prices belong to the interior of l^∞. Their no-arbitrage condition is weaker than in Dana and Le Van [5] $\bigcap_i^m P^i \neq \emptyset$ where P^i is the cone of no-arbitrage prices.

5. Ha-Huy, Le Van and Wooders [17] use also no-arbitrage prices a la Ha-Huy and Le Van [12].

3.2 Conditions on net trades

3. Hart [16] proposed the Weak No Market Arbitrage (*WNMA*) condition:

Definition 7 *The economy satisfies WNMA if $(w^1, w^2, \ldots, w^m) \in R^1 \times R^2 \times \ldots \times R^m$ satisfies $\sum_{i=1}^m w^i = 0$ then $w^i \in L_i$ for every i.*

4. Page [22] proposed the No Unbounded Arbitrage (*NUBA*) condition:

Definition 8 *The economy satisfies* NUBA *if* $(w^1, w^2, \ldots, w^m) \in R^1 \times R^2 \times \ldots \times R^m$ *satisfies* $\sum_{i=1}^m w^i = 0$ *then* $w^i = 0$ *for every* i.

5. In [24], Page, Wooders and Monteiro introduced the notion of *Inconsequential arbitrage* (IC).

Definition 9 *The economy satisfies* Inconsequential arbitrage condition *if for any* $(w_1, w_2, \ldots, w_m)$ *with* $w_i \in R_i$ *for all* i *and* $\sum_{i=1}^m w^i = 0$ *and* $(w^1, w^2, \ldots, w^m)$ *is the limit of* $\lambda_n(x^1(n), x^2(n), \ldots, x^m(n))$ *with* $(x^1(n), x^2(n), \ldots, x^m(n)) \in \mathcal{A}$ *and* λ_n *converges to zero when* n *tends to infinity, there exists* $\epsilon > 0$ *such that for* n *sufficiently big we have* $U^i(x^i(n) - \epsilon w^i) \geq U^i(x^i(n))$.

3.3 Condition on the utility set

For the finite dimension, Dana Le Van and Magnien [6], for the on infinite dimension, Brown and Werner [4] assume directlythe compactness of the individually rational utility set. They prove

$$\mathcal{U} \text{ is compact } \Rightarrow \text{Existence of equilibrium}$$

3.4 The results

Dana Le Van and Magnien [6], Brown and Werner [4] prove

$$\mathcal{U} \text{ is compact } \Rightarrow \text{Existence of equilibrium}$$

In Allouch et al. [1], we find these results

$$(NUBA) \Leftrightarrow \mathcal{A} \text{ is compact}$$

(NUBA) $\Rightarrow$ (WNMA) $\Rightarrow$ (IC) $\Rightarrow \mathcal{U}$ is compact $\Rightarrow$ Existence of an equilibrium

These conditions are equivalent if the economy has no half-line.

In Dana and Le Van [5], their no-arbitrage condition implies existence of an equilibrium. The converse holds if the economy has no half-line.

Ha-Huy and Le Van [12] in a model with a countably infinite number of states, prove that their no-arbitrage condition is equivalent to the existence of an equilibrium.

Ha-Huy, Le Van and Wooders [17] impose conditions on the utility functions to obtain equivalence between their no-arbitrage condition and existence of equilibrium. Their additional assumptions on the utility functions are satisfied if these latter are separable.

4 The Main Result of Our Paper

We now define the set of *weak no-arbitrage prices* for agent i which is the cone

$$P^i := \{p \in \mathbb{R}^S : \exists\, x \in \mathbb{R}^S, \lambda > 0 \text{ such that } p \in \lambda \partial U^{i\prime}(x)\}.$$

Our main result is

Theorem 1 *Suppose that U^i is concave, strictly increasing for any i. We have*

$$\bigcap_{i=1}^m P^i \neq \emptyset \Leftrightarrow \text{ there exists general equilibrium.}$$

Our idea of proof relies on the result that any concave function on $\mathbb{R}^S$ is bounded above by a family of affine functions.

Let $\mathcal{F}$ denote the set of affine functions $p \cdot x + q$ with $p \in \mathbb{R}^S, q \in \mathbb{R}$ such that $U(x) \leq p \cdot x + q$ for any $x \in \mathbb{R}^S$. Without loss of generality, we can write $\mathcal{F} = \{(p, q)\} \subset \mathbb{R}^S \times \mathbb{R}$. Since U is strictly increasing, we have $p \in \mathbb{R}_+^S$ for any $(p, q) \in \mathcal{F}$. The following result can be found in Rockafellar [26]

Lemma 1 *The set $\mathcal{F}$ is a closed convex set of $\mathbb{R}^{S+1}$. For any $x \in \mathbb{R}^S$, we have*

$$U(x) = \min_{(p,q)\in\mathcal{F}}(p \cdot x + q)$$

and

$$\partial U(x) = \overline{co}\{p \in \mathbb{R}^S \text{ s.t. there exists } q : (p, q) \in \mathcal{F} \text{ and } U(x) = p \cdot x + q\}$$

where $\overline{co}$ denotes the closure of the convex hull.

The proof of Theorem 1 will be done in three steps.

- *Step 1* We give preliminary results which will be used for the proof.

- *Step 2* We consider an economy with m agents, m consumption sets equal to $\mathbb{R}^S$, m endowments (e^i). The utility functions of agent i is defined by

$$\tilde{U}^i(x) = \inf_{(p,q)\in\mathcal{F}^i}(p \cdot x + q) \qquad (1)$$

where $\mathcal{F}^i$ is a finite set of affine functions $(p^i \cdot x + q^i)_i$.
We define, for any agent i, the cone of no arbitrage prices $\tilde{P}^i$ which are generated by the vectors $(p^i)_i$. Let $\tilde{U}$ denote the individually rational utility set of this economy. We prove

$$\cap_i \tilde{P}^i \neq \emptyset \Leftrightarrow \tilde{U} \text{ compact } \Leftrightarrow \text{Existence of an equilibrium}$$

- *Step 3* We construct a sequence of economies $\mathcal{E}^n$. In these economies the utility functions are defined as in (1). Under No-arbitrage condition, for each n, we prove there exists an equilibrium. After that, we prove there exists a sequence of equilibria of $\mathcal{E}^n$ which converges, when n converges to infinity, to an equilibrium of the initial economy. At this stage, we prove

$$\text{No-arbitrage condition} \Rightarrow \text{Existence of an equilibrium}$$

Hpwever, the converse is very easy to prove. We actually obtain the equivalence

$$\text{No-arbitrage condition} \Leftrightarrow \text{Existence of an equilibrium}$$

This equivalence does not involve an equivalence with the compactness of the individually rational utility set. More explicitly, it is not necessary that the individually rational utility set is compact to have an equilibrium.

