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A Decomposition for the Future: Closeness vs Distantness*

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Abstract

This article builds an axiomatization of inter-temporal trade-offs that makes an explicit account of the distant future and thus encompasses motives related to sustainability, transmission to offsprings and altruism. The focus is on separable representations and the approach is completed following a decision-theory index based approach that is applied to utility streams. The introduction of some new axioms is shown to lead to the emergence of two distinct orders that respectively relate to the *distant future* and *close future* components of some utility stream. This enlightens the limits of the commonly used fat tail intensity requesites for the evaluation of utility streams these are supersed and replaced by an axiomatic approach to optimal myopia degrees.

Keywords: Axiomatization, Myopia, Temporal Order Decompositions, Distant future sensitivities.

JEL Classification: D11, D90.

Contents

1. Introduction

Even though the long-run concerns for sustainability, conservation and the well-being of the future generations of offsprings nowadays go far beyond the boundaries of the academic circles and promptly come into the fore into most public agendas, it is not the least surprising that there seems to have been limited efforts towards a penetrating understanding of the actual meaning of *having an unbounded horizon* or *accounting for the infinite.*

The first endeavor towards an axiomatic approach to the topic was brought by Brown & Lewis [5] and explicitly anchored on *myopia*: it has nonetheless received the sparse echo that was due to what was perceived as a mere mathematical curiosity, *i.e.,* the identification of the *weight of the distant future.* This nevertheless raises a number of questions that may not have hitherto received sufficient attention. Is, together with most of the social welfare literature, an arbitrarily large finite future a satisfactory proxy for an unbounded horizon? Does the very fact of having some remote low orders tail for a stream of utils mean that it is negligible in not exerting any influence for finite dates? More precisely, are there some specificities attached to arbitrarily remote infinite horizon streams and is it reasonable to compare these through the same apparatus that is used for the finite parts of these streams? Otherwise stated, does order theory keep on being the appropriate apparatus for such elements and, assuming this is the case, how is it to be adaptated to simultaneously accomodate finite and infinite elements?

The purpose of this article is to provide an integrated picture of myopia and the valuation of inter-temporal utility streams as pictured by the properties of some index functions. The criteria used in the literature for comparing inter-temporal utility streams commonly rest upon intuitive properties such as *completeness*, *monotonicity*, *continuity*, *positive homogeneity* and *constant additivity* (the last three properties have different names in different works). While an order satisfying these properties can be represented by an *index function*, it is worthwhile emphasizing that these properties can be preserved through usual operators, *e.g.*, *summation*, *maximum*, *minimum*, or by any convex combination of these. As a matter of illustration, one can consider these following examples of index functions satisfying these fundamental properties.

$$I_1(x) = (1-\delta)\sum_{t=0}^{\infty} \delta^t x_t \text{ for some } 0 < \delta < 1,$$

$$I_2(x) = \liminf_{t\to\infty} x_t.$$

The first index function represents an order which is *very myopic*, *i.e*, the value of each stream is essentially defined by a finite number of dates or generations. In opposition to this, the second index/order belongs to an other extreme *non-myopic orders kind*, orders. The evaluation of the streams would not change if only the values of a *finite* number of dates were modified.

As this was mentioned before, a convex combination of these two index functions can also be considered:

$$I(x) = (1-\chi)I_1(x) + \chi I_2(x) \text{ for some } 0 < \chi < 1.$$

The evaluation is now decomposed into a convex combination of two components, a first relating to the *evaluation of the close future* through the index I_1 and a second pertaining to the *evaluation of the distant future* through the index I_2. Within this expression, the parameter χ can be understood as the *degree of myopia* that measures the weight of the *distant future.*

This article will aim at pursuing such a line of research by avoiding often hidden myopic negligible tail insensitivity requesites and by building an axiomatization of inter-temporal trade-offs that makes an explicit account of the distant future and ensuingly encompasses motives related to sustainability, transmission to offsprings and altruism.

From a general perspective, an elementary way of assessing the effect of the distant future proceeds by considering constant gains or losses in the asymptotic behaviour of inter-temporal utility streams. Given an order representing by some index function I, the weight of the distant future could, *e.g.*, measured through two simple parameters, *i.e.*, $\chi_1 = \lim_{T\to\infty} I(0,0,\cdots,0,1,1,1,\ldots)$, and $\chi_2 = -\lim_{T\to\infty} I(0,0,\cdots,0,-1,-1,-1,\ldots)$, both with T nil components, that respectively depict *remote constant gains* and *remote constant losses.* Building on such coefficients, two ranges of questioning naturally arise. *First, is a configuration where both of the distant future coefficients sum up to zero ($\chi_1 = \chi_2 = 0$) associated with some* tail-insensitivity *property where the distant future becomes negligible? Second, is there some scope for systematically decomposing the evaluation of inter-temporal streams between its distant future value and its close future value and, assuming this is the case, which form could such a function uncover?*

Surprisingly enough, the answer to the first question is negative and this can be checked, *e.g.*, by carefully considering the following index function I as:

$$I(x) = \min\left\{\hat{\omega}\cdot x, \max\left\{\omega\cdot x, \liminf_{s\to\infty} x_s\right\}\right\}, \text{ for } x \in \ell_\infty.$$

for ω and $\hat{\omega}$ two probability measures belonging to ℓ_1 and satisfying $\omega \neq \hat{\omega}$. This can be understood as a social welfare function for an economy with two agents evaluating intertemporal utility streams. While the first agent is very myopic and only considers the close future evaluation, the second one is partially myopic evaluates any utility stream using the maximum between its value in the close future and its distant future value. The criterion of the social planer eventually maximizes the welfare of the least favored agent, along the classical maximin criterion of Rawls [14] or some more recent argument in Chambers & Echenique [8]. For this example, letting the two coefficients χ_1 and χ_2 be reduced to zero keeps on preserving some scope for altering the evaluation of a sequence by the sole modification of its distant future components, whence some *tail sensitivity*. Otherwise stated, and as negligible as the value of the distant future may *sound*, it keeps on exerting some influence on the evaluation of the utility streams.[1]

In order to reach a thorough understanding of such a potentiality but also to introduce the scope for a systematic decomposition of the future and then answer to the second question, supplementary structures have to be superimposed on the preferences order relation. Two new axioms are presented. The first *distant future sensitivities* one states that given an utility streams and a constant stream, the decision maker can always says about her preference between the distant futures of these two streams. The second *close future sensitivities* axiom is similar but relates to a comparison that takes place between some close futures.

The introduction of this structure is shown to lead to the emergence of two distinct orders that respectively relate to the *distant future* and *close future* components of some utility stream. Both the *distant future* and the *close future* orders satisfy some fundamentals properties and respectively assume representations through some *distant future and close future index functions.* Moreover, and it is the main result of this article, the evaluation of an utility streams can be decomposed into a convex combination of its distant future and its close future components, the parameters of this convex combination changing as a function of the utility streams and lying between χ_1 and χ_2. Interestingly, these two values hence play a decisive role in the characterization of the eventual *myopia degrees* and they are chosen as a function of the utility streams is to be understood as represents two different sorts of behaviours about considering the distant future that directly relate to optimism and pessimism.

The way this study relates to the earlier literature can easily be understood with the above

[1]The associated calculus are detailed in the main text of this article.

discussion. Indeed, the notion of *strong myopia*, due to Brown & Lewis [5] coincides with the *upward myopia* notion of Saywer [15] and means, in its version presented by Becker & Boyd [4], that, for any $x \succ y$, one has for any z, $x \succ (y_0, y_1, \ldots, y_T, z_{T+1}, z_{T+2}, \ldots)$ for sufficiently large values of T. In the context of this article, these cases are equivalent to the *downward myopia* of by Saywer [15] where $x \succ y$ implies that, for any z, $(x_0, x_1, \ldots, x_T, z_{T+1}, z_{T+2}, \ldots) \succ y$ for sufficient large values of T. This corresponds to an extreme occurrence where $\chi_1 = \chi_2 = 0$. Another extreme, the *completely patient and time invariant* preferences in Marinacci [13], the Banach limits[2] correspond to the case $\chi_1 = \chi_2 = 1$. In parallel to this, Araujo [1] proves that, in order for the set of non trivial Pareto allocations to exist, consumers must exhibit some impatience in their preferences. Otherwise stated, this excludes the possibility of preferences being represented by Banach limits, or this is equivalent to at least of one of the two values χ_1, χ_2 to differ from 1. Following a very different approach and contemplating a social planner problem, Chichilnisky [7] associated charges to the *non-dictatorship of present* part of the social welfare criterion where the present would have no *per se* effect. Finally, and formerly related with the current study with an analysis completed over the set of bounded real sequences ℓ_∞, Chambers & Echenique [8] have recently put forth an axiomatic approach to multiple discounts. The current approach is complementary to theirs in focusing on myopia dimensions that *precede* discounting concerns and emerge as soon as are relaxed the *tail insensitivity* of the utility sequences.

