Optimal growth and *Ramsey-Rawls* criteria

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**ABSTRACT**

This work studies an inter-temporal optimization problem using a combination criteria between the Ramsey criteria and the Rawls criteria. A detailed description of the behavior of the economy through time is provided.

1. Introduction

In the classical work "Theory of justice", Rawls [11] poses the following question: what would be the choice for the outcome of the society if one is cached behind the *veil of ignorance*? In the total lack of information about the condition under which he∗ will be born, the economic agent should choose the maximization of the least favoured person (or generation). For example, given a inter-temporal consumption streams, his evaluation criteria of inter-temporal utilities streams should be

\[ U(c_0, c_1, c_2, \ldots) = \inf_{t \geq 0} u(c_t), \]

where \( u(c_t) \) is the instantaneous utility of the \( t^{th} \) generation.

Naturally, numerous attempts, for example Arrow [2] or Calvo [3] have been done to study the evolution of the economy if this criteria is used to evaluate inter-temporal welfare. Arrow [2] assumes constant productivity. Calvo [3] studies the maximin problem with uncertain technology. The result is pessimistic, while the initial accumulation of capital is low, the economy remains in this low capital accumulation situation forever.

The first part of this article studies the same question but with the difference that we allow the possibility for a growth to infinity, by excluding not the case that the productivity of every level of capital accumulation is sufficiently high. We consider...
a generalisation the configuration of Arrow [2] by imposing only the concavity to the production function. The result is the same: for any initial capital accumulation, the best choice in order to maximize the least favoured generation is remains to the initial state forever. In order to maximize the equality, one must sacrifice the efficiency.

A question raises: what happens then if we combine the famous Ramsey criteria, which evaluates the inter-temporal utilities streams using a constant discount rate \( \beta \in (0,1) \), and the Rawls criteria? Precisely, we can consider the evaluation criteria is as follows:

\[
U(c_0, c_1, c_2, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t) + a \inf_{t \geq 0} u(c_t),
\]

with some positive parameter \( a \), representing the important of equality in the choice of the economic agent.

There are always sacrificed generations with Ramsey criteria. If the productivity is high, the utility of present generations will be lowered for a rapid accumulation of capital, and in the contrary case, the generations in distant future will suffer the same treatment. And if we combine Ramsey criteria and Rawls criteria, by considering not only the efficiency but also the equality?

This is not the unique motivation which urges us to study the Ramsey-Rawls combination problem.

The link between the results in decision theory and time discounting literature is strong. The reason for this tight link is clear: by normalizing the time discounting system in order to obtain a probability and consider the set of time as the set of states, the inter-temporal choice is equivalent to an act in the world of Savage [12]. For example while the theorem of Savage [12] posed an axiomatic base for mean expected utility, the works of Koopmans in [7] and [8] gives conditions for inter-temporal representation in the later.

In recent decades, there is a vast literature which expands the world of Savage, by extending the theory in order to encompass the behaviors which do not satisfy Savage’s famous sure-thing principle. The classical work of Gilboa & Schmeidler [5] formulates the notion ambiguity averse, representing the behavior of an economic agent with always maximizes the worst scenario among the set of different possible probabilities.

In an parallel line of thinking, the same consideration can also be done in time discounting domain. Let \( \Delta \) the set of time discounting systems possibles:

\[
\Delta = \left\{ \pi = (\pi_0, \pi_1, \ldots) \text{ such that } \pi_s > 0, \forall s \text{ and } \sum_{s=0}^{\infty} \pi_s = 1 \right\}.
\]

The inter-temporal evaluation of economic agent, while he has only a vague idea about the appropriate time discounting system to choose, only knowing that the appropriate time discount system must belong to \( D \), a subset of \( \Delta \), can be represented
as follows, in the same spirit of Gilboa & Schmeidler [5]:

\[
U(c_0, c_1, c_2, \ldots) = \inf_{\pi \in D} \left[ \sum_{t=0}^{\infty} \pi_t u(c_t) \right].
\]

Recently, Chambers & Echenique [4] established axiomatic bases for the maximin criteria inter-temporal evaluation, with different discount rates. The corresponding set \( \Delta \) in the set up of Chamber & Echenique [4] is a convex hull of a set of time discounting systems which are geometrical sequences.

Imagine a situation where the ambiguity is complete, for example our agent is cached behind the *veil of ignorance*. Without any possible information to predict the future, the set of all possible time discounting systems should be \( \Delta \) and the inter-temporal evaluation becomes

\[
U(c_0, c_1, c_2, \ldots) = \inf_{\pi \in \Delta} \left[ \sum_{t=0}^{\infty} \pi_t u(c_t) \right] = \inf_{t \geq 0} u(c_t).
\]

We can push further the question about the criteria with multiple possible time discounting systems. Suppose the configuration where the agent is not completely ignorance but he always has doubts (even little) about his choice of this time discounting system. Our agent has "opinion" that the good constant discount rate to choose is \( \beta \in (0, 1) \) and the corresponding discount rates system is \( \pi_t^* = (1 - \beta)\beta^t \), for all \( t \geq 0 \).

The word "opinion" is used in the same spirit as Kopylov [9], to define a state of mind that is less rigid than "belief". The economic agent thinks that \( \pi^* \) is a good choice, but there are reasons suggesting him that this conclusion could be hasty. He should take into account the possibility for all other time discounting systems. Precisely, he should consider the set \( D = (1 - \lambda)\pi^* + \lambda \Delta \), with some \( 0 \leq \lambda \leq 1 \).

This formulation is very similar to \( \lambda \)-contamination literature, with the axiomatic foundation established in Alon [1], or Kopylov [9]. The parameter \( \lambda \) represents the lack of confident in the choice \( \pi^* \) of the agent. If \( \lambda = 1 \), the ambiguity is total. In contrast to this, if \( \lambda = 1 \), he believes without doubt that \( \pi^* \) is the good one.

Under the \( \lambda \)-contamination criteria, the inter-temporal evaluation becomes

\[
U(c_0, c_1, c_2, \ldots) = \inf_{\pi \in D} \left[ \sum_{t=0}^{\infty} \pi_t u(c_t) \right] = (1 - \lambda) \sum_{t=0}^{\infty} (1 - \beta)\beta^t u(c_t) + \lambda \inf_{\pi \in \Delta} \left[ \sum_{t=0}^{\infty} \pi_t u(c_t) \right] = (1 - \lambda) \sum_{t=0}^{\infty} (1 - \beta)\beta^t u(c_t) + \lambda \inf_{t \geq 0} u(c_t).
\]

\[\dagger\] The term \( 1 - \beta \) is just a normalizing term, to ensures the the sum \( \sum_{t=0}^{\infty} \pi_t^* = 1 \).
Taking \( a = \frac{4}{(1-\lambda)(1-\beta)} \), this is equivalent to the criteria

\[
U(c_0, c_1, c_2, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t) + a \inf_{t \geq 0} u(c_t).
\]

One more time, we find the Ramsey-Rawls combination. The second part of this work is devoted to study this problem. If the productivity is high, the utility of the early dates (or generations) are lowered as much as possible, for the sake of a rapid accumulation of capital. It is worth to sacrifice even a litter bit the value of the equality criteria, in order to have a better accumulation level of capital.