4.1 Step 1: some preliminary results

Let U be a concave function real valued over $\mathbb{R}^S$. We denote by R the cone of useful vectors associated with U. The following result is trivial but important to have it in mind.

Lemma 2 *Let w be a useful vector. Let x be in $\mathbb{R}^S$. The function $\lambda \in \mathbb{R} \to U(x + \lambda w)$ is non decreasing.*

Lemma 3 *Let w be a useful vector associated with U.*
If, for some $\tilde{x}$, $\sup_{\lambda \geq 0} U(\tilde{x} + \lambda w) = +\infty$, then for any x, $\sup_{\lambda \geq 0} U(x + \lambda w) = +\infty$.
Equivalently, if, for some $\tilde{x}$, $\sup_{\lambda \geq 0} U(\tilde{x} + \lambda w) < +\infty$, then for any x, $\sup_{\lambda \geq 0} U(x + \lambda w) < +\infty$.

Proof: See Appendix 1. ∎

Let w be a useful vector associated with U such that $\sup_{\lambda \geq 0} U(x + \lambda w) < +\infty, \forall x$. Define $V(x, w) = \sup_{\lambda \geq 0} U(x + \lambda w)$. It is easy to check

Lemma 4 *The function $V(., w)$ is concave.*

Lemma 5 *The vector w is useless for $V(\cdot, w)$. And for any $p \in \partial_x V(x, w)$ we have $p \cdot w = 0$.*

Proof: See Appendix 1. ∎

Lemma 6 *Suppose that U is concave and w is one of its useful vectors satisfying for any $x \in \mathbb{R}^S$, $\max_{\lambda \geq 0} U(x + \lambda w)$ exists. Define $V(x, w) = \sup_{\lambda \geq 0} U(x + \lambda w)$. We have $V(x, w) < +\infty$ for any $x \in \mathbb{R}^S$ and:*

$$(1) \ \partial_x V(x, w) = \partial U(x + \tilde{\lambda} w),$$

where $\tilde{\lambda}$ is big enough such that $U(x + \tilde{\lambda} w) = \max_{\lambda \geq 0} U(x + \lambda w)$,
Actually we have

$$\partial_x V(x, w) = \partial U(x + \hat{\lambda} w), \forall \hat{\lambda} > \tilde{\lambda} \ and \ U(x + \hat{\lambda} w) = \max_{\lambda \geq 0} U(x + \lambda w)$$

$$(2) \ \forall p \in \partial_x V(x, w), \ p \cdot w = 0$$

and

$$V(x, w) = U(x + \tilde{\lambda} w) \Leftrightarrow \forall p \in \partial_x V(x, w) = \partial U(x + \tilde{\lambda} w), p \cdot w = 0$$

(3) If u is a useful vector for U then it is useful for $V(., W)$

Proof: See Appendix 1. ∎

Let $\mathcal{F}$ be a set of affine functions $p \cdot x + q$, with $(p, q) \in \mathbb{R}_+^S \times \mathbb{R}$. Without loss of generality, we write $\mathcal{F} = \{(p, q)\} \subset \mathbb{R}_+^S \times \mathbb{R}$. With $\mathcal{F}$ we associate the function $\tilde{U}$ defined by

$$\forall x \in \mathbb{R}^s, \ \tilde{U}(x) = \inf\{f(x) : f \in \mathcal{F}\}$$

Lemma 7 *Suppose that $\mathcal{F}$ is the convex hull of finite number of elements: $\mathcal{F} = \text{convex}\{(p_1, q_1), (p_2, q_2), \cdots, (p_M, q_M)\}$. Assume w is a useful vector of $\tilde{U}$ which satisfies: there exists $\tilde{x} \in \mathbb{R}^S$ such that $\sup_{\lambda \geq 0} \tilde{U}(\tilde{x} + \lambda w) < +\infty$. In this case, for any $x \in \mathbb{R}^S$, $\max_{\lambda \geq 0} \tilde{U}(x + \lambda w)$ exists.*

Proof: See Appendix 1. ∎

4.2   Step 2

Consider an economy $\tilde{\mathcal{E}}$ with m agents, consumption sets equal to $\mathbb{R}^S$, endowments (e^i). The utility functions of the agents are defined by: for any i

$$\tilde{U}^i(x) = \min_{(p, q) \in \mathcal{F}^i} (p \cdot x + q)$$

where $\mathcal{F}^i = co\left\{(p_1^i, q_1^i), (p_2^i, q_2^i), \ldots, (p_{M^i}^i, q_{M^i}^i)\right\}$, each p^i belongs to $\mathbb{R}_+^S$, each q^i is in $\mathbb{R}$. Denote by $\tilde{R}^i$ the useful vectors set for agent i. Let $\tilde{P}^i = \text{convexcone}\{p_1^i, p_2^i, \ldots, p_{M^i}^i\}$. Let

$$\tilde{W} = \{(w^1, w^2, \ldots, w^m) \in \tilde{R}^1 \times \tilde{R}^2 \times \cdots \times \tilde{R}^m : \sum_{i=1}^m w^i = 0\}$$

Denote by $\tilde{\mathcal{A}}$ the set of individually rational allocations, by $\tilde{\mathcal{U}}$ the individually rational utility set of this economy.

Suppose there exists $(w^1, w^2, \ldots, w^m) \in \tilde{W}$ such that, for any i, $\sup_{\lambda \geq 0} \tilde{U}^i(x + \lambda w^i) < +\infty$ for any x. In this case we can define as before $V^i(x, w^i)$ for any x, any i. From Lemma 5, we have $V(x, w^i) = \tilde{U}^i(x + \tilde{\lambda} w^i)$ for some $\tilde{\lambda} \geq 0$ ($\tilde{\lambda}$ is independent of i, since the number of agents is finite).

The individually rational attainable allocations set $\mathcal{A}^V$ is defined as:

$$\mathcal{A}^V = \{(x^1, x^2, \ldots, x^m) \in (\mathbb{R}^S)^m \mid \sum_{i=1}^m x^i = \sum_{i=1}^m e^i \text{ and } V^i(x^i, w^i) \geq U^i(e^i)\}.$$

The individually rational utility set $\mathcal{U}^V$ is defined as:

$$\mathcal{U}^V = \{(v^1, v^2, \ldots, v^m) \in \mathbb{R}^m \mid \exists x \in \mathcal{A}^V : U^i(e^i) \leq v^i \leq V^i(x^i, w^i), \forall i\}.$$

We have the result:

Lemma 8 *Suppose there exists $(w^1, w^2, \ldots, w^m) \in \tilde{W}$ such that, for any i, $\sup_{\lambda \geq 0} \tilde{U}^i(x + \lambda w^i) < +\infty$ for any x. Consider the functions $V^i(., w^i)$. Then $\mathcal{U}^V = \tilde{\mathcal{U}}$.*

Proof: See Appendix 2. ■

Lemma 9 *Let $(w^1, w^2, \ldots, w^m) \in \tilde{W}$. If $p \in \cap_i \tilde{P}^i$ then $p \cdot w^i = 0, \forall i$.*

Proof: See Appendix 2. ■

The following proposition is crucial for the proof of Theorem 1.