2. Some basic Axioms and a Role for the Distant Future in the Evaluation of the Utility Streams

2.1 Fundamentals, Basic Axioms & the Construction of an Index Function

This study contemplates an axiomatization approach to the evaluation of infinite utility streams, the whole argument being cast for discrete time sequences. In order to avoid any confusion, letters like x, y, z will be used for sequences (of utils) with values in $\mathbb{R}$ while a notation $c\mathbb{1}$, $c'\mathbb{1}$, $c''\mathbb{1}$ will be used for constant sequences, the notation $\mathbb{1}$ being retained for the constant unitary sequence $(1, 1, \ldots,)$. In parallel to this, greek letters λ, η, μ will be preferred for constant scalars.

Recall first that the space of ℓ_∞ is defined as the set of real sequences $\{x_s\}_{s=0}^{\infty}$ such that

[2]For a careful definition of Banach limits, see page 55 in Becker & Boyd [4].

$\sup_{s\geq 0}|x_s| < +\infty$. For every $x \in \ell_\infty$ and $T \geq 0$, let $x_{[0,T]} = (x_0, x_1, \dots, x_T)$ denote its $T+1$ first components, $x_{[T+1,\infty[} = (x_{T+1}, x_{T+2}, \dots)$ its *tail* starting from date $T+1$ and, finally, $(x_{[0,T]}, y_{[T+1,+\infty[}) = (x_0, x_1, \dots, x_T, y_{T+1}, y_{T+2}, \dots)$ that considers the $T+1$ first elements of the sequence x and the $T+1$-tail of the sequence y. The following axiom introduces some fundamental properties for the order $\succeq$ on ℓ_∞.

Axiom F. The order $\succeq$ satisfies the following properties:

(i) *Completeness* For every $x, y \in \ell_\infty$, either $x \succeq y$ or $y \succeq x$.

(ii) *Transitivity* For every $x, y, z \in \ell_\infty$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$. Denote as $x \sim y$ the case where $x \succeq y$ and $y \succeq x$. Denote as $x \succ y$ the case where $x \succeq y$ and $y \not\succeq x$.

(iii) *Monotonicity* If $x, y \in \ell_\infty$ and $x_s \geq y_s$ for every $s \in \mathbb{N}$, then $x \succeq y$.

(iv) *Non-triviality* There exist $x, y \in \ell_\infty$ such that $x \succ y$.

(v) *Archimedeanity* For $x \in \ell_\infty$ and $b\mathbb{1} \succ x \succ b'\mathbb{1}$, there are $\lambda, \mu \in [0,1]$ such that

$$(1-\lambda)b\mathbb{1} + \lambda b'\mathbb{1} \succ x \text{ and } x \succ (1-\mu)b\mathbb{1} + \mu b'\mathbb{1}.$$

(vi) *Weak convexity* For every $x, y, b\mathbb{1} \in \ell_\infty$, and $\lambda \in]0,1]$,

$$x \succeq y \Leftrightarrow (1-\lambda)x + \lambda b\mathbb{1} \succeq (1-\lambda)y + \lambda b\mathbb{1}.$$

All of the properties (i), (ii), (iii) and (iv) are standardly used in decision theory. The *Archimedeanity* property (v) ensures that the order is continuous in the sup-norm topology of ℓ_∞. The eventual *Weak convexity* property (vi) is admittedly less immediate. It is referred to as *certainty independence* in the decision theory literature and ensures that direction $\mathbb{1}$ is *comparison neutral*: following that direction, the comparison does not change between two sequences.

Under these conditions, the order $\succeq$ can be represented by an index function which is homogeneous of degree 1 and constantly additive:[3]

(i) For $x \in \ell_\infty$, $\lambda > 0$, $I(\lambda x) = \lambda I(x)$.

(ii) For $x \in \ell_\infty$, constant $b \in \mathbb{R}$, $I(x + b\mathbb{1}) = I(x) + b$.

[3]For the details of demonstration, see [9].

Even though this directly compares with the conclusions reached in Gilboa & Schmeidler [11], and Ghirardato & al [10], it is worthwhile emphasizing that this article considers the total space ℓ_∞ as opposed to the space of *simple acts*—these are equivalent to sequences in ℓ_∞ which take a finite number of values—that was used by these authors.

2.2 Non-negligible distant future and non-negligible close future

In the literature, the notions of *impatience*[4] or *delay aversion*[5] are generally understood through the convergence of $\mathbb{1}_{\{T\}}$ to zero and as T tends to infinity. It is however to be stressed that such a property does not *per se* imply the convergence to zero of the effect of the associated tail, *i.e.*, some *constant distant future sequence* $\mathbb{1}_{[T,+\infty[}$. More generally, it is commonly assumed in the literature that the value of the distant future converges to zero when T converges to infinity. In the current framework and under Proposition 2.1, this is to mean that $I\left(0\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,\infty[}\right)$ and $I\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right)$ are to converge to zero when T tends to infinity[6]. To check upon such this property in the current environment, it is first useful to introduce the two following coefficients:[7]

$$\begin{aligned}
\chi_1 &= \lim_{T\to\infty} I\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right),\\
\chi_2 &= -\lim_{T\to\infty} I\left(0\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,\infty[}\right)\\
&= 1 - \lim_{T\to\infty} I\left(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right).
\end{aligned}$$

These two values χ_1 and χ_2, which will be considered extensively in the course of this study, will further play an important role in the definition of the *myopia degrees.* Under the above definition, the condition $\chi_1 = \chi_2 = 0$ is similar to the usual *negligible-tail* or *tail-insensitivity* conditions of the literature. A natural conjecture hence formulates as the satisfaction, under this condition and for any $x, z \in \ell_\infty$, of:

$$\lim_{T\to\infty} I\left(x_{[0,T]}, z_{[T+1,\infty[}\right) = I(x),$$

i.e, for sufficiently large values of T, the tail of the sequence z would become irrelevant and the whole evaluation of the utility stream would proceed from the sequence x. The following counter example will however provide an illustration where, in spite of a valuation of the

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[4] See Koopmans [12].

[5] See Bastianello & Chateauneuf [2].

[6] Observe that these two properties are not equivalent.

[7] From the *monotonicity* property, $I\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right)$ and $1 - I\left(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right)$ are decreasing as a function of T, so these limits are well defined.

distant future that could be nil,[8] this future remote component of the sequence could keep on exerting some influence on the evaluation of the whole sequence. This suggests a structure for the index function that could well be more complex than the one previously claimed on an intuitive intuitive basis.

Example 2.1. *Consider two probability measures belonging to the set ℓ_1, namely ω and $\hat{\omega}$, and satisfying $\omega \neq \hat{\omega}$.[9] Define the index function* I *as:*

$$\mathrm{I}(x) = \min\left\{\hat{\omega} \cdot x, \max\left\{\omega \cdot x, \liminf_{s\to\infty} x_s\right\}\right\}, \text{ for } x \in \ell_\infty.$$

This representation can be understood as a social welfare function for an economy with two agents. While the first agent would be highly myopic and only consider the close future of the utility stream $\hat{\omega} \cdot x$, the second one would be rely on a weaker form of myopia by considering the maximum between the close future value $\omega \cdot x$ and the infimum limit of the distant future value of the stream. The criterion of the social planer would eventually maximizes the welfare of the least favored agent along the classical maximin criteria of Rawls [14] or its more recent acception due to Chambers & Echenique [8].

