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Indeterminacy and Cycles in a Cash-in-Advance Economy with Production*

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Abstract

In this paper we study the determinacy of price level, nominal and real interest rate and income, in an economy with productive capital and cash-in-advance constraint on consumption purchases. In particular, we show that under mild assumptions on preferences and production sets there may arise a continuum of equilibria converging to the unique stationary solution, which is therefore locally indeterminate, as well as deterministic cycles. Similar results are observed when sustained growth, in view of the presence of aggregate externalities in capital utilization, is accounted for.

Keywords: cash-in-advance, indeterminacy, growth.
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1 Introduction

In this paper we study the determinacy of price level, nominal and real interest rate and income in an economy with productive capital and cash-in-advance (Clower) constraint on consumption purchases. In particular, we show that under mild assumptions on preferences and production sets, there may arise a continuum of equilibria converging to the unique stationary solution which is therefore indeterminate. Moreover, close to the steady state equilibrium, there may emerge deterministic cycles in correspondence to which all the relevant economic indicators fluctuate perpetually. Analogous features, in terms of indeterminacy and cycles, are observed when the presence of aggregate externalities in capital utilization allows for sustained growth.

Markets are competitive, agents are endowed with perfect foresight and there is no uncertainty. The representative household consumes, supplies labor inelastically, invests in physical equipment and holds money balances because of the presence of a cash-in-advance constraint on consumption purchases, as in previous studies by Wilson (1979), Stockman (1981), Abel (1985), Svensson (1985), Lucas and Stokey (1987), Coleman (1987) and Bloise (1995): current consumption purchases must be financed out of money balances held at the beginning of the period and previously accumulated. Under these hypotheses, one gets that money, although dominated by capital in terms of returns, is held in view of the liquidity services it provides and which can be seen as implicit dividends. Therefore, the value of money can be determined by an asset-pricing equation as the price of whatever asset. In each period agents choose the composition of their portfolio in terms of capital and money by comparing the opportunity cost (the nominal interest rate) of holding money with the value of its liquidity services. Indeed, capital yields higher proceedings which, however, may be transformed in consumption only through a previous reinvestment in money balances.

The presence of productive capital entails several relevant features. First, it requires income to be determined simultaneously with prices, in opposition to previous models *à la* Lucas in which equilibrium quantities were given by the endowments stream. Second, the return on investment is endogenously determined by the marginal productivity of capital, in contrast to models in which assets' yields were given exogenously (Svensson (1985), Lucas and Stokey (1987)). Third, as it has been already shown in Stockman (1981) and Abel (1985), although the equilibrium allocations are not efficient, the choice of a constant rate of growth at which government injects (or withdraws) money in the economy does not affect neither the stationary solution nor the speed of convergence towards it; it follows that money is superneutral in the long-run as well as along the transitional dynamics. Another important feature concerns the determinacy of equilibrium prices and quantities, and represents the main contribution of the paper. In particular, we show that when consumption is not very intertemporally substitutable (i.e. there are sufficiently strong income effects), near the unique stationary solution there may be a continuum of convergent equilibrium trajectories indexed by the initial conditions of the variables which are not pre-determined. Therefore, although no uncertainty affects fundamentals, the particular equilibrium path which will prevail will crucially depend upon agents' "states of expectations" and the induced coordination mechanism. Finally, in correspondence to a whole range of agents' preferences parameters, we observe that deterministic cycles may easily arise close to the steady state.

Similar results, in terms of equilibrium determinacy and deterministic cycles, are observed when the economy, in view of the presence of aggregate externalities in capital utilization, displays long-run growth. As a matter of fact, we observe a continuum of equilibrium trajectories for the growth rate, all of them approaching asymptotically the (unique) balanced one, where quantities grow at a common constant rate, while the real and nominal interest rate as well as inflation are constant.

The mechanisms at the origin of indeterminacy and cycles in both cases is the following. When prices raise, individuals are highly liquidity constrained and investment in capital must expand. This leads to a decline in prices in the following period. When prices decline, individuals consumption purchases are higher and production contracts, thus generating an increase in prices in the next period. If consumption were highly intertemporally substitutable, this mechanism would lead to exploding dynamics. On the contrary, a low degree of intertemporal substitution smooths consumption oscillations.