Proposition 1 *Suppose that for all $(w^1, w^2, \ldots, w^m) \in \tilde{W}$, there exists $\max_{\lambda \geq 0} \tilde{U}^i(x + \lambda w^i)$ for any $x \in \mathbb{R}^S$. Assume $\cap_i \tilde{P}^i \neq \emptyset$. Then $\tilde{\mathcal{U}}$ is compact and hence there exists equilibrium for economy $\tilde{\mathcal{E}}$.*

Proof: See Appendix 2. ■

Lemma 10 *Assume $\bigcap \tilde{P}^i \neq \emptyset$. Then $\tilde{\mathcal{U}}$ is compact*

Proof: See Appendix 2. ■

Proposition 2 *Suppose that for any i, $\mathcal{F}^i$ is a convex hull of finite number of elements. Then*

$$\bigcap \tilde{P}^i \neq \emptyset \Leftrightarrow \tilde{\mathcal{U}} \text{ is compact } \Leftrightarrow \text{ there exists general equilibrium.}$$

Proof: (1) Suppose there exists a general equilibrium $((x*i)_i, p^*)$. It is easy to show that $p^* \in \cap_i \tilde{P}^i$.

(2) Conversely, suppose $\bigcap_{i=1}^m \tilde{P}^i \neq \emptyset$. Since for any i, $\mathcal{F}^i$ is a convex hull of a finite number of elements, we will prove that the utility functions of the economy satisfy the conditions in Proposition 1. Hence $\tilde{\mathcal{U}}$ is compact and there exists a general equilibrium.

First from Lemma 10, $\tilde{\mathcal{U}}$ is bounded.

Now, let $(w^1, w^2, \ldots, w^m) \in W$ and $(x^1, x^2, \ldots, x^m) \in \tilde{\mathcal{A}}$. If for some i, $\lim_{\lambda \to +\infty} \tilde{U}^i(x^i + \lambda w^i) = +\infty$ then $\tilde{\mathcal{U}}$ is not bounded since $\sum_i (x^i + \lambda w^i) = \sum_i e^i$. Using Lemma 7, we obtain that for any $x^i \in \mathbb{R}^S$, $\max_{\lambda \geq 0} \tilde{U}^i(x^i + \lambda w^i)$ exists. In other words, the utility functions of the economy satisfy the conditions in Proposition 1. Hence $\tilde{\mathcal{U}}$ is compact and there exists a general equilibrium.

■

4.3   Step 3 : Proof of Theorem 1

Recall that $P^i = \{\lambda \partial U(x) : \lambda > 0, x \in \mathbb{R}^S\}$.

(1) We will prove that $\bigcap_{i=1}^m P^i \neq \emptyset$ implies existence of general equilibrium. Take any

$$p \in \bigcap_{i=1}^m P^i.$$

For any i, there exist $\lambda_1, \lambda_2, \cdots, \lambda_{T^i} \in \mathbb{R}_+, x_1^i, x_2^i, \cdots, x_{T^i}^i \in \mathbb{R}^S, p_1, p_2, \cdots, p_{T^i} \in \mathbb{R}_+^S$ with $p_k \in \partial U^i(x_k^i)$, for $1 \leq k \leq T^i$, such that

$$p = \sum_{k=1}^{T^i} \lambda_k p_k, \ \forall i.$$

It is easy to verify that $(p_k, U^i(x_k^i) - p_k \cdot x_k^i)$ belongs to $\mathcal{F}^i$.

For each i, define the sequence of sets $\mathcal{F}_1^i \subset \mathcal{F}_2^i \subset \cdots \subset \mathcal{F}_n^i \subset \cdots$ satisfying

- For any $1 \leq k \leq T^i$, for any n, $(p_k, U^i(x_k^i) - p_k \cdot x_k^i) \in \mathcal{F}_n^i$.

- For any n, $\mathcal{F}_n^i$ is convex hull of finite number of elements.

- For any $(p, q) \in \text{ri}\left(\mathcal{F}^i\right)$, there exists N such that for any $n \geq N$, $(p, q) \in \mathcal{F}_n^i$.

For each n, define $U_n^i(x) = \min_{(p,q) \in \mathcal{F}_n^i} (p \cdot x + q)$. Define P_n^i the convex cone generated by the elements generating $\mathcal{F}_n^i$. By the construction of $\mathcal{F}^i$, we have for any n

$$\bigcap_{i=1}^m P_n^i \neq \emptyset.$$

10

Using Lemmas 7, 1, the economy $\mathcal{E}_n = (U_n^i, e^i)_{i=1}^m$ has equilibrium. Denote by G_n the set of equilibrium allocations of this economy. We will prove that G_n is closed. Suppose that $\{x^*(k)\}_{k=1}^\infty \subset G_n$ and converges to x^*. Without loss of generality, we can suppose that the associated equilibrium prices $\{p^*(k)\}$ converges to $\hat{p}^*$.

We prove that $\limsup_{k\to\infty} U_n^i(x^{*i}(k)) \leq U_n^i(x^{*i})$, for any i[1]. Indeed, for any $(p, q) \in \mathcal{F}_n^i$, we have
$$p \cdot x^{*i}(k) + q \geq U_n^i(x^{*i}(k)),$$
for any k. Let k converges to infinity we have
$$p \cdot x^{*i} + q \geq \limsup_{k\to\infty} U_n^i(x^{*i}(k)).$$

Since this is true for any $(p, q) \in \mathcal{F}_n^i$, $U_n^i(x^{*i}) \geq \limsup_{k\to\infty} U_n^i(x_n^i(k))$.

Hence if $U_n^i(x) > U_n^i(x^{*i})$, then for k big enough we have $U_n^i(x) > U_n^i(x^{*i}(k))$, which implies $p^*(k) \cdot x > p^*(k) \cdot x^{*i}(k)$. Let k converges to infinity we get $U_n^i(x) > U_n^i(x^{*i})$ implies $p^* \cdot x \geq p^* \cdot x^{*i}$. Hence $(p^*, x^*))$ is a quasi-equilibrium of the economy $\mathcal{E}_n$. Since short-sales are allowed, quasi-equilibrium is equilibrium. See [7]. The set G_n is closed.