It is readily checked that I *satisfies the fundamental axiom* ***F****. Further observe that, for large enough values of* T, *both* $\omega \cdot \left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right)$ *and* $\hat{\omega} \cdot \left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right)$ *are bounded above by* 1, *which implies that the value of the index defined for asymptotically constant unitary gains, namely* χ_1*, sums up to:*

$$\mathrm{I}\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) = \min\left\{\sum_{s=T+1}^{\infty} \hat{\omega}_s, 1\right\},$$

an expression that converges to zero. Likewise, the value of the index defined for asymptotically constant unitary losses satisfies:

$$\lim_{T\to\infty} \mathrm{I}\left(0\mathbb{1}_{[0,T]}, -\mathbb{1}_{[T+1,\infty[}\right) = 0.$$

Remark that there however exist $x, z \in \ell_\infty$ *such that* $\lim_{T\to\infty} \mathrm{I}\left(x_{[0,T]}, z_{[T+1,\infty[}\right) \neq \mathrm{I}(x)$*. Indeed, the two sequences* $\hat{\omega}$ *and* ω *having been assumed to be distinct and both belonging to* ℓ_1*, there exists* $x \in \ell_\infty$ *such that* $\hat{\omega} \cdot x > \omega \cdot x > \liminf_{s\to\infty} x_s$*. Considering now* z *satisfying* $\hat{\omega} \cdot x > \liminf_{s\to\infty} z_s > \omega \cdot x > \liminf_{s\to\infty} x_s$*. It is finally obtained that:*

$$\lim_{T\to\infty} \mathrm{I}\left(x_{[0,T]}, z_{[T+1,\infty[}\right) = \liminf_{s\to\infty} z_s,$$

[8] One can prove that $\lim_{T\to\infty} \mathrm{I}\left(0\mathbb{1}_{[0,T]}, z_{[T+1,+\infty[}\right) = 0$ for any $z \in \ell_\infty$.

[9] ℓ_1 is the set of real sequences $\{\omega_s\}_{s=0}^{\infty}$ such that $\sum_{s=0}^{\infty} |\omega_s| < +\infty$. For $\omega \in \ell_1$ and $x \in \ell_\infty$, the scalar product $\omega \cdot x = \sum_{s=0}^{\infty} \omega_s x_s$. The word "probability" in the statement means that $\omega_s, \hat{\omega}_s$ are non-negative for any s and $\sum_{s=0}^{\infty} \omega_s = \sum_{s=0}^{\infty} \hat{\omega}_s = 1$.

i.e., *the infimum of the asymptotic behaviour of the sequence* z, *that differs from* $\mathrm{I}(x) = \omega \cdot x$.

Along Example 2.1 where the sole occurrence to two nil values for the myopia parameters, *i.e.*, $\chi_1 = \chi_2 = 0$ was not sufficient to ensure the negligibility of the distant future, the following example shows that, under a configuration $\chi_1 = \chi_2 = 1$ where the two myopia parameters assume unitary values, the close future can keep on influencing on the evaluation of the distant future.

Example 2.2. *Consider an order being represented by the following index function*

$$\hat{\mathrm{I}}(x) = \min\left\{\limsup_{s\to\infty} x_s, \max\left\{\omega \cdot x, \liminf_{s\to\infty} x_s\right\}\right\},$$

with ω *a probability measure in* ℓ_1. *Along the interpretation of Example , while the first agent in this economy is extremely non-myopic and evaluates utility streams by the sole consideration of the supremum of its asymptotic values, the second one is only partially myopic. Relying on the same arguments as in Example 2.2, it derives that:*

$$\lim_{\mathrm{T}\to\infty} \hat{\mathrm{I}}(0\mathbb{1}_{[0,\mathrm{T}]}, \mathbb{1}_{[\mathrm{T}+1,\infty[}) = 1,$$

$$-\lim_{\mathrm{T}\to\infty} \hat{\mathrm{I}}(0\mathbb{1}_{[0,\mathrm{T}]}, -\mathbb{1}_{[\mathrm{T}+1,\infty[}) = -\left(\lim_{\mathrm{T}\to\infty} \hat{\mathrm{I}}(\mathbb{1}_{[0,\mathrm{T}]}, 0\mathbb{1}_{[\mathrm{T}+1,\infty[}) - \hat{\mathrm{I}}(\mathbb{1})\right) = 1$$

whence the obtention of unitary values for the two myopia parameters $\chi_1 = \chi_2 = 1$.[10] *Considering again* x, z *satisfying* $\liminf_{s\to\infty} x_s < \omega \cdot x < \omega \cdot z < \limsup_{s\to\infty} x_s$. *It can be checked that*

$$\lim_{\mathrm{T}\to\infty} \hat{\mathrm{I}}(z_{[0,\mathrm{T}]}, x_{[\mathrm{T}+1,\infty[}) = \omega \cdot z,$$

which differs from $\hat{\mathrm{I}}(x) = \omega \cdot x$.

The consideration of Examples 2.1 and 2.2 urges the need for a deeper understanding of the problem at stake, *i.e.*, the precise influence of the remote components of a utility stream. As this shall be argued in the next section, a clear picture becomes available when the preferences order is apprehended through appropriate complementary structures.

3. A Decomposition for the Future: Closeness vs Distantness

3.1 Distant Future Order

The following axiom assumes that there exists an *evaluation of the distant future components of the utility stream* which is independent from the starting components—the *close future*—of

[10] This implies for any $x \in \ell_\infty$, $\lim_{\mathrm{T}\to\infty} \hat{\mathrm{I}}(x_{[0,\mathrm{T}]}, 0\mathbb{1}_{[\mathrm{T}+1,\infty[}) = 0$.

that utility stream.

Axiom G1. For any $x \in \ell_\infty$ and any constant $d \in \mathbb{R}$, either, for any $\epsilon > 0$, there exists $T_0(\epsilon)$ such that for any $z \in \ell_\infty$, for every $T \geq T_0(\epsilon)$:

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \succeq \left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1},$$

or, for any $\epsilon > 0$, there exists $T_0(\epsilon)$ such that for any $z \in \ell_\infty$, for every $T \geq T_0(\epsilon)$:

$$\left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) \succeq \left(z_{[0,T]}, x_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

For any sequence x and a constant sequence $d\mathbb{1}$, the distant future component of the sequence x will either overtake the sequence $(d-\epsilon)\mathbb{1}$ or be overtaken by the sequence $(d+\epsilon)\mathbb{1}$, and this is going to take place independently from the initial components—the *close future*—of the sequence z. Otherwise stated, either x or $d\mathbb{1}$ dominates in the distant future. This *distant future sensitivities* axiom contradicts with the usual *negligible-tail* or *tail-insensitivity* axioms in the literature. Along these considerations, the simplest conceivable order satisfying both **F** and **G1** relates to the infimum limit of a sequence in ℓ_∞ and is represented by $I(x) = \liminf_{s\to\infty} x_s$. It is associated with the occurrence of unitary values for both of the myopia parameters, *i.e.*, $\chi_1 = \chi_2 = 1$.

Definition 3.1. *Define the order* $\succeq_d$ *as, for any* $x, y \in \ell_\infty$, *the satisfaction of* $x \succeq_d y$ *if and only if, for any* $\epsilon > 0$, *there exists* $T_0(\epsilon)$ *such that, for any* $z \in \ell_\infty$ *and for every* $T \geq T_0(\epsilon)$*:*

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \succeq \left(z_{[0,T]}, y_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

The order in Example 2.2 also satisfies axioms **F** and **G1**.

