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A Not So Myopic Axiomatization of Discounting*

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Abstract

This article builds an axiomatization of inter-temporal trade-offs that makes an explicit account of the distant future and therefore encompasses motives related to sustainability, transmission to offsprings and altruism. The focus is on separable representations and the approach is completed following a decision-theory index based approach that is applied to utility streams. This enlightens the limits of the commonly used tail intensity requesites for the evaluation of utility streams: in this article, these are supersed and replaced by an axiomatic approach to optimal myopia degrees that in its turn precedes the determination of optimal discount. The overall approach is anchored in the new and explicit proof of a temporal decomposition of the preference orders between the distant future and the close future itself directly related to the determination of the optimal myopia degrees. The argument is shown to provide a novel understanding of temporal biases with the scope for a distant future bias when the finite dimensional gets influenced by the infinite dimensional. The reference to robust orders and pessimism-like axioms finally allows for determining tractable representations for the indexes.

Keywords: Axiomatization, Myopia, Discount, Temporal Order Decompositions, Infinite Dimensional Topologies.

JEL Classification: D11, D15, D90.

Contents

1. Introduction

Even though the long-run concerns for sustainability, conservation and the well-being of the future generations of offsprings nowadays go far beyond the limits of the academic circles and promptly come into the fore into most public agendas, it is not the least surprising that there seems to have been limited efforts towards a fine understanding of the actual meaning of *having an unbounded horizon* or *accounting for the infinite.* The first endeavor into the direction of an axiomatic approach to the topic was brought by Brown & Lewis [10] and explicitly anchored on *myopia*: it has nonetheless received the sparse echo that was due to what was perceived as a mere mathematical curosity, *i.e.,* the identification of the *weight of the infinite*. This nevertheless raises a number of questions that may not have hitherto received sufficient attention. Is, together with most of the social welfare literature, an arbitrarily large finite future a satisfactory proxy for an unbounded horizon? Does the very fact of having some remote low orders tail for a stream of utils mean that it is negligible in not exerting any influence for finite dates? More precisely, are there some specificities attached to arbitrarily remote infinite horizon streams and is it reasonable to compare these through the same apparatus that is used for the finite parts of these streams? Otherwise stated, does order theory keep on being the appropriate apparatus for such elements and, assuming this is the case, how is it to be adaptated to simultaneously accomodate finite and infinite elements? Finally, a large part of the recent evaluation of streams of utils has been dealing with the importance of *time perception paradoxes* and *temporal distances* that would modify over time and gives rise to various temporal biases: how does this relate to the above concerns and is there any specificity that is associated to infinite dimensional elements in this regard?

This article aims at pursuing such a line of research by building an axiomatization of inter-temporal trade-offs that makes an explicit account of the distant future and thus encompasses motives related to sustainability, transmission to offsprings and altruism by avoiding often hidden myopic negligible tail insensitivity requesites. The focus is on separable representations and the approach is completed following

a decision-theory index based approach that is applied to infinite dimension streams. This enlightens the limits of the commonly used flat tail intensity requesites for the evaluation of utility streams: in this article, these are supersed and replaced by an axiomatic approach to optimal myopia degrees that in its turn precedes the determination of optimal discount. The benchmark order in this article indeed satisfies *weak myopia* but does not satisfy *strong myopia* in the sense of Brown & Lewis [10], the key methological approach of this article stating as a *temporal decomposition of the initial preferences order* between a distant future order and a close future order.
A first part of the article, which provides the main results of the article, begins by reconsidering the *negligible tail* conditions of the earlier literature. It is indeed conceivable that, even though the value of the distant future *sounds* negligible, it keeps on influencing the evaluation of utility streams. In order to reach a thorough understanding of such a potentiality, supplementary structures have to be superimposed on the preferences order relation. In that perspective, this first section introduces two new axioms allowing for a *temporal decomposition* of the initial order between a *distant future* order and a *close future* order. The first *distant future sensitivities* axiom ensures that, for any two streams, one is always in position to compare their distant futures. The key feature of such a comparison states as the fact that it is invariant to change in only a finite number of values in these two streams[1]. The second, *close future sensitivities* axiom, contemplates a comparison between the close futures of two streams. Under this axiom, every given distant future is negligible for the close future and, its influence on the evaluation of the streams—according to the *close future order*—converging to zero for an arbitrarily large date. It is however to be stressed that such a convergence is not uniform, in the sense that its speed explicitly depends on the nature of the stream that is used to provide an account for the distant future.
Under such a system of axioms, it is shown that the evaluation of a utility stream can be decomposed into a first component that accounts for distant future concerns and a second component that accounts for close future ones. From the very possibility of such a decomposition in turn arises a multiplicity problem for the

[1]Consider, *e.g.,* the comparison between the $\liminf$ of the two sequences.

weights parameters associated with the distant future and close future components. Depending upon the utility stream under consideration, there are various possible choices for the weights that relate to the distant future and close future evaluations. Two possible scenarios emerge and can be identified as featuring *myopic* and *non-myopic* behaviours. The former depicts a behaviour where the distant future gain is not sufficiently valued so that it cannot compensate the loss undergone in the close future, the later representing an opposite behaviour. These weight parameters are further shown to provide an account of *optimal myopia degrees* for the agent. As a matter of illustration, having *low optimal myopia degrees* is to be understood as having a large account of the remote future. To sum up, a decomposition into two parts of the evaluation of utility streams has been put into evidence, with a first that features the evaluation of the distant future while the second accounts for the evaluation of the close future.

A natural direction of extension does emerge. It focuses on the potential role of time by introducing time dependencies into the benchmark order. The time dependent orders are first assumed not to be influenced by the past. At any given date, the agent then completes the evaluation of a utility stream. This evaluation consists of a *recursive convex sum* between the utility level at that date, and the evaluation at the subsequent date of the utility stream. Interestingly, a multitude of choices are shown to be admissible for the weight parameters of this convex sum. Present and future bias can then receive an original and integrated account. Some behaviour shall be labelled as *present biased* when, according to the perception of a given agent, the *temporal distance* between two successive dates it to decrease over time. This more specifically means that the optimal discount factor is increasing and this is shown to result from an specific axiom that further constrains the range of admissible time-dependent orders. Admittedly, taking a reverse configuration where the perception of the distance between two consecutive dates would become greater over time could provide some account of a future bias that would however be far less feasible. This article instead undertakes a separate approach of the issue at stake through the retainment of an axiom that can provide an account of a *distant future bias* in the evaluation of a given utility stream. The currently introduced *distant future bias* is to be understood on the account of the existence of some continuity between the time dependent orders and the distant future order. Otherwise stated and under further qualifications, there does exist some direct influences of the evaluation of an *infinite* distant future on the evaluations of some *finite* close future. Interestingly, it is therefore established that some given order may concomitantly have some *present bias* and *distant future bias* facets.

Another direction of extension explores robust or unanimous orders, a given utility stream being robustly better than an alternative one if and only if such a comparison is unanimous among a set of linear orders. These linear orders can be understood as a set of possible evaluations and can be considered as special cases of the orders of the first section: for each order in this set, the choice of optimal myopic parameters is reduced to one. More precisely, each sub-order is shown to ground upon two separate components: a first that belongs to the set of σ−additive measures on $\mathbb{N}$ and a second part that belongs to the set of *charges*[2] and is labelled a *purely finitely additive* set.

The third part of this article introduces some further concepts that relate to the *degree of optimism*. Assuming that this degree of optimism does not decrease with respect to some robustness comparisons, the order is proved to assume a so-called α-maximin criterion, the evaluation of a given utility stream depending only on its best and worst evaluations. Under the extra assumption that every sub-order satisfies some impatience and consistency properties, the σ−additive part of the sub-orders satisfies the stationary property, *i.e.,* the evaluation does not depend upon the date of evaluation. The σ−additive part of the sub-orders corresponds to an inter-temporal sum of utilities, the sequence of discount factors belonging to ℓ_1. The stationary property further implies that these discount factors assume a geometrical representation.

A significant methodological added value of this article states as the *temporal decomposition* of the benchmark order that is split between an *infinite dimentional* distant future component and a *finite dimensional* close future one. This will in its turn imply the possibility of formulating the preference index as a weighted sum of the

[2]For a detailed exposition, see Bhaskara Rao & Bhaskara Rao [8].

distant future index and the close future index, the associated weights being directly related to some optimal degree on myopia that is shown to be inherent to the definition of the preference index. Second and more substantially: this early decomposition is refined and strengthened by the subsequent focus on robust linear orders that puts into evidence the scope for decomposing the weights put on the future between a classical σ-additive measure and some new finitely additive measures that directly springs from the explicit consideration of infinite dimensional objects. This in its turn allows for a subtle decomposition of the robust orders and a precise description of α-minmax representations. The decomposition of the orders assumes specific facets for time-dependent orders in that temporal biases may result into some sort of continuity between finite time close future orders and infinite distant future orders. Far beyond the specifics of this contribution, it is the conviction of the authors that such a methodological approach could be used and would hence significantly facilitate the numerous areas where an order theoretic approach is the hallmark of any serious answer to the problem at stake.

The article organizes as follows. Section 2 introduces the setup with the key axioms that are detailed, justified and shown to allow for a temporal decomposition of the order. Section 3 introduces the possibility of time-dependencies for the orders and emphasizes the possibility of temporal biases: even though it proceeds through finite dimensions that presumably only invoke the close future order, it is shown that there is an explicit scope for *distant future biases* for which a finite time temporal order could be directly affected by the features of an order that involves an infinite time, *i.e.,* the distant future one. Section 4 is an attempt to push results a little further through a focus on robust linear orders: it results, under further lines of arguments involving some weak forms of stationarity and pessimism / optimism, into the possibility of explicit α-maxmin representations for the preferences indexes. Section 5 is finally interested in establishing a careful comparison between this article and the earlier literature. It is first therein shown that it relates to various strands of the classical discount literature but also, in some different ways, to some recent strands of the decision theory literature. In this regard, it is also emphasized how it provides a renewed picture of the recent and important literature on temporal biases. It is then also clarified how it relates to a sparse but important and influential literature that provides some axomiatic approach to myopia. Finally, it is argued that it provides the first axiomatic understanding of *charges* in a decision theory context while such notions have already been shown to have the potential to shed a new light is areas as diverse as general equilibrium theory, wariness, social choice or price bubbles.

2. Basic axioms and benchmark axioms for the decomposition of preferences

2.1 Fundamentals

This paper considers an axiomatization approach to the evaluation of infinite utility streams, the whole argument being cast for discrete time sequences. In order to avoid confusion, letters like x, y, z will be used for sequences (of utils) with values in $\mathbb{R}$; a notation $c\mathbb{1}$, $c'\mathbb{1}$, $c''\mathbb{1}$ will be used for constant sequences, where $\mathbb{1}$ denotes $(1, 1, \ldots,)$. A notation λ, η, μ will also be used for constant scalars.
Recall that the dual space of ℓ_∞ , *i.e.*, the set of real sequences such that $\sup_s |x_s| < +\infty$, can be decomposed into the direct sum of two subspaces, ℓ_1 and ℓ_1^d: $(\ell_\infty)^* = \ell_1 \oplus \ell_1^d$. The subspace ℓ_1 satisfies σ-additivity. The subspace ℓ_1^d, the *disjoint complement* of ℓ_1, is the one of finitely additive measures defined on $\mathbb{N}$. More precisely, for each measure $\phi \in \ell_1^d$, for any $x \in \ell_\infty$, the value of $\phi \cdot x$ depends only on the distant behaviour of x, and does not change if there only occurs a change in a finite number of values x_s, $s \in \mathbb{N}$.
Let $\mathscr{P}$ denote the set of *weights* and *charges* which can be considered as finitely additive probabilistic measures on $(\mathbb{N}, \Sigma)$. $\mathscr{P}$ is the set of $\big((1-\lambda)\underset{\sim}{\omega}, \lambda\phi\big)$ where

(i) $\lambda \in [0, 1]$.

(ii) $\underset{\sim}{\omega} = (\omega_0, \omega_1, \ldots) \in \ell_1$ with $\omega_s \geq 0$ for every $s \in \mathbb{N}$ and $\sum_{s=0}^{\infty} \omega_s = 1$.

(iii) ϕ is a *charge* belonging to ℓ_1^d. The charge ϕ can be considered as a *purely finitely additive measure* on $\mathbb{N}$: for every finite subset $A \subset \mathbb{N}$, $\phi(A) = 0$.

(iv) The two functions: $x \mapsto \sum_{s=0}^{\infty} \omega_s x_s$ and $x \mapsto \phi \cdot x$ are linear and continuous on ℓ_∞. For every $x, y \in \ell_\infty$, assume that there exists only a finite number of s such that $x_s \neq y_s$, then $\phi \cdot x = \phi \cdot y$.

Finally recall that the measure $(\underset{\sim}{\omega}, \phi)$ is countably additive if and only if $\lambda = 0$.

2.2 Elementary Axioms & Construction of the Index Function

The following axiom is imposed for the order $\succeq$ on ℓ_∞.

Axiom F1. The order $\succeq$ satisfies the following properties:

(i) *Completeness* For every $x, y \in \ell_\infty$, either $x \succeq y$ or $y \succeq x$.

(ii) *Transitivity* For every $x, y, z \in \ell_\infty$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$. Denote as $x \sim y$ the case $x \succeq y$ and $y \succeq x$. Denote as $x \succ y$ the case $x \succeq y$ and $y \not\succeq x$.

(iii) *Monotonicity* If $x, y \in \ell_\infty$ and $x_s \geq y_s$ for every $s \in \mathbb{N}$, then $x \succeq y$.

(iv) *Non-triviality* There exist $x, y \in \ell_\infty$ such that $x \succ y$.

(v) *Archimedeanity* For $x \in \ell_\infty$ and $b\mathbb{1} \succ x \succ b'\mathbb{1}$, there are $\lambda, \mu \in [0, 1]$ such that

$$(1 - \lambda)b\mathbb{1} + \lambda b'\mathbb{1} \succ x \text{ and } x \succ (1 - \mu)b\mathbb{1} + \mu b'\mathbb{1}.$$

(vi) *Weak convexity* For every $x, y, b\mathbb{1} \in \ell_\infty$, and $\lambda \in]0, 1]$,

$$x \succeq y \Leftrightarrow (1 - \lambda)x + \lambda b\mathbb{1} \succeq (1 - \lambda)y + \lambda b\mathbb{1}.$$

All of the properties (i), (ii), (iii) and (iv) are standardly used in decision theory. The *Archimedeanity* property (v) ensures that the order is continuous in the sup-norm topology of ℓ_∞. The eventual *Weak convexity* property (vi) is admittedly less immediate. It is referred to as *certainty independence* in the decision theory literature and ensures that direction $\mathbb{1}$ is *comparison neutral*: following that direction, the comparison between two sequences does not change. This is made precise in the following statement:

Proposition 2.1. *Assume that axiom* **F1** *is satisfied. There exists an index function* $\mathrm{I} : \ell_\infty \to \mathbb{R}$ *representing* $\succeq$ *and satisfying* completeness, positive homogeneity *and* constant additivity *properties:*

(i) *For every* $x, y \in \ell_\infty$*, any* $\lambda > 0$*,* $x \succeq y$ *if and only if* $\lambda x \succeq \lambda y$*.*

(ii) *For every* $x, y \in \ell_\infty$*, a constant* $b \in \mathbb{R}$*,* $x \succeq y$ *if and only if* $x + b\mathbb{1} \succeq y + b\mathbb{1}$*.*

(iii) *For* $x \in \ell_\infty$*,* $\lambda > 0$*,* $\mathrm{I}(\lambda x) = \lambda \mathrm{I}(x)$*.*

(iv) *For* $x \in \ell_\infty$*, constant* $b \in \mathbb{R}$*,* $\mathrm{I}(x + b\mathbb{1}) = \mathrm{I}(x) + b$*.*

The results in Proposition 2.1 can be compared to the conclusions reached in Gilboa & Schmeidler [20], and Ghirardato & al [19]. This article however considers the total space ℓ_∞ as opposed to the space of simple acts— —these are equivalent to sequences in ℓ_∞ which take a finite number of values. The order is homogeneous of degree 1 and *constantly additive*, a property which means that the direction $\mathbb{1}$ is *comparison neutral*.

2.3 A decomposition of preferences between the close future and the distant future

2.3.1 Non-negligible tail and the effect of the distant future

For every $x \in \ell_\infty$ and $0 \leq \mathrm{T} \leq \mathrm{T}'$, let

$$x_{[\mathrm{T},\mathrm{T}']} = (x_\mathrm{T}, x_{\mathrm{T}+1}, \ldots, x_{\mathrm{T}'}),$$
$$x_{[\mathrm{T}+1,\infty[} = (x_{\mathrm{T}+1}, x_{\mathrm{T}+2}, \ldots)$$
$$\mathbb{1}_{\{\mathrm{T}\}} = (\underbrace{0, 0, \ldots, 0,}_{\mathrm{T}\ \text{times}} 1, 0, 0, \ldots).$$

In the literature, the notions of *impatience*[3] or *delay aversion*[4] can generally be understood through the convergence to zero of $\mathbb{1}_{\{\mathrm{T}\}}$ as T tends to infinity. It is however worth emphasizing that such a property does not *per se* imply the convergence to zero of the effect of the *distant future sequence* $\mathbb{1}_{[\mathrm{T},+\infty[}$. More generally, it is commonly assumed in the literature that the value of the distant future converges to zero when T converges to infinity. In the current framework and under Proposition 2.1, this means that $\mathrm{I}\big(0\mathbb{1}^{\mathrm{T}}, \big(-\mathbb{1}_{[\mathrm{T}+1,\infty[}\big)\big)$ and $\mathrm{I}\big(0\mathbb{1}_{[0,\mathrm{T}]}, \mathbb{1}_{[\mathrm{T}+1,\infty[}\big)$ are to converge

[3]See Koopmans [23].

[4]See Bastianello & Chateauneuf [4].

to zero when T tends to infinity[5]. To check about such a possibility in the current environment, it is first useful to introduce the two following coefficients[6]:

$$\chi_d = \lim_{T\to\infty} I\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right),$$
$$\chi_c = -\lim_{T\to\infty} I\left(0\mathbb{1}^T, \left(-\mathbb{1}_{[T+1,\infty[}\right)\right)$$
$$= 1 - \lim_{T\to\infty} I\left(\mathbb{1}^T, 0\mathbb{1}_{[T+1,\infty[}\right).$$

The two values χ_d and χ_c will be considered extensively in this article and play an important role in the definition of the *myopia degrees*.

Proposition 2.2. *Assume that axiom* **F1** *is satisfied. Suppose that* $\chi_c = \chi_d = 0$. *Then for any constants* $c, d \in \mathbb{R}$,

$$\lim_{T\to\infty} I\left(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) = c.$$

The condition $\chi_d = \chi_c = 0$ is similar to the usual *negligible-tail* or *tail-insensitivity* conditions. Under this condition, a natural conjecture formulates as the holding, for any $x, z \in \ell_\infty$, of

$$\lim_{T\to\infty} I\left(x_{[0,T]}, z_{[T+1,\infty[}\right) = I(x).$$

The following counter example however suggests that, even though the valuation of the distant future could be nil[7], it could keep on exerting some influence on the evaluation of the sequences, the index function form being thereafter more complicate than the above guess.

Example 2.1. *Consider two probability measures belonging to* ℓ_1, $\underset{\sim}{\omega}$ *and* $\underset{\sim}{\hat{\omega}}$, *satisfying* $\underset{\sim}{\omega} \neq \underset{\sim}{\hat{\omega}}$. *Consider also a finitely additive measure* $\phi \in \ell_d^1$. *Define the index function* I *as:*

$$I(x) = \max\left\{\underset{\sim}{\hat{\omega}} \cdot x, \min\left\{\underset{\sim}{\omega} \cdot x, \phi \cdot x\right\}\right\}.$$

[5] Observe that these two properties are not equivalent.

[6] From the *monotonicity* property F1(iii), $I\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right)$ and $1 - I\left(\mathbb{1}^T, 0\mathbb{1}_{[T+1,\infty[}\right)$ are decreasing as a function of T, these limits being well-defined.

[7] It can be proved that $\lim_{T\to\infty} I\left(0\mathbb{1}_{[0,T]}, z_{[T+1,+\infty[}\right) = 0$ for any $z \in \ell_\infty$.