The results we obtain provide additional support to the widely shared view that the existence of multiple equilibria and deterministic fluctuations in competitive economies is not necessarily linked either to the presence of agents maximizing on horizons of life short compared to that of the economy, as in *OLG* models[1], or to high degrees of impatience, as in multi-sectors

[1] See, e.g., Grandmont (1985).

infinite horizon models[2]. Indeed, the cash-in-advance constraint represents a friction in the trade process which may prevent, and independently on the degree of impatience of agents as well as on the length of their life-spans, from the stabilizing smoothing consumption effects which strongly characterize the Walrasian paradigm.

The remainder of the paper is organized as follows. In section 2 we present the benchmark model. Section 3 is devoted to the study of the occurrence of indeterminacy and deterministic cycles. In section 4 we extend the analysis to the long-run growth case. Section 5 concludes the paper.

2 The model

The economy is populated by a continuum of identical agents and identical firms, whose sizes are normalized to unity. Time is discrete: at each period there is a stock of capital, a produced good that can be either consumed or invested, and a given amount of nominal outside money. The representative household is endowed with perfect foresight and maximizes the discounted stream of utility functions $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where $0 < \beta < 1$ is the discount factor and u the instantaneous utility of consumption which satisfies the following standard assumption.

Assumption 1 (Preferences). *The period utility function* $u : R_+ \rightarrow R$ *is smooth, strictly increasing and strictly concave. Moreover* $\lim_{c \rightarrow 0^+} u'(c) = +\infty$.

The boundary condition in assumption 1 is to ensure an interior solution in agents' maximization problem. In each period t, the representative household supplies inelastically one unit of labor, and chooses how much to consume c_t, invest in capital k_{t+1} and in money balances M_t, subject to the budget constraint

$$p_t c_t + p_t \left[k_{t+1} - (1-\delta) k_t\right] + M_t = p_t r_t k_t + p_t w_t + M_{t-1} + \tau_t, \qquad (1)$$

where p is the price of the good, r the real rental price of capital, w the real wage, $\delta \in [0, 1]$ the depreciation rate of capital and τ the nominal lump-sum transfers issued by the government. Purchases of the consumption good

[2] Among others, see Benhabib and Nishimura (1985) and Boldrin and Montrucchio (1986).

requires money balances accumulated in the previous periods, i.e. agents are subject to the cash-in-advance (Clower) constraint[3]

$$p_t c_t \leq M_{t-1}. \tag{2}$$

Constraint (2) binds as long as money is dominated by capital in terms of returns, i.e. when the nominal interest factor $i_t \equiv (1 - \delta + r_t)\ \pi_t$ is greater than one, where $\pi_t \equiv p_t/p_{t-1}$ is the inflation factor between period $t-1$ and period t. In this case, the first order condition for the consumer has the form

$$u'(c_t) = \beta R_t u'(c_{t+1}) \pi_t / \pi_{t+1} \tag{3}$$

where $R \equiv (1 - \delta + r)$ is the real gross interest rate. The arbitrage equation (3) reflects the fact that if one wishes to decrease one unit of consumption in period t, the induced increase in capital in the same period is $R_t \pi_t$ and thereby that of consumption in period $t+1$ amounts to $R_t \pi_t / \pi_{t+1}$. Optimal plans for the single household must also satisfy the transversality condition

$$\lim_{t \to +\infty} \beta^t u'(c_t) [k_{t+1} + \pi_{t+1} m_{t+1}] = 0 \tag{4}$$

where $m_t \equiv M_{t-1}/p_t$ denotes beginning-of-period real balances.

Firms produce the good by renting capital and labor, according to a constant returns to scale production function $F(k, l) = l f(a)$, where $a \equiv k/l$ denotes capital intensity, which exhibits the usual neoclassical features.

Assumption 2 (Technology). *The intensive production function* $f : R_+ \to R$ *is smooth, strictly increasing and strictly concave. Moreover* $f(0) = 0$, $\lim_{a \to 0^+} f'(a) = +\infty$ *and* $\lim_{a \to +\infty} f'(a) = 0$.