Once the capital accumulation level is sufficiently high, the economy follows a Ramsey path which does not violate the equality constraints, and converges to the steady state, or infinite, for the case such steady state does not exist. Thanks to the constraints imposed by the equality criteria of Rawls, the difference of utility between early dates and the later dates in distant future is not too high.

In the case of low productivity, if the pondering weight of the equality criteria is sufficiently high, the economy converges to higher steady state than the one of Ramsey problem. The difference between the lowest dates (in distant future) and the highest dates (in present) is diminished. The optimal choice in long term behaves as at a steady state of some Ramsey problem with a value of discount rate higher than the models one.

Moreover, if the pondering weight of the equality criteria is low, the optimal sequence coincides with the solution of Ramsey problem. In order to the equality criteria takes effect, its important in the overall criteria must be sufficiently strong.

The article is organized as follows. The section 2 considers the optimization under Rawls criteria, with a general production function and utility function. The section 3 analyses the Ramsey-Rawls problem. Using results of section 3, the section 4 studies the problem with linear production function and logarithmic utility function. The proofs are given in Appendix.

2. Optimal Solution for Rawls Criterion

We consider the following optimization problem under the Rawls criteria:

\[
\max \left[ \inf_{t \geq 0} u(c_t) \right],
\]

under the constraint \( c_t + k_{t+1} \leq f(k_t) \) for all \( t \), with \( k_0 > 0 \) given.

Let \( \Pi(k_0) \) the set of feasible paths \( \{k_t\}_{t=0}^{\infty} \) such that \( 0 \leq k_{t+1} \leq f(k_t) \) for any \( t \). This set is compact in the production topology. For each feasible sequence \( k = (k_0, k_1, k_2, \ldots) \), define

\[
\nu(k) = \inf_{t \geq 0} u(f(k_t) - k_{t+1}).
\]
The upper semi-continuity of the Rawls criteria in respect to this topology requires only the continuity of the utility function and the production function.

**Lemma 2.1.** Assume that the utility function $u$ and the production function $f$ are continuous,

i) The function $\nu$ is upper semi-continuous for the production topology.

ii) There exists $k^* \in \Pi(k_0)$ such that

$$\nu(k^*) = \max_{k \in \Pi(k_0)} \nu(k).$$

For the description of the solution of Rawls problem, we add the concavity of production function $f$, and the existence of non-trivial feasible sequence.

**Assumption A1.** The utility function $u$ is strictly concave, increasing and satisfies Inada condition. The production function $f$ is concave, strictly increasing. We assume also that $f(0) = 0$ and $f'(0) > 1$.

Denote by $\overline{k}$ the solution to $f'(k) = 1$, which maximizes $f(k) - k$. In the case $f'(k) > 1$ for any $k \geq 0$, let $\overline{k} = +\infty$.

Under the continuity of utility function and the concavity of production function, we can prove that for $k_0$ smaller than $\overline{k}$, the optimal choice for Rawls problem is remaining in status quo.

**Proposition 2.1.** i) Consider the case $0 \leq k_0 \leq \overline{k}$. The problem has a unique solution $k^* = (k_0, k_0, \ldots)$ and

$$\max_{k \in \Pi(k_0)} \nu(k) = u \left( f(k_0) - k_0 \right).$$

ii) Consider the case $\overline{k}$ is finite and $k_0 \geq \overline{k}$. The problem has an infinite number of solutions and

$$\max_{k \in \Pi(k_0)} \nu(k) = u \left( f(\overline{k}) - \overline{k} \right).$$

From now on, for the sake of simplicity, let $\hat{\nu}(k_0)$ be the best value possible for the Rawls criteria with initial state $k_0$:

$$\hat{\nu}(k_0) = \max_{k \in \Pi(k_0)} \nu(k).$$

‡Otherwise every feasible sequence converges to zero, and the problem becomes trivial.
3. **Optimal solution when Ramsey meets Rawls**

3.1 **The Ramsey-Rawls problem**

We consider in this section the criterion which a convex combination of the well-known criteria Ramsey and Rawls:

$$U(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t) + a \inf_{t \geq 0} u(c_t),$$

where $a$ is a positive constant. We will use the term "Ramsey part" to denote the sum $\sum_{t=0}^{\infty} \beta^t u(c_t)$ and "Rawls part" to denote $\inf_{t \geq 0} u(c_t)$.

Consider the following optimization problem $(P)$:

$$V(k_0) = \sup \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) + a \inf_{t \geq 0} u(c_t) \right]$$

s.c $c_t + k_{t+1} \leq f(k_t)$ for any $t \geq 0, 
\quad k_0 \geq 0$ is given.

In the section 2, we know that the Rawls part, $\nu(k) = \inf_{t \geq 0} u (f(k) - k_{t+1})$ is upper semi-continuous in respect to the product topology. It is well-known in the literature that under suitable conditions, the Ramsey part $\sum_{t=0}^{\infty} \beta^t u(c_t)$ is also upper semi-continuous. In order to simplify the exposition, we assume directly this upper semi-continuity. Curious readers can refer to the work of Le Van & Morhaim [6] for the details of the conditions ensuring this property, with the most important one is the tail-insensitivity condition.

**Assumption A2.** Assume that for any feasible sequence $\{k_t\}_{t=0}^{\infty}$, the function $\sum_{t=0}^{\infty} \beta^t u(c_t)$ is determined and satisfies the upper semi-continuity in respect to the product topology.

Under this assumption, the Ramsey part is also upper semi-continuous, and hence the same property is satisfied for the function $U$. Combining with the compactness of $\Pi(k_0)$ in respect to product topology, the problem $(P)$ always has optimal solution and we can write:

$$V(k_0) = \max_{k \in \Pi(k_0)} \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) + a \inf_{t \geq 0} u(c_t) \right].$$

The strictly concavity of utility function $u$ ensures the uniqueness of the optimal solution.

3.2 **The Ramsey problem**

In this subsection, we just like to evoke some well-known results in the literature of Ramsey model. Under the Assumption A2, the Ramsey problem always has an
optimal solution. By the strict concavity of utility function \( u \), the optimal solution is unique. Denote by \( v \) the value function and \( \{\hat{k}_t\}_{t=0}^{\infty} \) the optimal solution of the Ramsey problem:

\[
v(k_0) = \max \sum_{t=0}^{\infty} \beta^t u(c_t)
\]

s.c. \( c_t + k_{t+1} \leq f(k_t) \) for any \( t \geq 0 \), \( k_0 \) is given.

By the uniqueness of optimal solution, and the well-known result that \( v \) is solution to an functional Bellman equation, there exists an optimal policy function \( \sigma \) which is strictly increasing such that \( \hat{k}_{t+1} = \sigma(\hat{k}_t) \), for any \( t \).