Let $d_n = \inf_{x\in G_n} \sum_{i=1}^m \|x^i\|$. Let $\epsilon > 0$. The set $x^* \in G_n$ such that $\|x^*\| < d_n + \epsilon$ is non empty. Since G_n is closed this set is compact. Minimizing on this set we get $x_n^* \in G_n$ satisfying
$$\sum_{i=1}^m \|x_n^{*i}\| = \min_{x^*\in G_n} \sum_{i=1}^m \|x^{*i}\|.$$

We will prove that the set $\{x_n^*\}_{n=1}^\infty$ is bounded. Suppose the contrary,
$$\lim_{n\to\infty} \sum_{i=1}^m \|x_n^{*i}\| = +\infty.$$

Without loss of generality, we can suppose that for any i:
$$\lim_{n\to\infty} \frac{x_n^{*i}}{\sum_{i=1}^m \|x_n^{*i}\|} = w^i.$$

Since $\sum_{i=1}^m x_n^{*i} = \sum_{i=1}^m e^i$, $\sum_{i=1}^m w^i = 0$. We have also $\sum_{i=1}^m \|w^i\| = 1$.
For any $(p, q) \in \mathcal{F}_n^i$,
$$p \cdot x_n^{*i} + q \geq U_n^i(x_n^{*i}) \geq U_n^i(e^i) \geq U^i(e^i)^2.$$

---

[1]Obviously, we can have a better result, $\lim_{k\to\infty} U^i(x^{*i}(k)) = U^i(x^*)$. But for the sake of simplicity, we only use this.

[2]Recall that for any x, $U^i(x) = \inf_{(p,q)\in\mathcal{F}^i}\{p\cdot x + q\} \leq U_n^i(x) = \inf_{(p,q)\in\mathcal{F}_n^i}\{p\cdot x + q\}$ since $\mathcal{F}_n^i \subset \mathcal{F}^i$

Dividing the left-hand-side and the right-hand-side by $\sum_{i=1}^{m} \|x_n^{*i}\|$ and let n converges to infinity, we have for any $(p, q) \in \mathcal{F}_n^i$:

$$p \cdot w^i \geq 0.$$

By the definition of set sequence $\{\mathcal{F}_n^i\}$, we verify easily that for any $(p, q) \in \mathcal{F}^i$, $p \cdot w^i \geq 0$. This implies w^i is a useful vector of U_n^i, for any n, and of U^i.

Since $\sum_{i=1}^{m} w^i = 0$, for any n, and since (x^*) is a Pareto optimal, we have $p \cdot w^i = 0$ for any $p \in \partial U_n^i(x_n^{*i})$. Actually we can prove that for any Pareto optimum of $\mathcal{E}_n$, $(\overline{x})$, for any $(w^1, w2, \dots, w^m) \in W_n$ we have for any i, for any $p \in \partial U^i n(x^i)$, $p \cdot w^i = 0$.

Fix any n big enough such that if $w_s^i > 0$, then $x_{n,s}^{*i} > 0$ and if $w_s^i < 0$, then $x_{n,s}^{*i} < 0$.

Denote by $\tilde{\mathcal{F}}_n^i$ the set of extreme points $(p, q) \in \mathcal{F}_n^i$ such that $p \cdot x^{*i} + q = U^i(x_n^{*i})$. This set is non empty and contains a finite number of elements. From Rockafellar [26], $\tilde{\mathcal{F}}_n^i \subset \partial U_n^i(x^{*i})$. Define $\hat{\mathcal{F}}_n^i$ the set of extreme points $(p', q') \in \mathcal{F}_n^i$ such that $p' \cdot x_n^{*i} + q' > U^i(x_n^{*i})$. This set can be empty.

We have for any $(p, q) \in \tilde{\mathcal{F}}_n^i$, $p \cdot w^i = 0$.

Since $\tilde{\mathcal{F}}_n^i$ and $\hat{\mathcal{F}}_n^i$ contain each a finite number of elements, there exists $\epsilon > 0$ such that

- For any i, for any $(p, q) \in \tilde{\mathcal{F}}_n^i$, for any $(p', q') \in \hat{\mathcal{F}}_n^i$ we have

$$p' \cdot (x_n^{*i} - \epsilon w^i) + q' > p \cdot (x_n^{*i} - \epsilon w^i) + q.$$

  This implies
  $$\operatorname*{argmin}_{\mathcal{F}_n^i} \left(p \cdot (x_n^{*i} - \epsilon w^i) + q \right) \subset \tilde{\mathcal{F}}_n^i.$$

- For any i, s, if $w_s^i > 0$ then $x_{n,s}^{*i} - \epsilon w_s^i > 0$, and if $w_s^i < 0$, then $x_{n,s}^{*i} - \epsilon w_s^i < 0$.

Take $(p, q) \in \tilde{\mathcal{F}}_n^i$ such that $U_n^i(x_n^{*i} - \epsilon w^i) = p \cdot (x_n^{*i} - \epsilon w^i) + q$. Since $p \cdot w^i = 0$ (because $(x^{*i} - \epsilon w^i)_i$ is a Pareto optimum), we have $U_n^i(x_n^{*i} - \epsilon w^i) = p \cdot (x_n^{*i} - \epsilon w^i) + q = p \cdot x_n^{*i} + q = U_n^i(x_n^{*i})$.

Take p_n^* the associated equilibrium price with x_n^* of the economy $\mathcal{E}_n$. We have $U_n^i(x) > U_n^i(x_n^{*i} - \epsilon w^i) = U_n^i(x_n^{*i})$ implies $p \cdot x > p \cdot x_n^{*i} = p \cdot (x_n^{*i} - \epsilon w^i)$. Hence $(p_n^*, (x_n^{*i} - \epsilon w^i)_{i=1}^m)$ is also an equilibrium of the economy $\mathcal{E}_n$.

Since $\sum_{i=1}^{m} \|w^i\| = 1$, and $x_{n,s}^{*i} - \epsilon w^i$ always has the same sign as $x_{n,s}^{*i}$ and w_s^i, we have

$$\sum_{i=1}^{m} \|x_n^{*i} - \epsilon w^i\| < \sum_{i=1}^{m} \|x_n^{*i}\|.$$

This is a contradiction with the choice of x_n^*.

Hence the sequence $\{x_n^*\}_{n=1}^\infty$ is bounded. Without loss of generality, suppose that $\lim_{n\to\infty} x_n^* = x^*$ and $\lim_{n\to\infty} p_n^* = p^*$.