Assumption 2 ensures on the one hand firms' maximization problem to be well defined and on the other one the existence of a unique stationary solution. Profit maximization of the firms implies that the real interest rate and the real wage equalize, respectively, the marginal productivity of capital and the marginal productivity of labor:

$$r = f'(a), \tag{5}$$

$$w = f(a) - f'(a)\, a. \tag{6}$$

[3] One may also assume, as Stockman (1981) and Abel (1985) do, that current transfers τ_t can be used to buy consumption. However, the dynamic behavior of the economy is invariant to whether or not transfers τ_t can be used in period t.

Government issues in each period lump-sum transfers τ of money balances at the constant rate μ, so that in period t the supply $\hat{M}$ of money is $\hat{M}_t = (1+\mu)^t \hat{M}_0$, where $\hat{M}_0$ is the initial amount of nominal balances, and $\tau_t = \mu \hat{M}_{t-1}$.

Equilibrium in labor market implies conditions (5)-(6) to hold for $a = k$; money market equilibrium requires that all nominal balances are held, i.e. $\pi_{t+1} = m_t (1+\mu) / m_{t+1}$, and by Walras law good market clears according to $k_{t+1} + c_t = g(k_t)$, where $g(k) \equiv f(k) + (1-\delta) k$. Thereby, when constraint (2) binds, an intertemporal equilibrium with perfect foresight is a deterministic sequence $\{k_t, c_t, \pi_t\}_{t=0}^{\infty}$, $(k_t, c_t, \pi_t) >> 0$ for $t = 0, 1, \ldots$, satisfying for every t the following system of equations:

$$\begin{aligned} g(k_t) - c_t &= k_{t+1}, & (7) \\ (1+\mu) c_t &= \pi_{t+1} c_{t+1}, & (8) \\ u'(c_t) g'(k_t)^{-1} \pi_t^{-1} &= \beta u'(c_{t+1}) \pi_{t+1}^{-1} & (9) \end{aligned}$$

subject to the initial endowments of capital $k_0 > 0$ and to the transversality condition (4). In particular, identifying x with (k, c, π), we have

$$G_0(x_t) = G_1(x_{t+1}) \tag{10}$$

where G_0 and G_1 are defined, respectively, by the left-hand side and right-hand side of equations (7)-(9). Notice that both the inflation factor π_0 and consumption c_0 relative to period zero are not given and therefore must be opportunely chosen on the basis of expectations about the whole equilibrium path. Capital stock, consumption and real balances are constant at the steady state. Let us denote with a bar over a variable its steady state value. The stationary level $\bar{k}$ of capital stock is obtained by solving equation $f'(k) = \beta^{-1} - (1-\delta)$ which under assumption 2 has a unique solution corresponding to the Modified Golden Rule. The stationary consumption $\bar{c}$ is given by output minus investment $f(\bar{k}) - \delta \bar{k}$ and, being constraint (2) binding, one has $\bar{m} = \bar{c}$. The stationary inflation factor $\bar{\pi}$ is equal to $1+\mu$: it follows that the price level at the stationary equilibrium needs not to be constant. Actually, it is constant only when the rate of growth μ of nominal money is zero. When $\mu > 0$, there is an inflationary stationary equilibrium and when $\mu < 0$ the value of money increases perpetually. One also readily verifies that constraint (2) binds at the steady state if and only if the discount factor β is lower than the factor $1+\mu$ of money growth. Under this condition, system

(10) is consistent with equilibrium in a small neighborhood of the stationary solution. In addition, by a direct inspection of system (10) and of the steady state conditions, one may verify that the monetary rule μ does not affect the steady state as well as the transitional dynamics of the economy and money is then superneutral[4].