We would like to recall here an important feature of Ramsey problem. When the productivity is high (\( f'(k_0) > \frac{1}{\beta} \)), the economic agent prefers to sacrifice the welfare of the early dates (or early generations) for a rapid accumulation of capital. The economy saves. The consumption sequence is increasing in this case.

As opposition to this, when the productivity is low (\( f'(k_0) < \frac{1}{\beta} \)), the economy chooses to dissaving. The impatience imposed by the discount rate implies the welfare sacrifices of dates (or generations) in a distant future. In this configuration, the consumption sequence is decreasing.

Let \( k^i \) be a solution to

\[
f''(k) = \frac{1}{\beta}.
\]

If the solution is not unique, we can take any one in the set of solutions. If \( f'(x) > \frac{1}{\beta} \) for all \( x \geq 0 \), let \( k^i = \infty \), and if \( f'(x) \leq \frac{1}{\beta} \) for all \( x \geq 0 \), let \( k^s = 0 \).

**Lemma 3.1.**

i) If \( k_0 \leq k^s \), then the consumption sequence \( \{\hat{c}_t\}_{t=0}^{\infty} \) and capital accumulation \( \{\hat{k}_t\}_{t=0}^{\infty} \) sequence are increasing, converge respectively to \( c^s = f(k^s) - k^s \) and \( k^s \). As a consequence of this,

\[
v(\hat{k}) = u(\hat{c}_0).
\]

ii) If \( k_0 \geq k^i \), the consumption sequence \( \{\hat{c}_t\}_{t=0}^{\infty} \) and capital accumulation \( \{\hat{k}_t\}_{t=0}^{\infty} \) sequence are decreasing, converge respectively to \( c^i = f(k^i) - k^i \) and \( k^i \). As a consequence of this,

\[
v(\hat{k}) = u(f(k^s) - k^i).
\]

### 3.3 Ramsey-modified Problem

For \( \epsilon \geq 0 \), we first consider the following intermediary problem \((P^\epsilon)\):

\[
\max \sum_{t=0}^{\infty} \beta^t u(c_t)
\]
s.t. \( c_t + k_{t+1} \leq f(k_t), \forall t \geq 0 \),
\( u(c_t) \geq \hat{v}(k_0) - \epsilon, \forall t \geq 0 \),
\( k_0 \) is given.

The intuition for studying this problem runs as follows. We already know that the maximum value possible for the Rawls part is \( \hat{v}(k_0) \). Naturally, rises the following question: if we accept to a lower value of Rawls part up to \( \epsilon \), what is the best improvement we can obtain for the Ramsey part? And which is the optimal sacrifice level \( \epsilon \) acceptable?

In order to respond to these questions, we study the problem \( (P^\epsilon) \). The Proposition 3.1 states that the optimal solution of \( (P) \) is also the optimal solution of \( (P^\epsilon) \), for some optimal value \( \epsilon \).

**PROPOSITION 3.1.** For any \( k_0 \geq 0 \),

\[
V(k_0) = \max_{\epsilon \geq 0} [W(\epsilon) + a(\hat{v}(k_0) - \epsilon)].
\]

By the Proposition 3.1, in order to understand the behavior of the optimal solution of initial problem \( (P) \), we study the behavior of the optimal solution of problems \( (P^\epsilon) \), with \( \epsilon \geq 0 \).

For the sake of simplicity, from now on, we will use the term "equality constraints" to denote the constraints \( u(c_t) \geq \hat{v}(k_0) - \epsilon \). Let \( W(\epsilon) \) be the value of the problem \( (P^\epsilon) \) and \( \{c^\epsilon_t, k^\epsilon_{t+1}\}_{t=0}^\infty \) be its optimal solution. By the strict concavity of \( u \), this sequence is unique.

It is obvious that, if \( \epsilon \) is sufficiently big, the solution of Ramsey problem satisfied also the equality constraints, and solving problem \( (P^\epsilon) \) becomes trivial task. Let \( \tilde{\epsilon} \) the critical value for this property: if we accept to lower the Rawls part to \( \tilde{\epsilon} \), the solution of Ramsey problem satisfies also the constraint of Ramsey-modified problem, and becomes solution of the later one.

Define

\[
\tilde{\epsilon} = \begin{cases} 
  u(f(k_0) - k_0) - u(f(k_0) - \sigma(k_0)) & \text{if } 0 \leq k_0 \leq k^i, \\
  u(f(k_0) - k_0) - u(f(k^i) - k^i) & \text{if } k^i \leq k_0 \leq \bar{k} \\
  u(f(\bar{k}) - \bar{k}) - u(f(k^i) - k^i) & \text{if } k_0 \geq \bar{k}.
\end{cases}
\]

The proof for the Lemma 3.2 is easy, based on the fact that the solution of Ramsey problem satisfies the constraints of Ramsey-modified one for sufficiently high \( \epsilon \).

**LEMMA 3.2.** Assume that \( \epsilon \geq \tilde{\epsilon} \).

i) The optimal solution of problem \( (P^\epsilon) \) coincides with the solution of Ramsey problem.

ii) \( W(\epsilon) = W(\tilde{\epsilon}) = v(k_0) \).
If $\epsilon = 0$, by Proposition 2.1, the optimal solution is $(k_0, k_0, \ldots)$. We consider now the interesting case, where $0 < \epsilon \leq \bar{\epsilon}$.

If $0 \leq k_0 \leq k^i$, the equality constraints are bind in the early dates and the optimal solution behaves as a solution of Ramsey problem when the accumulation of capital reaches a sufficiently high level.

If $k_0 \geq k^i$, equality constraints are bind from some date $T$ sufficiently big and in the long run, every date (or generation) has the same utility level, which is equal exactly the lowest level acceptable.

**Proposition 3.2.**  
 i) Consider the case $0 < k_0 < k^i$. If $0 < \epsilon \leq \bar{\epsilon}$, there exists $T$ such that:

a) For $0 \leq t \leq T$, $u(c_t^\epsilon) = \hat{\nu}(k_0) - \epsilon$.

b) For $t \geq T + 1$, $u(c_t^\epsilon) > \hat{\nu}(k_0) - \epsilon$.

c) The sequence $\{k_t^\epsilon\}_{t=T+1}^{\infty}$ is the solution of Ramsey problem with initial state $k_{T+1}^\epsilon$.

ii) Consider the case $k^i > k_0$. If $0 < \epsilon \leq \bar{\epsilon}$, there exists $T$ such that

a) For $0 \leq t \leq T$, $u(c_t^\epsilon) > \hat{\nu}(k_0) - \epsilon$.

b) For $t \geq T + 1$, $u(c_t^\epsilon) = \hat{\nu}(k_0) - \epsilon$.