It is readily checked that I *satisfies all of the properties listed in Axiom* ***F1***. *Further observe that, for any* T,

$$\underset{\sim}{\hat{\omega}} \cdot \left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) = \sum_{s=T+1}^{\infty} \hat{\omega}_s,$$

$$\underset{\sim}{\omega} \cdot \left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) = \sum_{s=T+1}^{\infty} \omega_s,$$

$$\phi \cdot \left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) = 1.$$

This implies that, for large enough values of T,

$$\underset{\sim}{\omega} \cdot \left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) < 1 = \phi \cdot \left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right).$$

Therefore:

$$\mathrm{I}\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) = \max\left\{\sum_{s=T+1}^{\infty} \hat{\omega}_s, \sum_{s=T+1}^{\infty} \omega_s\right\},$$

that converges to zero. Making use of the same arguments,

$$\lim_{T\to\infty} \mathrm{I}\left(0\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,\infty[}\right) = 0.$$

There however exist $x, z \in \ell_\infty$ *such that* $\lim_{T\to\infty} \mathrm{I}\left(x_{[0,T]}, z_{[T+1,\infty[}\right) \neq \mathrm{I}(x)$. *Indeed, since* $\underset{\sim}{\hat{\omega}}$ *and* $\underset{\sim}{\omega}$ *are distinct and both belong to* ℓ_1, *there exists* $x \in \ell_\infty$ *such that:*

$$\underset{\sim}{\hat{\omega}} \cdot x < \underset{\sim}{\omega} \cdot x < \phi \cdot x.$$

Take z satisfying

$$\underset{\sim}{\hat{\omega}} \cdot x < \phi \cdot z < \underset{\sim}{\omega} \cdot x < \phi \cdot x.$$

For any T, $\phi \cdot \left(x_{[0,T]}, z_{[T+1,\infty[}\right) = \phi \cdot z$, *it derives that:*

$$\begin{aligned}\lim_{T\to\infty} \mathrm{I}\left(x_{[0,T]}, z_{[T+1,\infty[}\right) &= \lim_{T\to\infty} \max\Big\{\underset{\sim}{\hat{\omega}} \cdot \left(x_{[0,T]}, z_{[T+1,\infty[}\right),\\ &\qquad \min\Big\{\underset{\sim}{\omega} \cdot \left(x_{[0,T]}, z_{[T+1,\infty[}\right), \phi \cdot z\Big\}\Big\}\\ &= \max\Big\{\underset{\sim}{\hat{\omega}} \cdot x, \min\Big\{\underset{\sim}{\omega} \cdot x, \phi \cdot z\Big\}\Big\}\\ &= \phi \cdot z,\end{aligned}$$

that differs from $\mathrm{I}(x) = \underset{\sim}{\omega} \cdot x$.

This suggests the need for a further understanding of the problem at stake that should be apprehended through complementary structures.

2.3.2 Distant future order

The following axiom assumes that there exists an *evaluation of the distant future* which is independent from the past——the *close future*.

Axiom G1. For any $x \in \ell_\infty$ and any constant $d \in \mathbb{R}$, either, for any $\epsilon > 0$ and $z \in \ell_\infty$, there exists $T_0(\epsilon, z)$ such that, for every $T \geq T_0(\epsilon, z)$:

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \succeq \left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1},$$

or, for any $\epsilon > 0$, $z \in \ell_\infty$, there exists $T_0(\epsilon, z)$ such that, for every $T \geq T_0(\epsilon, z)$:

$$\left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) + \epsilon\mathbb{1} \succeq \left(z_{[0,T]}, x_{[T+1,\infty[}\right).$$

For any sequence x and a constant sequence $d\mathbb{1}$, the sequence x will either overtake the sequence $(d-\epsilon)\mathbb{1}$ or be overtaken by the sequence $(d+\epsilon)\mathbb{1}$, and this is going to take place independently from the past——the *close future*. Otherwise stated, either x dominates in the distant future, or $d\mathbb{1}$ dominates in the distant future. This *distant future sentitivities* axiom contradicts the usual *negligible-tail* or *tail-insensitivity* axioms in the literature.

Example 2.2. *Take,* e.g., *a binary relation* $\succeq$ *on* ℓ_∞ *defined as: for some* $0 < \lambda < 1$, $0 < \delta < 1$, $x \succeq y$ *if:*

$$(1-\lambda)(1-\delta)\sum_{t=0}^{\infty}\delta^t x_t + \lambda\liminf_{t\to\infty} x_t \geq (1-\lambda)(1-\delta)\sum_{t=0}^{\infty}\delta^t y_t + \lambda\liminf_{t\to\infty} y_t.$$

For any $x \in \ell_\infty$, $d \in \mathbb{R}$, $\epsilon > 0$, *either* $\liminf_{t\to\infty} x_t > d - \epsilon$, *or* $d + \epsilon > \liminf_{t\to\infty} x_t$ *for any* $\epsilon > 0$. *Consider, for example, the case* $\liminf_{t\to\infty} x_t \geq d$. *For any* $\epsilon > 0$, *take* $T_0(\epsilon)$ *large enough such that, for any* $T \geq T_0(\epsilon)$,

$$(1-\lambda)(1-\delta)\sum_{t=T}^{\infty}\delta^t |x_t| < \lambda\frac{\epsilon}{2},$$

$$(1-\lambda)(1-\delta)\sum_{t=T}^{\infty}\delta^t |d-\epsilon| < \lambda\frac{\epsilon}{2}.$$

Therefore, and for any $T \geq T_0(\epsilon)$,

$$\begin{aligned}\lambda\liminf_{t\to\infty} x_t &\geq \lambda d - \lambda\epsilon + \lambda\epsilon \\ &\geq \lambda(d-\epsilon) - (1-\lambda)(1-\delta)\sum_{t=T}^{\infty}\delta^t x_t + (1-\lambda)(1-\delta)\sum_{t=T}^{\infty}\delta^t(d-\epsilon),\end{aligned}$$

which implies

$$(1-\lambda)(1-\delta)\sum_{t=0}^{T-1}\delta^t z_t + (1-\lambda)(1-\delta)\sum_{t=T}^{\infty} x_t + \lambda \liminf_{t\to\infty} x_t$$
$$\geq (1-\lambda)(1-\delta)\sum_{t=0}^{T-1}\delta^t z_t + (1-\lambda)(1-\delta)\sum_{t=T}^{\infty}(d-\epsilon) + \lambda(d-\epsilon),$$

and is equivalent to $(z_0, z_1, \ldots, z_T, x_{T+1}, x_{T+2}, \ldots) \geq (z_0, z_1, \ldots, z_T, d-\epsilon, d-\epsilon, \ldots)$.

The order in the above example satisfies axiom **G1**. Further observe that $\chi_d = \lambda$ and $\chi_c = 1-\lambda$ are therein positive, whence the obtention of a critical time $T_0(\epsilon)$ that solely depends on ϵ and can be defined independently from z.

Lemma 2.1. *Assume that axioms* **F1** *and* **G1** *are satisfied. For any* $x, y \in \ell_\infty$*, either, for any* $\epsilon > 0$ *and* $z \in \ell_\infty$*, there exists* $T_0(\epsilon, z)$ *such that, for every* $T \geq T_0(\epsilon, z)$*:*

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \geq \left(z_{[0,T]}, y_{[T+1,\infty[}\right) - \epsilon\mathbb{1},$$

or, for any $\epsilon > 0$*,* $z \in \ell_\infty$*, there exists* $T_0(\epsilon, z)$ *such that, for every* $T \geq T(\epsilon, z)$*:*

$$\left(z_{[0,T]}, y_{[T+1,\infty[}\right) \geq \left(z_{[0,T]}, x_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

From Lemma 2.1, for $x, y \in \ell_\infty$, define $x \geq_d y$ if, for any $\epsilon > 0$ and $z \in \ell_\infty$, there exists $T_0(\epsilon, z)$ such that, for every $T \geq T_0(\epsilon, z)$:

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \geq \left(z_{[0,T]}, y_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

Proposition 2.3 proves that it suffices for one of the two values χ_d and χ_c to differ from zero for the order $\geq_d$ to satisfy every property of axiom **F1**. This in its turn assumes as its most immediate consequence that there also exists an index function satisfying any of the properties listed in Proposition 2.1.

Proposition 2.3. *Assume that axioms* **F1** *and* **G1** *are satisfied.*

(i) *The order* $\geq_d$ *is complete.*

(ii) *If at least one of the two values* χ_d, χ_c *differs from zero, the order* $\geq_d$ *is non-trivial and satisfies every property in axiom F1.*

(iii) *If at least one of the two values* χ_d *and* χ_c *differs from zero, there exists an index function* $\mathrm{I}_d : \ell_\infty \to \mathbb{R}$ *representing the distant future that satisfies* the positive homogeneity and constant additivity *properties of Proposition 2.1.*

The main properties of the *distant future* order $\succeq_d$ are presented in Proposition 2.4.

Proposition 2.4. *Assume that axioms* **F1** *and* **G1** *are satisfied.*

(i) *For any* $x, z \in \ell_\infty$*, for every* $\mathrm{T} \in \mathbb{N}$, $\mathrm{I}_d\big(z_{[0,\mathrm{T}]}, x_{[\mathrm{T}+1,\infty[}\big) = \mathrm{I}_d(x)$.

(ii) *For any constants* $c, d \in \mathbb{R}$ *such that* $c \leq d$,

$$\lim_{\mathrm{T}\to\infty} \mathrm{I}\big(c\mathbb{1}_{[0,\mathrm{T}]}, d\mathbb{1}_{[\mathrm{T}+1,\infty[}\big) = (1 - \chi_d)c + \chi_d d.$$

(iii) *For* $x \in \ell_\infty$*, if the sequence* x *converges, then* $\mathrm{I}_d(x) = \lim_{\mathrm{T}\to\infty} x_\mathrm{T}$.

(iv) *Consider* $x \in \ell_\infty$ *such that the sequence* x *converges. For any* $y \in \ell_\infty$, $\mathrm{I}_d(x + y) = \mathrm{I}_d(x) + \mathrm{I}_d(y)$.

Otherwise stated and from (i), the value of the index function does not depend upon the past— —the *close future*. Further and from (ii), a simple decomposition becomes available when the sequence assumes constant values for both the close future and the distant future. As this is made clear in (iii), the evaluation of a given sequence x according to I_d is in its turn provided by the limit of that sequence when the later is well-defined. Finally and in (iv), the index function satisfies some form of the *constant additivity* property.

2.3.3 Close future order

Recall however that example 2.1 has established that the sole *distant future sensitivities* axiom **G1** does not suffice to disentangle the *distant future* from the past— —- the *close future*. In order to enable such a decomposition, consider the *close future sensitivities* axiom **G2**, that is to be understood as *the complement* of axiom **G1**.

Axiom G2. For any $x \in \ell_\infty$, a constant $c \in \mathbb{R}$, either, for any $\epsilon > 0$ and $z \in \ell_\infty$, there exists $\mathrm{T}_0(\epsilon, z)$ such that, for every $\mathrm{T} \geq \mathrm{T}_0(\epsilon, z)$,

$$\big(x_{[0,\mathrm{T}]}, z_{[\mathrm{T}+1,\infty[}\big) \succeq \big(c\mathbb{1}_{[0,\mathrm{T}]}, z_{[\mathrm{T}+1,\infty[}\big) - \epsilon\mathbb{1},$$

or, for any $\epsilon > 0$, $z \in \ell_\infty$, there exists $T_0(\epsilon, z)$ such that, for every $T \geq T_0(\epsilon, z)$,

$$\left(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}\right) + \epsilon\mathbb{1} \succeq \left(x_{[0,T]}, z_{[T+1,\infty[}\right).$$

This assumption reads as follows: for any sequence x and a constant sequence $d\mathbb{1}$, either the sequence x will overtake the sequence $(c-\epsilon)\mathbb{1}$ or it will be dominated by the sequence $(c+\epsilon)\mathbb{1}$, both of these occurrences being defined whatever the distant future. Otherwise stated, either x or $d\mathbb{1}$ dominates in the close future.

Example 2.3. *Consider the order defined in Example 2.1, $x \succeq y$ if and only if*

$$(1-\lambda)(1-\delta)\sum_{t=0}^{\infty}\delta^t x_t + \lambda \liminf_{t\to\infty} x_t \geq (1-\lambda)(1-\delta)\sum_{t=0}^{\infty}\delta^t y_t + \lambda \liminf_{t\to\infty} y_t.$$

Consider the case

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t x_t \geq c.$$

For any $\epsilon > 0$, any $z \in \ell_\infty$, take $T_0(\epsilon)$ such that for any $T \geq T_0(\epsilon)$,

$$(1-\delta)\sum_{t=0}^{T}\delta^t x_t > (1-\delta)\sum_{t=0}^{T}\delta^t (c-\epsilon).$$

But

$$(1-\delta)\sum_{t=0}^{T-1}\delta^t x_t + (1-\delta)\sum_{t=T+1}^{\infty}\delta^t z_t \geq (1-\delta)\sum_{t=0}^{T}\delta^t (c-\epsilon) + (1-\delta)\sum_{t=T+1}^{\infty}\delta^t z_t,$$

that implies

$$\begin{aligned}&(1-\lambda)(1-\delta)\sum_{t=0}^{T}\delta^t x_t + (1-\lambda)(1-\delta)\sum_{t=T+1}^{\infty}\delta^t z_t + \lambda \liminf_{t\to\infty} z_t \\ &\geq (1-\lambda)(1-\delta)\sum_{t=0}^{T}\delta^t (c-\epsilon) + (1-\lambda)(1-\delta)\sum_{t=T+1}^{\infty}\delta^t z_t + \lambda \liminf_{t\to\infty} z_t,\end{aligned}$$

or

$$(x_0, x_1, \ldots, x_T, z_{T+1}, z_{T+2}, \ldots) \succeq (\underbrace{c-\epsilon, c-\epsilon, \ldots, c-\epsilon}_{T+1 \text{ times}}, z_{T+1}, z_{T+2}, \ldots).$$

Again observe that, in this configuration, $\chi_d = \lambda$ and $\chi_c = 1-\lambda$ are both strictly smaller than 1, hence the critical time $T_0(\epsilon)$ can be defined independently from z.

Usual conditions in the literature assume that the effect of the distant future converges to zero— —*e.g.*, the *Continuity at infinity* of Chambers & Echenique [14], or the axioms ensuring insensitivity to the distant future, or some sort of *flat tail* for the distribution. Remark that, in opposition to this, the *close future sensitivities* Axiom **G2** merely assumes that the distant future does not alter the evaluation of the close future.

LEMMA 2.2. *Assume that axioms* **F1** *and* **G2** *are satisfied. For any* $x, y \in \ell_\infty$*, either, for any* $\epsilon > 0$*,* $z \in$*, there exists* $T_0(\epsilon, z)$ *such that, for every* $T \geq T_0(\epsilon, z)$*,*

$$\left(x_{[0,T]}, z_{[T+1,\infty[}\right) \succeq \left(y_{[0,T]}, z_{[T+1,\infty[}\right) - \epsilon\mathbb{1},$$

or, for any $\epsilon > 0$*,* $z \in \ell_\infty$*, there exists* $T_0(\epsilon, z)$ *such that, for* $T \geq T_0(\epsilon, z)$*,*

$$\left(y_{[0,T]}, z_{[T+1,\infty[}\right) \succeq \left(x_{[0,T]}, z_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

From Lemma 2.2, the close order $\succeq_c$ can be defined as follows: for $x, y \in \ell_\infty$, $x \succeq_c y$ if, for any $\epsilon > 0$, $z \in \ell_\infty$, there exists $T_0(\epsilon, z)$ such that, for every $T \geq T_0(\epsilon, z)$,

$$\left(x_{[0,T]}, z_{[T+1,\infty[}\right) \succeq \left(y_{[0,T]}, z_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

PROPOSITION 2.5. *Assume that axioms* **F1** *and* **G2** *are satisfied.*

(i) *The close order* $\succeq_c$ *is complete.*

(ii) *If at least one of two values* χ_d, χ_c *differs from* 1*, then the order* $\succeq_c$ *is non-trivial and satisfies every property in axiom* **F1**.

(iii) *If at least one of two values* χ_d, χ_c *differs from* 1*, then there exists an index function for the close future* $I_c : \ell_\infty \to \mathbb{R}$ *satisfying the* positive homogeneity and constant additivity *properties of Proposition 2.1.*

The following Proposition then provides another simple decomposition for constant sequences and between the close and the distant futures. It moreover proves that the close future order recovers the usual *tail-insensitivity* property, the corresponding distant future order of $\succeq_c$ being indeed trivial.

PROPOSITION 2.6. *Assume that axioms* **F1** *and* **G2** *are satisfied. Suppose that at least one of the two values* χ_d, χ_c *is different from 1.*

(i) *For every constants* $c, d \in \mathbb{R}$ *such that* $c \geq d$,

$$\lim_{T\to\infty} I\Big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\Big) = (1-\chi_c)c + \chi_c d.$$

(ii) *For any* $x, y \in \ell_\infty$,

$$\lim_{T\to\infty} I_c\Big(x_{[0,T]}, y_{[T+1,\infty[}\Big) = I_c(x).$$

The following preparation lemma finally provides a useful clarification of the boundary cases for which $\chi_d = \chi_c = 0$, or $\chi_d = \chi_c = 1$.

Lemma 2.3. *Assume that axiom* **F1** *is satisfied. Fix two different measures in* ℓ_1, $\underset{\sim}{\omega}$ *and* $\hat{\underset{\sim}{\omega}}$, *and two different measures in* ℓ^1_d, ϕ *and* $\hat{\phi}$.

(i) *There exists a complete order* $\succeq$ *satisfying axiom* **G1** *such that the order* $\succeq_d$ *is non trivial and satisfies every property of axiom* **F1**, *and* $\chi_d = \chi_c = 0$. *It is represented by the following index function:*

$$I(x) = \max\Big\{\hat{\underset{\sim}{\omega}} \cdot x, \min\Big\{\underset{\sim}{\omega} \cdot x, \phi \cdot x\Big\}\Big\}$$

(ii) *There exists a complete order* $\hat{\succeq}$ *satisfying axiom* **G2** *such that the order* $\succeq_c$ *is non trivial and satisfies every property of axiom* **F1**, *and* $\chi_d = \chi_c = 1$. *It is represented by the following index function:*

$$\hat{I}(x) = \max\Big\{\hat{\phi} \cdot x, \min\Big\{\underset{\sim}{\omega} \cdot x, \phi \cdot x\Big\}\Big\}.$$

(iii) *By adding axioms* **G1** *and* **G2** *for the order* $\succeq$:

a) *If* $\chi_d = \chi_c = 0$, *then the order* $\succeq_d$ *is trivial: for any* $x, y \in \ell_\infty$, $x \sim_d y$.

b) *If* $\chi_d = \chi_c = 1$, *then the order* $\succeq_c$ *is trivial: for any* $x, y \in \ell_\infty$, $x \sim_c y$.

From Lemma 2.3, the two axioms **G1** and **G2** are therefore not equivalent.

2.3.4 A decomposition between the distant and close future orders

From the previous developments and under axioms **F1** and **G1**, **G2**, it may be surmised that there is some strong potential for the index function I to restate as a convex sum of the two index functions I_d and I_c, *e.g.*,

$$I(x) = (1-\chi^*)I_c(x) + \chi^* I_d(x),$$

for some value $\chi^* \in [0,1]$. Observe that if the chosen parameter χ^* does not change over time, such a decomposition implies that:

$$\lim_{T\to\infty} I\Big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big) + \lim_{T\to\infty} I\Big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\Big) = 1,$$

which is equivalent to $\chi_c = \chi_d$, and therefore $\chi^* = \chi_d = \chi_c$. Observe however that, under axioms **F1** and **G1**, **G2**, the holding of such an equality cannot be guaranteed. This also indicates that, when $\chi_d \neq \chi_c$, the decomposition parameter must change as a function of the sequence x.

The configuration

$$\lim_{T\to\infty} I\Big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big) + \lim_{T\to\infty} I\Big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\Big) \leq 1,$$

which is equivalent to $\chi_d \leq \chi_c$, can first be understood as a *pessimistic*, or a *very myopia-bending* situation. Otherwise stated, the value brought the distant future is not sufficiently large to compensate the loss that is incurred in the close future.

Likewise, the configuration

$$\lim_{T\to\infty} I\Big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big) + \lim_{T\to\infty} I\Big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\Big) \geq 1,$$

which is equivalent to $\chi_d \geq \chi_c$, can be understood as a *optimistic*, or a *non very myopia-bending* situation. The gain in the distant future is valued more than the lost that is incurred in the close future.

The following theorem, which is one of the main results of this article, will prove that there exists a multiplicity of *possible myopia degrees*. This theorem also clarifies how it is the choice of the *optimal myopia degree* χ that determines an optimal share between the close future and the distant future indexes.

Theorem 2.1. *Assume that axioms* **F1** *and* **G1**, **G2** *are satisfied.*

(i) *For* $\chi_d \leq \chi_c$, *let* $\underline{\chi} = \chi_d$, $\overline{\chi} = \chi_c$. *For any* $x \in \ell_\infty$,

$$I(x) = \min_{\underline{\chi}\leq\chi\leq\overline{\chi}} \Big[(1-\chi)I_c(x) + \chi I_d(x)\Big].$$

(ii) *For* $\chi_d \geq \chi_c$, *let* $\underline{\chi} = \chi_c$, $\overline{\chi} = \chi_d$. *For any* $x \in \ell_\infty$,

$$I(x) = \max_{\underline{\chi}\leq\chi\leq\overline{\chi}} \Big[(1-\chi)I_c(x) + \chi I_d(x)\Big].$$

The Minimum operator in part (i) represents a configuration where the evaluation the distant future is lower than the one of the close future. The Maximum operator part (ii) represents an opposite occurrence.
Lastly, it may be wondered why an α−maximin decomposition between the close future and the distant future indexes is not also available. The following corollary proves that such an α−maximin behaviour just features another version of the max and min operators.