3 Indeterminacy and cycles

Deterministic equilibrium dynamics is said to be indeterminate if there exists a continuum of sequences $\{x_t\}_{t=0}^{\infty}$ satisfying system (10) for all t, subject to the initial condition k_0, and all of which converging to the steady state $\bar{x}$. Following the usual procedure, the study of (local) indeterminacy requires an exam of the linear operator[5]

$$A = \left[DG_1\left(\bar{x}\right)\right]^{-1} DG_0\left(\bar{x}\right)$$

which regulates the linear tangent motion to (10) near the steady state. Since system (10) involves only one pre-determined variable, the initial stock k_0 of capital, indeterminacy arises if and only if the dimension n of the stable manifold of A is greater than or equal to two, i.e. if and only if A possesses at least two eigenvalues lying inside the unit circle. In this case, for every initial stock k_0 of capital, there will exist a $n-1$ dimensional space in which it is possible to place the initial conditions c_0 and π_0 in such a way that the equilibrium dynamics converges to the stationary solution and respects the transversality condition (4). In the opposite case, equilibrium of system (10) will be determinate.

To this purpose, it is useful to define here $\epsilon = -\left(u'\right)^{-1} u''\bar{c}$ the elasticity of the marginal utility, $s = f^{-1}f'\bar{k}$ the share of profit in total income, $\rho = -\left(f'\right)^{-1} f''\bar{k}$ the elasticity of the real interest rate, all evaluated at the

[4] Superneutrality issues have been deeply studied in Stockman (1981) and Abel (1985). The fact that money is superneutral along the transitional dynamics can be easily verified once one solves equations (7) and (8) for c and π and replace them in equation (9). In this way, one obtains a third-order difference equation in $(k_t, k_{t+1}, k_{t+2}, k_{t+3})$ in which the monetary rule μ does not enter. In view of (3) and of the equilibrium conditions, one has that inefficiency of the transitional capital formation process may be removed only by pegging the nominal interest factor equal to one. This implies that the demand of money in each period must be accommodated according to the flexible monetary rule $\mu_t = R_{t+1}^{-1} m_{t+1} m_t^{-1} - 1$.

[5] $DG_i\left(x\right)$, with $i = 1, 2$, denotes the matrix of the derivatives of G_i with respect to x.

steady state equilibrium, and the structural parameter $\phi \equiv 1-\beta(1-\delta) > 0$. After some straightforward but tedious computations, we obtain the following expression for[6] A:

$$A = \begin{bmatrix} \beta^{-1} & -1 & 0 \\ \phi\rho(1-\epsilon)^{-1}\bar{c}\bar{k}^{-1} & 1 & -(1-\epsilon)^{-1}\bar{c}\bar{\pi}^{-1} \\ -\phi\rho(1-\epsilon)^{-1}\bar{\pi}\bar{k}^{-1} & 0 & (1-\epsilon)^{-1} \end{bmatrix}.$$

The next proposition characterizes the modulus of the eigenvalues of A when the elasticity ϵ of the marginal utility varies in its interval of definition. As a matter of fact, it is shown that when consumption is relatively intertemporally substitutable (low ϵ), then there is only one stable eigenvalue and the steady state exhibits the classical saddle-path stability and, so, equilibrium is determinate. Conversely, an additional stable roots emerges when ϵ is made sufficiently large.

Proposition 1 *The linear operator A has always a real eigenvalue $0 < \xi_1 < 1$. In addition:*

(i) if $0 < \epsilon < 1$, then there are either two real eigenvalues $\xi_2 > 1$, $\xi_3 > 1$ or two complex conjugate eigenvalues ξ_2, $\widehat{\xi}_2$ with modulus greater than one;

(ii) if $1 < \epsilon < \epsilon_0$, where

$$\epsilon_0 \equiv 2 + 2^{-1}\beta\phi\rho(1+\beta)^{-1}\left(\theta s^{-1} - \delta\right)$$

and[7] $\theta \equiv \beta^{-1} - (1-\delta) > 0$, then there are two real eigenvalues $\xi_2 > 1$, $\xi_3 < -1$;

(iii) if $\epsilon > \epsilon_0$ then there are two real eigenvalue $\xi_2 > 1$ and $-1 < \xi_3 < 0$.