Consider one more time the case $k_0 \geq k^i$, define $\bar{k}$ the solution to

$$u \left( f(\bar{k}) - \bar{k} \right) = \hat{\nu}(k_0) - \epsilon.$$ 

It is easy to verify that the $k_t^\epsilon = \bar{k}$ for $T$ sufficiently high. Let $\bar{\beta}$ the discount rate satisfying

$$f'(\bar{k}) = \frac{1}{\bar{\beta}}.$$ 

By the Proposition 2.1 and the choice of $\bar{\epsilon}$, we have $k^i < \bar{k} < k^s$. Hence $\bar{\beta} > \beta$. In the long run, the optimal solution for the case $k_0 \geq k^i$ behaves as a solution of a Ramsey problem with discount rate $\bar{\beta}$, which is higher than $\beta$.

The Lemma 3.3 is a direct consequence of Proposition 3.2. The function $W$ is strictly concave in respect to $\epsilon$ belonging to $[0, \bar{\epsilon}]$. This concavity implies the existence of the right derivative of $W$ at 0 and the left derivative of $W$ at $\bar{\epsilon}$. In section 3.4, these two values will play the role of critical thresholds for the equality parameter $a$. The behavior of the optimal solution depends strongly in the relative position of $a$ and $W'(0)$, $W'('\bar{\epsilon})$. The details will be presented in the subsection 3.4.

**Lemma 3.3.**  
 i) For any $k_0$, the function $W$ is strictly concave on $[0, \bar{\epsilon}]$.

ii) If $0 \leq k_0 < k^i$, then $W'(0) = +\infty$ and $W'(\bar{\epsilon}) = 0$.

iii) If $k_0 > k^i$, then $W'(0) < +\infty$. 

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3.4 Optimal solution of Ramsey-Rawls problem

It is worth nothing that if the equality parameter $a$ is very low, there is no change in the behavior of the economy. The optimal solution follows the same behavior as a solution of Ramsey problem.

Denote by $\epsilon^*$ the optimal level in Proposition 3.1:

$$e^* = \arg\max_{\epsilon \geq 0} [W(\epsilon) + a(\hat{v}(k_0) - \epsilon)].$$

Let $\{k_t^*\}_{t=0}^\infty$ the corresponding optimal solution of the Ramsey-modified problem. It is easy to verify that $\{k_t^*\}_{t=0}^\infty$ is also the solution of Ramsey-Rawls problem.

In the case the productivity is high ($f'(k_0) > \frac{1}{\beta}$), the utility of the early dates (or generations) are lowered as much as possible, for the sake of a rapid accumulation of capital. It is worth to sacrifice even a litter bit the value of the equality criteria, in order to have a better accumulation level of capital.

Once the capital accumulation level is sufficiently high, the economy follows a Ramsey path which does not violate the equality constraints, and converges to the steady state $k^s$. Thanks to the constraints imposed by the equality criteria of Rawls, the difference of utility between early dates and the later dates in distant future is not too high. This difference depends negatively with the equality parameter $a$, which imposes a trade-off between equality and the speed of convergence to the steady state.

**Proposition 3.3.** Consider the case $0 \leq k_0 \leq k^s$. For any $a > 0$, we have $0 < e^* < \bar{\epsilon}$ and there exists $T$ such that

i) For $0 \leq t \leq T$, $u(c^*_t) = u(f(k_0) - k_0) - e^*$.

ii) For $t \geq T + 1$, $u(c^*_t) > u(f(k_0) - k_0) - e^*$.

iii) The sequence $\{k_t^*\}_{t=T+1}^\infty$ is the solution of Ramsey problem with initial state $k^*_{T+1}$.

In the case of low productivity ($f'(k_0) < \frac{1}{\beta}$), the equality criteria (if sufficiently high) causes the economy to converge to higher steady state than the one of Ramsey problem. The difference between the lowest dates (in distant future) and the highest dates (in present) is diminished. The optimal choice in long term behaves as at a steady state of some Ramsey problem with a value of discount rate $\bar{\beta}$ higher than $\beta$.

Moreover, there exists a threshold for equality parameter $a$. Beyond this threshold, the optimal sequence remains the same and every date (or generations) enjoys the same utility level.

If the equality parameter $a$ is too low, there is no change in the behavior of the economy, comparing with the Ramsey problem.

**Proposition 3.4.** Consider the case $k_0 \geq k^s$. 


i) For $W'(\bar{\epsilon}) < a < W'(0)$, we have $0 < \epsilon^* < \bar{\epsilon}$ and there exists $T$ such that:

a) For $0 \leq t \leq T$, $u(c^*_t) > \hat{\nu}(k_0) - \epsilon^*$.

b) For $t \geq T + 1$, $u(c^*_t) = \hat{\nu}(k_0) - \epsilon^*$.

ii) For $a \geq W'(0)$, $\epsilon^* = 0$ and for any $t$, $k^*_t = k_0$.

iii) For $0 \leq a \leq W'(\bar{\epsilon})$, we have $\epsilon^* = \bar{\epsilon}$ and the optimal solution of problem (P) coincides with the solution of the Ramsey problem with initial state $k_0$.

4. CONSTANT PRODUCTIVITY AND LOGARITHMIC
UTILITY FUNCTION

In this section, we make some calculus for the case the productivity is constant ($f(k) = Ak$ and the utility function is logarithmic $u(\epsilon) = \ln \epsilon$. The optimal policy function is

$$\sigma(k) = \beta Ak.$$ 

Assume that $A > 1$. Hence $\overline{k} = \infty$.

By induction, one has

$$\hat{k}_t = (\beta A)^t k_0,$$

$$\hat{c}_t = A(1 - \beta) (\beta A)^t k_0.$$ 

The value function is defined as

$$v(k_0) = \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$= \frac{\ln A + \ln(1 - \beta) + \ln k_0}{1 - \beta} + (\ln \beta + \ln A) \sum_{t=0}^{\infty} t \beta^t.$$ 

1. Consider the case $A > \frac{1}{\beta}$. For this case, $k^\ell = \infty$. Hence for any $k_0$ we have $0 < k_0 < k^\ell$. By Lemma 3.3, $W'(0) = \infty$ and $W'(\bar{\epsilon}) = 0$. For any $a$ there is an optimal sacrifice level $\epsilon^*$ satisfying $W'(\epsilon^*) = a$. There is $T$ such that for $0 \leq t \leq T$,

$$u \left( f(k^*_t) - k^*_t \right) = u \left( f(k_0) - k_0 \right) - \epsilon,$$

which is equivalent to

$$\ln (Ak^*_t - k^*_{t+1}) = \ln(A - 1) + \ln k_0 - \epsilon.$$ 

§See Stokey & Lucas, with Prescott [10].
For $0 \leq t \leq T$,
\[ k_{t+1}^* = A k_t^* - \frac{(A - 1)k_0}{e^\epsilon}. \]

The value $T$ is the smallest which satisfy
\[ u \left( f(k_{T+1}^*) - \sigma(k_{T+1}^*) \right) \geq u \left( f(k_0) - k_0 \right) - \epsilon, \]
which is equivalent to
\[ \ln \left( A k_{T+1}^* - \beta A k_{T+1}^* \right) \geq \ln (A k_0 - k_0) - \epsilon. \]
This is equivalent to
\[ \ln A + \ln(1 - \beta) + \ln k_{T+1}^* \geq \ln(1 - \beta) + \ln k_0 - \epsilon. \]

The value $T$ is the first which satisfy
\[ k_{T+1}^* \geq \frac{A - 1}{A(1 - \beta)} \times \frac{k_0}{e^\epsilon}. \]

The sequence $\{k_{t+1}^*\}_{t=0}^\infty$ is the solution of Ramsey problem with initial state $k_{T+1}^*$.