Example 3.1. *Consider again the order represented by the index function* I *in Example 2.1.*

$$I(x) = \min\left\{\hat{\omega} \cdot x, \max\left\{\omega \cdot x, \liminf_{s\to\infty} x_s\right\}\right\}, \text{ for } x \in \ell_\infty.$$

For any $x \in \ell_\infty$ *and some scalar* $d \in \mathbb{R}$, *consider first the configuration* $\liminf_{s\to\infty} x_s \geq d$. *Fixing any* $\epsilon > 0$, *select* $T_0(\epsilon)$ *such that, for any* $T \geq T_0(\epsilon)$*:*

$$\sum_{s=T+1}^{\infty} \omega_s x_s \geq d \sum_{s=T+1}^{\infty} \omega_s - \epsilon,$$
$$\sum_{s=T+1}^{\infty} \hat{\omega}_s x_s \geq d \sum_{s=T+1}^{\infty} \hat{\omega}_s - \epsilon.$$

This translates, for any $z \in \ell_\infty$ *and any* $\mathrm{T} \geq \mathrm{T}_0(\epsilon)$, *as the satisfaction of:*

$$\begin{aligned}
\mathrm{I}\Big(z_{[0,\mathrm{T}]}, x_{[\mathrm{T}+1,\infty[}\Big) &\geq \min\Big\{\hat{\omega} \cdot (z_{[0,\mathrm{T}]}, d\mathbb{1}_{[\mathrm{T}+1,\infty[}) - \epsilon, \\
&\qquad \max\Big\{\omega \cdot \Big(z_{[0,\mathrm{T}]}, d\mathbb{1}_{[\mathrm{T}+1,\infty[}\Big) - \epsilon, \liminf_{s\to\infty} x_s - \epsilon\Big\}\Big\} \\
&\geq \mathrm{I}\Big(z_{[0,\mathrm{T}]}, d\mathbb{1}_{[\mathrm{T}+1,\infty[}\Big) - \epsilon.
\end{aligned}$$

Whence, and again for any $\mathrm{T} \geq \mathrm{T}_0(\epsilon)$, *the behaviour described by Axiom **G1**:*

$$\Big(z_{[0,\mathrm{T}]}, x_{[\mathrm{T}+1,\infty[}\Big) \succeq (z_{[0,\mathrm{T}]}, d\mathbb{1}_{[\mathrm{T}+1,\infty[}) - \epsilon\mathbb{1},$$

a similar line of argument being available for the remaining configuration $\liminf_{s\to\infty} x_s \leq d$. *Moreover, and even though* $\chi_1 = \chi_2 = 0$, *the order* $\succeq_d$ *is not trivial. Select indeed* z^* *satisfying* $\hat{\omega} \cdot z^* > 0 > \omega \cdot z^*$. *It derives that:*

$$\begin{aligned}
&\lim_{\mathrm{T}\to\infty} \mathrm{I}\Big(z^*_{[0,\mathrm{T}]}, 0\mathbb{1}_{[\mathrm{T}+1,\infty[}\Big) = \min\Big\{\hat{\omega} \cdot z^*, \max\{\omega \cdot z^*, 0\}\Big\} = 0, \\
&\lim_{\mathrm{T}\to\infty} \mathrm{I}\Big(z^*_{[0,\mathrm{T}]}, \mathbb{1}_{[\mathrm{T}+1,\infty[}\Big) = \min\Big\{\hat{\omega} \cdot z^*, \max\{\omega \cdot z^*, 1\}\Big\} = \min\Big\{\hat{\omega} \cdot z^*, 1\Big\} > 0.
\end{aligned}$$

Whence $\mathbb{1} \succeq_d 0\mathbb{1}$ *and* $0\mathbb{1} \not\succeq_d \mathbb{1}$, *or* $\mathbb{1} \succ_d 0\mathbb{1}$*: he order* $\succeq$ *is not trivial.*

Proposition 3.1 proves that it suffices for one of the two myopia parameters χ_1 and χ_2 to differ from zero for the order $\succeq_d$ to satisfy axiom **F**. This in its turn assumes as its most immediate consequence that there also exists an index function satisfying any axiom **F**.

Proposition 3.1. *Assume that the initial order* $\succeq$ *satisfies axioms* **F** *and* **G1**.

(i) *The order* $\succeq_d$ *is complete.*

(ii) *If at least one of the two values* χ_1, χ_2 *differs from zero, the order* $\succeq_d$ *is non-trivial, satisfies axiom* **F** *and can be represented by an index function* I_d, *which is positively homogeneous, constantly additive, satisfying:*

$$\mathrm{I}_d\Big(z_{[0,\mathrm{T}]}, x_{[\mathrm{T}+1,\infty[}\Big) = \mathrm{I}_d(x) \textit{ for any } x, z \in \ell_\infty, \mathrm{T} \in \mathbb{N}.$$

Otherwise stated and from (ii), the value of the index function does not depend upon the starting components of the sequence z—the *close future.* More generally and, upon a change in a mere finite number of values of the inter-temporal stream, the distant future evaluation of that stream is let unmodified.

3.2 Close future order

In order to enable a decomposition between the *distant future* and the *close future*, consider a *close future sensitivities* axiom **G2**, that is to be understood as *the complement* of axiom **G1**.

Axiom G 2. For any $x \in \ell_\infty$, a constant $c \in \mathbb{R}$, either, for any $\epsilon > 0$, there exists $T_0(\epsilon)$ such that, for any $z \in \ell_\infty$ and for every $T \geq T_0(\epsilon)$,

$$\left(x_{[0,T]}, z_{[T+1,\infty[}\right) \succeq \left(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}\right) - \epsilon\mathbb{1},$$

or there exists $T_0(\epsilon)$ such that, for any $z \in \ell_\infty$ and for every $T \geq T_0(\epsilon)$,

$$\left(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}\right) \succeq \left(x_{[0,T]}, z_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

This assumption reads as follows: for any sequence x and a constant sequence $d\mathbb{1}$, either the sequence x will overtake the sequence $(c-\epsilon)\mathbb{1}$ or it will be dominated by the sequence $(c+\epsilon)\mathbb{1}$, both of these occurrences being defined whatever the behaviour in the distant future. Otherwise stated, either x or $d\mathbb{1}$ dominates in the close future.

Usual conditions in the literature typically assume that the effect of the distant future converges to zero—*e.g.*, the *Continuity at infinity* of Chambers & Echenique [8], or the axioms ensuring insensitivity to the distant future, or some sort of *negligible tail* for the distribution. Remark that, in opposition to this, the *close future sensitivities* Axiom **G2** merely assumes that the distant future does not alter the evaluation of the close future.

As a basic illustration, consider the order represented by the index function $I(x) = (1-\delta)\sum_{s=0}^{\infty}\delta^s x_s$, for some $0 < \delta < 1$. Such an order satisfies both **F** and **G2**, its myopia parameters being both nil $\chi_1 = \chi_2 = 0$. A more elaborated formulation is provided in the following example:

Example 3.2. *Consider the order represented by the index function $\hat{I}$ in Example 2.2:*

$$\hat{I}(x) = \min\left\{\limsup_{s\to\infty} x_s, \max\left\{\omega\cdot x, \liminf_{s\to\infty} x_s\right\}\right\}, \text{ for } x \in \ell_\infty.$$

Fixing any $x \in \ell_\infty$ and some constant $c \in \mathbb{R}$, consider first the configuration $\omega\cdot x \geq c$. For any given $\epsilon > 0$, select a date $T_0(\epsilon)$ such that, for any $T \geq T_0(\epsilon)$, one has:

$$\sum_{s=0}^{T}\omega_s x_s \geq c\sum_{s=0}^{T}\omega_s - \epsilon.$$

For any sequence $z \in \ell_\infty$ and any date $T \geq T_0(\epsilon)$, the value of the index $\hat{I}$ satisfies:

$$\begin{aligned}\hat{I}\left(x_{[0,T]}, z_{[T+1,\infty[}\right) &\geq \min\left\{\limsup_{s\to\infty} z_s, \max\left\{\omega\cdot(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}) - \epsilon, \liminf_{s\to\infty} z_s\right\}\right\}\\ &\geq \hat{I}\left(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}\right) - \epsilon.\end{aligned}$$

Whence, and for any $T \geq T_0(\epsilon)$, *the behaviour described by Axiom* ***G2***:

$$\left(x_{[0,T]}, z_{[T+1,\infty[}\right) \succeq \left(c\mathbb{1}_{[0,T]}, z_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

The configuration $\omega \cdot x \leq c$ *could be understood following the same line of arguments. Moreover, and even though* $\chi_1 = \chi_2 = 1$, *the close order* $\succeq_c$ *is not trivial. Indeed, select* $z^* \in \ell_\infty$ *such that* $\liminf_{s\to\infty} z^*_s < 0 < \limsup_{s\to\infty} z^*_s$.