Corollary 1. *Assume that axioms* **F1** *and* **G1, G2** *are satisfied. For any* $\alpha \in [0, 1]$, *consider the index function of the* α−*maximin criterion*

$$\mathrm{I}_\alpha(x) = \alpha \min_{\underline{\chi}\leq\chi\leq\overline{\chi}}\Big[(1-\chi)\mathrm{I}_c(x) + \chi \mathrm{I}_d(x)\Big] + (1-\alpha) \max_{\underline{\chi}\leq\chi\leq\overline{\chi}}\Big[(1-\chi)\mathrm{I}_c(x) + \chi \mathrm{I}_d(x)\Big].$$

There exists $0 \leq \underline{\chi}_\alpha \leq \overline{\chi}_\alpha \leq 1$ *such that one of the two following assertions is true:*

(i) *For any* $x \in \ell_\infty$,

$$\mathrm{I}_\alpha(x) = \min_{\underline{\chi}_\alpha\leq\chi\leq\overline{\chi}_\alpha}\Big[(1-\chi)\mathrm{I}_c(x) + \chi \mathrm{I}_d(x)\Big].$$

(ii) *For any* $x \in \ell_\infty$,

$$\mathrm{I}_\alpha(x) = \max_{\underline{\chi}_\alpha\leq\chi\leq\overline{\chi}_\alpha}\Big[(1-\chi)\mathrm{I}_c(x) + \chi \mathrm{I}_d(x)\Big].$$

2.4 The case of the $\ell_\infty(a)$ spaces

A limit of the previous argument however springs from its focus on bounded sequences. This section will however illustrate how it can straightforwardly be extended to deal with unbounded sequences. For $a \geq 1$, define

$$\ell_\infty(a) = \left\{ x = (x_0, x_1, x_2, \dots, x_s, \dots) \text{ such that } \sup_s \left|\frac{x_s}{a^s}\right| < +\infty. \right\}$$

The space ℓ_∞ is thus the space corresponding to $a = 1$. For each a, define $\|x\|_{\ell_\infty(a)} = \sup_s |x_s/a^s|$. The coefficient a can be understood as an actualization rate, and $\ell_\infty(a)$ as the associated set of feasible sequences.[8]

[8]For the sake of simplicity, this article works on the case discount rate is constant. The same arguments and results can be applied for more general spaces $\ell_\infty(1, a_1, a_2, \dots,)$ with $a_s > 0$ for any s

Consider the actualization function $\psi_a : \ell_\infty(a) \to \ell_\infty$:

$$\psi_a(x) = \left(x_0, \frac{x_1}{a}, \frac{x_2}{a^2}, \ldots, \frac{x_s}{a^s}, \ldots\right).$$

For a given discount rate a, the economic agents make comparisons between actualised sequences, which belong to ℓ_∞.
Along Dolmas [15], the isomorphism ψ_a preserves the *natural* pre-order: if $x_s \geq y_s$ for any s, then $\psi_a(x) \geq \psi_a(y)$. In addition to this, it preserves the norm: for any $x \in \ell_\infty(a)$, $\left\|\psi_a(x)\right\|_{\ell_\infty} = \|x\|_{\ell_\infty(a)}$.
This article then considers a family of complete orders $\succeq^a$ defined on $\ell_\infty(a)$, which are *preserved* by the functions ψ_a: for any $\ell_\infty(a)$, and for any $x, y \in \ell_\infty(a)$,

$$x \succeq^a y \text{ if any only if } \psi_a(x) \succeq^1 \psi_a(y).$$

The axiom **F2** links the orders $\succeq^a$ on $\ell_\infty(a)$ and the usual order $\succeq$ on ℓ_∞.

Axiom F2. The $\succeq^a$ –orders satisfy:
For any space $\ell_\infty(a)$, for any $x, y \in \ell_\infty(a)$, $x \succeq^a y$ if and only if

$$\left(x_0, \frac{x_1}{a}, \frac{x_2}{a^2}, \ldots, \frac{x_s}{a^s}, \ldots\right) \succeq \left(y_0, \frac{y_1}{a}, \frac{y_2}{a^2}, \ldots, \frac{y_s}{a^s}, \ldots\right).$$

Under axiom **F2**, the order $\succeq^a$ has the same properties as the order $\succeq$, with the mere qualification the reference direction becomes $\mathbb{1}^a$ in lieu of $\mathbb{1}$.

Proposition 2.7. *Let* $\mathbb{1}^a = (1, a, a^2, \ldots, a^s, \ldots)$. *Assume that axioms* **F1** *and* **F2** *are satisfied.*

(i) *The orders* $\succeq^a$ *satisfy the* transitivity, monotonicity, *and* non-triviality *properties in axiom* ***F1****.*

(ii) $\ell_\infty(a)$ – weak convexity: *For any* $x, y \in \ell_\infty(a)$ *and a constant* $b \in \mathbb{R}$, $\lambda \in [0, 1[$,

$$x \succeq^a y \textit{ if and only if } (1-\lambda)x + \lambda b \mathbb{1}^a \succeq^a (1-\lambda)y + \lambda b \mathbb{1}^a.$$

and the norm is defined as:

$$\|x\|_{\ell_\infty(1, a_1, a_2, \ldots)} = \sup_{s \geq 0} \left| \frac{x_s}{a_1 a_2 \cdots a_s} \right|.$$

(iii) $\ell_\infty(a)$ – Archimedeanity: *For any* $x \in \ell_\infty(a)$, *if* $b\mathbb{1}^a \succ x \succ b'\mathbb{1}^a$, *then there exists* $\lambda, \mu \in [0,1]$ *such that*

$$\Big((1-\lambda)b + \lambda b'\Big)\mathbb{1}^a \succ x \succ \Big((1-\mu)b + \mu b'\Big)\mathbb{1}^a.$$

(iv) *There exists an index function* $I^a : \ell_\infty(a) \to \mathbb{R}$ *such that:*

a) *For any* $x, y \in \ell_\infty(a)$,

$$x \succeq^a y \text{ if and only if } I^a(x) \geq I^a(y).$$

b) *For any* $\lambda \geq 0$, $x \in \ell_\infty(a)$ *and a constant* b,

$$I^a\Big(\lambda x + b\mathbb{1}^a\Big) = \lambda I^a(x) + b.$$

Under axioms **G1** and **G2**, using the same methods as in the previous sections, there exist a distance future order $\succeq^a_d$ and a close future order $\succeq^a_c$ which are respectively represented by index functions I^a_d and I^a_c satisfying every property in Proposition 2.7.

For any $\ell_\infty(a)$ space,

$$\chi^a_d = \lim_{T\to\infty} I^a\Big(\underbrace{0,0,\dots,0}_{T+1 \text{ times}}, a^{T+1}, a^{T+2}, \dots\Big),$$

$$\chi^a_c = 1 - \lim_{T\to\infty} I^a\Big(1, a, a^2, \dots, a^T, 0, 0, \dots\Big),$$

where χ^a_d, χ^a_c are two critical values.[9]

Theorem 2.2. *Assume that axioms* ***F1, F2****, and* ***G1, G2*** *are satisfied.*

(i) *Consider the case* $\chi^a_d \leq \chi^a_c$. *Let* $\underline{\chi}^a = \chi^a_d$, $\overline{\chi}^a = \chi^a_c$. *For any* $x \in \ell_\infty(a)$,

$$I^a(x) = \min_{\underline{\chi}^a \leq \chi \leq \overline{\chi}^a} \Big[(1-\chi)I^a_c(x) + \chi I^a_d(x)\Big].$$

(ii) *Consider the case* $\chi^a_d \geq \chi^a_c$. *Let* $\underline{\chi}^a = \chi^a_c$, $\overline{\chi}^a = \chi^a_d$. *For any* $x \in \ell_\infty(a)$,

$$I^a(x) = \max_{\underline{\chi}^a \leq \chi \leq \overline{\chi}^a} \Big[(1-\chi)I^a_c(x) + \chi I^a_d(x)\Big].$$

[9] These two values exists, thanks to the monotonicity property of the $\succeq^a$ orders.

3. Time-dependent orders and preference biases

A natural direction of extension does emerge. It focuses on the potential role of time by introducing time dependencies into the benchmark order and illustrates how this may provide some renewed picture of the much discussed *temporal biases*.

3.1 Time-dependent orders

Axiom B1. Consider $T \geq 1$, $x \in \ell_\infty$ and a constant $c \in \mathbb{R}$.
Either for any $\epsilon > 0$, $z, z' \in \ell_\infty$, there exists $T_0(\epsilon, z, z') \geq T$ such that for any $T' \geq T_0(\epsilon, z, z')$:

$$\left(z_{[0,T-1]}, x_{[T,T']}, z'_{[T'+1,\infty[}\right) \succeq \left(z_{[0,T-1]}, c\mathbb{1}_{[T,T']}, z'_{[T'+1,\infty[}\right) - \epsilon\mathbb{1},$$

or, for any $\epsilon > 0$, $z, z' \in \ell_\infty$, there exists $T_0(\epsilon, z, z') \geq T$ such that, for any $T' \geq T_0(\epsilon, z, z')$:

$$\left(z_{[0,T-1]}, c\mathbb{1}_{[T,T']}, z'_{[T'+1,\infty[}\right) + \epsilon\mathbb{1} \succeq \left(z_{[0,T-1]}, x_{[T,T']}, z'_{[T'+1,\infty[}\right).$$

This axiom contemplates a variation of the classical *limited independence* condition of Koopmans [23] where the close future evaluation of some date T does neither depend on the past nor on the distant future. Either the sequence $x_{[T,+\infty[}$ dominates the constant sequence independently from the beginning and the tail of the sequences, or it is dominated by the constant sequence independently from the beginning and the tail of the sequences.

Lemma **3.1.** *Consider axioms* **F1***,* **G1***,* **G2***, and* **B1***. Consider* $T \geq 1$*,* $x \in \ell_\infty$*. For any constant c, either, for any* $z \in \ell_\infty$*:*

$$\left(z_{[0,T-1]}, x_{[T,\infty[}\right) \succeq_c \left(z_{[0,T-1]}, c\mathbb{1}_{[T,\infty[}\right),$$

or, for any $z \in \ell_\infty$*:*

$$\left(z_{[0,T-1]}, c\mathbb{1}_{[T,\infty[}\right) \succeq_c \left(z_{[0,T-1]}, x_{[T,\infty[}\right).$$

With Lemma 3.1, the analysis of temporal biases rests upon the one of the properties of the future order $\succeq_c$. Define

$$\chi_d^T = I_c\left(0\mathbb{1}_{[0,T-1]}, \mathbb{1}_{[T,\infty[}\right),$$
$$\chi_c^T = 1 - I_c\left(\mathbb{1}_{[0,T-1]}, 0\mathbb{1}_{[T,\infty[}\right).$$

Under axiom **B1**, the close future evaluation after some date T can be represented by the order $\succeq_T$ and the index function I_T, satisfying every property of Proposition 2.1.

PROPOSITION 3.1. *Consider axioms* **F1**, **G1**, **G2** *and* **B1**.

(i) *For every* $T \geq 1$*, there exists a complete order* $\succeq_T$ *on* ℓ_∞ *such that, for any* $x, y \in \ell_\infty$*,* $x \succeq_T y$ *if and only if for any* $z_0, z_1, \ldots, z_{T-1}$*,*

$$\left(z_{[0,T-1]}, x\right) \succeq_c \left(z_{[0,T-1]}, y\right).$$

(ii) *The order* $\succeq_T$ *satisfies the* transitivity, monotonicity *and* weak convexity *properties in axiom* **F1**.

(iii) *If at least one of the two values* χ_d^T *and* χ_c^T *differs from zero, then the order* $\succeq_T$ *satisfies the* non-triviality *and* Archimedeanity *properties in axiom* **F1**.

(iv) *If at least one of the two values* χ_d^T *and* χ_c^T *differs from zero, then there exists an index function of the* T*−order,* $I_T : \ell_\infty \to \mathbb{R}$*, such that* $x \succeq_T y$ *if and only if* $I_T(x) \geq I_T(y)$ *satisfies:*

a) $I_T(\lambda x) = \lambda I_T(x)$*, for every* $\lambda \geq 0$.

b) $I_T(x + c\mathbb{1}) = I_T(x) + c$.

In the same way as in Section 2, the following lemma characterizes a configuration under which the order $\succeq_T$ becomes trivial: for any $x \in \ell_\infty$, the evaluation of x at T merely depends on the past before T and the distant future of x.

LEMMA 3.2. *Consider axioms* **F1**, **G1**, **G2** *and* **B1**. *Suppose that there exists* T_0 *such that* $\chi_d^{T_0} = \chi_c^{T_0} = 0$*; then, for any* $T \geq T_0$*, the order* $\succeq_T$ *is trivial and* $x \sim_T y$ *for any* $x, y \in \ell_\infty$*. Also note that, for every* $T' \geq T$*, the order* $\succeq_{T'}$ *would then be similarly trivial.*

Proposition 3.2. *Consider axioms* **F1**, **G1**, **G2** *and* **B1**. *Suppose that, for any* T, *at least one of the two values* χ_d^T, χ_c^T *is different from zero. Then, and for every* $T \geq 0$,

(i) *Let* $\chi_d^T \leq \chi_c^T$ *and define* $\underline{\delta}_T = \chi_d^T$ *et* $\overline{\delta}_T = \chi_c^T$. *For any* $x \in \ell_\infty$:

$$I_T\left(x_{[T,\infty[}\right) = \min_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \left[(1-\delta)x_T + \delta I_{T+1}(x_{[T+1,\infty[})\right].$$

(ii) *Let* $\chi_d^T \geq \chi_c^T$ *and define* $\underline{\delta}_T = \chi_c^T$ *et* $\overline{\delta}_T = \chi_d^T$. *For any* $x \in \ell_\infty$:

$$I_T\left(x_{[T,\infty[}\right) = \max_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \left[(1-\delta)x_T + \delta I_{T+1}(x_{[T+1,\infty[})\right].$$

At any given date, the evaluation of a utility stream consists of a *recursive convex sum* between the utility level at that date, and the evaluation at the subsequent date of the utility stream. Interestingly, a multitude of choices are shown to be admissible for the weight parameters of this convex sum. The min solution here represents a configuration where the value of the future beyond some date T is not large enough to compensate the lost that is incurred in present, the max solution representing the opposite occurrence.

3.2 Temporal Distances, Present Bias and Distant Future Bias

3.2.1 Temporal Distances & Present Bias

Axiom B2. For any $T \geq 1$, constant $c \in \mathbb{R}$,

(i) If there exists T'_0 such that, for any $T' \geq T'_0$,

$$\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right) \succeq \left(0\mathbb{1}_{[0,T-1]}, c\mathbb{1}_{[T,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right),$$

then there exists T'_1 such that, for any $T' \geq T'_1$,

$$\left(0\mathbb{1}_{[0,T+1]}, \mathbb{1}_{[T+2,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right) \succeq \left(0\mathbb{1}_{[0,T]}, c\mathbb{1}_{[T+1,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right).$$

(ii) If there exists T'_0 such that, for any $T' \geq T'_0$,

$$\left(\mathbb{1}_{[0,T-1]}, c\mathbb{1}_{[T,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right) \succeq \left(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right),$$

then there exists T'_1 such that, for any $T' \geq T'_1$,

$$\left(\mathbb{1}_{[0,T]}, c\mathbb{1}_{[T+1,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right) \succeq \left(\mathbb{1}_{[0,T+1]}, 0\mathbb{1}_{[T+2,\infty[}\right).$$

The *supremum*—the greatest of the minorants—of the values of the parameter c in part (i) and the *infimum*—the smallest of the majorants—of the values of the parameter c in part (ii) can both be used to figure out the *perception of the temporal distance* between date T and date $T+1$. These *extremum* values are evaluated using the perception at date T of the two sequences $(0,\mathbb{1})$ and $(1,0\mathbb{1})$. Axiom **B2** means that this *temporal distance* is decreasing as a function of T. Another intuition formulates as follows: the evaluation of the sequences $(0,\mathbb{1})$ and $(0,(-\mathbb{1}))$—these are determinant in the formulations of the index functions—at time $T+1$ is influenced by the evaluation at the time T.
The following lemma provides a straigthforward implication of axiom **B2**.

Lemma 3.3. *Consider axioms* **F1**, **G1**, **G2** *and* **B1**, **B2**. *Suppose that, for any* T*, at least one of the two values* χ_d^T *and* χ_c^T *differs from* 1 *and at least one of the two values* χ_d^T *and* χ_c^T *differs from zero. For any* $T \geq 1$ *and a constant* $c \in \mathbb{R}$,

(i) *If* $(0,\mathbb{1}) \succeq_T c\mathbb{1}$, *then* $(0,\mathbb{1}) \succeq_{T+1} c\mathbb{1}$.

(ii) *If* $c\mathbb{1} \succeq_T (1,0\mathbb{1})$, *then* $c\mathbb{1} \succeq_{T+1} (1,0\mathbb{1})$.

Otherwise stated, the *temporal distance* that is perceived between dates T and $T+1$ is greater that the one that is perceived between dates $T+1$ and $T+2$: at date T, the valuation of a constant sequence from tomorrow on is lower than its corresponding valuation at date $T+1$.

Remark 1. *The two axioms* **B1** *and* **B2** *are not equivalent because axiom* **B1** *does not imply axiom* **B2**. *In order to parallely establish that axiom* **B2** *does neither imply axiom* **B1**, *consider the order* $\succeq$ *that is represented by the index function*

$$I(x) = \max\left\{\hat{\phi}\cdot x, \min\left\{\underset{\sim}{\omega}\cdot x, \phi\cdot x\right\}\right\},$$

where $\hat{\phi}, \phi$ *are two different charges in the* ℓ_1^d *space, and* $\underset{\sim}{\omega}$ *is a weights measure that belongs to* ℓ_1 *and satisfies for any* $T \geq 0$,

$$\omega_T = \delta_0\delta_1\cdots\delta_{T-1}(1-\delta_T),$$

with a sequence $\left\{\delta_T\right\}_{T=0}^{\infty}$ *that is increasing and* $\lim_{T\to\infty}\left(\Pi_{s=0}^{T}\delta_s\right) = 0$. *The order* $\succeq$ *satisfies axiom* **B2** *but cannot satisfy axiom* **B1**.

Proposition 3.3. *Consider axioms* **F1**, **G1**, **G2** *and* **B1**, **B2**. *The order* $\succeq$ *is* present-biased *in that the two sequences* $\left\{\underline{\delta}_T\right\}_{T=0}^{\infty}$ *and* $\left\{\overline{\delta}_T\right\}_{t=0}^{\infty}$ *of the* min *or* max *occurrences in Proposition 3.2 are increasing according to* $\underline{\delta}_T \le \underline{\delta}_{T+1}$ *and* $\overline{\delta}_T \le \overline{\delta}_{T+1}$ *for any* T.

3.2.2 Distant Future Bias

Taking a reverse conjunction from Axiom B2 and a temporal distance between two consecutive dates that would be greater in the future than in the present, could admittedly provide a version of a future bias in preferences; it however sounds somewhat artificial.

This article rather tries to shed another light on the potential for future bias in preferences. Exhibiting a future bias here means the emergence, for large values of T, of some sort of continuity between the temporal orders $\succeq_T$ and the distant order $\succeq_d$. More explicitly, there does emerge a range of influences that spring from the evaluation of the distant future—the *infinite*—on the $\succeq_T$-evaluations of the close future— —the *finite*.

Axiom B3. Take $x, z \in \ell_\infty$ and some constants $d, d' \in \mathbb{R}$: if there exists T_0 such that, for any $T \ge T_0$,

$$\left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) \succeq \left(z_{[0,T]}, x_{[T+1,\infty[}\right) \succeq \left(z_{[0,T]}, d'\mathbb{1}_{[T+1,\infty[}\right),$$

then, for any $\epsilon > 0$, there exist $T_0'(\epsilon, z)$ such that, for any $T \ge T_0'(\epsilon, z)$, there exists s_0 that depends on T, such that, for any large enough $s \ge s_0$,

$$\left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) + \epsilon\mathbb{1} \succeq \left(z_{[0,T]}, x_{[T+1,T+s]}, d\mathbb{1}_{[T+s+1,\infty[}\right),$$
$$\left(z_{[0,T]}, x_{[T+1,T+s]}, d'\mathbb{1}_{[T+s+1,\infty[}\right) \succeq \left(z_{[0,T]}, d'\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

This axiom means that, for a large finite time T, the evaluation of the sequence in a future that is close from T is influenced by the distant future evaluation. Even though its formulation is complex, the underlying intuition is rather simple and presented in the following proposition.

Proposition 3.4. *Consider axioms* **F1**, **G1**, **G2** *and* **B1**, **B3**.

(i) *For any* $x \in \ell_\infty$, $d, d' \in \mathbb{R}$, *if*

$$d\mathbb{1} \succeq_d x \succeq_d d'\mathbb{1},$$

then, for any $\epsilon > 0$*, there exists* $T_0(\epsilon)$ *such that, for* $T \geq T_0(\epsilon)$*,*

$$(d+\epsilon)_{[T,\infty[}\mathbb{1} \succeq_T x_{[T,\infty[} \succeq_T (d'-\epsilon)\mathbb{1}_{[T,\infty[}.$$

(ii) *For any* $x \in \ell_\infty$*,* $\lim_{T\to\infty} I_T\left(x_{[T,\infty[}\right) = I_d(x)$*.*

(iii) *The sequence* $\left\{\bar{\delta}_T\right\}_{t=0}^{\infty}$ *is such that* $\limsup_{T\to\infty} \bar{\delta}_T = 1$*.*

(iv) *Adding axiom* **B2** *and present bias to distant future bias, the sequence* $\left\{\bar{\delta}_T\right\}_{t=0}^{\infty}$ *further satisfies* $\lim_{T\to\infty} \bar{\delta}_T = 1$*.*

Otherwise stated and in (i), the sequence x will either always dominate or be dominated by a constant sequence under the order $\succeq_T$ and for sufficiently large values of T. In these regards, a similitude should be pointed out with the former *distant future sensitivities* axiom **G1**, where a sequence x either always dominates or is always dominated by a constant sequence in the distant future. (ii), in its turn, enlightens the scope for some sort of continuity between I_T and I_d for sufficiently large values of T. Even though the underlying intuition behind (iii) and (iv) is not necessarily very clear, the subsequent example illustrates how they closely articulate with (ii):

Example 3.1. *Consider an example in which, for* T *sufficiently large,* $\bar{\delta}_T = 1$*. For* $s = 0$*, define the index functions* I_T^0 *as:*

$$I_T^0\left(x_{[T,\infty[}\right) = \liminf_{T\to\infty} x_t.$$

For each $s \geq 0$*, let*

$$I_T^{s+1}\left(x_{[T,\infty[}\right) = \min_{\underline{\delta}_T \leq \delta \leq 1} \left[(1-\delta)x_T + \delta I_{T+1}^s(x_{[T+1,\infty[})\right].$$

For each T*, let finally* $I_T\left(x_{[T,\infty[}\right) = \liminf_{s\to\infty} I_T^s\left(x_{[T,\infty[}\right)$*. By construction and using a recurrence argument, for any* s*,*

$$I_T^s\left(x_{[T,\infty[}\right) \leq \liminf_{t\to\infty} x_t,$$

whence, for any T*, the satisfaction of* $I_T\left(x_{[T,\infty[}\right) \leq \liminf_{t\to\infty} x_t$*. Further noticing that* $\liminf_{T\to\infty} I_T\left(x_{[T,\infty[}\right) \geq \liminf_{t\to\infty} x_t$ *and finally taking* $I_d(x) = \liminf_{t\to\infty} x_t$*, it follows that*

$$\lim_{T\to\infty} I_T\left(x_{[T,\infty[}\right) = I_d(x).$$

The order represented by the index function I *therefore satisfies the* future bias *property.*

4. The robustness Pre-orders $\succeq^*$, $\succeq_c^*$, $\succeq_d^*$

4.1 Representation of the robustness pre-order $\succeq^*$

In order to reach more explicit properties for the index functions I, I_c and I_d, consider a *pre-order,* as opposed to the earlier complete order, $\succeq^* \subseteq \succeq$, featuring the robustness of the order $\succeq$: whatever the convex modifications with a common component, the comparison would ensuingly not be modified.