Proof. The eigenvalues of A correspond to the roots of the characteristic polynomial of A

$$\begin{aligned} P(\xi) &= \xi^3 - \left[1 + \beta^{-1} + (1-\epsilon)^{-1}\right]\xi^2 \\ &\quad + \left\{\beta^{-1} + (1-\epsilon)^{-1}\left[1 + \beta^{-1} + \phi\rho\left(\theta s^{-1} - \delta\right)\right]\right\}\xi - \beta^{-1}(1-\epsilon)^{-1}. \end{aligned}$$

[6]Provided that $\epsilon \neq 1$, DG_1 is a linear isomorphism and, so, the formula defining A above is meaningful. In addition, we assume that A has no eigenvalues on the unit circle.

[7]Observe that at the steady state equilibrium one has $\overline{c}/\overline{k} = f\left(\overline{k}\right)\overline{k}^{-1} - \delta$ and, since $f'\left(\overline{k}\right) = \theta$, it is possible to write $\overline{c}/\overline{k} = \theta s^{-1} - \delta$.

Performing simple computations, we obtain

$$\begin{aligned}P(0) &= -\beta^{-1}(1-\epsilon)^{-1},\\ P(1) &= \phi\rho(1-\epsilon)^{-1}\left(\theta s^{-1}-\delta\right),\\ P(-1) &= -(1-\epsilon)^{-1}\left[\left(1+\beta^{-1}\right)(4-2\epsilon)+\phi\rho\left(\theta s^{-1}-\delta\right)\right].\end{aligned}$$

One can easily verify that $P(0) < 0$ and $P(1) > 0$ if and only if $\epsilon < 1$ and that $P(-1) > 0$ if and only if $1 < \epsilon < \epsilon_0$. Moreover, notice that, if $1 < \epsilon$, $P(-1) < 0$ whenever $\left(1+\beta^{-1}\right)(4-2\epsilon)+\phi\rho\left(\theta s^{-1}-\delta\right) < 0$, i.e.

$$\epsilon > \epsilon_0. \tag{11}$$

Finally, observe that $\lim_{\xi\to+\infty} P(\xi) = +\infty$, $\lim_{\xi\to-\infty} P(\xi) = -\infty$ and the polynomial is a continuous function and its domain is connected. Let us now consider separately the three cases (i), (ii) and (iii) of the proposition.

(i) $0 < \epsilon < 1$. Then the polynomial is negative for all $\xi \leq 0$ and, being $P(1) > 0$, there is a real root ξ_1 belonging to $(0,1)$. In addition, by observing that the determinant $D = -P(0)$ corresponds to the product of the characteristic roots, one may immediately draw the conclusion that there exists at least one root with modulus greater than one. If such a root is real, by the continuity of the polynomial, one has $\xi_i > 1$ for $i = 1, 2$. If it is complex, one has obviously $|\xi_2| = \left|\widehat{\xi}_2\right| > 1$.

(ii) $1 < \epsilon < \epsilon_0$. Then, from the fact that $P(0) > 0$ and $P(1) < 0$, it follows that one root ξ_1 belongs to the interval $(0,1)$ and one root ξ_2 to the interval $(1,+\infty)$. Moreover, since $P(-1) > 0$, the third root ξ_3 belongs to $(-\infty,-1)$.

(iii) $\epsilon > \epsilon_0$. Then it is immediate to verify that ξ_1 and ξ_2 belongs to the same intervals of point *(ii)*. In addition, since $P(-1) < 0$, ξ_3 must be in $(-1,0)$. ■

Proposition 1 deserves some further comments. Since the elasticity ϵ of the marginal utility represents the inverse of the elasticity of intertemporal substitution in consumption, inequality (11) is actually satisfied when consumption is not very substitutable across periods (i.e., when the income effect is strong enough). Indeterminacy, conversely, does not arise for any other configuration of the structural parameters which do not meet inequality (11). In fact, when intertemporal substitutability prevails, the steady state equilibrium always presents the classical saddle-path stability and, so, there exists only one convergent path. Let us also observe that the elasticity ρ of the real interest rate can be expressed as $(1-s)/\sigma$, where σ denotes the elasticity of capital-labor substitution[8]. It follows that the higher substitutable the production inputs are (high σ), the more likely inequality (11) is to be satisfied, thus indeterminacy to arise.