We will prove that $\lim_{n\to\infty} U_n^i(x_n^*) = U^i(x^*)$.

Since $U_n^i(x_n^{*i}) \geq U^i(x_n^{*i})$ for any n, we have

$$\liminf_{n\to\infty} U_n^i(x_n^{*i}) \geq \lim_{n\to\infty} U^i(x_n^{*i}) = U^i(x^{*i}).$$

For any $(p,q) \in \text{ri}\left(\mathcal{F}^i\right)$, for n sufficiently big, $(p,q) \in \mathcal{F}_n^i$, we have $p \cdot x_n^{*i} + q \geq U_n^i(x_n^{*i})$. Let n converges to infinity we get:

$$p \cdot x^{*i} + q \geq \limsup_{n\to\infty} U_n^i(x_n^{*i}).$$

Since the inequality is true for any $(p,q) \in \text{ri}\left(\mathcal{F}^i\right)$, we have

$$U^i(x^{*i}) = \inf_{\mathcal{F}}\left(p \cdot x^{*i} + q\right) \geq \limsup_{n\to\infty} U_n^i(x_n^{*i}).$$

We have proved $U^i(x^{*i}) = \lim_{n\to\infty} U_n^i(x_n^{*i})$.

Suppose that $U^i(x) > U^i(x^{*i})$. Using the same arguments as above, one has $U^i(x) = \lim_{n\to\infty} U_n^i(x)$. This implies for n big enough, we have $U_n^i(x) > U_n^i(x_n^{*i})$. Hence for n big enough we have $p_n^* \cdot x > p_n^* \cdot x_n^{*i}$. Let n converges to infinity we get $U^i(x) > U^i(x^{*i})$ implies $p^* \cdot x \geq p^* \cdot x^{*i}$.

Hence (p^*, x^*) is a quasi-equilibrium of the initial economy. Using result of [7], when the consumption sets equal $\mathbb{R}^S$, a quasi-equilibrium is anequilibrium.

(2) We now prove the converse. Suppose that the economy has an equilibrium. It is easy to prove that the equilibrium price belongs to $\bigcap_{i=1}^m P^i$. Therefore, $\bigcap_{i=1}^m P^i \neq \emptyset$. ∎

5 Appendix 1

5.1 Proof of Lemma 3

Let $x \in \mathbb{R}^S$. Then there exist $y \in \mathbb{R}^S$, $\theta \in (0,1)$, such that $x = \theta\tilde{x} + (1-\theta)y$. Suppose $\lim_{\lambda\to+\infty} U(\tilde{x} + \lambda w) = +\infty$. We then get

$$
\begin{aligned}
U(x + \lambda w) &= U(\theta(\tilde{x} + \lambda w) + (1-\theta)(y + \lambda w)) \\
&\geq \theta U(\tilde{x} + \lambda w) + (1-\theta)U(y + \lambda w) \\
&\geq \theta U(\tilde{x} + \lambda w) + (1-\theta)U(y) \\
\Rightarrow \lim_{\lambda\to+\infty} U(x + \lambda w) &\geq \theta \lim_{\lambda\to+\infty} U(\tilde{x} + \lambda w) + (1-\theta)U(y) = +\infty
\end{aligned}
$$

∎

5.2 Proof of Lemma 5

Let $\mu \in \mathbb{R}$, $x \in \mathbb{R}^S$. We have

$$
\begin{aligned}
V(x + \mu w, w) &= \sup_{\lambda \geq 0} U(x + \mu w + \lambda w) \\
&= \sup_{\lambda \geq 0} U(x + \lambda w) = V(x, w)
\end{aligned}
$$

Let $p \in \partial_x V(x, w)$. Then

$$
\forall \lambda \in \mathbb{R}, 0 = V(x, w) - V(x + \lambda w, w) \geq -\lambda p \cdot w
$$

Hence $p \cdot w = 0$. ∎

5.3 Proof of Lemma 6

(1) First observe if $x \in \mathbb{R}^S$ and $U(x + \hat{\lambda} w) = V(x, w) = \max_{\lambda \geq 0} U(x + \lambda w)$, then from Lemma 2, for any $\lambda > \hat{\lambda}$, we have $U(x + \lambda w) = U(x + \hat{\lambda} w) = V(x, w)$. Let $x \in \mathbb{R}^S$, $y \in \mathbb{R}^S$. Choose $\tilde{\lambda}$ large enough such that $V(x, w) = U(x + \tilde{\lambda} w)$, $V(y, w) = U(y + \tilde{\lambda} w)$. We have, for any $p \in \partial U(x + \tilde{\lambda} w)$,

$$
\begin{aligned}
V(x, w) - V(y, w) &= U(x + \tilde{\lambda} w) - U(y + \tilde{\lambda} w) \\
&\geq p \cdot (x + \tilde{\lambda} w - y - \tilde{\lambda} w) \\
&= p \cdot (x - y),
\end{aligned}
$$

Hence $p \in \partial_x V(x, w)$. We have proved $\partial U(x + \tilde{\lambda} w) \subset \partial_x V(x, w)$.

Let us prove the converse. Take $p \in \partial_x V(x, w)$. Since w is a useless vector of $V(\cdot, w)$, we have $p \cdot w = 0$ (Lemma 5). Recall that $U(x + \tilde{\lambda} w) = V(x, w)$. For any $y \in \mathbb{R}^S$ we have

$$
\begin{aligned}
U(x + \tilde{\lambda} w) - U(y) &\geq V(x, w) - V(y, w) \\
&\geq p \cdot (x - y) \\
&= p \cdot (x + \tilde{\lambda} w - y),
\end{aligned}
$$

since $p \cdot w = 0$. Hence $p \in \partial U(x + \tilde{\lambda} w)$.
We have proved $\partial_x V(x, w) \subset \partial U(x + \tilde{\lambda} w)$.
Now take $\hat{\lambda} > \tilde{\lambda}$. We have

$$
\begin{aligned}
V(x, w) &= U(x + \tilde{\lambda}) = U(x + \hat{\lambda}) \\
V(y, w) &= U(y + \tilde{\lambda}) = U(y + \hat{\lambda})
\end{aligned}
$$

The same computations as above give $\partial_x V(x, w) = \partial U(x + \hat{\lambda} w)$.