Definition 1. *Let the pre-order $\succeq^*$ be defined by*

$$x \succeq^* y \text{ if, for every } 0 \leq \lambda \leq 1, z \in \ell_\infty, \lambda x + (1-\lambda)z \succeq \lambda y + (1-\lambda)z.$$

It is first to be noticed that, in the general case, the pre-order $\succeq^*$ is not complete. The completeness of $\succeq^*$ is equivalent to the linearity of the index function I or, more precisely, the existence of $\left((1-\lambda)\underset{\sim}{\omega}, \lambda\phi\right)$ in $\left(\ell_\infty\right)^*$ with $\underset{\sim}{\omega} \in \ell_1$, $\phi \in \ell_d^1$, and $0 \leq \lambda \leq 1$ such that $I(x) = (1-\lambda)\sum_{s=0}^{\infty} \omega_s x_s + \lambda\phi \cdot x$.
Lemma 4.1 then gathers the fundamental properties of the pre-order $\succeq^*$.

Lemma **4.1.** *Assume that axiom* **F1** *is satisfied. For every x, y: $x \succeq^* y$ if and only if either of the two following assertions is satisfied:*

(i) *For every $z \in \ell_\infty, x + z \succeq y + z$.*

(ii) *There exists $z \in \ell_\infty$, $x + z \succeq^* y + z$.*

The understanding of the properties of the pre-order $\succeq^*$ is important in the analysis of the order $\succeq$ and proposition 4.1 will clarify its precise status. The initial order $\succeq$ can be considered as a family of linear sub-orders, the pre-order $\succeq^*$ featuring the particular one that deals with *robustness* or *unanimity*. This pre-order $\succeq^*$ can be considered as depicting an *unanimous class of preferences*: a given sequence x is *robustly preferred* to another sequence y if and only if any sub-preference to the order $\succeq$ prefers x to y. These sub-preferences are a convex set with a measure belonging to $(\ell_\infty)^*$, defined as the normalized positive polar cone of the set x such that $x \succeq^* 0\mathbb{1}$.

Proposition 4.1. *Assume that axiom* **F1** *is satisfied. There exists a convex set Ω of weights $\left((1-\lambda)\underset{\sim}{\omega}, \lambda\phi\right)$ which can be considered as finitely additive probabilistic measures on $\mathbb{N}$ where:*

(i) $0 \leq \lambda \leq 1$,

(ii) $\underset{\sim}{\omega} = (\omega_0, \omega_1, \omega_2, \ldots)$ *is a probability measure, i.e., a sequence of weights, belonging to* ℓ^1,

$$\sum_{s=0}^{\infty} \omega_s = 1.$$

(iii) ϕ *is a charge in* ℓ^1_d *satisfying* $\phi(\mathbb{N}) = 1$, *such that, for every* $x, y \in \ell_\infty$, $x \succeq^* y$ *if and only if, for any* $\big((1-\lambda)\underset{\sim}{\omega}, \lambda\phi\big) \in \Omega$,

$$(1-\lambda)\sum_{s=0}^{\infty} \omega_s x_s + \lambda\phi \cdot x \geq (1-\lambda)\sum_{s=0}^{\infty} \omega_s y_s + \lambda\phi \cdot y.$$

It is worth emphasizing that the value λ can change between different measures. Upon the addition of axioms **G1** and **G2**, the set Ω can be considered as the set of measures that subsume an index function along Proposition 2.6, the set of *possible myopia degrees* being then reduced to a unique value. The family of *weights* can here be considered as a family of *finitely additive probabilities*.
Since the robustness order $\succeq^*$ can be represented by a set of possible evaluations, it is will prove fruitful to consider the best evaluation and the worst evaluation values. For each $x \in \ell_\infty$, define

$$\gamma^*_x = \sup\big\{\gamma \text{ such that } x \succeq^* \gamma\mathbb{1}\big\},$$
$$\gamma^{*x} = \inf\big\{\gamma \text{ such that } \gamma\mathbb{1} \succeq^* x\big\}.$$

The values γ^{*x} and γ^*_x represent the best and worst scenario,s or the best and worst possible evaluations of x. A first clarification is then in order.

Lemma **4.2.** *Assume that axiom* **F1** *is satisfied.*

(i) *The coefficients values* γ^{*x} *and* γ^*_x *restate as:*

$$\gamma^{*x} = \sup_{((1-\lambda)\underset{\sim}{\omega}, \lambda\phi)\in\Omega} \left[(1-\lambda)\sum_{s=0}^{\infty} \omega_s x_s + \lambda\phi \cdot x\right],$$
$$\gamma^*_x = \inf_{((1-\lambda)\underset{\sim}{\omega}, \lambda\phi)\in\Omega} \left[(1-\lambda)\sum_{s=0}^{\infty} \omega_s x_s + \lambda\phi \cdot x\right].$$

(ii) *For any* $x \in \ell_\infty$,

$$\gamma_x^* \leq \mathrm{I}(x) \leq \gamma^{*x}.$$

Since $\gamma^{*x} \geq \mathrm{I}(x) \geq \gamma_x^*$, there exists $a_x \in [0,1]$ such that

$$\mathrm{I}(x) = a_x \gamma_x^* + (1 - a_x)\gamma^{*x}.$$

If $\gamma_x^* < \gamma^{*x}$, then a_x is unique. The value a_x can be considered as the pessimism degree associated with the sequence $x \in \ell_\infty$, as $1 - a_x$ the optimism degree associated with x. It is natural to study the case where the optimism degree does not decrease in respect to the robustness order $\succeq^*$.

The following axiom imposes that if y is robustly better than x, then the pessimism degree associated with y must not be larger than the one associated with x.

Axiom G3. Consider $x, y \in \ell_\infty$ satisfying $\gamma_x^* < \gamma^{*x}$ and $\gamma_y^* < \gamma^{*y}$. If $y \succeq^* x$ then $a_x \geq a_y$.

The following Proposition proves that the only situation for which the degree of optimism does not decrease in respect to the robustness order $\succeq^*$ is the well-known case α–maximin in the literature.

Proposition 4.2. *Assume that axioms* **F1** *and* **G3** *are satisfied. For every* $x \in \ell_\infty$ *such that* $\gamma_x^* < \gamma^{*x}$, a_x *is equal to a constant* a^*. *For every* $x \in \ell_\infty$,

$$\begin{aligned}
\mathrm{I}(x) &= a^* \gamma_x^* + (1 - a^*)\gamma^{*x} \\
&= a^* \inf_{((1-\lambda)\underline{\omega}, \lambda\phi) \in \Omega} \left[(1-\lambda) \sum_{s=0}^{\infty} \omega_s x_s + \lambda \phi \cdot x \right] \\
&\qquad + (1 - a^*) \sup_{((1-\lambda)\underline{\omega}, \lambda\phi) \in \Omega} \left[(1-\lambda) \sum_{s=0}^{\infty} \omega_s x_s + \lambda \phi \cdot x \right].
\end{aligned}$$

4.2 Representation of the distant future pre-order $\succeq_d^*$

Following the same idea about the robustness order, one can define the robustness order $\succeq_d^*$ for the order $\succeq_d$. Since the order $\succeq_d$ does not take into account the present and the close future, the pre-order $\succeq_d^*$ satisfies the same property.

Definition 2. *Let* $\succeq_d^*$ *be defined as*

$$x \succeq_d^* y \ \text{ if and only if } \forall\ \lambda \in [0,1], z \in \ell_\infty, \lambda x + (1-\lambda)z \succeq_d \lambda y + (1-\lambda)z.$$

Proposition 4.3. *There exists a weights set* $\Omega_d \subset \ell_d^1$ *such that* $x \succeq_d^* y$ *if and only if* $\phi \cdot x \geq \phi \cdot y$ *for every* $\phi \in \Omega_d$.

As in subsection 4.1, define, for each $x \in \ell_\infty$, define d_x^*, d^{*x} as the best and worst evaluation of sequence x:

$$d_x^* = \sup\left\{d \in \mathbb{R} \text{ such that } x \succeq_d^* d\mathbb{1}\right\} = \inf_{\phi \in \Omega_d} (\phi \cdot x),$$

$$d^{*x} = \inf\left\{d \in \mathbb{R} \text{ such that } d\mathbb{1} \succeq_d^* x\right\} = \sup_{\phi \in \Omega_d} (\phi \cdot x).$$

For any $x \in \ell_\infty$, define the degree of pessimism in distant future associated with x: the value $a_{d,x}$ satisfying

$$I_d(x) = a_{d,x} d_x^* + (1 - a_{d,x}) d^{*x}.$$

The value $a_{d,x}$ is unique if $d_x^* < d^{*x}$.

Axiom G4. Consider $x, y \in \ell_\infty$ satisfying $d_x^* < d_x^*$, and $d_x^* < d^{*,y}$. If $y \succeq_d^* x$ then $a_{d,x} \geq a_{d,y}$.

Along Proposition 4.2, under the assumption that the degree of pessimism cannot increase with respect to the robust pre-order $\succeq_d^*$, the index of distant future order assumes a α–maximin representation.

Proposition 4.4. *Assume that axioms* **F1**, **G1**, **G2** *and* **G4** *are satisfied. For any* $x \in \ell_\infty$ *such that* $d_x^* < d^{*x}$, $a_{d,x}$ *is equal to a constant* a_d^*. *For any* x, *the distant index assumes the following representation:*

$$I_d(x) = a_d^* d_x^* + (1 - a_d^*) d^{*x}.$$

4.3 Representation of the close future pre-order $\succeq_c^*$

4.3.1 Fundamental properties

Definition 3. *Let* $\succeq_c^*$ *be defined as*

$$x \succeq_c^* y \ \text{ if and only if, for every } \ \lambda \in [0,1], z \in \ell_\infty, \lambda x + (1-\lambda)z \succeq_c \lambda y + (1-\lambda)z.$$

Using the same arguments as the ones developed for the proof of Lemma 4.1, the following characterization of the robustness order $\succeq^*$ becomes available:

LEMMA 4.3. *Assume that axioms* **F1** *and* **G2** *are satisfied. For every* $x, y \in \ell_\infty$, $x \succeq_c^* y$ *if and only if, for every* $z \in \ell_\infty$, $x + z \succeq_c y + z$.

For every $x, y \in \ell_\infty$ that have the same limit at the infinity, which implies that they have the same distant future, the comparison between the two according to the pre-orders $\succeq^*$ and $\succeq_c^*$ is equivalent.

LEMMA 4.4. *Assume that axioms* **F1**, **G1** *and* **G2** *are satisfied and let* $x, y \in \ell_\infty$ *satisfy* $\lim_{T\to\infty} x_T = \lim_{T\to\infty} y_T$. *Then*

$$x \succeq^* y \Leftrightarrow x \succeq_c^* y.$$

Proposition 2.6 proves that, for any sequence x, the value $I\left(x_{[0,T]}, z_{[T+1,\infty[}\right)$ converges to $I(x)$ when T tends to infinity. This convergence is not uniform: indeed, even-though the distant order of $\succeq_c$ is trivial, the order $\succeq_c$ does not necessarily satisfy the usual *tail-insensitivity* condition of the literature. To ensure this property, the article considers axiom **A1**. Axiom **A1** is the *close future* version of well-know axioms—the *continuity at infinity* axiom of Chambers and Echenique [14] or other axioms in the literature—ensuring a strong version of myopia and, moreover, the compactness of the weights set Ω_c when it belongs to ℓ_1.

AXIOM A1. For every $0 < c < 1$, there exists $T_0(c)$ such that, for every $T \geq T_0(c)$,

$$\left(\mathbb{1}_{[0,T_0(c)]}, 0\mathbb{1}_{[T_0(c)+1,\infty[}\right) \succeq^* \left(c\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right).$$

Under axiom **A1**, Lemma 4.5 is a direct implication of Lemma 4.4 and proves that $\succeq_c$ satisfies a *tail-insensitivy* property.

LEMMA 4.5. *For every* $0 < c < 1$, *there exists* $T_0(c)$ *such that:*

$$\left(\mathbb{1}_{[0,T_0(c)]}, 0\mathbb{1}_{[T_0(c)+1,\infty[}\right) \succeq_c^* c\mathbb{1}.$$

Under axiom **A1**, the weights set Ω_c is *tight*, or *weakly compact* in ℓ_1.

Lemma 4.6. *Assume that axioms* **F1**, **G1**, **G2** *and* **A1** *are satisfied. There exists a set* $\Omega_c \subset \ell_1$ *that is weakly compact and such that, for* $x, y \in \ell_\infty$, $x \succeq_c^* y$ *if and only if, for every* $\underset{\sim}{\omega} \in \Omega_c$,

$$\sum_{s=0}^{\infty} \omega_s x_s \geq \sum_{s=0}^{\infty} \omega_s y_s.$$

4.3.2 Pessimism and optimism degrees

As in subsection 4.1, for each $x \in \ell_\infty$, define c_x^*, c^{*x} as

$$c_x^* = \sup\left\{c \in \mathbb{R} \text{ such that } x \succeq_c^* c\mathbb{1}\right\} = \inf_{\underset{\sim}{\omega} \in \Omega_c} \sum_{s=0}^{\infty} \omega_s x_s,$$

$$c^{*x} = \inf\left\{c \in \mathbb{R} \text{ such that } c\mathbb{1} \succeq_c^* x\right\} = \sup_{\underset{\sim}{\omega} \in \Omega_c} \sum_{s=0}^{\infty} \omega_s x_s.$$

For each sequence $x \in \ell_\infty$, there exists $a_{c,x}$ such that

$$I_c(x) = a_{c,x} c_x^* + (1 - a_{c,x}) c^{*x}.$$

Again, the value $a_{c,x}$ is unique if $c_x^* < c^{*x}$.

Axiom G5. Consider $x, y \in \ell_\infty$ satisfying $c_x^* < c^{*x}$ and $c_y^* < c^{*y}$. If $y \succeq^* x$ then $a_{c,x} \geq a_{c,y}$.

Proposition 4.5. *Assume that axioms* **F1** *and* **G1**, **G2** *and* **G5** *are satisfied. For any* $x \in \ell_\infty$ *such that* $c_x^* < c^{*x}$, a_x *is equal to a constant* a_c^*. *For any* x,

$$I_c(x) = a_c^* c_x^* + (1 - a_c^*) c^{*x}.$$

4.3.3 Geometrical representation

In order to better characterize the set Ω, consider axiom **G6** which characterizes the impatience and the stability properties of the pre-order $\succeq^*$.

Axiom G6. *Impatience and stationarity* Given $x \in \ell_\infty$ and a constant c. For every $T \in \mathbb{N}$,

$$\left(x_{[0,T]}, c\mathbb{1}_{[T+1,\infty[}\right) \succeq^* c\mathbb{1} \Rightarrow \left(x_{[0,T]}, c\mathbb{1}_{[T+1,\infty[}\right) \succeq^* \left(c, x_{[0,T]}, c\mathbb{1}_{[T+2,\infty[}\right) \succeq^* c\mathbb{1}.$$

In axiom **G6**, the first $\succeq^*$ characterizes impatience whereas the second one features stability. Otherwise stated, if a combination is robustly better than a constant sequence, it remains robustly better if it is moved forward into the future, the effect according to the order $\succeq^*$ becoming lower over time.

Proposition 4.6. *Assume that axioms* **F1**, **G1**, **G2**, **G6** *and* **A1** *are satisfied. For any* $x \in \ell_\infty$, *a constant* c, $x \succeq_c^* c\mathbb{1}$ *implies:*

(i) $x \succeq_c^* (c,x) \succeq_c^* (c,c,x) \succeq_c^* \cdots \succeq_c^* \left(c\mathbb{1}_{[0,T]}, x\right) \succeq_c^* \cdots \succeq_c^* c\mathbb{1}$;

(ii) *for any* T, $c^*_{\left(c^*_x \mathbb{1}_{[0,T]}, x\right)} = c^*_x$;

(iii) *for any* T, $c^*_{(1/2)x+(1/2)\left(c^*_x \mathbb{1}_{[0,T]}, x\right)} = c^*_x$.

In Proposition 4.6, under axiom **G6**, for each sequence $x \in \ell_\infty$, the value of the worst scenario corresponding to x, evaluated under the order $\succeq_c$, does neither change with the shift of the sequence to the future nor with a convex combination with this shift.

Lemma 4.7. *For each weight* $\underset{\sim}{\omega} \in \ell_1$ *and* $T \in \mathbb{N}$, *define* $\underset{\sim}{\omega}^T$ *as*

$$\omega_s^T = \frac{\omega_{T+s}}{\sum_{s'=0}^{\infty} \omega_{T+s'}}, \forall\, s \geq 0.$$

If, for every T, $\underset{\sim}{\omega} = \underset{\sim}{\omega}^T$, *then there exists* $0 < \delta < 1$ *such that* $\omega_s = (1-\delta)\delta^s$ *for every* s.

Lemma 4.7 provides a characterization of the exposed points of the set Ω_c. From Theorem 4 in Amir & Lindenstrauss [1], a weakly compact convex set is indeed the convex hull of its exposed points.

Proposition 4.7. *Consider axioms* **F1**, **G1**, **G2**, **G6** *and* **A1**. *Then there exists a subset* $\mathscr{D}$ *of* $]0,1[$ *such that* Ω_c *is the convex hull of* $\left\{\left((1-\delta),(1-\delta)\delta,\cdots,(1-\delta)\delta^s,\cdots\right)\right\}_{\delta\in\mathscr{D}}$.

Chambers & Echenique [14] impose instead an *indifference stationarity* axiom, which supposes that any x which is equivalent to a constant sequence $c\mathbb{1}$, x is equivalent to any convex combination between x and $\left(c\mathbb{1}_{[0,T]}, x\right)$, for any T. This article supposes another property, namely the axiom **G6**. The difference between the two articles essentially springs from the fact that, while Chambers & Echenique work on a complete order $\succeq$ and complete a min –representation of the index function, this article works on a partial order $\succeq^*$, corresponding to a larger family of possible orders and index functions, for example the $\alpha-$ maximin representation. Unsurprisingly, the two different approaches involve two rather different systems of axioms.

4.4 A summary representation for the robust orders $\succeq^*$, $\succeq^*_d$, and $\succeq^*_c$

Even for small values of a^*_c , the current index representation can be fairly realistic. The decision maker could indeed be optimistic for some close future but oppositely cautious for some more distant future, thereby selecting a large enough value for a^*_d. The following statement is then a direct consequence of Propositions 4.4 and 4.5:

Proposition 4.8. *Consider axioms* **F1**, **G1**, **G2**, **G4** *and* **G5**. *For any* $x \in \ell_\infty$,

$$I_c(x) = a^*_c \inf_{\underset{\sim}{\omega}\in\Omega_c} \sum_{s=0}^{\infty} \omega_s x_s + \left(1 - a^*_c\right) \sup_{\underset{\sim}{\omega}\in\Omega_c} \sum_{s=0}^{\infty} \omega_s x_s,$$

$$I_d(x) = a^*_d \inf_{\phi\in\Omega_d} \phi \cdot x + \left(1 - a^*_d\right) \sup_{\phi\in\Omega_d} \phi \cdot x.$$

Moreover, there exist $0 \le \underline{\chi} \le \overline{\chi} \le 1$ *such that one of the two following assertions is true:*

(i) *For any* $x \in \ell_\infty$,

$$I(x) = \min_{\underline{\chi}\le\chi\le\overline{\chi}} \left[(1-\chi)I_c(x) + \chi I_d(x)\right],$$

the boundaries on the myopia degree χ *being defined by* $\chi_d = \underline{\chi}$ *and* $\chi_c = \overline{\chi}$.

(ii) *For any* $x \in \ell_\infty$,

$$I(x) = \max_{\underline{\chi}\le\chi\le\overline{\chi}} \left[(1-\chi)I_c(x) + \chi I_d(x)\right],$$

the boundaries on the myopia degree χ *being defined by* $\chi_d = \overline{\chi}$ *and* $\chi_c = \underline{\chi}$.