The change in stability which leads to indeterminacy occurs though a flip bifurcation: indeed, when ϵ is continuously increased and goes through ϵ_0, there is one characteristic root that goes through -1. As it is shown, e.g., in Grandmont (1988), this implies that when ϵ is close to ϵ_0, there will generically emerge, according to the direction of the bifurcation, a stable or unstable two-period deterministic cycle, in which inflation and quantities fluctuate perpetually.

In the next section we extend the analysis of the effects of the cash-in-advance constraint to the case in which, in view of the presence of externalities from physical capital, the economy exhibits long-run growth. We show that in close analogy to the benchmark case, equilibrium is indeterminate when the elasticity of the marginal utility is sufficiently high, and that the change in stability occurs through a flip bifurcation.

4 Cash-in-advance and growth

It is well known that one popular specification for the technology which allows for sustained growth emphasizes the role played by the externalities and assumes the aggregate production set to be non-convex. One of the first contributions of this type is that of Romer (1986) which assumes an externality from physical capital and is related to some of the previous work on learning-by-doing (Arrow (1962)). In such a spirit, we now assume that each firm produces the good according to the production function $\psi\left(\widetilde{k}\right)F(k,l)$, $\psi'>0$, where $\widetilde{k}$ is the average stock of capital available in the economy. Normalizing to one the inelastic labor supply, since firms are identical at symmetric equilibrium one has $\widetilde{k}=k$; production is therefore given by $\psi(k)f(k)$,whereas the real interest rate and real wage are given by, respectively, $r=\psi(k)f'(k)$ and $w=\psi(k)[f(k)-kf'(k)]$. Firms ignore indeed the impact of their own investment decisions on the productivity of other firms through their effect on

[8]By definition, $1/\sigma$ is the elasticity of the ratio of the rental prices of capital and labor and satisfies $1/\sigma=\Omega+\rho$, where $\Omega\equiv\omega'k/\omega$ is the elasticity of the real wage $\omega(a)$.

$\psi(k)$. In order for balanced growth to be possible, the interest rate r must be constant, condition which requires $\psi(k) f(k)$ to have the form $Ak^{\psi}k^{\alpha}$, $A > 0$, $\alpha > 0$, $\psi > 0$, $\alpha + \psi = 1$. In this way, the interest rate results to be constantly equal to $r \equiv A\alpha$ whereas the real wage is linear in capital, $w_t = A(1-\alpha) k_t$. In order to have a balanced consumption growth rate, we chose preferences to be of the constant elasticity form

$$u(c) = \begin{cases} c^{1-\epsilon}/(1-\epsilon) & if\ \epsilon > 0,\ \epsilon \neq 1, \\ \ln c & if\ \epsilon = 1 \end{cases} \tag{12}$$

where ϵ is the (constant) elasticity of the marginal utility or, equivalently, the inverse of the elasticity of intertemporal substitution in consumption. Assuming constraint (2) to be binding and setting $z_t \equiv c_t/k_t$ the consumption-capital ratio, equations (7)-(9) may be opportunely rearranged in order to obtain the first-order difference system in (z, π):

$$(1-\delta+A) z_t^{-1} - 1 = (1+\mu) z_{t+1}^{-1} \pi_{t+1}^{-1}, \tag{13}$$

$$(1+\mu)^{-\epsilon} \beta (1-\delta+A\alpha) \pi_t = \pi_{t+1}^{1-\epsilon}. \tag{14}$$

In particular, identifying x with (z, π), we have

$$V_0(x_t) = V_1(x_{t+1}) \tag{15}$$

where V_0 and V_1 are defined, respectively, by the left-hand side and right-hand side of equations (13) and (14). Then, an equilibrium with perfect foresight is a sequence $\{x_t\}_{t=0}^{+\infty}$ satisfying system (15) and such that the level variables k, m and π satisfy the transversality condition (4). Setting $a \equiv 1-\delta+A$ and $b \equiv \beta(1-\delta+A\alpha)$, it is easy to verify that the (unique) stationary solution of (15) is given by $\bar{z} = a - b^{1/\epsilon}$ and $\bar{\pi} = (1+\mu)/b^{1/\epsilon}$ to which corresponds the balanced growth rate

$$\bar{\gamma} = [\beta(1-\delta+A\alpha)]^{1/\epsilon} - 1.$$

Therefore, strictly positive growth requires $\beta(1-\delta+A\alpha) > 1$. Moreover, under the assumption of positive growth, in order the transversality conditions (4) to be satisfied along a balanced path, the curvature of the utility function must be not very low, i.e.