(2) Let $\hat{\lambda} > \tilde{\lambda}$. Let $p \in \partial U(x + \tilde{\lambda}w) = \partial U(x + \hat{\lambda}w)$. We have on the one hand

$$0 = U(x + \tilde{\lambda}) - U(x + \hat{\lambda}) \geq (\tilde{\lambda} - \hat{\lambda})p \cdot w \Rightarrow p \cdot w \geq 0$$

and on the other hand

$$0 = U(x + \hat{\lambda}) - U(x + \tilde{\lambda}) \geq (\hat{\lambda} - \tilde{\lambda})p \cdot w \Rightarrow p \cdot w \leq 0$$

Conversely, let $\lambda > \tilde{\lambda}$. Let $p \in \partial U(x + \tilde{\lambda}w)$. It satisfies $p \cdot w = 0$. We have

$$0 \geq U(x + \tilde{\lambda}w) - U(x + \lambda w) \geq (\tilde{\lambda} - \lambda)p \cdot w = 0$$

Hence $U(x + \tilde{\lambda}w) \geq U(x + \lambda w)$, $\forall \lambda \geq \tilde{\lambda}$. From Lemma 2, we actually have $U(x + \tilde{\lambda}w) = U(x + \lambda w)$, $\forall \lambda \geq \tilde{\lambda}$. For $\lambda < \tilde{\lambda}$, we have $U(x + \lambda w) \leq U(x + \tilde{\lambda}w)$ from Lemma 2.

(3) Let u be useful for U. We have

$$\begin{aligned}
\forall x, \forall \lambda \geq 0, \forall \mu \geq 0, \ U(x + \lambda u + \mu w) &\geq \ U(x + \mu w) \\
\Rightarrow V(x + \lambda u, w) = \sup_{\mu \geq 0} U(x + \lambda u + \mu w) &\geq \ \sup_{\mu \geq 0} U(x + \mu w) = V(x, w)
\end{aligned}$$

∎

5.4 Proof of Lemma 7

Firstly, observe that since $\sup_{\lambda \geq 0} \tilde{U}(\tilde{x} + \lambda w) < +\infty$, for any $x \in \mathbb{R}^S$, $\sup_{\lambda \geq 0} \tilde{U}(x + \lambda w) < +\infty$ (see Lemma 3). It is easy to verify that for any k, $p_k \cdot w \geq 0$.

We will prove that there exists $p_k \in \{p_1, p_2, \cdots, p_M\}$ such that $p_k \cdot w = 0$. Indeed, suppose the contrary, for any $1 \leq k \leq M$, $p_k \cdot w > 0$. Then for any $x \in \mathbb{R}^S$, for any $(p_k, q_k) \in \mathcal{F}$,

$$\lim_{\lambda \to \infty} (p_k \cdot (x + \lambda w) + q_k) = +\infty,$$

which implies

$$\begin{aligned}
\lim_{\lambda \to \infty} \tilde{U}(x + \lambda w) &= \lim_{\lambda \to \infty} \min_{(p_k, q_k) \in \mathcal{F}} (p_k(x + \lambda w) + q_k) \\
&= +\infty,
\end{aligned}$$

a contradiction.

Define $\overline{\mathcal{F}} = \{(p_k, q_k) \text{ such that } p_k \cdot w = 0\}$.

Denote by C the supremum value $\sup_{\lambda \geq 0} \tilde{U}(x + \lambda w)$. Fix $\tilde{\lambda}$ big enough such that for any $(p_{k'}, q_{k'}) \notin \overline{\mathcal{F}}$, $p_{k'} \cdot (x + \tilde{\lambda}w) + q_{k'} > C$. This implies for any $(p_{k'}, q_{k'}) \notin \overline{\mathcal{F}}$:

$$\begin{aligned}
p_{k'} \cdot (x + \tilde{\lambda}w) + q_{k'} &> \tilde{U}(x + \tilde{\lambda}w) \\
&= \min_{(p_k, q_k) \in \mathcal{F}} \left(p_k(x + \tilde{\lambda}w) + q_k \right).
\end{aligned}$$

15

This implies
$$\tilde{U}(x + \tilde{\lambda}w) = \min_{(p_k, q_k) \in \overline{\mathcal{F}}} \left(p_k(x + \tilde{\lambda}w) + q_k \right).$$

Since $p_k \geq 0, \forall k$ and since the number of elements is finite, there exists $(p_k, q_k) \in \overline{\mathcal{F}}$ such that $\tilde{U}(x + \tilde{\lambda}w) = p_k \cdot x + q_k$.

Let $J = \{k : (p_k, q_k) \in \overline{\mathcal{F}} \text{ and } \tilde{U}(x + \tilde{\lambda}w) = p_k \cdot x + q_k\}$. From Rockafellar [26], $\partial \tilde{U}(x + \tilde{\lambda}w) = \text{conv}\{p_k, k \in J\}$. Hence for any $p \in \partial \tilde{U}(x + \tilde{\lambda}w)$, we have that p belongs to the convex hull of $\{p_k \text{ such that } (p_k, q_k) \in \overline{\mathcal{F}}\}$. This implies $p \cdot w = 0$. From Lemma 6, $\tilde{U}(x + \tilde{\lambda}w) = \sup_{\lambda \geq 0} \tilde{U}(x + \lambda w)$. ∎

6 Appendix 2

6.1 Proof of Lemma 8

Evidently, since for any x^i, $\tilde{U}^i(x^i) \leq V^i(x^i, w^i)$, we have $\tilde{\mathcal{U}} \subset \mathcal{U}^V$.

Take $(v^1, v^2, \ldots, v^m) \in \mathcal{U}^V$. There exists $(x^1, x^2, \ldots, x^m) \in \mathcal{A}^V$ such that $U^i(e^i) \leq v^i \leq V^i(x^i, w^i)$, $\forall i$. There exists $\lambda \geq 0$ big enough such that $V^i(x^i, w^i) = U^i(x^i + \lambda w^i)$, for any i. Observe that $(x^1 + \lambda w^1, x^2 + \lambda w^2, \ldots, x^m + \lambda w^m) \in \mathcal{A}^V$. This implies $(v^1, v^2, \ldots, v^m) \in \tilde{\mathcal{U}}$. ∎

6.2 Proof of Lemma 9

Let $p \in \cap_i \tilde{P}^i$. If $p = 0$ the claim is true. Assume $p = \mu^i p'^i$ where $\mu^i > 0$ and $p'^i \in \text{co}\{p_1^i, p_2^i, \ldots, p_{M^i}^i\}$. For any i, any x^i, any $\lambda > 0$, we have

$$\tilde{U}^i(x^i) \leq \tilde{U}^i(x^i + \lambda w^i) \leq p'^i \cdot (x^i + \lambda w^i) + q^i, q^i \in \text{co}\{q_1^i, \ldots, q_{M^i}^i\}$$

Hence $p'^i \cdot w^i \geq 0, \forall i \Leftrightarrow p \cdot w^i \geq 0, \forall i$. Since $\sum_i w^i = 0$ we get $\forall i, p \cdot w^i = 0$. ∎

6.3 Proof of Proposition1

If W is linear space, then the condition *WNMA* is satisfied. This implies the compactness of $\tilde{\mathcal{U}}$ and the existence of an equilibrium.