$$\lim_{T\to\infty} \hat{I}\left(0\mathbb{1}_{[0,T]}, z^*_{[T+1,\infty[}\right) = \min\left\{\limsup_{s\to\infty} z^*_s, \max\left\{0, \liminf_{s\to\infty} z^*_s\right\}\right\} = 0,$$
$$\lim_{T\to\infty} \hat{I}\left(\mathbb{1}_{[0,T]}, z^*_{[T+1,\infty[}\right) = \min\left\{\limsup_{s\to\infty} z^*_s, \max\left\{1, \liminf_{s\to\infty} z^*_s\right\}\right\} = \min\left\{\limsup_{s\to\infty} z^*_s, 1\right\} > 0.$$

Whence $\mathbb{1} \succeq_c 0\mathbb{1}$ *and* $0\mathbb{1} \not\succeq_c \mathbb{1}$, *or* $\mathbb{1} \succ_c 0\mathbb{1}$ *and the order* $\succeq_c$ *is not trivial.*

Definition 3.2. *Define the* close future *order* $\succeq_c$ *as, for any* $x, y \in \ell_\infty$, *the satisfaction of* $x \succeq_c y$ *if and only if for any* $\epsilon > 0$, *there exists* $T_0(\epsilon)$ *such that, for any sequence* $z \in \ell_\infty$ *and for every date* $T \geq T_0(\epsilon)$,

$$\left(x_{[0,T]}, z_{[T+1,\infty[}\right) \succeq \left(y_{[0,T]}, z_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

Proposition 3.2. *Assume that the initial order* $\succeq$ *satisfies axioms* **F** *and* **G2**.

(i) *The close order* $\succeq_c$ *is complete.*

(ii) *If at least one of the two values* χ_1, χ_2 *differs from* 1, *then the order* $\succeq_c$ *is non-trivial, satisfies axiom* **F** *and be represented by an index function* I_c *which is positively homogeneous, constantly additive and satisfies:*

$$\lim_{T\to\infty} I_c\left(x_{[0,T]}, z_{[T+1,\infty[}\right) = I_c(x) \text{ for any } x, z \in \ell_\infty.$$

The property (ii) illustrates the close future order recovers a *tail-insensitivity* related property, the corresponding distant future order of $\succeq_c$ being indeed trivial.

The results in Propositions 3.1 and 3.2 as well as the characterizations in Examples 3.1 and 3.2, suggest the need for a more achieved characterization of the configurations where the two myopia parameters assume boundary values, *i.e.*, $\chi_1 = \chi_2 = 0$, or $\chi_1 = \chi_2 = 1$. The following statement aims at providing a clarified view of the way these relate with the triviality of either the distant or the future order:

Proposition 3.3. *Assume that the order* $\succeq$ *satisfies axioms* ***F*** *and* ***G1***, ***G2***.

a) *If* $\chi_1 = \chi_2 = 0$, *then the order* $\succeq_d$ *is trivial: for any* $x, y \in \ell_\infty$, $x \sim_d y$.

b) *If* $\chi_1 = \chi_2 = 1$, *then the order* $\succeq_c$ *is trivial: for any* $x, y \in \ell_\infty$, $x \sim_c y$.

Otherwise stated, it is only in the case where the initial order satisfies *both* axioms **G1** and **G2**, *i.e.*, the one where the decomposition between the distant and the close components future is fully completed, that the boundary values for the myopia coefficients may result into the triviality of one of the two orders.

3.3 A decomposition between the distant and close future orders

From the previous developments and under axioms **F**, **G1** and **G2**, one may surmise that there is some potential for the index function I to be decomposed into a convex sum of two index functions I_d and I_c, *e.g.*,

$$\mathrm{I}(x) = (1-\chi^*)\mathrm{I}_c(x) + \chi^*\mathrm{I}_d(x),$$

for some value $\chi^* \in [0,1]$. First observe that, would the selected parameter χ^* not modify over time, such a decomposition ought to imply that:

$$\lim_{\mathrm{T}\to\infty} \mathrm{I}\left(\mathbb{1}_{[0,\mathrm{T}]}, 0\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) + \lim_{\mathrm{T}\to\infty} \mathrm{I}\left(0\mathbb{1}_{[0,\mathrm{T}]}, \mathbb{1}_{[\mathrm{T}+1,\infty[}\right) = 1,$$

which is equivalent to $\chi_2 = \chi_1$, and therefore $\chi^* = \chi_1 = \chi_2$. Remark however that, under axioms **F**, **G1** and **G2**, the satisfaction of such an equality cannot be guaranteed. This also indicates that, when $\chi_1 \neq \chi_2$, the decomposition parameter must change as a function of the involved sequence x.

The configuration

$$\lim_{\mathrm{T}\to\infty} \mathrm{I}\left(\mathbb{1}_{[0,\mathrm{T}]}, 0\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) + \lim_{\mathrm{T}\to\infty} \mathrm{I}\left(0\mathbb{1}_{[0,\mathrm{T}]}, \mathbb{1}_{[\mathrm{T}+1,\infty[}\right) \leq 1,$$

which is equivalent to the holding of $\chi_1 \leq \chi_2$, can first be understood as a *pessimistic*, or a mainly *myopia-bending* occurrence: the value brought by the distant future is not sufficiently large to compensate the loss that is incurred in the close future.

Likewise, the configuration

$$\lim_{\mathrm{T}\to\infty} \mathrm{I}\left(\mathbb{1}_{[0,\mathrm{T}]}, 0\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) + \lim_{\mathrm{T}\to\infty} \mathrm{I}\left(0\mathbb{1}_{[0,\mathrm{T}]}, \mathbb{1}_{[\mathrm{T}+1,\infty[}\right) \geq 1,$$

which is equivalent to the holding of $\chi_1 \geq \chi_2$, can be understood as an *optimistic*, or an essentially *nonmyopia-bending* situation: the gain in the distant future is valued more than the lost that is incurred in the close future.

The following theorem, which is the main result of this article, will prove that there exists a multiplicity of *admissible myopia degrees.* This theorem also clarifies how it is the very choice of the *myopia degree* χ that determines an optimal share between the close future and the distant future indexes.

THEOREM 3.1. *Assume that the initial order* $\succeq$ *satisfies axioms* **F** *and* **G1**, **G2**.

(i) *For any* $x \in \ell_\infty$,

a) *Let* $c_x = \mathrm{I}_c(x)$, $d_x = \mathrm{I}_d(x)$,

$$\mathrm{I}(x) = \lim_{\mathrm{T}\to\infty} \mathrm{I}(c_x \mathbb{1}_{[0,\mathrm{T}]}, d_x \mathbb{1}_{[\mathrm{T}+1,\infty[}).$$

b) *If* $c_x \leq d_x$, *then*

$$\mathrm{I}(x) = (1-\chi_1)c_x + \chi_1 d_x.$$

c) *If* $c_x \geq d_x$, *then*

$$\mathrm{I}(x) = (1-\chi_2)c_x + \chi_2 d_x.$$

(ii) *Let* $\underline{\chi} = \min\{\chi_1, \chi_2\}$, *and* $\overline{\chi} = \max\{\chi_1, \chi_2\}$.

a) *If* $\chi_1 \leq \chi_2$, *then and for every* $x \in \ell_\infty$,

$$\mathrm{I}(x) = \min_{\underline{\chi}\leq\chi\leq\overline{\chi}} \Big[(1-\chi)\mathrm{I}_c(x) + \chi \mathrm{I}_d(x)\Big].$$

b) *If* $\chi_1 \geq \chi_2$, *then and for every* $x \in \ell_\infty$,

$$\mathrm{I}(x) = \max_{\underline{\chi}\leq\chi\leq\overline{\chi}} \Big[(1-\chi)\mathrm{I}_c(x) + \chi \mathrm{I}_d(x)\Big].$$

First remark that, from Theorem 3.1(i), the evaluation can be expressed as a function of the distant and close future values. The weight of the convex combination being provided by the remote gains myopia coefficient χ_1 for the case where the close future is less valued than the distant future and by the remote losses myopia coefficient χ_2 in the opposite case. Theorem 3.1(ii) is a direct consequence of Theorem 3.1(i). For $\chi_1 \leq \chi_2$, the decision maker will always assign the highest possible parameter to the smallest value between $\mathrm{I}_c(x)$ and $\mathrm{I}_d(x)$; he indeed always selects the minimum value of a convex combination whose weight is given by χ. It should finally be pointed out that neither the operator min can prevail under $\chi_1 > \chi_2$, nor the operator max under $\chi_1 < \chi_2$, the behaviour of decision maker being thus appropriately described by the comparison between χ_1 and χ_2.