2. Consider the case $A < \frac{1}{\beta}$. In this case, $k_t = 0$ and every solution of Ramsey problem converges to zero. The critical value $\tilde{\epsilon}$ is then
\[ \tilde{\epsilon} = u \left( f(k_0) - k_0 \right) - u(0) = \infty. \]

We will then determine $W'(0)$. For $\epsilon$ close to zero, the critical time $T$ from which $u(c^*_1) = u \left( f(k_0) - k_0 \right) - \epsilon$ is $T = 1$.

The capital level $k_1^\epsilon$ is solution to
\[ u \left( f(k_1) - k_1 \right) = u \left( f(k_0) - k_0 \right) - \epsilon. \]
This implies
\[ \ln \left( A k_1^\epsilon - k_1^\epsilon \right) = \ln(A - 1) + \ln k_0 - \epsilon. \]
Hence
\[ k_1^\epsilon = \frac{k_0}{e^\epsilon}. \]
We have
\[ W(\epsilon) = u \left( f(k_0) - k_1^\epsilon \right) + \frac{\beta}{1 - \beta} \left( u \left( f(k_0) - k_0 \right) - \epsilon \right) \]
\[
= \ln \left( A k_0 - \frac{k_0}{e^\epsilon} \right) + \frac{\beta}{1 - \beta} (\ln (A k_0 - k_0) - \epsilon) \\
= \ln \left( A - \frac{1}{e^\epsilon} \right) + \frac{\beta}{1 - \beta} (\ln (A - 1) + \ln k_0 - \epsilon).
\]

Hence for \( \epsilon \) close to zero,
\[
W'(\epsilon) = \frac{e^{-\epsilon}}{A - e^{-\epsilon}} - \frac{\beta}{1 - \beta}.
\]

Let \( \epsilon \) converges to zero, we get
\[
W'(0) = \frac{1 - \beta A}{(A - 1)(1 - \beta)}.
\]

We then have the following Proposition. The equality parameter has effect if and only if it is sufficiently high. Otherwise, there is no difference between the behaviour following Ramsey-Rawls criteria and the one following Ramsey criteria.

**Proposition 4.1.**

i) For \( a \leq \frac{1 - \beta A}{(A - 1)(1 - \beta)} \), we have \( \epsilon^* \geq 0 \), and there exists \( T \) such that:

a) For \( 0 \leq t \leq T \), \( u(c_t^*) > \ln (A - 1) + \ln k_0 - \epsilon^* \).

b) For \( t \geq T + 1 \), \( u(c_t^*) = \ln (A - 1) + \ln k_0 - \epsilon^* \).

ii) For \( a \geq \frac{1 - \beta A}{(A - 1)(1 - \beta)} \), \( \epsilon^* = 0 \). The optimal path is constant: \( k_t^* = k_0 \) for any \( t \geq 0 \).

**A. Proof of Lemma 2.1**

(i) Consider the sequence of feasible paths \( k^n \) which converges to \( k \) in the product topology.

Fix any \( \epsilon > 0 \). By the definition of \( \nu(k) \), there exists \( T \) such that \( u(c_T) < \inf_{t \geq 0} u(c_t) + \epsilon \).

By the convergence of the sequence \( \{k^n\}_{n=0}^\infty \) in product topology, we get \( \lim_{n \to \infty} c^n_T = c_T \).

Hence for \( n \) sufficient large, it is true that \( u(c^n_T) < u(c_T) + \epsilon \). This implies
\[
\inf_{t \geq 0} u(c_t^n) \leq u(c_T^n) < u(c_T) + \epsilon
\]
\[
< \inf_{t \geq 0} u(c_t) + 2\epsilon.
\]

We get
\[
\limsup_{n \to \infty} \nu(k^n) < \nu(k) + 2\epsilon.
\]

Let \( \epsilon \) converges to zero, we get the upper semi-continuity of \( \nu \).

(ii) The part (i) is a consequence of the upper semi-continuity of \( \nu \) and the compactness of \( \Pi(k_0) \) in respect to product topology.
(i) Denote $\mathbf{k}^*$ as a solution to the problem. For any $t \geq 0$, 
\[ u \left( f(k_t^*) - f(k_{t+1}^*) \right) \geq \nu(\mathbf{k}^*) \]
\[ \geq \nu(k_0, k_0, \ldots) \]
\[ = u \left( f(k_0) - k_0 \right) . \]

We then have $f(k_0) - k_t^* \geq f(k_0) - k_0$, which is equivalent to $k_t^* \leq k_0$.

Suppose that $k_t^* \leq k_0$ for some $t$. Then
\[ k_0 - k_{t+1}^* \geq f(k_0) - f(k_t^*) \]
\[ \geq f'(k_0)(k_0 - k_t^*) \]
\[ \geq k_0 - k_t^* , \]

which implies $k_{t+1}^* \leq k_t^*$. By induction, $k_0 \geq k_t^*$ for all $t$. Furthermore, the sequence $(k_t^*)$ is decreasing and then converges to $\hat{k} \leq k_0$.

From the continuity of $f$, we have that $f(\hat{k}) - \hat{k} \geq f(k_0) - k_0$. But the function $f(x) - x$ is increasing in $[0, \overline{k}]$, thus, $f(\hat{k}) - \hat{k} \leq f(k_0) - k_0$, then $\hat{k} = k_0$, and $k_t^* = k_0$ for all $t$, because $k_0 \geq k_t^* \downarrow k_0$.