Consider the case $\tilde{W}$ is not a linear space. Take $w \in \tilde{W}$ which satisfies $-w \notin \tilde{W}$.

Define: $V^i(x, w^i) = \sup_{\lambda \geq 0} \tilde{U}^i(x + \lambda w^i)$.

We will prove that for any i there exists $p_{k(i)}^i \in \{p_1^i, \ldots, p_{M^i}^i\}$ which satisfies $p_{k(i)}^i \cdot w^i = 0$. Indeed, it is easy to check that for any i, for any $p \in \{p_1^i, \ldots, p_{M^i}^i\}$ we have $p \cdot w^i \geq 0$. Suppose there exists i which satisfies $p \cdot w^i > 0$ for any $p \in \{p_1^i, \ldots, p_{M^i}^i\}$. Take $p \in \cap_i \tilde{P}^i$. Then $p \cdot w^j \geq 0, \forall j \neq i$ and $p \cdot w^i > 0$. We have a contradiction $0 = p \cdot \sum_i w^i > 0$. Now, without loss of generality,

we can assume that for some N^i which satisfies $1 \leq N^i \leq M^i$, we have that $p_1^i, p_2^i, \ldots, p_{N^i}^i$ are orthogonal to w^i, and for all $N^i + 1 \leq k \leq M^i$, the scalar product $p_k^i \cdot w^i$ is strictly positive.

We claim that for any x,

$$V^i(x, w^i) = \min \left\{ p \cdot x + q : (p, q) \in \{(p_1^i, q_1^i), \ldots, (p_{N^i}^i, q_{N^i}^i)\} \right\}$$

Indeed, we have

$$
\begin{aligned}
V^i(x, w^i) &= \sup_{\lambda \geq 0} \tilde{U}^i(x + \lambda w^i) \\
&\leq \sup_{\lambda \geq 0} \left\{ p \cdot (x + \lambda w^i) + q : \forall (p, q) \in \{p_1^i, \ldots, p_{N^i}^i\} \right\} \\
&\leq \left\{ p \cdot x + q : \forall (p, q) \in \{p_1^i, \ldots, p_{N^i}^i\} \right\}, \text{ since } p \cdot w^i = 0
\end{aligned}
$$

Then, there exists $\tilde{\lambda} > 0$ such that

$$
\begin{aligned}
V^i(x, w^i) &= \tilde{U}^i(x + \tilde{\lambda} w^i) \\
&= \min \left\{ p \cdot (x + \tilde{\lambda} w^i) + q : (p, q) \in \{(p_1^i, q_1^i), \ldots, (p_{N^i}^i, q_{N^i}^i)\} \right\} \\
&= \min \left\{ p \cdot x + q : (p, q) \in \{(p_1^i, q_1^i), \ldots, (p_{N^i}^i, q_{N^i}^i)\} \right\} \text{ since } p \cdot w^i = 0
\end{aligned}
$$

We now prove the claim:

Claim: Let $(u^1, u^2, \ldots, u^m)$ satisfy: for any i, u^i is a useful vector of V^i, $\forall i$, and $\sum_{i=1}^m u^i = 0$. Then there exists $\max_{\lambda \geq 0} V^i(x^i + \lambda u^i, w^i)$, $\forall x^i \in \mathbb{R}^S$, $\forall i$.

Denote by R_V^i the set of useful vectors of $V^i(., w^i)$. We will prove the following assertion:

$$R_V^i = \tilde{R}^i + \{\lambda w^i\}_{\lambda \in \mathbb{R}}.$$

Since $\tilde{R}^i \subset R_V^i$ (see Lemma 6, statement 3) and w^i is a useless vector of $V^i(.w^i)$, we have $\tilde{R}^i + \{\lambda w^i\}_{\lambda \in \mathbb{R}} \subset R_V^i$. Take any vector $u \in R_V^i$. We prove that for all $1 \leq k \leq N^i$, $p_k^i \cdot u \geq 0$. Indeed, take p_k^i for $1 \leq k \leq N^i$. We have for any x, any $\lambda > 0$

$$V^i(x, w^i) \leq V^i(x + \lambda u, w^i) = \sup_{\mu \geq 0} \tilde{U}^i(x + \lambda u + \mu w^i) \leq \sup_{\mu \geq 0} \{p_k^i \cdot (x + \lambda u + \mu w^i) + q_k^i = p_k^i \cdot (x + \lambda u) + q_k^i$$

since $p_k^i \cdot w^i = 0$. This leads to:

$$\frac{1}{\lambda} V^i(x, w^i) \leq \frac{p_k^i \cdot x + q_k^i}{\lambda} + p_k^i \cdot u$$

Let $\lambda \to +\infty$. We get $p_k^i \cdot u \geq 0$.

Consider the vector $u + \lambda w^i$, with $\lambda \geq 0$. For $1 \leq k \leq N^i$, $p_k^i \cdot (u + \lambda w^i) \geq 0$. For $k \geq N^i + 1$, since $p_k^i \cdot w^i > 0$, we have $p_k^i \cdot (u + \lambda w^i) = p_k^i \cdot u + \lambda p_k^i \cdot w^i > 0$ with λ big enough. With this λ, we have $p_k^i \cdot (u + \lambda w^i) \geq 0$ for all k. This

17

implies $u + \lambda w^i \in \tilde{R}^i$, since for any x, $\partial \tilde{U}^i(x) \subset \text{co}\{p_k^i\}_k$. We have proved that $R_V^i \subset \tilde{R}^i + \{\lambda w^i\}_{\lambda \in \mathbb{R}}$.

Suppose that $(u^1, u^2, \ldots, u^m)$ are such that $u^i \in R_V^i$ for all i, and $\sum_{i=1}^m u^i = 0$. We want to prove $\max_{\lambda \geq 0} V^i(x + \lambda u^i, w^i)$ exists for any x.