While a convex combination between Examples 3.1 and 3.2 could have been conjectured to provide an interesting illustration of this decomposition, it is readily checked that such a formulation is inappropriate would satisfy neither Axiom **G1**, nor Axiom **G2**. The following illustration will however provide an elementary of the properties at stake:

EXAMPLE 3.3. *As a basic illustration, consider the two orders represented by the two following index functions:*

(i) *The index function*

$$\mathrm{I}(x) = \min_{\underline{\chi} \le \chi \le \overline{\chi}} \left[(1-\chi) \sum_{s=0}^{\infty} (1-\delta)\delta^s x_s + \chi \liminf_{s \to \infty} x_s \right],$$

is such that $\mathrm{I}_c(x) = \sum_{s=0}^{\infty}(1-\delta)\delta^s x_s$ *and* $\mathrm{I}_d(x) = \liminf_{s\to\infty} x_s$ *with* $\chi_1 = \underline{\chi}$, *and* $\chi_2 = \overline{\chi}$. *Fixing indeed any sequence* $x \in \ell_\infty$ *and a constant* $c \in \mathbb{R}$, *or any scalar* $\epsilon > 0$. *Consider the case* $\sum_{s=0}^{\infty}(1-\delta)\delta^s x_s \ge c$ *and fix a date* $\mathrm{T}_0(\epsilon)$ *such that for any date* $\mathrm{T} \ge \mathrm{T}_0(\epsilon)$ *and any* $z \in \ell_\infty$, *one has* $\sum_{s=0}^{\mathrm{T}}(1-\delta)\delta^s x_s \ge c \sum_{s=0}^{\mathrm{T}} \omega_s - \epsilon$. *This in turn implies that, for any* $z \in \ell_\infty$ *and for any* $\underline{\chi} \le \chi \le \overline{\chi}$, *the following inequality is satisfied:*

$$\begin{aligned}(1-\chi)\left(\sum_{s=0}^{\mathrm{T}}(1-\delta)\delta^s x_s + \sum_{s=\mathrm{T}+1}^{\infty}(1-\delta)\delta^s z_s\right) + \chi \liminf_{s\to\infty} z_s \\ \ge (1-\chi)\left(c\sum_{s=0}^{\mathrm{T}} \omega_s + \sum_{s=\mathrm{T}+1}^{\infty} \omega_s z_s\right) + \chi \liminf_{s\to\infty} z_s - \epsilon.\end{aligned}$$

Such an inequality holding for any χ, *it derives that, for any* $\mathrm{T} \ge \mathrm{T}_0(\epsilon)$ *and* $z \in \ell_\infty$, *the index* I *satisfies:*

$$\mathrm{I}\Big(x_{[0,\mathrm{T}]}, z_{[\mathrm{T}+1,\infty[}\Big) \ge \mathrm{I}\Big(d\mathbb{1}_{[0,\mathrm{T}]}, z_{[\mathrm{T}+1,\infty[}\Big) - \epsilon.$$

The remaining occurrence $\sum_{s=0}^{\infty}(1-\delta)\delta^s x_s \le c$ *can be analysed with a related argument. The order thus satisfies the* close future sensitivities *axiom **G2**, its close future order being represented by the function* $\mathrm{I}_c(c) = \sum_{s=0}^{\infty}(1-\delta)\delta^s x_s$. *Relying on the same arguments, it is readily proved that this order also satisfies the* distant future sensitivities *axiom **G1**, its distant future index function being given by* $\mathrm{I}_d(x) = \liminf_{s\to\infty} x_s$. *The properties* $\chi_1 = \underline{\chi}$ *and* $\chi_2 = \overline{\chi}$ *finally result from the minimum form of the operator.*

(ii) *Likewise and following the same line of arguments, it is readily checked that the index function*

$$\hat{\mathrm{I}}(x) = \max_{\underline{\chi} \le \chi \le \overline{\chi}} \left[(1-\chi) \sum_{s=0}^{\infty} (1-\delta)\delta^s x_s + \chi \liminf_{s \to \infty} x_s \right]$$

assumes a decomposition with $\mathrm{I}_c(x) = \sum_{s=0}^{\infty}(1-\delta)\delta^s x_s$ *and* $\mathrm{I}_d(x) = \liminf_{s\to\infty} x_s$ *and for* $\chi_1 = \overline{\chi}$ *and* $\chi_2 = \underline{\chi}$.

Lastly, one may wonder about the absence of an α–maximin decomposition between the close future and the distant future indexes.

Corollary 3.1. *Assume that the initial order $\succeq$ satisfies axioms* **F**, **G1** *and* **G2**. *For any $\alpha \in [0,1]$, consider the index function of the α–maximin criterion*

$$\mathrm{I}_\alpha(x) = \alpha \min_{\underline{\chi}\leq\chi\leq\overline{\chi}}\Big[(1-\chi)\mathrm{I}_c(x)+\chi\mathrm{I}_d(x)\Big]+(1-\alpha)\max_{\underline{\chi}\leq\chi\leq\overline{\chi}}\Big[(1-\chi)\mathrm{I}_c(x)+\chi\mathrm{I}_d(x)\Big].$$

Let $\underline{\chi}_\alpha = \min\Big\{\alpha\underline{\chi}+(1-\alpha)\overline{\chi},(1-\alpha)\underline{\chi}+\alpha\overline{\chi}\Big\}$ *and* $\overline{\chi}_\alpha = \max\Big\{\alpha\underline{\chi}+(1-\alpha)\overline{\chi},(1-\alpha)\underline{\chi}+\alpha\overline{\chi}\Big\}$.

(i) *Let α satisfy $1/2\leq\alpha\leq 1$. For any $x\in\ell_\infty$,*

$$\mathrm{I}_\alpha(x) = \min_{\underline{\chi}_\alpha\leq\chi\leq\overline{\chi}_\alpha}\Big[(1-\chi)\mathrm{I}_c(x)+\chi\mathrm{I}_d(x)\Big].$$

(ii) *Let α satisfy $0\leq\alpha\leq 1/2$. For any $x\in\ell_\infty$,*

$$\mathrm{I}_\alpha(x) = \max_{\underline{\chi}_\alpha\leq\chi\leq\overline{\chi}_\alpha}\Big[(1-\chi)\mathrm{I}_c(x)+\chi\mathrm{I}_d(x)\Big].$$

Otherwise stated, Corollary 3.1 clarifies how the choice under an α–maximin behaviour can be fully described by an immediate reformulation of the max or min operators of Theorem 3.1.

A. Proof of Proposition 3.1

(i) For $x\in\ell_\infty$, define $\mathrm{D}(x)$ as the set of values d such that for any $\epsilon>0$, there exists $\mathrm{T}_0(\epsilon)$ such that, for any $z\in\ell_\infty$, for any $\mathrm{T}\geq\mathrm{T}_0(\epsilon)$, one has

$$\Big(z_{[0,\mathrm{T}]},x_{[\mathrm{T}+1,\infty[}\Big)\succeq\Big(z_{[0,\mathrm{T}]},d\mathbb{1}_{[\mathrm{T}+1,\infty[}\Big)-\epsilon\mathbb{1}.$$

Define $\mathrm{D}(y)$ accordingly. Without loss of generality, suppose that $\sup\mathrm{D}(x)\geq\sup\mathrm{D}(y)$ and first let $\sup\mathrm{D}(y)<+\infty$. Then define $d_y=\sup\mathrm{D}(y)$, that is finite. Fix any $\epsilon>0$: since $d_y+(\epsilon/2)\mathbb{1}$ does not belong to $\mathrm{D}(y)$ and $d-(\epsilon/2)\mathbb{1}$ belongs to $\mathrm{D}(y)$, there exists $\mathrm{T}_0(\epsilon)$ such that, for any $z\in\ell_\infty$, for $\mathrm{T}\geq\mathrm{T}_0(\epsilon)$:

$$\begin{aligned}\left(\left(z+\frac{\epsilon}{2}\mathbb{1}\right)_{[0,\mathrm{T}]},\left(d_y+\frac{\epsilon}{2}\mathbb{1}\right)_{[\mathrm{T}+1,\infty[}\right)+\frac{\epsilon}{2}\mathbb{1}&\succeq\Big(z_{[0,\mathrm{T}]},y_{[\mathrm{T}+1,\infty[}\Big)\\&\succeq\left(\left(z-\frac{\epsilon}{2}\mathbb{1}\right)_{[0,\mathrm{T}]},\left(d_y-\frac{\epsilon}{2}\mathbb{1}\right)_{[\mathrm{T}+1,\infty[}\right)-\frac{\epsilon}{2}\mathbb{1}.\end{aligned}$$