4.5 The robust temporal pre-orders $\succeq^*_T$

Define the robust time-dependent order $\succeq^*_T$ as the satisfaction of $x \succeq^*_T y$ if and only if, for any z, $x + z \succeq^*_T y + z$. Lemma 4.8 then provides a characterization of the weights set Ω_T that represents the robustness order $\succeq^*_T$.

Lemma **4.8.** *Consider axioms* **F1**, **G1**, **G2** *and* **B1**.

(i) *For any* T, $x, y \in \ell_\infty$ *and some constant* $c \in \mathbb{R}$,

$$x_{[T+1,\infty[} \succeq^*_{T+1} y_{[T+1,\infty[} \text{ if and only if } \left(c, x_{[T+1,\infty[}\right) \succeq^*_T \left(c, y_{[T+1,\infty[}\right).$$

(ii) *There exists a weights set* $\Omega_T \subset \ell_1$ *such that* $x \succeq^*_T y$ *if and only if, for any* $\underset{\sim}{\omega} \in \Omega_T$,

$$\sum_{s=0}^{\infty} \omega_s x_s \geq \sum_{s=0}^{\infty} \omega_s y_s.$$

(iii) *For any* T, $\Omega_T = \left\{\underset{\sim}{\omega}^T\right\}_{\underset{\sim}{\omega} \in \Omega_c}$.

Proposition 4.9 then equips the analysis with a representation of the weights sets Ω_c and Ω_T.

Proposition 4.9. *Consider axioms* **F1**, **G1**, **G2** *and* **B1**.

(i) *The weights set* Ω_c *is the convex hull of the set*

$$\left\{\left(1-\delta_0, \delta_0(1-\delta_1), \delta_0\delta_1(1-\delta_2), \ldots, \delta_0\delta_1\ldots\delta_T(1-\delta_{T+1}), \ldots\right)\right\},$$

where $\delta_T \in \left\{\underline{\delta}_T, \overline{\delta}_T\right\}$ *for any* T.

(ii) *The weights set* Ω_T *is the convex hull of the set*

$$\left\{\left(1-\delta_T, \delta_T(1-\delta_{T+1}), \delta_T\delta_{T+1}(1-\delta_{T+2}), \ldots, \delta_T\delta_{T+1}\ldots\delta_{T+s}(1-\delta_{T+s+1}), \ldots\right)\right\},$$

where $\delta_{T+s} \in \left\{\underline{\delta}_{T+s}, \overline{\delta}_{T+s}\right\}$ *for any* s.

5. Related Literature

5.1 Axiomatization of discounting

Efforts towards the understanding of discounting and choice date back the breakthrough contributions of Tjalling Koopmans. The classical axiomatization of discounted utility, and, noticeably, the first formulation of the *stationarity* postulate on the preference ordering, was indeed completed by Koopmans [23, 24] and provided the hallmark of all of the subsequent developments in the theory of inter-temporal choice, a recent contribution due to Bleichrodt, Rohde & Wakker [9] having extended its argument to unbounded utility while an earlier argument of Dolmas [15] had provided some steps in that direction. Another influential line of axiomatizations was inaugurated by Fishburn & Rubinstein [17] with a focus on the realization of a single outcome at a given date ; it has more recently been extended and generalized by Masatlioglu & Ok [30]. Following parallel roads and the decision theory multiple priors axiomatizations of Gilboa & Schmeidler [20] but relying upon a different system of axioms based upon *time-variability aversion,* Wakai [33] has provided an insightful account of smoothing behaviours where the optimal discount assumes an maxmin representation. This has just been comforted by Chambers & Echenique [14] who, following a multiple priors approach, have introduced new *Invariance to stationary relabeling* and *compensation* axioms and established a larger scope for a maxmin based geometric representation of the preference order that would then provide an optimal determination of discount.

The contribution with the closest concerns from the ones of this article is a recent work due to Lapied & Renault [25] who consider a decision maker facing alternatives that are defined on a very distant future, *i.e.,* a time horizon that exceeds his life-time horizon. They emphasize the emergence of an *asymptotic patience* property, meaning that, for some remote date, no time tradeoff between alternative any longer prevails. Mention should also be made of the recent and independently completed contribution of Gabaix & Laibson [18] with a subtle articulation between forecasting accuracy, discounting and myopia in an imperfect information environment that relies upon a recent literature on experiments.

Montiel Olea & Strzalecki [29] have already completed an axiomatic approach to the *quasi-hyperbolic discounting* representation of Phelps & Pollack [31], Laibson [26] and, more generally, to *present biased* preferences. They suppose that, for any two equivalent future sequences, a patient one and an impatient one, pushing both of them towards the present will distort the preference towards the impatient choice. This article assumes the *present bias* notion for every date T and not only the initial one. The index functions I_T are determined from a set of multiple discount rates. Hence the notion *present bias* in this article must contain two parts, one part for the upper bound of discount rates, and one part for the lower bound of discount rates. The axiom 10 in Montiel Olea & Strzalecki [29] correspond to the section part of the axiom **B2**, or the second part of Lemma 3.3. Finally, Chakraborty [11] has just completed a generalized appraisal of present bias within the Fishburn & Rubinstein [17] approach where preferences are defined on the realization of a single outcome at a given date. Even though it builds from a from an approach that differs from the current *utility streams* appraisal, his *weak present bias* axiom **A4** shares some similarity with the current axiom **B2**.

Ghirardato, Marinacci & Maccheroni [19] introduced the robustness orders. A robustness order is characterized by a set of finitely additive measures. Under the axiom stated that two sequences are robustly equivalents if they have the same evaluations of the best scenarios, the index function assumes the well known α-maxmin representation, its value being decomposed as a convex combination—with constant weight parameters—of the values of the worst and best scenarios.

5.2 Axiomatization of myopia & charges

The concept of *impatience,* that is due to Koopmans [23], is generally defined as letting the *effect* of a unit of consumption, or utils, to diminish as it is moved forward in time. A weaken version of this notion, *delay aversion*, was initiated by Benoit & Ok [5], and has recently been considered in the works of Bastianello [3], and Bastianello & Chateauneuf [4]: it pre-supposes that the effect of a given unit of consumption, or utils, is to converge towards zero as the consumption, or utility, is pushed to infinity. On other concerns, the first approach to myopia is due to Brown & Lewis [10] and considers the implications of the infinite postponment of a sequence of consumptions. The main contribution of this article closely relates with the advances of this myopia literature.

The notion of *strong myopia*, due to Brown & Lewis [10], means, in the version presented by Becker & Boyd [6], that for any $x \succ y$, for any z, $x \succ \left(y_{[0,T]}, z_{[T+1,\infty[}\right)$ is satisfied for sufficiently large values of T. This coincides with the notion *upward myopia* of Saywer [32]. In the context of this article, these cases are equivalent to *downward myopia* of by Saywer [32] where $x \succ y$ implies that, for any z, $\left(x_{[0,T]}, z_{[T+1,\infty[}\right) \succ y$ for T sufficiently big. This corresponds to an extreme occurrence where $\chi_d = \chi_c = 0$. Another extreme, the *completely patient and time invariant* preferences in Marinacci [28], Banach limits correspond to the case $\chi_d = \chi_c = 1$. The Banach limits[10], a special case of the *charges*, were first introduced by Bewley [7] in a general equilibrium context. A *charge* is a linear function on ℓ_∞ whose value solely depends on the long-run behaviour of the sequence. Bewley [7] proves the existence of a price belonging to the *ba*[11] set for an economy with an infinite number of dimensions and that is endowed with a ℓ_∞ topology. Following these ideas, Gilles [21] considered the possibility of charges in studying the asset bubbles. The ℓ_1 part of the price system represents the *fundamental value* of the asset, while the bubble is represented by the purely finitely additive part in ℓ_1^d.[12] Araujo [2] proves that, in order for the set of non trivial Pareto allocations to exist, consumers must exhibit some impatience in their preferences. Otherwise stated, this excludes the possibility of preferences being represented by Banach limits, or this is equivalent to at least of one of the two values χ_d, χ_c to differ from 1. Following a very different approach and contemplating a social planner problem, Chichilnisky [13] associated charges to the *non-dictatorship of present* part of the social welfare criterion where the present would have no *per se* effect. Kahn & Stinchcombe [22] is another recent example of the use of Banach limits in the context of social welfare functions for that treat present and future people equally and respect the Pareto criterion.

A. Proofs for section 2

A.1 Proof of Proposition 2.1

(i) Suppose that $x \succeq y$. First, and for $0 \le \lambda \le 1$, $x \succeq y$ is equivalent to $\lambda x + (1-\lambda)0\mathbb{1} \succeq \lambda y + (1-\lambda)0\mathbb{1}$.

Considering then the configuration $\lambda > 1$, $\lambda x \succeq \lambda y$ then prevails if and only if $(1/\lambda)\lambda x \succeq (1/\lambda)\lambda y$, or $x \succeq y$, also prevails.

(ii) Suppose that $x \succeq y$. By the *weak convexity* property, $x \succeq y$ implies $(1/2)x + (1/2)b\mathbb{1} \succeq (1/2)y + (1/2)b\mathbb{1}$. Multiplying the two sides by 2, it follows that $x + b\mathbb{1} \succeq$

[10] For a careful definition, see page 55 in Becker & Boyd [6].

[11] The dual of ℓ_∞ is given by $(\ell_\infty)^* = \ell_1 \oplus \ell_1^d$, where ℓ_1^d is the set of *purely finitely additive* measures.

[12] A short and excellent review of the theory of charges and of their link with bubbles can be found in Gilles [21].

$y+b\mathbb{1}$. Finally, and if $x+b\mathbb{1} \geq y+b\mathbb{1}$, then $x+b\mathbb{1}+(-b\mathbb{1}) \geq y+b\mathbb{1}+(-b\mathbb{1})$, or $x \geq y$.

(iii) and (iv) For $x \in \ell_\infty$, define $b_x = \sup\left\{b \in \mathbb{R} \text{ such that } x \geq b\mathbb{1}\right\}$. By the *Archimedeanity* property, it follows that $x \sim b_x\mathbb{1}$. Let then $\mathrm{I}(x) = b_x$. But $x \geq y$ if and only if $\mathrm{I}(x) \geq \mathrm{I}(y)$. Making use of (i) and (ii), for any $\lambda > 0$ and a constant $b \in \mathbb{R}$, $\mathrm{I}(\lambda x) = \lambda \mathrm{I}(x)$ and $\mathrm{I}(x+b\mathbb{1}) = \mathrm{I}(x)+b$ for every constant b. QED

A.2 Proof of Proposition 2.2

First consider the parametric configuration $c \leq d$. It follows that:

$$\begin{aligned}\mathrm{I}\left(c\mathbb{1}_{[0,\mathrm{T}]}, d\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) &= c + \mathrm{I}\left(0\mathbb{1}_{[0,\mathrm{T}]}, (d-c)\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) \\ &= c + (d-c)\mathrm{I}\left(0\mathbb{1}_{[0,\mathrm{T}]}, \mathbb{1}_{[\mathrm{T}+1,\infty[}\right),\end{aligned}$$

that converges to zero when T tends to infinity. Then observe that

$$\begin{aligned}\lim_{\mathrm{T}\to\infty} \mathrm{I}\left(0\mathbb{1}_{[0,\mathrm{T}]}, -\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) &= \lim_{\mathrm{T}\to\infty} \mathrm{I}\left(\mathbb{1}_{[0,\mathrm{T}]}, 0\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) - \mathrm{I}(\mathbb{1}) \\ &= 1 - \chi_c - 1 \\ &= -\chi_c \\ &= 0.\end{aligned}$$

This also implies that, for $c \geq d$:

$$\begin{aligned}\mathrm{I}\left(c\mathbb{1}_{[0,\mathrm{T}]}, d\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) &= c + \mathrm{I}\left(0\mathbb{1}_{[0,\mathrm{T}]}, (d-c)\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) \\ &= c + (c-d)\mathrm{I}\left(0\mathbb{1}_{[0,\mathrm{T}]}, -\mathbb{1}_{[\mathrm{T}+1,\infty[}\right),\end{aligned}$$

that converges to zero. QED

A.3 Proof of Lemma 2.1

For $x \in \ell_\infty$, define $\mathrm{D}(x)$ as the set of values d such that for any $\epsilon > 0$, for any $z \in \ell_\infty$, there exists $\mathrm{T}_0(\epsilon, z)$ such that, for any $\mathrm{T} \geq \mathrm{T}_0(\epsilon, z)$, one has

$$\left(z_{[0,\mathrm{T}]}, x_{[\mathrm{T}+1,\infty[}\right) \geq \left(z_{[0,\mathrm{T}]}, d\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) - \epsilon\mathbb{1}.$$

Define $\mathrm{D}(y)$ accordingly, consider the case $\sup \mathrm{D}(x) \geq \sup \mathrm{D}(y)$ and first let $\sup \mathrm{D}(y) < +\infty$. Then define $d_y = \sup \mathrm{D}(y)$, that is finite. Fix any $\epsilon > 0$: since $d_y + (\epsilon/2)\mathbb{1}$ does not belong to $D(y)$ and $d - (\epsilon/2)\mathbb{1}$ belongs to $D(y)$, for any $z \in \ell_\infty$, there exists $T_0(\epsilon, z)$ such that, for $T \geq T_0(\epsilon, z)$:

$$\left(\left(z + \frac{\epsilon}{2}\mathbb{1}\right)_{[0,T]}, \left(d_y + \frac{\epsilon}{2}\mathbb{1}\right)_{[T+1,\infty[}\right) + \frac{\epsilon}{2}\mathbb{1} \geq \left(z_{[0,T]}, y_{[T+1,\infty[}\right)$$
$$\geq \left(\left(z - \frac{\epsilon}{2}\mathbb{1}\right)_{[0,T]}, \left(d_y - \frac{\epsilon}{2}\mathbb{1}\right)_{[T+1,\infty[}\right) - \frac{\epsilon}{2}\mathbb{1}.$$

This implies, for $T \geq T_0(\epsilon, z)$, the satisfaction of:

$$\left(z_{[0,T]}, d_y\mathbb{1}_{[T+1,\infty[}\right) + \epsilon\mathbb{1} \geq \left(z_{[0,T]}, y_{[T+1,\infty[}\right) \geq \left(z_{[0,T]}, d_y\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

Since $d_x \geq d_y$, for every $\epsilon > 0$ and $z \in \ell_\infty$, there exists $T_0(\epsilon, z)$ such that

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \geq \left(z_{[0,T]}, d_y\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1}$$
$$\geq \left(z_{[0,T]}, y_{[T+1,\infty[}\right) - 2\epsilon\mathbb{1}.$$

Consider now the case $\sup D(y) = +\infty$. This implies that $\sup D(x) = +\infty$. Take $d > \sup_s y_s$: since $d \in D(x)$, for every $\epsilon > 0$ and $z \in \ell_\infty$, there exists $T_0(\epsilon, z)$ such that, for $T \geq T_0(\epsilon, z)$:

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \geq \left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1}$$
$$\geq \left(z_{[0,T]}, y_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

For the remaining configuration $\sup D(y) \geq \sup D(x)$, making use of the same arguments, for every $\epsilon > 0$ and $z \in \ell_\infty$, there exists $T_0(\epsilon, z)$ such that, for $T \geq T_0(\epsilon, z)$:

$$\left(z_{[0,T]}, y_{[T+1,\infty[}\right) \geq \left(z_{[0,T]}, x_{[T+1,\infty[}\right) - \epsilon\mathbb{1},$$

whence the statement. QED

A.4 Proof of Proposition 2.3

(i) Using Lemma 2.1, the order $\succeq_d$ is complete.
(ii) It must be proveed that there exists $x, y \in \ell_\infty$ such that $x \succ_d y$. Chose by example $\mathbb{1}$ and $0\mathbb{1}$. Obviously, $\mathbb{1} \succeq_d 0\mathbb{1}$ is first satisfied. Suppose now that $0\mathbb{1} \succeq_d \mathbb{1}$. Consider first the configuration $\chi_d > 0$. Then, and for $0 < \epsilon < \chi_d$, there exists $T_0(\epsilon, 0\mathbb{1})$ such that for $T \geq T_0(\epsilon, 0\mathbb{1})$,

$$I\left(0\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right) \geq I\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) - \epsilon.$$

Letting T tend to infinity, it follows that $0 \geq \chi_d - \epsilon$, a contradiction. Consider then the configuration $\chi_c > 0$. For $0 < \epsilon < \chi_c$, there exists $T_0(\epsilon, \mathbb{1})$ such that, for $T \geq T_0(\epsilon, \mathbb{1})$:

$$\left(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right) \geq I\left(\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) - \epsilon.$$

Letting T tend to infinity, it follows that $\epsilon \geq \chi_c$, a contradiction. The distant order $\succeq_d$ is hence not trivial.

Further observe that, if $x \succeq_d d\mathbb{1}$, then, for every $d' \in \mathbb{R}$, $x + d'\mathbb{1} \succeq_d (d+d')\mathbb{1}$. Indeed, for $\epsilon > 0$ and $z \in \ell_\infty$, there exists $T_0(\epsilon, z)$ such that, for $T \geq T_0$,

$$\left((z - d'\mathbb{1})_{[0,T]}, x_{[T+1,\infty[}\right) \geq \left((z - d'\mathbb{1})_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

From Proposition 2.1 and for $T \geq T_0(\epsilon, z)$,

$$\left(z_{[0,T]}, (x + d'\mathbb{1})_{[T+1,\infty[})\right) \geq \left(z_{[0,T]}, (d + d')\mathbb{1}_{[T+1,\infty[})\right) - \epsilon\mathbb{1}.$$

Then consider $x \in \ell_\infty$ and a constant d such that, for any $z \in \ell_\infty$, $\epsilon > 0$, there exists $T_0(\epsilon, z)$ with, for $T \geq T_0(\epsilon, z)$,

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \geq \left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[})\right) - \epsilon\mathbb{1}.$$

Fix then any $\lambda > 0$. From axiom **G1**, there exists $T_0'(\epsilon, z)$ such that, for $T \geq T_0'(\epsilon, z)$,

$$\left(\left(\frac{1}{\lambda}z\right)_{[0,T]}, x_{[T+1,\infty[}\right) \geq \left(\left(\frac{1}{\lambda}z\right)_{[0,T]}, d\mathbb{1}_{[T+1,\infty[})\right) - \frac{1}{\lambda}\epsilon\mathbb{1},$$

that in its turn implies, for $T \geq T_0'(\epsilon, z)$,

$$\left(z_{[0,T]}, \left(\lambda x_{[T+1,\infty[}\right)\right) \geq \left(z_{[0,T]}, \lambda d\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

Hence, for $x \succeq_d y$ and for every $\lambda > 0$, the occurrence of $\lambda x \succeq_d \lambda y$.

Consider now $x, y \in \ell_\infty$ such that $x \succeq_d y$. For every $0 < \lambda < 1$, using the same arguments as in the proof of Proposition 2.1, $(1-\lambda)x + \lambda d\mathbb{1} \succeq_d (1-\lambda)y + d\mathbb{1}$.

The order $\succeq_d$ having been proved to be non trivial, the value $d_x = \sup D(x)$ is finite and, for every $d > d_x > d'$, the relation $d\mathbb{1} \succ_d x \succ_d d'\mathbb{1}$ is to hold. There thus obviously exists $\lambda, \mu \in [0, 1]$ such that $(1-\lambda)d + \lambda d' > d_x > (1-\mu)d + \mu d'$ and the order $\succeq_d$ satisfies the *Archimedeanity* property.

(iii) Define $I_d(x) = \sup_x D(x)$. The order $\succeq_d$ satisfying every property in axiom **F1**, from Proposition 2.1, I_d satisfies every property asserted in the statement. QED

A.5 Proof of Proposition 2.4

(i) This property is the direct consequence of the definition.

(ii) For every T, since $d - c \geq 0$, the index function restates along:

$$\begin{aligned} I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) &= c + I\big(0\mathbb{1}_{[0,T]}, (d-c)\mathbb{1}_{[T+1,\infty[}\big) \\ &= c + (d-c)I\big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\big). \end{aligned}$$

Letting T converge to infinity,

$$\lim_{T\to\infty} I\big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) = (1-\chi_d)c + \chi_d d.$$

(iii) Take indeed any $d > \lim_{T\to\infty} x_T$; for every $z \in \ell_\infty$ and for large enough values of T,

$$\big(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\big) \succeq \big(z_{[0,T]}, x_{[T+1,\infty[}\big),$$

that implies $d \geq d_x$. Making use of the same argument for $d < \lim_{T\to\infty} x_T$, it follows that $d \leq d_x$. Whence the obtention of $d_x = \lim_{T\to\infty} x_T$.