(ii) First, consider the sequence $\mathbf{k} = (k_0, \overline{x}, \overline{x}, \ldots)$ which is feasible. We get
\[
\max_{\mathbf{k} \in \Pi(k_0)} \nu(\mathbf{k}) \geq f(\overline{k}) - \overline{k}.
\]

Let $\mathbf{k}'$ be an optimal solution. Since for all $t \geq 0$, $f(k_t^*) - k_{t+1}^* \geq f(\overline{k}) - \overline{k}$,

\[
\overline{k} - k_{t+1}^* \geq f(\overline{k}) - f(k_t^*) \]
\[ \geq f'(\overline{k})(\overline{k} - k_t^*) \]
\[ = \overline{k} - k_t^* . \]

This implies $k_{t+1}^* \leq k_t^*$ for any $t$. The sequence $\mathbf{k}'$ is decreasing and converges to some $\hat{k}$. By the continuity of $f$, $f(\hat{k}) - \hat{k} \geq f(\overline{k}) - \overline{k}$. Since $\overline{k}$ maximizes $f(x) - x$, this implies $\hat{k} = \overline{k}$. Hence
\[
\hat{\nu}(k_0) = f(\overline{k}) - \overline{k}.
\]

Since $k_0 > \overline{x}$, by induction, we can construct a sequence $\mathbf{k}$ which satisfies: for all $t$, $\overline{k} < k_{t+1}^* < f(k_t) - f(\overline{x}) + \overline{k}$. With this sequence, we have $f(k_t) - k_{t+1}^* > f(\overline{k}) - \overline{k}$, and $k_{t+1}^* < k_t^*$, since $f(k_t) - k_t < f(\overline{k}) - \overline{k}$. So the sequence $(k_t)_{t=0}^{\infty}$ converges to $\overline{k}$ and $\hat{\nu}(k_0) = f(\overline{k}) - \overline{k}$. We have an infinity number of sequences satisfying this property. The problem has an infinite number of solutions.
C. Proof of Proposition 3.1

Recall that for $0 \leq k_0 \leq \overline{k}$, for any feasible sequence $\{k_t\}_{t=0}^\infty$,

$$\inf_{t \geq 0} u(f(k_t) - k_{t+1}) \leq \hat{v}(k_0).$$

Let $\{k_t^*\}_{t=0}^\infty$ be the optimal solution of problem (P). Define

$$\epsilon^* = \hat{v}(k_0) - \inf_{t \geq 0} u(c_t^*).$$

We have

$$V(k_0) = \sum_{t=0}^\infty \beta^t u(c_t^*) + a \inf_{t \geq 0} u(c_t^*)$$

$$= \sum_{t=0}^\infty \beta^t u(c_t^*) + a (\hat{v}(k_0) - \epsilon^*)$$

$$\leq W(\epsilon^*) + a (\hat{v}(k_0) - \epsilon^*).$$

Conversely, for any $\epsilon \geq 0$,

$$W(\epsilon) + a (\hat{v}(k_0) - \epsilon) = \sum_{t=0}^\infty \beta^t u(c_t^*) + a (\hat{v}(k_0) - \epsilon)$$

$$\leq \sum_{t=0}^\infty \beta^t u(c_t^*) + a \inf_{t \geq 0} u(c_t^*)$$

$$\leq V(k_0).$$

The proof is completed.

D. Proof of Proposition 3.2

Obviously $W$ is increasing. The concavity of $W$ comes from the concavity of utility function $u$ and production function $f$.

We consider first the case $0 \leq k_0 \leq \overline{k}$. For each $\epsilon > 0$, let $x^*(\epsilon)$ be the smallest $x \geq k_0$ such that

$$u(f(x) - x) \geq u(f(k_0) - k_0) - \epsilon.$$

If the strict inequality is satisfied for any $x \geq k_0$, let $x^*(\epsilon) = \infty$.

(i) We consider the case

$$f'(k_0) > \frac{1}{\beta}.$$ 

First, observe that $k^t > 0$ and $k_0 < k^t$. 

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Since the function $x - \sigma(x)$ is strictly increasing in $(k_0, k^i)$, either $x^*(\epsilon) = \infty$, either $x^*$ is finite and $0 < x^*(\epsilon) < k^i$. Indeed, it is obvious that $k^i = \infty$ implies $x^* = \infty$. Suppose that $k^i$ is finite. Then $k^i = \sigma(k^i)$ and hence $u(f(k^i)) < u(f(k_0) - k_0) - \epsilon$. This implies $x^*$ is finite and $k_0 < x^* < k^i$. 

We will prove the following claim: for any $t$, $k_t^\epsilon < k^i$. This is true if for any $t$, $k_t^\epsilon < x^*(\epsilon)$. Consider the case there exists $T$ satisfying $k_T^\epsilon < x^*(\epsilon) \leq k_{T+1}^\epsilon$.

We have

$$u(f(k_T^\epsilon) - k_{T+1}^\epsilon) \geq u(f(k_0) - k_0) - \epsilon$$

$$> u(f(k^i) - k^i).$$

Then $k_{T+1}^\epsilon < k^i$.

Let $\tilde{k}_t^\epsilon t = T + 1$ the solution of Ramsey problem with initial state $k_{t+1}^\epsilon$. Since $k_{t+1}^\epsilon < k^i$, $\tilde{k}_t < k^i$ for any $t \geq T + 1$ and

$$\inf_{t \geq T + 1} u(\tilde{k}_t) = u(f(k_{t+1}^\epsilon) - \sigma(k_{t+1}^\epsilon))$$

$$\geq u(f(x^*(\epsilon)) - \sigma(x^*(\epsilon)))$$

$$= u(f(k_0) - k_0) - \epsilon.$$

Hence the sequence $\{k_0, k_1^\epsilon, \ldots, k_T^\epsilon, k_T^\epsilon, k_{T+2}^\epsilon, \ldots\}$ is the optimal solution for the problem $(P^\epsilon)$, or $\tilde{k}_t = k_t^\epsilon$ for any $t \geq T + 1$. The prove that $k_t^\epsilon < k^i$ for any $t$ is completed.

Consider the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) - \sum_{t=0}^{\infty} \beta^t \lambda_t [c_t + k_{t+1} - f(k_t)]$$

$$- \sum_{t=0}^{\infty} \beta^t \mu_t [u(f(k_0) - k_0) - \epsilon - u(c_t)].$$

By the Inada condition of $u$, at optimal the consumption and capital level are strictly positive. The Lagrangian parameters for these constraints are hence zero. For any $t$:

$$(1 + \mu_t)u'(c_t^\epsilon) = \lambda_t,$$

$$\lambda_t = \beta \lambda_{t+1} f'(k_{t+1}^\epsilon).$$

This implies for any $t$:

$$(1 + \mu_t)u'(c_t^\epsilon) = \beta(1 + \mu_{t+1})u'(c_{t+1}^\epsilon) f'(k_{t+1}^\epsilon)$$

$$\geq \beta f'(k_{t+1}^\epsilon) u'(c_{t+1}^\epsilon).$$

Suppose that $u(c_T^\epsilon) > u(f(k_0) - k_0) - \epsilon$. The constraint does not bind and hence $\mu_T = 0$. 

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Since \( f'(k_{T+1}^\epsilon) \geq \frac{1}{\beta} \), then \( u'(c_{T+1}^\epsilon) \geq u'(c_T^\epsilon) \), and hence \( c_{T+1}^\epsilon \geq c_T^\epsilon \). The \((T+1)\)th constraint also does not bind: \( u(c_{T+1}^\epsilon) > u(f(k_0) - k_0) - \epsilon \).

By induction, for any \( t \geq T + 1 \), \( u(c_t^\epsilon) > u(f(k_0) - k_0) - \epsilon \) and \( \mu_t = 0 \). The sequence \( \{(c_t^\epsilon, k_{T+1}^\epsilon)\}_{t=T}^\infty \) is increasing and satisfies Euler equations. Hence \( \{k_t^\epsilon\}_{t=T}^\infty \) is the solution for Ramsey problem with initial state \( k_T^\epsilon \). We also have \( \lim_{t \to \infty} k_t^\epsilon = k^\epsilon \).