For each i, we can write $u^i = r^i + \mu_i w^i$, with $r^i \in \tilde{R}^i$, $\mu_i \in \mathbb{R}$. Choose $\mu > 0$ big enough such that for all i, $\lambda_i = \mu + \mu_i > 0$. We have:

$$\sum_{i=1}^m (r^i + \lambda_i w^i) = \sum_{i=1}^m (r^i + \mu_i w^i) + \mu \sum_{i=1}^m w^i = 0.$$

For all i, $\lambda_i > 0$, so $r^i + \lambda_i w^i \in \tilde{R}^i$. Observe that since w^i is a useless vector of $V^i(., w^i)$, we have $V^i(x^i + \lambda u^i, w^i) = V(x^i + \lambda r^i, w^i)$. From the assumption in the statement of the proposition, $\max_{\lambda \geq 0} \tilde{U}^i(x + \lambda(r^i + \lambda_i w^i))$ exists. Then there exists $\lambda^1 \geq 0$ such that for all $\lambda \geq \lambda^1$ we have:

$$\tilde{U}^i(x + \lambda(r^i + \lambda_i w^i)) = \tilde{U}^i(x + \lambda^1(r^i + \lambda_i w^i)) \leq V^i(x + \lambda^1 r^i, w^i).$$

Fix any $\lambda^2 \geq \lambda^1$. For all $\lambda > \lambda^2$, we have:

$$\tilde{U}^i(x + \lambda^2 r^i + \lambda \lambda_i w^i) \leq \tilde{U}^i(x + \lambda(r^i + \lambda_i w^i)) \leq V^i(x + \lambda^1 r^i, w^i).$$

Let $\lambda \to +\infty$ the left hand side $\tilde{U}^i(x + \lambda^2 r^i + \lambda \lambda_i w^i)$ tends to $V^i(x + \lambda^2 r^i, w^i)$ so

$$V^i(x + \lambda^2 r^i, w^i) \leq V^i(x + \lambda^1 r^i, w^i).$$

This inequality is true for any $\lambda^2 \geq \lambda^1$. Since r^i is also a useful vector of $V^i(., w)$, $\max_{\lambda \geq 0} V^i(x + \lambda r^i, w^i)$ exists, or equivalently $\max_{\lambda \geq 0} V^i(x + \lambda u^i, w^i)$ exists.

Define $\tilde{U}_1^i(x^i) = V^i(x^i, w^i)$, $\tilde{P}_1^i = \text{convexcone}\{p_1^i, p_2^i, \ldots, p_{N^i}^i\}$. We have proved that $\tilde{U}_1^i(x^i) = V^i(x^i, w^i) = \min \{p \cdot x + q : (p, q) \in \{(p_1^i, q_1^i), \ldots, (p_{N^i}^i, q_{N^i}^i)\}\}$
Define a new economy $\tilde{\mathcal{E}}_1$ which is characterized by $\{(\tilde{U}_1^i, e^i, X^i)\}$ with $X^i = \mathbb{R}^S$. Denote, by $\tilde{R}_1^i$ the set of useful vectors associated with $\tilde{U}_1^i$. Let

$$\tilde{W}_1 = \{(w^1, \ldots, w^m) \in \tilde{R}_1^i \times \tilde{R}_2^i \times \cdots \times \tilde{R}_m^i : \sum_{i=1}^m w^i = 0\}$$

It is easy to prove that any useful vector of $\tilde{U}^i$ is also a useful vector of $\tilde{U}_1^i$. So, $\tilde{W} \subset \tilde{W}_1$. From the very definition of $V^i(., w)$, we have that, for all i, w^i is a useless vector of $\tilde{U}_1^i$. Thus, $(w^1, w^2, \ldots, w^m) \in L(\tilde{W}_1)$. Hence we have $\dim L(\tilde{W}_1) \geq \dim L(\tilde{W}) + 1$.

The economy $\tilde{\mathcal{E}}_1$ satisfies the property saying that for any $(u^1, u^2, \cdots, u^m) \in \tilde{W}_1$, there exists $\max_{\lambda \geq 0} \tilde{U}_1(x + \lambda u^i)$ for any $x \in \mathbb{R}^S$. Moreover, the weak no-arbitrage prices cone $\tilde{P}_1^i$ is generated by a finite number of vectors in $\mathbb{R}^S$. Hence

the utility functions of $\tilde{\mathcal{E}}_1$ satisfy the same conditions as for the utility functions of the economy $\tilde{\mathcal{E}}$.

Observe we have actually $\tilde{P}_1^i = \tilde{P}^i \cap \{w^i\}^\perp$. From Lemma 9, we have $\cap_i \tilde{P}^i = \cap_i \tilde{P}_1^i$, hence $\cap_i P_1^i \neq \emptyset$.

By induction and the same arguments, suppose that we arrive to an economy $\tilde{\mathcal{E}}_t$, with $t \geq 1$. If $\tilde{W}_t$ is not a linear space, we can construct a new economy $\tilde{\mathcal{E}}_{t+1}$, with $\dim L(\tilde{W}_{t+1}) \geq \dim L(\tilde{W}_t) + 1$.

The utility sets are all the same $\tilde{\mathcal{U}}_t = \tilde{\mathcal{U}}$, $\forall t$ (see Lemma 8). For all t, the set $\tilde{W}_t \subset \mathbb{R}^{S \times m}$, so there must exist some T such that $\tilde{W}_T$ is linear space. This implies economy $\tilde{\mathcal{E}}_T$ satisfies the $WNMA$ condition. Then $\tilde{\mathcal{U}}_T$ is compact. Hence $\tilde{\mathcal{U}}$ is compact and our initial economy $\tilde{\mathcal{E}}$ has an equilibrium. ∎

6.4 Proof of Lemma 10

Assume $\bigcap_{i=1}^m \tilde{P}^i \neq \emptyset$. We prove that the individually rational set $\tilde{U}$ is bounded. Let $\bar{p} \in \bigcap_{i=1}^m \tilde{P}^i$. We can write

$$\bar{p} = \lambda^i \sum_{k=1}^{M^i} \theta_k^i p_k^i, \text{ where } \lambda^i > 0, \theta_k^i \geq 0, \sum_k \theta_k^i = 1, \forall i.$$

Let $(x^1, x^2, \ldots, x^m) \in \tilde{\mathcal{A}}$. Then

$$\forall i, \ \lambda^i \tilde{U}^i(x^i) \ \leq \ \bar{p} \cdot x^i + \lambda^i \sum_{k=1}^{M^i} \theta_k^i q_k^i$$

$$\Rightarrow \lambda^i \tilde{U}^i(x^i) + \sum_{j \neq i} \lambda^j \tilde{U}^j(e^j) \ \leq \ \bar{p} \cdot \sum_i e^i + \sum_{i=1}^m \lambda^i \sum_{k=1}^{M^i} \theta_k^i q_k^i$$

That shows that $\left(U^i(x^i)\right)_i$ is bounded for any $(x^1, x^2, \ldots, x^m) \in \tilde{\mathcal{A}}$. Hence, $\tilde{\mathcal{U}}$ is bounded. ∎

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