(iv) Consider $x, y \in \ell_\infty$ such that $\lim_{T\to\infty} x_T$ does exist. Take then any $d > d_x$; for every z and for every large enough values of T,

$$\big(z_{[0,T]}, (d\mathbb{1} + y)_{[T+1,\infty[}\big) \succeq \big(z_{[0,T]}, (x+y)_{[T+1,\infty[}\big).$$

This in its turn implies that $d + d_y = I_d(d\mathbb{1} + y) \geq I_d(x+y)$. It follows that $d_x + d_y \geq I_d(x+y)$. For every $d < d_x$, and relying upon the same line of arguments, it similarly follows that $d_x + d_y \leq I_d(x+y)$. QED

A.6 Proof of Lemma 2.2

For $x \in \ell_\infty$, define $C(x)$ as the set of values c such that, for any $\epsilon > 0$ and for $z \in \ell_\infty$, there exists $T_0(\epsilon, z)$ such that, for $T \geq T_0(\epsilon, z)$,

$$\big(x_{[0,T]}, z_{[T+1,\infty[}\big) \succeq \big(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}\big) - \epsilon.$$

Follow then the same line of arguments as the ones developed for the proof of Proposition 2.3 and consider first the configuration $\sup C(x) \geq \sup C(y)$. For any

$\epsilon > 0$ and for $z \in \ell_\infty$, there exists $T_0(\epsilon, z)$ such that, for $T \geq T_0(\epsilon, z)$,

$$\left(x_{[0,T]}, z_{[T+1,\infty[}\right) \geq \left(y_{[0,T]}, z_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

For the remaining configuration with $\sup C(y) \geq \sup C(x)$, for any $\epsilon > 0$ and $z \in \ell_\infty$, there similarly exists $T_0(\epsilon, z)$ such that, for $T \geq T_0(\epsilon, z)$,

$$\left(y_{[0,T]}, z_{[T+1,\infty[}\right) \geq \left(x_{[0,T]}, z_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

The proof is complete. QED

A.7 Proof of Proposition 2.5

(i) From Lemma 2.2, the order $\succeq_c$ is complete.

(ii) It can first be proved that $\mathbb{1} \succ_c 0\mathbb{1}$. Suppose the opposite and $0\mathbb{1} \succeq_c \mathbb{1}$ and consider the case $\chi_d < 1$. For $0 < \epsilon < 1 - \chi_d$, there exists $T_0(\epsilon, \mathbb{1})$ such that, for $T \geq T_0(\epsilon, \mathbb{1})$,

$$I\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) \geq I\left(\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) - \epsilon.$$

Letting T tend to infinity, one gets $\chi_d \geq 1 - \epsilon$: a contradiction. For the remaining case $\chi_c < 1$, make use of the same arguments. For the proof of the other properties in axiom **F1**, follow the arguments developed for the proof of Proposition 2.3.

(iii) Define $I_c(x) = \sup C(x)$. Follow the arguments of the proof of Proposition 2.3. QED

A.8 Proof of Proposition 2.6

(i) Observe that, for $c - d \geq 0$,

$$\begin{aligned}
\lim_{T\to\infty} I\left(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) &= \lim_{T\to\infty} I\left((c-d)\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right) + d \\
&= (c-d)\lim_{T\to\infty} I\left(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right) + d \\
&= (1-\chi_c)(c-d) + d \\
&= (1-\chi_c)c + \chi_c d.
\end{aligned}$$

(ii) First observe that $\lim_{T\to\infty} I_c\left(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right) = 1$. Suppose indeed the opposite and let $\lim_{T\to\infty} I_c\left(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right) < 1$. There hence exists $c > 0$ such that, for every T, $(1-c)\mathbb{1} \succeq_c \left(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right)$. But this in its turn implies that, for every $\epsilon > 0$ and for every T, there exists $T_0(\epsilon, 0\mathbb{1}) \geq T$ such that, for every $T' \geq T_0(\epsilon, 0\mathbb{1})$,

$$\left((1-c)\mathbb{1}_{[0,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right) \succeq \left(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1},$$

or, for large enough values of T′,

$$I\Big((1-c)\mathbb{1}_{[0,T']}, 0\mathbb{1}_{[T'+1,\infty[}\Big) \geq I\Big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big) - \epsilon.$$

Letting T′ tend to infinity, it follows that, for every $\epsilon > 0$ and for every T,

$$(1-\chi_c)(1-c) \geq I\Big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big) - \epsilon.$$

Letting ϵ converge to zero and letting T tend to infinity, it follows that $(1-\chi_c)(1-c) \geq 1-\chi_c$, a contradiction. Consequently, either $\lim_{T\to\infty} I_c\Big(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big) = 1$ or $\lim_{T\to\infty} I_c\Big(0\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,\infty[}\Big) = 0$.
Relying to the same arguments, $\lim_{T\to\infty} I_c\Big(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\Big) = 0$. These two limits being equal to zero, for any constants c and d,

$$\lim_{T\to\infty} I_c\Big(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\Big) = c.$$

Consider any $x \in \ell_\infty$ and fix a constant d. For every $\epsilon > 0$ and for large enough values of T,

$$I_c\Big(c_x\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\Big) + \epsilon \geq I_c\Big(x_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\Big) \geq I_c\Big(c_x\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\Big) - \epsilon.$$

Letting T tend to infinity and ϵ converge to zero,

$$\lim_{T\to\infty} I_c\Big(x_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\Big) = I_c(x).$$

For every $x, y \in \ell_\infty$, fix then $d \geq \sup_s y_s \geq \inf_s y_s \geq d'$. Whence, for every T,

$$I_c\Big(x_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\Big) \geq I_c\Big(x_{[0,T]}, y_{[T+1,\infty[}\Big) \geq I_c\Big(x_{[0,T]}, d'\mathbb{1}_{[T+1,\infty[}\Big).$$

Letting T tend to infinity, if eventually follows that $\lim_{T\to\infty} I_c\Big(x_{[0,T]}, y_{[T+1,\infty[}\Big) = I_c(x)$, that completes the proof. QED

A.9 Proof of Lemma 2.3

(i) Observe that, for the index function $I(x)$, it first derives that $\chi_d = \chi_c = 0$. Moreover, and for every $x, z \in \ell_\infty$:

$$\lim_{T\to\infty} I\Big(z_{[0,T]}, x_{[T+1,\infty[}\Big) = \max\Big\{\hat{\underset{\sim}{\omega}} \cdot z, \min\Big\{\underset{\sim}{\omega} \cdot z, \phi \cdot x\Big\}\Big\}.$$

This implies that the order $\succeq$ represented by the index function I satisfies axiom **G1**. Following the same arguments developed for the proof of Proposition 2.3, it can readily be checked that $\succeq$ satisfies every property of axiom **F1** but the *non-triviality* and *Archimedeanity* properties. In order to prove that the *non-triviality* property is satisfied, take z^* satisfies $\hat{\underset{\sim}{\omega}} \cdot z^* < 0 < 1 < \underset{\sim}{\omega} \cdot z^*$. First notice that such a sequence z^* does exist: take any z satisfying $\hat{\underset{\sim}{\omega}} \cdot z < \underset{\sim}{\omega} \cdot z$ and define $\hat{z} = z - (1/2)\left(\hat{\underset{\sim}{\omega}} \cdot z + \underset{\sim}{\omega} \cdot z\right)$. The inequalities $\hat{\underset{\sim}{\omega}} \cdot \hat{z} < 0 < \underset{\sim}{\omega} \cdot \hat{z}$ are satisfied. Define then $z^* = (1/2\underset{\sim}{\omega} \cdot \hat{z})\hat{z}$. It derives that $\hat{\underset{\sim}{\omega}} \cdot z^* < 0$ and $\underset{\sim}{\omega} \cdot z^* = 1/2$. Consequently:

$$\lim_{T\to\infty} I\left(z^*_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right) = \max\left\{\hat{\underset{\sim}{\omega}} \cdot z^*, \min\left\{\underset{\sim}{\omega} \cdot z^*, \phi \cdot 0\mathbb{1}\right\}\right\} = 0,$$

$$\lim_{T\to\infty} I\left(z^*_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) = \max\left\{\hat{\underset{\sim}{\omega}} \cdot z^*, \min\left\{\underset{\sim}{\omega} \cdot z^*, \phi \cdot \mathbb{1}\right\}\right\} = \frac{1}{2}.$$

Whence $\mathbb{1} \succeq_d 0\mathbb{1}$ and $0\mathbb{1} \not\succeq_d \mathbb{1}$, or $\mathbb{1} \succ_d 0\mathbb{1}$. Once the *non-triviality* property has been proved to be satisfied, the obtention of the *Archimedeanity* property follows as a direct consequence. For the details of the argument, consult the proof of Proposition 2.3. Finally, and for every $x, y \in \ell_\infty$, $x \succeq_d y$ if and only if $\phi \cdot x \geq \phi \cdot y$.
Take any $x \in \ell_\infty$ such that $\hat{\underset{\sim}{\omega}} \cdot x < \phi \cdot x < \underset{\sim}{\omega} \cdot x$. Take then d and d' such that $\hat{\underset{\sim}{\omega}} \cdot x < d' < d < \phi \cdot x < \underset{\sim}{\omega} \cdot x$. Finally take $z^1 = d'\mathbb{1}$ and $z^2 = x$. It follows that:

$$\lim_{T\to\infty} I\left(x_{[0,T]}, z^1_{[T+1,\infty[}\right) = \max\left\{\hat{\underset{\sim}{\omega}} \cdot x, \min\left\{\underset{\sim}{\omega} \cdot x, \phi \cdot z^1\right\}\right\} = \phi \cdot z^1,$$

$$\lim_{T\to\infty} I\left(d\mathbb{1}_{[0,T]}, z^1_{[T+1,\infty[}\right) = \max\left\{\hat{\underset{\sim}{\omega}} \cdot d\mathbb{1}, \min\left\{\underset{\sim}{\omega} \cdot d\mathbb{1}, \phi \cdot z^1\right\}\right\} = d,$$

where $\phi \cdot z^1 < d$. For $z^2 = x$, it derives that:

$$\lim_{T\to\infty} I\left(x_{[0,T]}, z^2_{[T+1,\infty[}\right) = \max\left\{\hat{\underset{\sim}{\omega}} \cdot x, \min\left\{\underset{\sim}{\omega} \cdot x, \phi \cdot x\right\}\right\} = \phi \cdot x,$$

$$\lim_{T\to\infty} I\left(d\mathbb{1}_{[0,T]}, z^2_{[T+1,\infty[}\right) = \max\left\{\hat{\underset{\sim}{\omega}} \cdot d\mathbb{1}, \min\left\{\underset{\sim}{\omega} \cdot d\mathbb{1}, \phi \cdot x\right\}\right\} = d,$$

where $\phi \cdot x > d$. The close future comparison between x and $d\mathbb{1}$ is hence influenced by the choice of z, that in its turn implies that $\succeq$ cannot satisfy axiom **G2**.
(ii) For the case of the order $\hat{\succeq}$ and the index function $\hat{I}$, making using of the same arguments, it follows that $\chi_c = \chi_d = 1$, the close future order $\hat{\succeq}_c$ satisfying every property of axiom **F1** and, for every $x, y \in \ell_\infty$, $x \hat{\succeq}_c y$ if and only if $\underset{\sim}{\omega} \cdot x \geq \underset{\sim}{\omega} \cdot y$.
Observe that, for every $x, z \in \ell_\infty$,

$$\lim_{T\to\infty} \hat{I}\left(x_{[0,T]}, z_{[T+1,\infty[}\right) = \max\left\{\hat{\phi} \cdot z, \min\left\{\phi \cdot z, \underset{\sim}{\omega} \cdot x\right\}\right\}.$$

This implies that the order $\hat{\succsim}$ satisfies axiom **G2**. In order to prove that $\hat{\succsim}$ does oppositely not satisfy **G1**, consider $x \in \ell_\infty$, $d, d' \in \mathbb{R}$ such that $\hat{\phi} \cdot x < d' < d < \underset{\sim}{w} \cdot x < \phi \cdot x$. Take then $z^1 = d'\mathbb{1}$ and $z^2 = x$. It follows that:

$$\lim_{T\to\infty} \hat{I}\left(z^1_{[0,T]}, x_{[T+1,\infty[}\right) = \max\left\{\hat{\phi} \cdot x, \min\left\{\phi \cdot x, \underset{\sim}{\omega} \cdot z^1\right\}\right\} = d',$$
$$\lim_{T\to\infty} \hat{I}\left(z^1_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) = \max\left\{\hat{\phi} \cdot d\mathbb{1}, \min\left\{\phi \cdot d\mathbb{1}, \underset{\sim}{\omega} \cdot d'\mathbb{1}\right\}\right\} = d,$$

where $d' < d$. For the sequence $z^2 = x$, it follows that:

$$\lim_{T\to\infty} \hat{I}\left(z^2_{[0,T]}, x_{[T+1,\infty[}\right) = \max\left\{\hat{\phi} \cdot x, \min\left\{\phi \cdot x, \underset{\sim}{\omega} \cdot x\right\}\right\} = \underset{\sim}{\omega} \cdot x,$$
$$\lim_{T\to\infty} \hat{I}\left(z^2_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) = \max\left\{\hat{\phi} \cdot x, \min\left\{\phi \cdot x, \underset{\sim}{\omega} \cdot d\mathbb{1}\right\}\right\} = d,$$

where $d < \underset{\sim}{\omega} \cdot x$. The distant future comparison between x and $d\mathbb{1}$ depending upon the specific choice of z, the order $\hat{\succsim}$ cannot satisfy **G1**

(iii) Suppose that **G1** and **G2** are satisfied and first consider the configuration $\chi_d = \chi_c = 0$. Define then the set $D(x)$ as in the proof of Proposition 2.3. Recall that, for every $x, y \in \ell_\infty$, $x \succeq_d y$ if and only if $\sup D(x) \geq \sup D(y)$. It is then to be proved that, for every $x \in \ell_\infty$, $\sup D(x) = +\infty$. Making use of part (ii) of Proposition 2.4 and Proposition 2.6, the following holds for any constants $c, d \in \mathbb{R}$:

$$\lim_{T\to\infty} I\left(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) = c.$$

This implies that, for every d, d' in $\mathbb{R}$ and for any $\epsilon > 0$, there exists a large enough $T(\epsilon)$ such that

$$\left(c\mathbb{1}_{[0,T]}, d'\mathbb{1}_{[T+1,\infty[}\right) \succeq \left(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

Fix now any $x, z \in \ell_\infty$, $d \in \mathbb{R}$ and any $\epsilon > 0$. Define $c_z = I_c(z)$, that is finite since neither χ_d nor χ_c is equal to zero. Finally fix any d' such that $d' \leq \inf_s x_s$.
There then exists some $T_0(\epsilon, x)$ such that, for $T \geq T_0(\epsilon, x)$,

$$\begin{aligned}\left(z_{[0,T]}, x_{[T+1,\infty[}\right) &\succeq \left(c_z\mathbb{1}_{[0,T]}, x_{[T+1,\infty[}\right) - \frac{\epsilon}{3}\mathbb{1}\\ &\succeq \left(c_z\mathbb{1}_{[0,T]}, d'\mathbb{1}_{[T+1,\infty[}\right) - \frac{\epsilon}{3}\mathbb{1}.\end{aligned}$$

Since, for large enough values of T,

$$\left(c\mathbb{1}_{[0,T]}, d'\mathbb{1}_{[T+1,\infty[}\right) \succeq \left(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \frac{\epsilon}{3}\mathbb{1},$$

for such values of T, the following also holds:

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \geq \left(c_z \mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \frac{2\epsilon}{3}\mathbb{1}.$$

But, by the very definition of c_z and for large enough values of T,

$$\left(c_z \mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) \geq \left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \frac{\epsilon}{3}\mathbb{1},$$

that implies

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \geq \left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

Hence, for every $x, y \in \ell_\infty$, $\sup \mathrm{D}(x) = \sup \mathrm{D}(y) = +\infty$, or $x \sim_d y$. Finally and for the remaining case $\chi_d = \chi_c = 1$, making use of the same arguments, for every $x, y \in \ell_\infty$, the holding $x \sim_c y$ is eventually established. QED

A.10 Proof of Theorem 2.1

(i) First suppose that $\chi_d \leq \chi_c$, define $c_x = \mathrm{I}_c(x)$ and $d_x = \mathrm{I}_d(x)$ and fix $\epsilon > 0$. From the definition of c_x and d_x, for large enough values of T,

$$\begin{aligned} x &= \left(x_{[0,T]}, x_{[T+1,\infty[}\right) \\ &\geq \left(c_x \mathbb{1}_{[0,T]}, x_{[T+1,\infty[}\right) - \epsilon\mathbb{1} \\ &\geq \left(c_x \mathbb{1}_{[0,T]}, d_x \mathbb{1}_{[T+1,\infty[}\right) - 2\epsilon\mathbb{1}. \end{aligned}$$

Therefore

$$\mathrm{I}(x) \geq \limsup_{T\to\infty} \mathrm{I}\left(c\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - 2\epsilon\mathbb{1}.$$

This inequality being further true for any arbitrary $\epsilon > 0$,

$$\mathrm{I}(x) \geq \limsup_{T\to\infty} \mathrm{I}\left(c_x\mathbb{1}_{[0,T]}, d_x\mathbb{1}_{[T+1,\infty[}\right).$$

Likewise,

$$\mathrm{I}(x) \leq \liminf_{T\to\infty} \mathrm{I}\left(c_x\mathbb{1}_{[0,T]}, d_x\mathbb{1}_{[T+1,\infty[}\right).$$

Therefore

$$\mathrm{I}(x) = \lim_{T\to\infty} \mathrm{I}\left(c_x\mathbb{1}_{[0,T]}, d_x\mathbb{1}_{[T+1,\infty[}\right).$$

First consider the configuration $c_x \leq d_x$ or, equivalently, $\mathrm{I}_c(x) \leq \mathrm{I}_d(x)$. As $d_x - c_x \geq 0$, it follows that:

$$\begin{aligned}
\mathrm{I}(x) &= \lim_{\mathrm{T}\to\infty} \mathrm{I}\Big(c_x \mathbb{1}_{[0,\mathrm{T}]}, d_x \mathbb{1}_{[\mathrm{T}+1,\infty[}\Big) \\
&= c_x + \lim_{\mathrm{T}\to\infty} \mathrm{I}\Big(0 \mathbb{1}_{[0,\mathrm{T}]}, (d_x - c_x) \mathbb{1}_{[\mathrm{T}+1,\infty[}\Big) \\
&= c_x + (d_x - c_x) \lim_{\mathrm{T}\to\infty} \mathrm{I}\Big(0 \mathbb{1}_{[0,\mathrm{T}]}, \mathbb{1}_{[\mathrm{T}+1,\infty[}\Big) \\
&= (1 - \chi_d) c_x + \chi_d d_x \\
&= (1 - \chi_d) \mathrm{I}_c(x) + \chi_d \mathrm{I}_d(x).
\end{aligned}$$

But $\mathrm{I}_c(x) \leq \mathrm{I}_d(x)$, that implies the holding of:

$$(1 - \chi_d)\mathrm{I}_c(x) + \chi_d \mathrm{I}_d(x) \leq (1 - \chi)\mathrm{I}_c(x) + \chi \mathrm{I}_d(x),$$

for any $\chi \in [\underline{\chi}, \overline{\chi}]$, with $\underline{\chi} = \chi_d$, $\overline{\chi} = \chi_c$. As for the remaining configuration $\mathrm{I}_c(x) \geq \mathrm{I}_d(x)$, and making use of the same arguments

$$\begin{aligned}
\mathrm{I}(x) &= \chi_c \mathrm{I}_c(x) + (1 - \chi_c)\mathrm{I}_d(x) \\
&\leq (1 - \chi)\mathrm{I}_c(x) + \chi \mathrm{I}_d(x),
\end{aligned}$$

for any $\chi \in [\underline{\chi}, \overline{\chi}]$. Whence, finally

$$\mathrm{I}(x) = \min_{\underline{\chi} \leq \chi \leq \overline{\chi}} \Big[(1 - \chi)\mathrm{I}_c(x) + \chi \mathrm{I}_d(x)\Big].$$

(ii) For the other configuration $\chi_d \geq \chi_c$ and making use of the same line of arguments, it similarly follows that:

$$\mathrm{I}(x) = \max_{\underline{\chi} \leq \chi \leq \overline{\chi}} \Big[(1 - \chi)\mathrm{I}_c(x) + \chi \mathrm{I}_d(x)\Big],$$

where $\underline{\chi} = \chi_c$, $\overline{\chi} = \chi_d$. QED

A.11 Proof of Corollary 1

First assume that $\mathrm{I}_c(x) \leq \mathrm{I}_d(x)$. The value of $\mathrm{I}_\alpha(x)$ is therefore defined as:

$$\begin{aligned}
\mathrm{I}_\alpha(x) &= \alpha\Big[(1 - \underline{\chi})\mathrm{I}_c(x) + \underline{\chi}\mathrm{I}_d(x)\Big] + (1 - \alpha)\left[(1 - \overline{\chi})\mathrm{I}_c(x) + \overline{\chi}\mathrm{I}_d(x)\right] \\
&= \Big[1 - \Big(\alpha\underline{\chi} + (1 - \alpha)\overline{\chi}\Big)\Big]\mathrm{I}_c(x) + \Big[\alpha\underline{\chi} + (1 - \alpha)\overline{\chi}\Big]\mathrm{I}_d(x).
\end{aligned}$$

Consider then the remaining configuration $I_c(x) \geq I_d(x)$. The value of $I_\alpha(x)$ is therefore defined as:

$$\begin{aligned} I_\alpha(x) &= \alpha\Big[(1-\overline{\chi})I_c(x) + \overline{\chi}I_d(x)\Big] + (1-\alpha)\Big[(1-\underline{\chi})I_c(x) + \underline{\chi}I_d(x)\Big] \\ &= \Big[1 - \big(\alpha\overline{\chi} + (1-\alpha)\underline{\chi}\big)\Big]I_c(x) + \Big[\alpha\overline{\chi} + (1-\alpha)\underline{\chi}\Big]I_d(x). \end{aligned}$$

For $1/2 \leq \alpha \leq 1$, define

$$\begin{aligned} \underline{\chi}_\alpha &= \alpha\underline{\chi} + (1-\alpha)\overline{\chi}, \\ \overline{\chi}_\alpha &= \alpha\overline{\chi} + (1-\alpha)\underline{\chi}. \end{aligned}$$