This implies, for $\mathrm{T}\geq\mathrm{T}_0(\epsilon)$, the satisfaction of:

$$\Big(z_{[0,\mathrm{T}]},d_y\mathbb{1}_{[\mathrm{T}+1,\infty[}\Big)+\epsilon\mathbb{1}\succeq\Big(z_{[0,\mathrm{T}]},y_{[\mathrm{T}+1,\infty[}\Big)\succeq\Big(z_{[0,\mathrm{T}]},d_y\mathbb{1}_{[\mathrm{T}+1,\infty[}\Big)-\epsilon\mathbb{1}.$$

Since $d_x \geq d_y$, for every $\epsilon > 0$ and $z \in \ell_\infty$, there exists $T_0(\epsilon)$ such that

$$\begin{aligned}\left(z_{[0,T]}, x_{[T+1,\infty[}\right) &\geq \left(z_{[0,T]}, d_y \mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1}\\ &\geq \left(z_{[0,T]}, y_{[T+1,\infty[}\right) - 2\epsilon\mathbb{1}.\end{aligned}$$

This implies that $x \succeq_d y$.

Consider now the case $\sup D(y) = +\infty$. This implies that $\sup D(x) = +\infty$. Take $d > \sup_s y_s$. Since $d \in D(x)$, for every $\epsilon > 0$, there exists $T_0(\epsilon)$ such that, for any $z \in \ell_\infty$, for $T \geq T_0(\epsilon)$:

$$\begin{aligned}\left(z_{[0,T]}, x_{[T+1,\infty[}\right) &\geq \left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1}\\ &\geq \left(z_{[0,T]}, y_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.\end{aligned}$$

(ii) First, one must prove the existence of $x, y \in \ell_\infty$ such that $x \succ_d y$. Chose by example $\mathbb{1}$ and $0\mathbb{1}$. Obviously, $\mathbb{1} \succeq_d 0\mathbb{1}$ is first satisfied. Suppose now that $0\mathbb{1} \succeq_d \mathbb{1}$. Consider first the case $\chi_1 > 0$. Then, and for $0 < \epsilon < \chi_1$, there exists $T_0(\epsilon)$ such that for $T \geq T_0(\epsilon)$,

$$I\left(0\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right) \geq I\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) - \epsilon.$$

Letting T tend to infinity, it follows that $0 \geq \chi_1 - \epsilon$, a contradiction. Consider then the case $\chi_2 > 0$. For $0 < \epsilon < \chi_2$, there exists $T_0(\epsilon)$ such that, for $T \geq T_0(\epsilon)$:

$$\left(\mathbb{1}_{[0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right) \geq I\left(\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) - \epsilon.$$

Letting T tend to infinity, it follows that $\epsilon \geq \chi_2$, a contradiction. The distant order $\succeq_d$ is hence not trivial.

Further observe that, if $x \succeq_d d\mathbb{1}$, then, for every $d' \in \mathbb{R}$, $x + d'\mathbb{1} \succeq_d (d + d')\mathbb{1}$. Indeed, for $\epsilon > 0$, there exists $T_0(\epsilon)$ such that, for any $z \in \ell_\infty$, for $T \geq T_0(\epsilon)$,

$$\left((z - d'\mathbb{1})_{[0,T]}, x_{[T+1,\infty[}\right) \geq \left((z - d'\mathbb{1})_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

From the *constantly additive* property, for $T \geq T_0(\epsilon)$,

$$\left(z_{[0,T]}, (x + d'\mathbb{1})_{[T+1,\infty[})\right) \geq \left(z_{[0,T]}, (d + d')\mathbb{1}_{[T+1,\infty[})\right) - \epsilon\mathbb{1}.$$

Hence $x + d'\mathbb{1} \succeq_d (d + d')\mathbb{1}$.

Then consider $x \in \ell_\infty$ and a constant d such that, $\epsilon > 0$, there exists $T_0(\epsilon)$ with, for any $z \in \ell_\infty$, for $T \geq T_0(\epsilon)$,

$$\left(z_{[0,T]}, x_{[T+1,\infty[}\right) \geq \left(z_{[0,T]}, d\mathbb{1}_{[T+1,\infty[})\right) - \epsilon\mathbb{1}.$$

Fix then any $\lambda > 0$. From axiom **G1**, there exists $T_0'(\epsilon)$ such that, for $T \geq T_0'(\epsilon)$,

$$\left(\left(\frac{1}{\lambda}z\right)_{[0,T]}, x_{[T+1,\infty[}\right) \geq \left(\left(\frac{1}{\lambda}z\right)_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \frac{1}{\lambda}\epsilon\mathbb{1},$$

that in its turn implies, for $T \geq T_0'(\epsilon)$,

$$\left(z_{[0,T]}, \left(\lambda x_{[T+1,\infty[}\right)\right) \geq \left(z_{[0,T]}, \lambda d\mathbb{1}_{T+1,\infty[}\right) - \epsilon\mathbb{1}.$$

Hence, for $x \succeq_d y$ and for every $\lambda > 0$, the occurrence of $\lambda x \succeq_d \lambda y$.

Consider now $x, y \in \ell_\infty$ such that $x \succeq_d y$. For every $0 < \lambda < 1$, one has $(1-\lambda)x + \lambda d\mathbb{1} \succeq_d (1-\lambda)y + \lambda d\mathbb{1}$.

The order $\succeq_d$ having been proved to be non trivial, the value $d_x = \sup D(x)$ is finite and, for every $d > d_x > d'$, the relation $d\mathbb{1} \succ_d x \succ_d d'\mathbb{1}$ is to hold. There thus obviously exists $\lambda, \mu \in [0,1]$ such that $(1-\lambda)d + \lambda d' > d_x > (1-\mu)d + \mu d'$ and the order $\succeq_d$ satisfies the *Archimedeanity* property.

Since $\succeq_d$ satisfies **F**, there exists an index function I_d which is homogeneous and constantly additive. The last property is a direct consequence of the definition of the order $\succeq_d$. QED

B. Proof of Proposition 3.2

(i) Using the same arguments as in the proof of Proposition 3.1, the order $\succeq_c$ is complete.

(ii) It can first be proved that $\mathbb{1} \succ_c 0\mathbb{1}$. Suppose the opposite and $0\mathbb{1} \succeq_c \mathbb{1}$ and consider the case $\chi_1 < 1$. For $0 < \epsilon < 1 - \chi_1$, there exists $T_0(\epsilon)$ such that, for $T \geq T_0(\epsilon)$,

$$I\left(0\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) \geq I\left(\mathbb{1}_{[0,T]}, \mathbb{1}_{[T+1,\infty[}\right) - \epsilon.$$

Letting T tend to infinity, one gets $\chi_1 \geq 1 - \epsilon$: a contradiction. For the remaining case $\chi_2 < 1$, make use of the same arguments. For the proof of the other properties in axiom **F**, follow the arguments developed for the proof of Proposition 3.1.