\( (ii) \) Consider the case
\[
\frac{1}{\beta} < f''(k_0). 
\]

Necessary condition for this is \( 0 \leq k^\epsilon < \infty \). Recall that we are working in the case \( k_0 \leq k \). We first prove that \( k_t^\epsilon > k^\epsilon \) for any \( t \geq 0 \). Assume that there exists \( T \) such that \( k_T^\epsilon \leq k^\epsilon \). We have
\[
u(f(k_T^\epsilon) - k_{T+1}^\epsilon) \geq \nu(k_0) - \epsilon
= u(f(k_0) - k_0) - \epsilon
> u(f(k_0) - k_0) - \nu(k_0)
= u(f(k^\epsilon) - k^\epsilon),
\]
which implies \( k_{T+1}^\epsilon < k_T^\epsilon < k^\epsilon \), since \( f(x) - x \) is strictly increasing in \((0, k^\epsilon)\). By induction, the sequence \( \{k_T^\epsilon\}_{T=0}^\infty \) is decreasing and converges to \( k < k^\epsilon \). Taking the limit, we get
\[
u(f(k^\epsilon) - k^\epsilon) > u(f(k) - \sigma(k))
\geq \nu(k_0) - \epsilon
\geq u(f(k_0) - k_0) - \epsilon
> u(f(k^\epsilon) - k^\epsilon),
\]
a contradiction.

Once the property that \( k_t^\epsilon > k^\epsilon \) for any \( t \geq 0 \) established, we re-utilise the Lagrangian:
\[
\mathcal{L} = \sum_{t=0}^\infty \beta^t u(c_t) - \sum_{t=0}^\infty \beta^t \lambda_t [c_t + k_{t+1} - f(k_t)]
- \sum_{t=0}^\infty \beta^t \mu_t [u(f(k_0) - k_0) - \epsilon - u(c_t)].
\]

For any \( t \):
\[
(1 + \mu_t)u'(c_t^\epsilon) = \lambda_t, \\
\lambda_t = \beta \lambda_{t+1} f'(k_{t+1}^\epsilon).
\]

This implies for any \( t \):
\[
u(c_t^\epsilon) \leq (1 + \mu_t)u'(c_t^\epsilon)
\]

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\[ \beta(1 + \mu_{t+1})u'(c_{t+1}^\varepsilon)f'(k_{t+1}^\varepsilon). \]

If \( u(c_T^\varepsilon) > u(f(k_0) - k_0) - \varepsilon \), then the constraint does not bind, and \( \mu_T = 0 \). Since \( f(k_T^\varepsilon) < \frac{1}{\beta} \), we get \( u'(c_{T-1}^\varepsilon) < u'(c_T^\varepsilon) \), which implies \( c_{T-1}^\varepsilon > c_T^\varepsilon \), with the direct consequence

\[ u(c_{T-1}^\varepsilon) > u(f(k_0) - k_0) - \varepsilon. \]

By induction, we get for any \( 0 \leq t \leq T \)

\[ u(c_t^\varepsilon) > u(f(k_0) - k_0) - \varepsilon. \]

If this property is ensured for any \( t \geq 0 \), the sequence \( \{k_t^\varepsilon\}_{t=0}^{\infty} \) satisfies Euler equations and transversality condition, hence it is the optimal solution for Ramsey problem and converges to \( k^\varepsilon \): a contradiction, since

\[ u(f(k^\varepsilon) - k^\varepsilon) < u(f(k_0) - k_0) - \varepsilon. \]

Hence there exists \( T \) such that for any \( t \geq T \)

\[ u(c_T^\varepsilon) = u(f(k_0) - k_0) - \varepsilon. \]

Obviously, for any \( t \geq 0 \), we have

\[ u(c_{T+t}^\varepsilon) = u(f(k_0) - k_0) - \varepsilon, \]

otherwise using the same arguments in the induction, we get \( u(c_T^\varepsilon) > u(f(k_0) - k_0) - \varepsilon \), a contradiction.

\section*{E. Proof of Lemma 3.3}

(i) We prove that \( W'(0) = +\infty \). Consider \( T(\varepsilon) \) in the proof of Proposition 3.2.

For any \( 0 \leq t \leq T(\varepsilon) \):

\[ \varepsilon = u(f(k_0) - k_0) - u(f(k_t^\varepsilon) - k_{t+1}^\varepsilon) \]

\[ \geq u'(f(k_0) - k_0)(f(k_0) - k_0 - f(k_t^\varepsilon) + k_{t+1}^\varepsilon) \]

\[ \geq u'(f(k_0) - \sigma(k_0))(f'(k_0)(k_0 - k_t^\varepsilon) + k_{t+1}^\varepsilon - k_0). \]

This implies

\[ k_{t+1}^\varepsilon - k_0 \leq \frac{\varepsilon}{u'(f(k_0) - k_0)} + f'(k_0)(k_t^\varepsilon - k_0). \]

By induction, we get for any \( t \geq 0 \),

\[ k_{t+1}^\varepsilon - k_0 \leq \frac{[f'(k_0)]^{t+1} - 1}{f'(k_0) - 1} \times \frac{\varepsilon}{u'(f(x^*) - x^*)}. \]
Hence

\[ x^*(\epsilon) - k_0 \leq k_{T(\epsilon) + 1} - k_0 \]
\[ \leq \frac{[f'(k_0)]^{T(\epsilon) + 1} - 1}{f'(k_0) - 1} \times \frac{\epsilon}{u'(f(k_0) - k_0)}. \]

\[ W(\epsilon) = \sum_{t=0}^{T(\epsilon)} \beta^t u(\epsilon_t^\epsilon) + \sum_{t=T(\epsilon) + 1}^{\infty} \beta^t u(\epsilon_t^\epsilon) \]
\[ = (u(f(k_0) - k_0) - \epsilon) \sum_{t=0}^{T(\epsilon)} \beta^t + \beta^{T(\epsilon) + 1} v(k_T^\epsilon) \]
\[ = (u(f(k_0) - k_0) - \epsilon) \sum_{t=0}^{T(\epsilon)} \beta^t + \beta^{T(\epsilon) + 1} v(k^\epsilon). \]

Hence

\[ W(\epsilon) - W(0) = -\epsilon \sum_{t=0}^{T(\epsilon)} \beta^t + \beta^{T(\epsilon) + 1} \left( v(k^\epsilon) - \frac{u(f(k_0) - k_0)}{1 - \beta} \right) \]
\[ = -\epsilon \frac{1 - \beta^{T(\epsilon) + 1}}{1 - \beta} + \beta^{T(\epsilon) + 1} \left( \frac{1}{1 - \beta} \right). \]

Now we prove that

\[ \lim_{\epsilon \to 0} \frac{\beta^{T(\epsilon)}}{\epsilon} = +\infty. \]