It obviously holds that $\underline{\chi}_\alpha \leq \overline{\chi}_\alpha$. Hence, for $I_c(x) \leq I_d(x)$,

$$I_\alpha(x) = \big(1 - \underline{\chi}_\alpha\big)I_c(x) + \underline{\chi}_\alpha I_d(x),$$

while, for $I_c(x) \geq I_d(x)$,

$$I_\alpha(x) = \big(1 - \overline{\chi}_\alpha\big)I_c(x) + \overline{\chi}_\alpha I_d(x).$$

These properties are equivalents to the holding, for any $x \in \ell_\infty$, of

$$I_\alpha(x) = \min_{\underline{\chi}_\alpha \leq \chi \leq \overline{\chi}_\alpha} \Big[(1-\chi)I_c(x) + \chi I_d(x)\Big].$$

For $0 \leq \alpha \leq 1/2$ and making use of the same arguments for

$$\begin{aligned} \underline{\chi}_\alpha &= \alpha\overline{\chi} + (1-\alpha)\underline{\chi}, \\ \overline{\chi}_\alpha &= \alpha\underline{\chi} + (1-\alpha)\overline{\chi}, \end{aligned}$$

it similarly follows that:

$$I_\alpha(x) = \max_{\underline{\chi}_\alpha \leq \chi \leq \overline{\chi}_\alpha} \Big[(1-\chi)I_c(x) + \chi I_d(x)\Big].$$

The statement follows. QED

B. Proofs for section 3

B.1 Proof of Lemma 3.1

Taking advantage of the decomposition of the order index I into the order indexes I_c and I_d through Theorem 2.1 and from axiom **B1**, letting T′ tends to infinity, either

$$I_c\big(z_{[0,T-1]}, x_{[T,\infty[}\big) \geq I_c\big(z_{[0,T-1]}, c\mathbb{1}_{[T,\infty[}\big) - \epsilon,$$

or

$$I_c\Big(z_{[0,T-1]}, c\mathbb{1}_{[T,\infty[}\Big) + \epsilon \geq I_c\Big(z_{[0,T-1]}, x_{[T,\infty[}\Big).$$

For any constant c, this is equivalent to the holding of either, for any $z \in \ell_\infty$, of:

$$\Big(z_{[0,T-1]}, x_{[T,\infty[}\Big) \succeq_c \Big(z_{[0,T-1]}, c\mathbb{1}_{[T,\infty[}\Big),$$

or, for any $z \in \ell_\infty$, of:

$$\Big(z_{[0,T-1]}, c\mathbb{1}_{[T,\infty[}\Big) \succeq_c \Big(z_{[0,T-1]}, x_{[T,\infty[}\Big),$$

that completes the argument of the proof. QED

B.2 Proof of Proposition 3.1

(i) Define $C_T(x)$ the set of values c such that, for any $z_0, z_1, \ldots, z_{T-1}$,

$$\Big(z_{[0,T-1]}, x\Big) \succeq_c \Big(z_{[0,T-1]}, c\mathbb{1}_{[T,\infty[}\Big).$$

Define the order $\succeq_T$ as the holding of $x \succeq_T y$ if and only if $\sup C(x) \geq \sup C(y)$. Fix $x, y \in \ell_\infty$ and suppose that for any $z_0, z_1, \ldots, z_{T-1}$,

$$\Big(z_{[0,T-1]}, x\Big) \succeq_c \Big(z_{[0,T-1]}, y\Big).$$

This implies that $C_T(y) \subset C_T(x)$, or $x \succeq_T y$. First consider the case $\sup C_T(y) < +\infty$ and take $c_y^T = \sup C_T(y)$. It is readily checked that $C_T(y)$ is closed, whence the satisfaction of $c_y^T \in C_T(y) \subset C_T(x)$. Further and from the definition of c_y^T, which is finite, for any $z_0, z_1, \ldots, z_{T-1}$,

$$\Big(z_{[0,T-1]}, x\Big) \succeq_c \Big(z_{[0,T-1]}, c_y^T\mathbb{1}_{[T,\infty[}\Big) \succeq_c \Big(z_{[0,T-1]}, y\Big).$$

Secondly consider the case $\sup C_T(y) = +\infty$, that implies the holding of $\sup C_T(x) = +\infty$. Whence, for any $c \geq \sup_s y_s$ and for any $z_0, z_1, \ldots, z_{T-1}$:

$$\Big(z_{[0,T-1]}, x\Big) \succeq_c \Big(z_{[0,T-1]}, c\mathbb{1}_{[T,\infty[}\Big) \succeq_c \Big(z_{[0,T-1]}, y\Big).$$

(ii) For the *transitivity*, *monotonicity* and *weak convexity* properties, replicate the arguments used for the proof of Proposition 2.3.

(iii) Suppose that at least one of two values $\chi_d^{\mathrm{T}}, \chi_c^{\mathrm{T}}$ differs from zero. It is to be proved that the order $\succeq_{\mathrm{T}}$ satisfies the technical *non-triviality* property. The *Archimedeanity* property would then follow as a direct corollary. But, and from the definition of χ_d^{T}, if $\chi_d^{\mathrm{T}} > 0$, then $\mathrm{I}_c\left(0\mathbb{1}_{[0,\mathrm{T}-1]}, \mathbb{1}\right) > \mathrm{I}_c\left(0\mathbb{1}_{[0,\mathrm{T}-1]}, 0\mathbb{1}\right)$. This implies $\mathbb{1} \succ_{\mathrm{T}} 0\mathbb{1}$. Likewise and from the definition of χ_c^{T}, if $\chi_c^{\mathrm{T}} > 0$, then $\mathrm{I}_c\left(\mathbb{1}_{[0,\mathrm{T}-1]}, \mathbb{1}\right) > \mathrm{I}_c\left(\mathbb{1}_{[0,\mathrm{T}-1]}, 0\mathbb{1}\right)$. This in its turn implies $\mathbb{1} \succ_{\mathrm{T}} 0\mathbb{1}$.

(iv) Suppose that at least one of two values $\chi_d^{\mathrm{T}}, \chi_c^{\mathrm{T}}$ is different from zero. From (i), (ii) and (iii), $\sup \mathrm{C}_{\mathrm{T}}(x) < +\infty$ for any x and the order $\succeq_{\mathrm{T}}$ satisfies every property in axiom **F1**. The index function $\mathrm{I}_{\mathrm{T}}(x) = \sup \mathrm{C}(x)$ therefore satisfies every property listed in Proposition 2.1. QED

B.3 Proof of Lemma 3.2

First suppose that $\chi_d^{\mathrm{T}_0} = 0$. This implies that $\mathrm{I}_c\left(0\mathbb{1}_{[0,\mathrm{T}_0-1]}, 0\mathbb{1}_{[\mathrm{T}_0,\infty[}\right) = 0$ but also, and from the *monotonicity* property, the alike holding of $\mathrm{I}_c\left(0\mathbb{1}_{[0,\mathrm{T}-1]}, 0\mathbb{1}_{[\mathrm{T},\infty[}\right) = 0$ for any $\mathrm{T} \geq \mathrm{T}_0$, or $\chi_d^{\mathrm{T}} = 0$ for any $\mathrm{T} \geq \mathrm{T}_0$. Then consider the holding of $\chi_c^{\mathrm{T}_0} = 0$, which is equivalent, from the definition of χ_c, to $\mathrm{I}_c\left(\mathbb{1}_{[0,\mathrm{T}_0-1]}, 0\mathbb{1}_{[\mathrm{T}_0,\infty[}\right) = 1$. The sequence $\left\{\mathrm{I}_c\left(\mathbb{1}_{[0,\mathrm{T}-1]}, 0\mathbb{1}_{[\mathrm{T},\infty[}\right)\right\}_{\mathrm{T}=0}^{\infty}$ being further increasing in T, this implies the holding of $\mathrm{I}_c\left(\mathbb{1}_{[0,\mathrm{T}-1]}, 0\mathbb{1}_{[\mathrm{T},\infty[}\right) = 1$ for any $\mathrm{T} \geq \mathrm{T}_0$.

Now and for any constant c, as $\chi_d^{\mathrm{T}} = \chi_c^{\mathrm{T}} = 0$, $c\mathbb{1} \sim_{\mathrm{T}} (-c)\mathbb{1}$. From the *weak convexity* property, this also implies that $c\mathbb{1} \sim_{\mathrm{T}} 0\mathbb{1}$. Then take any $x \in \ell_\infty$ and consider a constant c such that $c \geq \sup_s |x_s|$: from the *monotonicity* property, this also implies the holding of $c\mathbb{1} \succeq_{\mathrm{T}} x \succeq_{\mathrm{T}} (-c)\mathbb{1}$, or the one of $x \sim_{\mathrm{T}} 0\mathbb{1}$. The statement follows.

B.4 Proof of Proposition 3.2

Fix $x \in \ell_\infty$, let $c = \mathrm{I}_{\mathrm{T}+1}\left(x_{[\mathrm{T}+1,\infty[}\right)$ and consider the case $x_{\mathrm{T}} \leq c$. From Proposition 3.1 and as $d - x_{\mathrm{T}} \geq 0$,

$$\begin{aligned}
\mathrm{I}_{\mathrm{T}}\left(x_{[\mathrm{T},\infty[}\right) &= \mathrm{I}_{\mathrm{T}}\left(x_{\mathrm{T}}, c\mathbb{1}\right) \\
&= x_{\mathrm{T}} + \mathrm{I}_{\mathrm{T}}\left(0, (c - x_{\mathrm{T}})\mathbb{1}\right) \\
&= x_{\mathrm{T}} + (c - x_{\mathrm{T}})\mathrm{I}_{\mathrm{T}}(0, \mathbb{1}) \\
&= \left(1 - \mathrm{I}_{\mathrm{T}}(0, \mathbb{1})\right)x_{\mathrm{T}} + \mathrm{I}_{\mathrm{T}}(0, \mathbb{1})c.
\end{aligned}$$

Likewise and for $x_T \geq d$:

$$\begin{aligned} I_T\left(x_{[T,\infty[}\right) &= I_T\left(x_T, c\mathbb{1}\right) \\ &= I_T\left(x_T - c, 0\mathbb{1}\right) + d \\ &= \left(x_T - c\right) I_T(1, 0\mathbb{1}) + d \\ &= I_T(1,0)x_T + \left(1 - I_T(1, 0\mathbb{1})\right)c. \end{aligned}$$

First suppose that $\chi_d^T \leq \chi_c^T$, or $I_T(0, \mathbb{1}) + I_T(1, 0\mathbb{1}) \leq 1$, and let $\underline{\delta}_T = \chi_d^T = I(0, \mathbb{1})$ and $\overline{\delta}_T = \chi_c^T = 1 - I(1, 0\mathbb{1})$. It follows that $0 < \underline{\delta}_T \leq \overline{\delta}_T < 1$ and

$$I_T\left(x_{[T,\infty[}\right) = \min_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \left[(1-\delta)x_T + \delta I_{T+1}\left(x_{[T+1,\infty[}\right)\right].$$

Consider the remaining case $\chi_d^T \geq \chi_c^T$, or $I_T(0, \mathbb{1}) + I_T(1, 0\mathbb{1}) \geq 1$ and let $\underline{\delta}_T = \chi_c^T = 1 - I_T(1, 0\mathbb{1})$, $\overline{\delta}_T = \chi_d^T = I_T(0, \mathbb{1})$. It follows that $0 < \underline{\delta}_T \leq \overline{\delta}_T < 1$ and

$$I_T\left(x_{[T,\infty[}\right) = \max_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \left[(1-\delta)x_T + \delta I_{T+1}\left(x_{[T+1,\infty[}\right)\right],$$

which establishes the statement. QED

B.5 Proof of Lemma 3.3

(i) Consider the constant c such that $(0, \mathbb{1}) \sim_T c\mathbb{1}$. From Proposition 3.1, this implies that:

$$\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) \sim_c \left(0\mathbb{1}_{[0,T-1]}, c\mathbb{1}_{[T,\infty[}\right).$$

The order $\succeq_c$ further satisfying every property in axiom **F1**, for any $c' < c$, the following is to hold:

$$\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) \succ_c \left(0\mathbb{1}_{[0,T-1]}, c'\mathbb{1}_{[T,\infty[}\right).$$

From Proposition 2.6, there then exists a large enough T_0 such that, for $T' \geq T_0$,

$$\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right) \succeq_c \left(0\mathbb{1}_{[0,T-1]}, c'\mathbb{1}_{[T,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right).$$

The left hand side and the right hand side of the above equation assuming the same distant future valuation, this implies that:

$$\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right) \succeq \left(0\mathbb{1}_{[0,T-1]}, c'\mathbb{1}_{[T,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right).$$

From axiom **B2**, this can be strengthened to:

$$\left(0\mathbb{1}_{[0,T+1]}, \mathbb{1}_{[T+2,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right) \succeq \left(0\mathbb{1}_{[0,T]}, c'\mathbb{1}_{[T+1,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right),$$

which is equivalent to

$$\left(0\mathbb{1}_{[0,T+1]}, \mathbb{1}_{[T+2,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right) \succeq_c \left(0\mathbb{1}_{[0,T]}, c'\mathbb{1}_{[T+1,T']}, 0\mathbb{1}_{[T'+1,\infty[}\right).$$

Letting T' tends to infinity, it follows that:

$$(0, \mathbb{1}) \succeq_{T+1} c'\mathbb{1}.$$

As this is true for any $c' < c$, letting c' converge to c, it derives that:

$$(0, \mathbb{1}) \succeq_{T+1} c\mathbb{1}.$$

(ii) Follow the same line of arguments as for (i). QED

B.6 Proof of Proposition 3.3

First observe that, for any T,

$$\underline{\delta}_T = \min\left\{I_T(0, \mathbb{1}), 1 - I_T(1, 0\mathbb{1})\right\},$$
$$\overline{\delta}_T = \max\left\{I_T(0, \mathbb{1}), 1 - I_T(1, 0\mathbb{1})\right\}.$$

But and from Lemma 3.3, both of the two sequences $\left\{I_T(0, \mathbb{1})\right\}_{T=0}^{\infty}$ and $\left\{1-I_T(1, 0\mathbb{1})\right\}_{T=0}^{\infty}$ are increasing. This in its turn implies that the two sequences $\left\{\underline{\delta}_T\right\}_{T=0}^{\infty}$ and $\left\{\overline{\delta}_T\right\}_{T=0}^{\infty}$ are also increasing.

B.7 Proof of Proposition 3.4

(i) Suppose that $x \in \ell_\infty$, $d, d' \in \mathbb{R}$ such that

$$d\mathbb{1} \succeq_d x \succeq_d d'\mathbb{1}.$$

Fix $z \in \ell_\infty$ and $\epsilon > 0$. From axiom **G1**, there exists $T_0(\epsilon, z)$ such that, for $T \geq T_0(\epsilon, z)$ it holds that:

$$\left(z_{[0,T]}, (d+\epsilon)\mathbb{1}_{[T+1,\infty[}\right) \succeq \left(z_{[0,T]}, x_{[T+1,\infty[}\right) \succeq \left(z_{[0,T]}, (d'-\epsilon)\mathbb{1}_{[T+1,\infty[}\right).$$

From axiom **B3**, there then exists some date $T_0'(\epsilon, z)$ such that, for any $T \geq T_0'(\epsilon, z)$ and for any $s \geq s_0$ there exists $s_0(T)$ such that, for $s \geq s_0(T)$,

$$\left(z_{[0,T]}, (d+\epsilon)\mathbb{1}_{[T+1,\infty[}\right) \geq \left(z_{[0,T]}, x_{[T+1,T+s]}, (d+\epsilon)\mathbb{1}_{[T+s+1,\infty[}\right),$$
$$\left(z_{[0,T]}, x_{[T+1,T+s]}, (d'-\epsilon)\mathbb{1}_{[T+s+1,\infty[}\right) \geq \left(z_{[0,T]}, (d'-\epsilon)\mathbb{1}_{[T+1,\infty[}\right).$$

From axiom **B1** and for any $T \geq T_0(\epsilon, z)$, letting s tends to infinity, it holds that:

$$(d+\epsilon)\mathbb{1}_{[T,\infty[} \succeq_T x_{[T,\infty[} \succeq_T (d'-\epsilon)\mathbb{1}_{[T,\infty[}.$$

(ii) Take $d = I_d(x)$. For any $\epsilon > 0$, there exists $T_0(\epsilon)$ such that, for $T \geq T_0(\epsilon)$,

$$(d+\epsilon)\mathbb{1}_{[T,\infty[} \succeq_T x_{[T,\infty[} \succeq_T (d-\epsilon)\mathbb{1}_{[T,\infty[},$$

which is equivalent to

$$d+\epsilon \geq I_T\left(x_{[T,\infty[}\right) \geq d-\epsilon.$$

Letting ϵ converge to zero, it follows that:

$$\lim_{T\to\infty} I_T\left(x_{[T,\infty[}\right) = I_d(x).$$

(iii) Take any sequence $\left\{T_k\right\}_{k=0}^{\infty} \subset \mathbb{N}$ converging to infinity and satisfying, for any k, $T_{k-1} < T_k - 1$. Then define $\hat{x}$ as $\hat{x}_T = 1$ for any $T \notin \left\{T_k\right\}_{k=0}^{\infty}$ and $\hat{x}_{T_k} = 0$ for any k. It follows that $0 \leq I_d(\hat{x}) \leq 1$. From Proposition 3.2 and for every T, there exists $\delta_T \in \left[\underline{\delta}_T, \overline{\delta}_T\right]$ such that:

$$I_T\left(\hat{x}_{[T,\infty[}\right) = \left(1-\delta_T\right)\hat{x}_T + \delta_T I_{T+1}\left(\hat{x}_{[T+1,\infty[}\right).$$

Recall then that $\lim_{T\to\infty} I_T\left(\hat{x}_{[T,\infty[}\right) = I_d(\hat{x})$ and consider the case $I_d(\hat{x}) = 0$. For any k and as $T_{k-1} < T_k - 1$, $\hat{x}_{T_k - 1} = 1$, one has, for any k:

$$\begin{aligned} I_{T_k-1}\left(\hat{x}_{[T_k-1,\infty[}\right) &= \left(1-\delta_{T_k-1}\right)\hat{x}_{T_k-1} + \delta_{T_k-1} I_{T_k}\left(\hat{x}_{[T_k,\infty[}\right) \\ &= \left(1-\delta_{T_k-1}\right)\hat{x}_{T_k-1} + \delta_{T_k-1} I_{T_k}\left(\hat{x}_{[T_k,\infty[}\right). \end{aligned}$$

Letting k tends to infinity, $I_{T_k-1}\left(\hat{x}_{[T_k-1,\infty[}\right)$ and $I_{T_k}\left(\hat{x}_{[T_k,\infty[}\right)$ converge to $I_d(\hat{x})$, which sums up to zero. This implies that $1-\delta_{T_k-1}$ converges to zero, or that δ_{T_k-1} converges to 1. As $\delta_{T_k-1} \leq \overline{\delta}_{T_k-1} \leq 1$, it derives that $\overline{\delta}_{T_k-1}$ converges to 1.