Consider any $x \in \ell_\infty$ and fix a constant d. For every $\epsilon > 0$ and for large enough values of T,

$$I_c\left(c_x\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) + \epsilon \geq I_c\left(x_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) \geq I_c\left(c_x\mathbb{1}_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) - \epsilon.$$

Letting T tend to infinity and ϵ converge to zero,

$$\lim_{T\to\infty} I_c\left(x_{[0,T]}, d\mathbb{1}_{[T+1,\infty[}\right) = I_c(x).$$

For every $x, y \in \ell_\infty$, fix then $d \geq \sup_s y_s \geq \inf_s y_s \geq d'$. Whence, for every T,

$$\mathrm{I}_c\big(x_{[0,\mathrm{T}]}, d\mathbb{1}_{[\mathrm{T}+1,\infty[}\big) \geq \mathrm{I}_c\big(x_{[0,\mathrm{T}]}, y_{[\mathrm{T}+1,\infty[}\big) \geq \mathrm{I}_c\big(x_{[0,\mathrm{T}]}, d'\mathbb{1}_{[\mathrm{T}+1,\infty[}\big).$$

Letting T tend to infinity, if eventually follows that $\lim_{\mathrm{T}\to\infty} \mathrm{I}_c\big(x_{[0,\mathrm{T}]}, y_{[\mathrm{T}+1,\infty[}\big) = \mathrm{I}_c(x)$, that completes the proof. QED

C. Proof of Theorem 3.1

(i) First suppose that $\chi_1 \leq \chi_2$, define $c_x = \mathrm{I}_c(x)$ and $d_x = \mathrm{I}_d(x)$ and fix $\epsilon > 0$. From the definition of c_x and d_x, for large enough values of T,

$$\begin{aligned} x &= \big(x_{[0,\mathrm{T}]}, x_{[\mathrm{T}+1,\infty[}\big) \\ &\geq \big(c_x\mathbb{1}_{[0,\mathrm{T}]}, x_{[\mathrm{T}+1,\infty[}\big) - \epsilon\mathbb{1} \\ &\geq \big(c_x\mathbb{1}_{[0,\mathrm{T}]}, d_x\mathbb{1}_{[\mathrm{T}+1,\infty[}\big) - 2\epsilon\mathbb{1}. \end{aligned}$$

Therefore

$$\mathrm{I}(x) \geq \limsup_{\mathrm{T}\to\infty} \mathrm{I}\big(c\mathbb{1}_{[0,\mathrm{T}]}, d\mathbb{1}_{[\mathrm{T}+1,\infty[}\big) - 2\epsilon\mathbb{1}.$$

This inequality being further true for any arbitrary $\epsilon > 0$,

$$\mathrm{I}(x) \geq \limsup_{\mathrm{T}\to\infty} \mathrm{I}\big(c_x\mathbb{1}_{[0,\mathrm{T}]}, d_x\mathbb{1}_{[\mathrm{T}+1,\infty[}\big).$$

Likewise,

$$\mathrm{I}(x) \leq \liminf_{\mathrm{T}\to\infty} \mathrm{I}\big(c_x\mathbb{1}_{[0,\mathrm{T}]}, d_x\mathbb{1}_{[\mathrm{T}+1,\infty[}\big).$$

Therefore

$$\mathrm{I}(x) = \lim_{\mathrm{T}\to\infty} \mathrm{I}\big(c_x\mathbb{1}_{[0,\mathrm{T}]}, d_x\mathbb{1}_{[\mathrm{T}+1,\infty[}\big).$$

First consider the configuration $c_x \leq d_x$ or, equivalently, $\mathrm{I}_c(x) \leq \mathrm{I}_d(x)$. As $d_x - c_x \geq 0$, it is obtained that:

$$\begin{aligned} \mathrm{I}(x) &= \lim_{\mathrm{T}\to\infty} \mathrm{I}\big(c_x\mathbb{1}_{[0,\mathrm{T}]}, d_x\mathbb{1}_{[\mathrm{T}+1,\infty[}\big) \\ &= c_x + \lim_{\mathrm{T}\to\infty} \mathrm{I}\big(0\mathbb{1}_{[0,\mathrm{T}]}, (d_x - c_x)\mathbb{1}_{[\mathrm{T}+1,\infty[}\big) \\ &= c_x + (d_x - c_x)\lim_{\mathrm{T}\to\infty} \mathrm{I}\big(0\mathbb{1}_{[0,\mathrm{T}]}, \mathbb{1}_{[\mathrm{T}+1,\infty[}\big) \\ &= (1-\chi_1)c_x + \chi_1 d_x \\ &= (1-\chi_1)\mathrm{I}_c(x) + \chi_1\mathrm{I}_d(x). \end{aligned}$$

For the case $I_c(x) \geq I_d(x)$, using similar arguments,

$$I(x) = (1-\chi_2)I_c(x) + \chi_2 I_d(x).$$

(iia) Consider first the configuration $\chi_1 \leq \chi_2$. This implies $\underline{\chi} = \chi_1$, $\overline{\chi} = \chi_2$.

For the case $I_c(x) \leq I_d(x)$, for any $\underline{\chi} \leq \chi \leq \overline{\chi}$,

$$I(x) = (1-\chi_1)I_c(x) + \chi_1 I_d(x) \leq (1-\chi)I_c(x) + \chi I_d(x).$$

As for the remaining case $I_c(x) \geq I_d(x)$, and making use of the same arguments

$$\begin{aligned} I(x) &= (1-\chi_2)I_c(x) + \chi_2 I_d(x) \\ &\leq (1-\chi)I_c(x) + \chi I_d(x), \end{aligned}$$

for any $\chi \in [\underline{\chi}, \overline{\chi}]$. Whence, finally

$$I(x) = \min_{\underline{\chi} \leq \chi \leq \overline{\chi}} \Big[(1-\chi)I_c(x) + \chi I_d(x)\Big].$$

(iib) For the other configuration $\chi_1 \geq \chi_2$ and making use of the same line of arguments, it is similarly obtained that:

$$I(x) = \max_{\underline{\chi} \leq \chi \leq \overline{\chi}} \Big[(1-\chi)I_c(x) + \chi I_d(x)\Big],$$

where $\underline{\chi} = \chi_2$, $\overline{\chi} = \chi_1$. QED

C.1 Proof of Corollary 3.1

First assume that $I_c(x) \leq I_d(x)$. The value of $I_\alpha(x)$ is therefore defined as:

$$\begin{aligned} I_\alpha(x) &= \alpha\Big[(1-\underline{\chi})I_c(x) + \underline{\chi} I_d(x)\Big] + (1-\alpha)\left[(1-\overline{\chi})I_c(x) + \overline{\chi} I_d(x)\right] \\ &= \Big[1 - (\alpha\underline{\chi} + (1-\alpha)\overline{\chi})\Big] I_c(x) + \Big[\alpha\underline{\chi} + (1-\alpha)\overline{\chi}\Big] I_d(x). \end{aligned}$$

Consider then the remaining configuration $I_c(x) \geq I_d(x)$. The value of $I_\alpha(x)$ is therefore defined as:

$$\begin{aligned} I_\alpha(x) &= \alpha\Big[(1-\overline{\chi})I_c(x) + \overline{\chi} I_d(x)\Big] + (1-\alpha)\Big[(1-\underline{\chi})I_c(x) + \underline{\chi} I_d(x)\Big] \\ &= \Big[1 - \Big((1-\alpha)\underline{\chi} + \alpha\overline{\chi}\Big)\Big] I_c(x) + \Big[(1-\alpha)\underline{\chi} + \alpha\overline{\chi}\Big] I_d(x). \end{aligned}$$

For $1/2 \leq \alpha \leq 1$, $\underline{\chi}_\alpha \leq \overline{\chi}_\alpha$. Hence, for $I_c(x) \leq I_d(x)$,

$$I_\alpha(x) = \left(1 - \underline{\chi}_\alpha\right) I_c(x) + \underline{\chi}_\alpha I_d(x),$$

while, for $\mathrm{I}_c(x) \geq \mathrm{I}_d(x)$,

$$\mathrm{I}_\alpha(x) = \left(1 - \overline{\chi}_\alpha\right)\mathrm{I}_c(x) + \overline{\chi}_\alpha \mathrm{I}_d(x).$$

These properties are equivalents to the holding, for any $x \in \ell_\infty$, of

$$\mathrm{I}_\alpha(x) = \min_{\underline{\chi}_\alpha \leq \chi \leq \overline{\chi}_\alpha} \left[(1-\chi)\mathrm{I}_c(x) + \chi \mathrm{I}_d(x)\right].$$

For $0 \leq \alpha \leq 1/2$ and making use of the same arguments, it is similarly obtained that:

$$\mathrm{I}_\alpha(x) = \max_{\underline{\chi}_\alpha \leq \chi \leq \overline{\chi}_\alpha} \left[(1-\chi)\mathrm{I}_c(x) + \chi \mathrm{I}_d(x)\right].$$

The statement follows. QED

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