Indeed, recall that

\[ \frac{[f'(k_0)]^{T(\epsilon) + 1} - 1}{f'(k_0) - 1} \times \frac{\epsilon}{u'(f(k_0) - k_0)} \sim x^*(\epsilon) - k_0. \]

This implies

\[ (f'(k_0))^{T(\epsilon)} \epsilon \sim O(1). \]

Hence

\[ T(\epsilon) \ln(f'(k_0)) \sim -\ln(\epsilon), \]

which is equivalent to

\[ T(\epsilon) \sim -\frac{\ln(\epsilon)}{\ln(f'(k_0))}. \]

We have

\[ \beta^{T(\epsilon)} \sim \left( e^{\ln \beta} \right)^{-\frac{\ln(\epsilon)}{\ln(f'(k_0))}} \]
\begin{align*}
\sim \epsilon \frac{\ln \beta}{\ln (f'(k_0))} \\
\sim \epsilon^{\ln (f'(k_0))}.
\end{align*}

Since \( f'(k_0) > 1 \), we have

\[
\lim_{\epsilon \to 0} \frac{\beta^{T(\epsilon)}}{\epsilon} = \lim_{\epsilon \to 0} \epsilon^{\ln (f'(k_0)) - 1} = \infty,
\]

which implies \( W'(0) = +\infty \).

\((ii)\) First assume that \( k^s < k_0 \leq \bar{k} \). Now we prove that \( W'(0) < +\infty \). For \( \epsilon \) small:

\[
W'(\epsilon) - W'(0) = \sum_{t=0}^{\infty} \beta^t \left[ u(f(k_0^\epsilon) - k_{t+1}^\epsilon) - u(f(k_0) - k_0) \right]
\]

\[
\leq u'(f(k_0) - k_0) \sum_{t=0}^{\infty} \beta^t \left[ f(k_0^\epsilon) - f(k_0) - k_{t+1}^\epsilon + k_0 \right]
\]

\[
\leq u'(f(k_0) - k_0) \sum_{t=0}^{\infty} \beta^t \left[ f'(k_0^\epsilon)(k_t^\epsilon - k_0) - k_{t+1}^\epsilon + k_0 \right]
\]

\[
\leq u'(f(k_0) - k_0) \sum_{t=0}^{\infty} \beta^t \left[ f'(k_0^\epsilon)(k_t^\epsilon - k_0) \right]
\]

\[
\leq u'(f(k_0) - k_0) f'(k_0) \sum_{t=0}^{\infty} \beta^t \left[ k_t^\epsilon - k_0 \right]
\]

\[
\leq u'(f(k_0) - k_0) f'(k_0) \sum_{t=0}^{\infty} \beta^t \left[ f'(k_0) \right]^{t+1} \frac{1}{f'(k_0) - 1} \times \frac{\epsilon}{u'(f(k_0) - k_0)}
\]

\[
= f'(k_0) \sum_{t=0}^{\infty} \beta^t \left[ f'(k_0) \right]^{t+1} \frac{1}{f'(k_0) - 1} \times \frac{\epsilon}{1 - \epsilon}
\]

\[
= O(\epsilon),
\]

since \( \beta f'(k_0) < 1 \).

This implies \( W'(\epsilon) - W'(0) = O(\epsilon) \), or \( W'(0) < +\infty \).

Now assume that \( \bar{k} \) is finite and \( k_0 \geq \bar{k} \). We use exactly the same arguments in the proof of part \((ii)\), by changing the constrains \( u(c_t) \geq u(f(k_0) - k_0) - \epsilon \) by \( u(c_t) \geq u \left( f(\bar{k}) - \bar{k} \right) \).

Now we prove that \( W'(\bar{\epsilon}) = 0 \). For \( \epsilon \) close enough to \( \bar{\epsilon} \), the critical time \( T(\epsilon) \) from which the optimal path behaves as a solution of Ramsey problem with initial state \( k_{T(\epsilon)}^\epsilon \) is \( T(\epsilon) = 1 \). We then have

\[
u(f(k_0) - k_1^\epsilon) = u(f(k_0) - k_0) - \epsilon,
\]

and the sequence \( \{k_t^\epsilon\}_{t=0}^{\infty} \) is the solution of Ramsey problem with initial state \( k_1^\epsilon \).
This implies
\[ W(\epsilon) = u(f(k_0) - k_0) - \epsilon + \beta v(k_1^\epsilon), \]
and
\[ W'(\epsilon) = -1 + \beta v'(k_1^\epsilon) \times \frac{dk_1^\epsilon}{d\epsilon}. \]

By the implicit function theorem, we have
\[ \frac{dk_1^\epsilon}{d\epsilon} = \frac{1}{u'(f(k_0) - k_1^\epsilon)}. \]

Observe that by letting \( \epsilon \) converges to \( \tilde{\epsilon} \) we have
\[ \lim_{\epsilon \to \tilde{\epsilon}} k_1^\epsilon = \sigma(k_0). \]

This implies
\[ W'(\tilde{\epsilon}) = -1 + \beta v'(\sigma(k_0)) \times \frac{1}{u'(f(k_0) - \sigma(k_0))}. \]

Recall that it is well-known in dynamic programming literature that
\[ v(k_0) = \max_{0 \leq k_1 \leq f(k_0)} \left[ u \left( f(k_0) - k_1 \right) + \beta v(k_1) \right] = u \left( f(k_0) - \sigma(k_0) \right) + \beta v(\sigma(k_0)). \]

Combining with Inada condition, this implies
\[ -u'(f(k_0) - \sigma(k_0)) + \beta v'(\sigma(k_0)) = 0, \]
which is equivalent to
\[ W'(\tilde{\epsilon}) = 0. \]

\( (iii) \) Since for any \( 0 \leq \epsilon \leq \tilde{\epsilon} \), there exists \( T \) such that the equality constraint corresponding to \( T \) bind. Hence the solutions corresponding to difference values of \( \epsilon \) are different. Combining this with the strict concavity of \( u \), we get \( W \) is strictly concave in \( [0, \tilde{\epsilon}] \). This implies the existence of an unique left derivative of \( W \).

\[ \text{F. PROOF OF PROPOSITION 3.3} \]

For any \( 0 \leq \epsilon \leq \tilde{\epsilon} \), the optimal solution satisfies the following property: there exists \( t \) such that \( u(c_1^t) = u(f(k_0) - k_0) \). Hence the solutions corresponding to difference values of \( \epsilon \) are also difference. Combining with the strictly concavity of \( u \), the function \( W \) is strictly concave in \( [0, \tilde{\epsilon}] \). This implies the existence of an unique left derivative of \( W \).

Since for any \( a > 0 \), we have \( W'(\tilde{\epsilon}) = 0 < a < W'(0) = \infty \), there exists unique \( 0 < \epsilon < \tilde{\epsilon} \) such that \( W'(\epsilon^*) = a \). The statement of the Lemma is a consequence of Propositions 3.1 and 3.2.

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REFERENCES


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