Considering now the case $I_d(\hat{x}) > 0$ and for any k,

$$\begin{aligned} I_{T_k}\left(\hat{x}_{[T_k-1,\infty[}\right) &= (1-\delta_{T_k})\hat{x}_{T_k} + \delta_{T_k} I_{T_k+1}\left(\hat{x}_{[T_k+1,\infty[}\right) \\ &= \delta_{T_k} I_{T_k+1}\left(\hat{x}_{[T_k+1,\infty[}\right). \end{aligned}$$

Letting k tends to infinity, both $I_{T_k}\left(\hat{x}_{[T_k,\infty[}\right)$ and $I_{T_k+1}\left(\hat{x}_{[T_k+1,\infty[}\right)$ do converge to $I_d(\hat{x})$, which is strictly positive, whence the convergence to 1 of δ_{T_k} and $\overline{\delta}_{T_k}$.
(iv) Adding axiom **B2**, the sequence $\left\{\overline{\delta}_T\right\}_{T=0}^{\infty}$ becomes increasing, whence and from (iii), the satisfaction of:

$$\lim_{T\to\infty} \overline{\delta}_T = 1,$$

which establishes the statement. QED

C. Proofs for section 4

C.1 Proof of Lemma 4.1

Suppose that $x \succeq^* y$, then and for every z, $(1/2)x + (1/2)z \succeq (1/2)y + (1/2)z$. Recall that this is equivalent to $x + z \succeq y + z$. Suppose that for every z, $x + z \succeq y + z$. Fix any $0 \leq \lambda < 1$. Fix any $z \in \ell_\infty$. One has

$$x + \frac{\lambda}{1-\lambda} z \succeq y + \frac{\lambda}{1-\lambda} z,$$

which implies the holding of $(1-\lambda)x + \lambda z \succeq (1 - \lambda y) + \lambda z$, whence the one of $x \succeq^* y$. QED

C.2 Proof of Proposition 4.1

Define $\mathscr{P}^*$ as the positive polar cone of $\mathscr{P} = \left\{x \in \ell_\infty \text{ such that } x \succeq^* 0\mathbb{1}\right\}$ in the dual space $\left(\ell_\infty\right)^*$:

$$\mathscr{P}^* = \left\{P \in (\ell_\infty)^* \text{ such that } P \cdot x \geq 0 \text{ for every } x \succeq^* 0\right\}.$$

Observe that by the very definition of the order $\succeq^*$, $\mathscr{P}$ is convex and separable by the vector $-\mathbb{1}$, the cone $\mathscr{P}^*$ does not degenerate to $\{0\}$.
For each $P \in \mathscr{P}^*$, define

$$\pi(P) = \frac{1}{P \cdot \mathbb{1}} P.$$

Since $x \succeq^* 0\mathbb{1}$ for every $x \in \ell_\infty$ satisfying $x_s \geq 0$ for all s, it follows that $P \cdot x \geq 0$ for every x such that $x_s \geq 0$ for every s. Let then $\Omega = \pi(\mathscr{P})$. As $P \cdot x \geq 0$ if and only if $\pi(P) \cdot x \geq 0$, $x \succeq^* 0\mathbb{1}$ is equivalent to $\pi(P) \cdot x \geq 0$ for every $P \in \mathscr{P}$. For every P, $\pi(P)$ can be decomposed as $\pi(P) = \lambda_c \underset{\sim}{\omega} + \lambda_d \phi$, where $\underset{\sim}{\omega} = (\omega_0, \omega_1, \cdots, \omega_s, \cdots) \in \ell_1$ and $\phi \in \ell_1^d$ is a finite additive measure: considering ϕ as a measure on $\mathbb{N}$, $\phi(A) = 0$ for every finite subset of $\mathbb{N}$. From the definition of Ω, for every $\left(\lambda_c \underset{\sim}{\omega}, \lambda_d \phi\right) \in \Omega$, $\lambda_c \sum_{s=0}^{\infty} \omega_s + \lambda_d \phi \cdot \mathbb{1} = 1$. The set Ω can be considered as a set of finite additive probabilities on $\mathbb{N}$. QED

C.3 Proof of Proposition 4.2

Consider $x, y \in \ell_\infty$ satisfying $\gamma_x^* < \gamma^{*x}$, $\gamma_y^* < \gamma^{*y}$: it is then to proved that $a_x = a_y$. Take a constant γ sufficiently big such that $x + \gamma\mathbb{1} \succeq^* y$. One gets $\gamma_{x+\gamma\mathbb{1}} = \gamma_x^* + \gamma$, $\gamma^{*,x+\gamma\mathbb{1}} = \gamma^{*x} + \gamma$ and $I(x + \gamma\mathbb{1}) = I(x) + \gamma$. This implies $a_{x+\gamma\mathbb{1}} = a_x$, whence $a_x = a_{x+\gamma\mathbb{1}} \leq a_y$. Take then a constant γ' such that $y + \gamma'\mathbb{1} \succeq^* x$. Relying to the same arguments, $a_y = a_{y+\gamma'\mathbb{1}} \leq a_x$, whence for every $x, y \in \ell_\infty$ such that $\gamma_x^* < \gamma^{*x}$ and $\gamma_y^* < \gamma^{*y}$, the satisfaction of $a_x = a_y$. QED

C.4 Proof of Proposition 4.3

Define $\mathscr{P}^d$ the set of $x \in \ell_\infty$ such that $x \succeq_d^* 0\mathbb{1}$, denote by $\mathscr{P}^{d*}$ its positive polar cone and let

$$\Omega_d = \left\{ \frac{1}{P \cdot \mathbb{1}} P \text{ with } P \in \mathscr{P}^{d*} \right\}.$$

It is first claimed that, for all $\left((1-\lambda)\underset{\sim}{\omega}, \lambda\phi\right) \in \Omega_d$, $(1-\lambda)\underset{\sim}{\omega} = 0$. Suppose the opposite; then there exists T such that $\omega_T > 0$. Take a constant $c > 0$ such that $(1-\lambda)\omega_T c > \lambda$ and let $x = \left(-c\mathbb{1}_{[0,T]}, {}^{T+1}\mathbb{1}\right)$. But, and from Proposition 2.4, x converges for every $z \in \ell_\infty$ and $I_d(x+z) = I_d(x) + I_d(z) = 1 + I_d(z) > I_d(z)$, whence $x \succeq_d^* 0\mathbb{1}$. Since

$$(1-\lambda)\underset{\sim}{\omega} \cdot x + \lambda\phi \cdot x \leq -(1-\lambda)\omega_T c + \lambda < 0,$$

this is however a contradiction, whence the satisfaction of $(1-\lambda)\underset{\sim}{\omega} = 0$, which also implies the holding of $\lambda = 1$. To sum up, the weights set Ω_d can therefore be considered as a *charges* subset belonging to ℓ_d^1. QED

C.5 Proof of Proposition 4.4

Rely to the same arguments as for the proof of Proposition 4.2. QED

C.6 Proof of Lemma 4.4

First suppose that $\chi_d + \chi_c \leq 1$ and recall that, for every $z \in \ell_\infty$, $x+z \geq y+z$ is equivalent to $I(x+z) \geq I(y+z)$. From Proposition 2.4 and the holding of $\lim_{T\to\infty} x_T = \lim_{T\to\infty} y_T$, then observe that $I_d(x+z) = I_d(y+z)$ and

$$\begin{aligned} I(x+z) &= \min_{\underline{\chi}\leq\chi\leq\overline{\chi}}\Big[(1-\chi)I_c(x+z)+\chi I_d(x+z)\Big] \\ &= \min_{\underline{\chi}\leq\chi\leq\overline{\chi}}\Big[(1-\chi)I_c(x+z)+\chi I_d(y+z)\Big]. \end{aligned}$$

This implies that, for every $z \in \ell_\infty$, $I(x+z) \geq I(y+z)$ if and only if $I_c(x+z) \geq I_c(y+z)$, or $x \succeq^* y$ if and only if $x \succeq^*_c y$. This line of arguments extends to the remaining configuration $\chi_d + \chi_c \geq 1$. QED

C.7 Proof of Lemma 4.6

Relying upon the same arguments as in the proof of Proposition 4.1, there exists a probability set $\Omega_c \subset \ell_1 \oplus \ell_d^1$ such that

$$x \succeq^*_c y \Leftrightarrow (1-\lambda)\sum_{s=0}^{\infty}\omega_s x_s + \lambda\phi\cdot x \geq (1-\lambda)\sum_{s=0}^{\infty}\omega_s x_s + \lambda\phi\cdot x,$$

for every $\big((1-\lambda)\underline{\omega}, \lambda\phi\big) \in \Omega_c$. Suppose that there exists $\big((1-\lambda)\underline{\omega}, \lambda\phi\big) \in \Omega_c$ satisfying $\lambda\phi \neq 0$, or equivalently $\lambda\phi\cdot\mathbb{1} > 0$, and fix $d > 0$ and $0 < \epsilon < 1$ such that for every T, $\epsilon + (1-\lambda)\sum_{s=0}^{T}\omega_s < d\lambda\phi\cdot\mathbb{1}$. Recollect that, from Lemma 1, there exists a large enough T_0 such that for $T \geq T_0$,

$$\Big(\epsilon\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big) \succeq^* \Big(0\mathbb{1}_{[0,T_0]}, d\mathbb{1}_{[T_0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big),$$

or

$$\Big(0\mathbb{1}_{[0,T_0]}, \big(-d\mathbb{1}_{[T_0,T]}\big), 0\mathbb{1}_{[T+1,\infty[}\Big) \succeq^* \Big(-\epsilon\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big).$$

Whence, for every $z \in \ell_\infty$ and from the definition of the pre-order $\succeq^*$,

$$\begin{aligned} &\Big(0\mathbb{1}_{[0,T_0]}, -d\mathbb{1}_{[T_0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big) + \Big(z_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big) \\ &\qquad\qquad \succeq \Big(-\epsilon\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big) + \Big(z_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\Big). \end{aligned}$$

But this is true for every $T \geq T_0$, so that, for every z,

$$\left(0\mathbb{1}_{[0,T_0]}, -d\mathbb{1}_{[T_0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right) + z \succeq_c -\epsilon\mathbb{1} + z.$$

In its turn, this implies that:

$$\left(0\mathbb{1}_{[0,T_0]}, -d\mathbb{1}_{[T_0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right) \succeq_c^* -\epsilon\mathbb{1},$$

or, equivalently,

$$(1-\lambda)\omega \cdot \left(0\mathbb{1}_{[0,T_0]}, -d\mathbb{1}_{[T_0,\infty[}\right) + \lambda\phi \cdot \left(0\mathbb{1}_{[0,T_0]}, -d\mathbb{1}_{[T_0,\infty[}\right) \geq -\epsilon,$$

that gives

$$(1-\lambda)\sum_{s=T_0}^{\infty} \omega_s - \lambda d \geq -\epsilon,$$

which is a contradiction. To sum up and for every $\left((1-\lambda), \lambda\phi\right) \in \Omega_c$, $\phi = 0$. Since Ω_c is a set of probabilities, this implies that $\lambda = 0$, and Ω_c can be considered as a subset of probabilites that is included in ℓ_1. With axiom **A1**, the set Ω_c can be considered as a set of tight measures ; it is therefore compact in the weak topology. Otherwise stated, Ω_c is weakly compact in ℓ_1. QED

C.8 Proof of Proposition 4.5

Rely to the same arguments as the ones used in the proof of Proposition 4.4. QED

C.9 Proof of Proposition 4.6

(i) First suppose that $x \succeq_c^* c\mathbb{1}$ and fix any $c' < c$: there exists a $\epsilon > 0$ such that $x \succeq_c^* (c' + \epsilon)\mathbb{1}$. But, and from Lemma 4.6, the set Ω_c is weakly compact, so that there exists T_0 such that for every $T \geq T_0$ and for every $\underset{\sim}{\omega} \in \Omega_c$, $c' \sum_{s=T+1}^{\infty} \omega_s < \epsilon$. This implies that, for every $T \geq T_0$,

$$\left(x_{[0,T]}, c'\mathbb{1}_{[T+1,\infty[}\right) \succeq_c^* c'\mathbb{1}.$$

From Lemma 4.4, this implies that

$$\left(x_{[0,T]}, c'\mathbb{1}_{[T+1,\infty[}\right) \succeq^* c'\mathbb{1},$$

whence

$$\left(x_{[0,T]}, c'\mathbb{1}_{[T+1,\infty[}\right) \succeq^* \left(c', x_{[0,T]}, c'\mathbb{1}_{[T+1,\infty[}\right) \succeq^* c'\mathbb{1}.$$

Relying again upon Lemma 4.4, it derives that:

$$\left(x_{[0,T]}, c'\mathbb{1}_{[T+1,\infty[}\right) \succeq_c^* \left(c', x_{[0,T]}, c'\mathbb{1}_{[T+1,\infty[}\right) \succeq_c^* c'\mathbb{1}.$$

Since this is true for every large enough value of T, this simplifies to:

$$x \succeq_c^* (c', x) \succeq_c^* c'\mathbb{1}.$$

Finally, and as c' was arbitrarily selected to be strictly smaller than c, by continuity, it eventually derives that:

$$x \succeq_c^* (c, x) \succeq_c^* c\mathbb{1}.$$

Referring to a classical recurrence argument, the result is available.

(ii) Since $x \succeq_c^* c_x^* \mathbb{1}$, from (i),

$$x \succeq_c^* \left(c_x^* \mathbb{1}_{[0,T]}, x\right) \succeq^* c_x^* \mathbb{1},$$

that implies $c^*_{\left(c_x^* \mathbb{1}_{[0,T]}, x\right)} \geq c_x^*$. Fixing now $c > c_x^*$ and from the definition of c_x, $x \not\succeq^* c\mathbb{1}$, or, equivalently there exists $z \in \ell_\infty$ such that $c\mathbb{1} + z \succ_c x + z$. But $x \succeq_c^* \left(c_x^* \mathbb{1}_{[0,T]}, x\right)$, so that it eventually holds that $x + z \succeq_c \left(c_x^* \mathbb{1}_{[0,T]}, x\right) + z$, or, equivalently, $c\mathbb{1} + z \succ_c \left(c_x^* \mathbb{1}_{[0,T]}, x\right) + z$. The coefficient c having been chosen arbitrarily larger than c_x^*, it derives that $c^*_{\left(c_x^* \mathbb{1}_{[0,T]}, x\right)} = c_x$.

(iii) Consider the sequences x and $\left(c_x^* \mathbb{1}_{[0,T]}, x\right)$. Since $x \succeq_c^* \left(c^*_{\left(c_x^* \mathbb{1}_{[0,T]}, x\right)}, x\right) \succeq_c^* c_x \mathbb{1}$, it follows that:

$$\begin{aligned} x + \left(c_x^* \mathbb{1}_{[0,T]}, x\right) &\succeq_c^* c_x^* \mathbb{1} + \left(c_x^* \mathbb{1}_{[0,T]}, x\right) \\ &\succeq_c^* c_x^* \mathbb{1}_{[0,T]} + c_x^* \mathbb{1} \\ &= 2c_x^* \mathbb{1}, \end{aligned}$$

whence the satisfaction of $c^*_{(1/2)x + (1/2)\left(c_x^* \mathbb{1}_{[0,T]}, x\right)} \geq 2c_x^*$. Fix then $c > c_x^*$. From the definition of c_x^*, there exists $z \in \ell_\infty$ such that $c\mathbb{1} + z \succ_c x + z$. This in its turn implies that $2c\mathbb{1} + 2z \succ_c 2x + 2z$. Since $x \succeq_c^* \left(c_x^* \mathbb{1}_{[0,T]}, x\right) \succeq_c^*$,

$$\begin{aligned} 2c\mathbb{1} + 2z &\succ_c 2x + 2z \\ &= x + (x + 2z) \\ &\succeq_c \left(c_x^* \mathbb{1}_{[0,T]}, x\right) + x + 2z. \end{aligned}$$

This implies that $c > c^*_{(1/2)x+(1/2)(c^*_x \mathbb{1}_{[0,T]},x)}$. Since c was chosen arbitrarily bigger than c^*_x, it finally holds that $c^*_x \geq c^*_{(1/2)x+(1/2)(c^*_x \mathbb{1}_{[0,T]},x)}$. QED

C.10 Proof of Lemma 4.7

Since $\underset{\sim}{\omega} = \underset{\sim}{\omega}^{T}$, for every $T \in \mathbb{N}$, it follows that

$$\omega_s = \frac{\omega_{T+s}}{\sum_{s'=0}^{\infty} \omega_{T+s'}} \text{ and } \omega_{s+1} = \frac{\omega_{T+s+1}}{\sum_{s'=0}^{\infty} \omega_{T+s'}}.$$

This implies, for every T, s, that:

$$\frac{\omega_{s+1}}{\omega_s} = \frac{\omega_{T+s+1}}{\omega_{T+s}},$$

But this is equivalent, for some $\delta > 0$ and for every $s \geq 0$, to

$$\frac{\omega_{s+1}}{\omega_s} = \delta,$$

or to $\omega_s = \delta^s \omega_0$ for every $s \geq 0$. Since $\sum_{s=0}^{\infty} \omega_s = 1$, it eventually follows that $0 < \delta < 1$ and $\omega^*_s = \left(1 - \delta^*\right)\left(\delta^*\right)^s$ for $s \geq 0$. QED

C.11 Proof of Proposition 4.7

The main part of this proof establishes that for every $\underset{\sim}{\omega}^*$ that corresponds to an exposed point of Ω_c, $\underset{\sim}{\omega}^* = \underset{\sim}{\omega}^{*,T}$ for all T. Since $\underset{\sim}{\omega}^*$ is an exposed point of Ω_c, which is a subset of ℓ_1, for every $\underset{\sim}{\omega} \in \Omega_c \setminus \{\underset{\sim}{\omega}^*\}$, there exists $x \in \ell_\infty$ such that $\underset{\sim}{\omega}^* \cdot x < \underset{\sim}{\omega} \cdot x$. This in its turn implies that $c^*_x = \underset{\sim}{\omega}^* \cdot x$. But and from the definition of Ω_c, $x \succeq^*_c c^*_x \mathbb{1}$; fixing any $T \in \mathbb{N}$ and from Proposition Proposition 4.6,

$$c^*_{(1/2)x+(1/2)(c^*_x \mathbb{1}_{[0,T]},x)} = c^*_x.$$

This implies that there exists $\underset{\sim}{\omega}'$ such that

$$c^*_x = \underset{\sim}{\omega}' \cdot \left(\frac{1}{2}x + \frac{1}{2}(c^*_x \mathbb{1}_{[0,T]}, x)\right) = \min_{\underset{\sim}{\omega} \in \Omega_c} \underset{\sim}{\omega} \cdot \left(\frac{1}{2}x + \frac{1}{2}(c^*_x \mathbb{1}_{[0,T]}, x)\right).$$

But $x^* \succeq^*_c (c^*_x \mathbb{1}, x) \succeq^* c^*_x \mathbb{1}$, for every $\underset{\sim}{\omega} \in \Omega_c$, $\underset{\sim}{\omega} \cdot x \geq c^*_x$ and $\underset{\sim}{\omega} \cdot (c^*_x \mathbb{1}, x^*)) \geq c^*_x$. This implies that:

$$\underset{\sim}{\omega}' \cdot x = c^*_x,$$

$$\underset{\sim}{\omega}' \cdot \left(c^* \mathbb{1}_{[0,T]}, x\right) = c^*_x.$$

$\underset{\sim}{\omega}^*$ being an exposed point of Ω_c, the first equality implies that $\underset{\sim}{\omega}' = \underset{\sim}{\omega}^*$. Then observe that $\underset{\sim}{\omega}^* \cdot \left(c_x^* \mathbb{1}_{[0,T]}, x\right) = c_x^*$ is equivalent to $\underset{\sim}{\omega}^{*,T} \cdot x = c_x^*$. But, and for every T, $\underset{\sim}{\omega}^{*,T}$ belongs to Ω_c. Indeed, suppose the contrary: Ω_c being weakly compact, there exists $\epsilon > 0$ such that the intersection between Ω and the open set $\left\{\underset{\sim}{\omega} \text{ such that } \left\|\underset{\sim}{\omega} - \underset{\sim}{\omega}^{*,T}\right\|_{\ell_1} < \epsilon\right\}$ is empty. By the Hahn-Banach theorem, there exists x' and a constant c such that $\underset{\sim}{\omega} \cdot x' > c > \underset{\sim}{\omega}^{*,T} \cdot x'$ for every $\underset{\sim}{\omega} \in \Omega_c$. This implies that $x' \succeq_c^* c\mathbb{1}$ and therefore that $x' \succeq_c^* \left(c\mathbb{1}_{[0,T]}, x'\right) \succeq_c^* c\mathbb{1}$, whence $\underset{\sim}{\omega}^* \cdot \left(c\mathbb{1}_{[0,T]}, x'\right) \geq c$, which is equivalent to $\underset{\sim}{\omega}^{*,T} \cdot x' \geq c$, a contradiction.

The vector $\underset{\sim}{\omega}^{*,T}$ belongs to Ω, and satisfies $\underset{\sim}{\omega}^{*,T} \cdot x = c_x^*$. From the definition of $\underset{\sim}{\omega}^*$ and x, $\underset{\sim}{\omega}^* = \underset{\sim}{\omega}^{*,T}$ for every $T \in \mathbb{N}$: by Lemma 4.7, there therefore exists $0 < \delta^* < 1$ such that for every s, $\omega_s^* = (1-\delta^*)(\delta^*)^s$. To sum up, every exposed point of Ω assumes a geometrical representation. The set Ω_c being weakly compact, by Theorem 4 in Amis & Lindenstrauss [1], Ω_c is the convex hull of its exposed points. In its turn, this implies the existence of a subset $\mathscr{D} \in]0,1[$ such that

$$\Omega_c = \text{convex}\left\{(1-\delta, (1-\delta)\delta, \cdots, (1-\delta)\delta^s, \cdots)\right\}_{\delta \in \mathscr{D}},$$

that establishes the statement. QED

C.12 Proof of Lemma 4.8

(i) Fix T, $x, y \in \ell_\infty$, $c \in \mathbb{R}$. From the recursive form of the time-dependent index function I_T, $\left(c, x_{[T+1,\infty[}\right) \succeq_T^* \left(c, y_{[T+1,\infty[}\right)$ if and only if, for any $z \in \ell_\infty$,

$$\left(c + z_T, x_{[T+1,\infty[} + z_{[T+1,\infty[}\right) \succeq_T \left(c + z_T, y_{[T+1,\infty[} + z_{[T+1,\infty[}\right),$$

which is equivalent to

$$x_{[T+1,\infty[} + z_{[T+1,\infty[} \succeq_{T+1} y_{[T+1,\infty[} + z_{[T+1,\infty[}.$$

Whence the satisfaction of $\left(c, x_{[T+1,\infty[}\right) \succeq_T^* \left(c, y_{[T+1,\infty[}\right)$ if and only if $x_{[T+1,\infty[} \succeq_{T+1}^* y_{[T+1,\infty[}$.

(ii)-(iii). From (i), $x_{[T,\infty[} \succeq_T^* y_{[T,\infty[}$ if and only if

$$\left(0\mathbb{1}_{[0,T-1]}, x_{[T,\infty[}\right) \succeq_c^* \left(0\mathbb{1}_{[0,T-1]}, y_{[T,\infty[}\right).$$

This is equivalent, for every $\underset{\sim}{\omega} \in \Omega_c$, to:

$$\underset{\sim}{\omega} \cdot \left(0\mathbb{1}, x_{[T,\infty[}\right) \geq \underset{\sim}{\omega} \cdot \left(0\mathbb{1}, y_{[T,\infty[}\right).$$

Whence the satisfaction of $x_{[T,\infty[} \succeq_T^* y_{[T,\infty[}$ if any only if, for any $\underset{\sim}{\omega} \in \Omega_c$,

$$\underset{\sim}{\omega}^T \cdot x_{[T,\infty[} \geq \underset{\sim}{\omega}^T \cdot y_{[T,\infty[}.$$

The set Ω_T therefore exists and $\Omega_T = \left\{\underset{\sim}{\omega}^T\right\}_{\underset{\sim}{\omega}\in\Omega_c}$. QED

C.13 Proof of Proposition 4.9

This is a direct consequence of Lemma 4.8.

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