$$\epsilon > \ln[\beta(1-\delta+A\alpha)] / \ln(1-\delta+A\alpha). \tag{16}$$

Constraint (2) is binding, under positive growth and $\mu > (1 - \delta + A\alpha)^{-1} - 1$, if and only if

$$\epsilon > \ln\left[\beta\left(1 - \delta + A\alpha\right)\right] / \ln\left[\left(1 - \delta + A\alpha\right)\left(1 + \mu\right)\right]. \tag{17}$$

Notice, since $A\alpha < A$, that condition (16) ensures $\bar{z} > 0$. Moreover, observe that condition (17) is satisfied for $\epsilon = 1$ if and only if $\beta < 1 + \mu$. Finally, it is easily verifiable that condition implies condition (17) when $\mu \geq 0$, and conversely that condition (17) entails (16) when $(1 - \delta + A\alpha)^{-1} - 1 < \mu \leq 0$.

The local dynamics of system (15) is topologically equivalent to that induced by the linear operator

$$B = \left[DV_1\left(\bar{x}\right)\right]^{-1} DV_0\left(\bar{x}\right).$$

Since neither z nor π do represent pre-determined variables, the stationary solution of (15) will be locally indeterminate if and only if the dimension n of the stable manifold of B is greater or equal to one, i.e. if and only if there is at least one eigenvalue which belongs to the unit disk. Indeed, in this case there will be a n-dimensional subspace in which it is possible to place the initial conditions z_0 and π_0. Performing simple computations, we find that B coincides with the following triangular matrix[9]:

$$B = \begin{bmatrix} a/b^{1/\epsilon} & -b^{1/\epsilon}\left(a - b^{1/\epsilon}\right)\left(1 + \mu\right)^{-1}\left(1 - \epsilon\right)^{-1} \\ 0 & \left(1 - \epsilon\right)^{-1} \end{bmatrix}.$$

One immediately verify that the two eigenvalues of B are $\zeta_1 = a/b^{1/\epsilon}$ and $\zeta_2 = (1 - \epsilon)^{-1}$ and do not depend upon μ; therefore money is superneutral. In view of condition (16), $a/b^{1/\epsilon}$ is always greater than one. Conversely, ζ_2 is greater than one for $0 < \epsilon < 1$, it is negative and lower than -1 when $1 < \epsilon < 2$, and belongs to the interval $(-1, 0)$ when

$$\epsilon > 2. \tag{18}$$

All this proves the following proposition.

Proposition 2 *The linear operator B has always a real eigenvalue ζ_1 greater than one. In addition:*

(i) if $0 < \epsilon < 1$, then the other (real) eigenvalue is greater than one;
(ii) if $1 < \epsilon < 2$, then the other eigenvalue is lower than -1;
(iii) If $\epsilon > 2$, the other eigenvalue belongs to $(-1, 0)$.

[9] The matrix B is defined when $\epsilon \neq 1$.

Condition $\epsilon > 2$ for indeterminacy requires again a low substitutability in intertemporal consumption, i.e. relatively strong income effects. When ϵ goes through 2, one eigenvalue goes through -1 and undergoes a flip bifurcation: accordingly, for ϵ arbitrarily close to 2, a two-period cycle will emerge in correspondence to which inflation and the consumption to capital ratio fluctuate forever. Let us finally observe that inequality (18) is perfectly consistent with its analogous in (11): indeed, the elasticity of the interest rate, comprehensive of the externalities, under the Cobb-Douglas specification of the technology here assumed, is equal to zero and so inequality (11) reduces to (18).

5 Conclusion

In this paper we have presented a cash-in-advance economy with productive capital and studied the demand of money and the determinacy of equilibrium prices, interest rates and quantities. Money is dominated by capital in terms of returns but it is held in view of the liquidity services it provides and which can be seen as implicit dividends on the basis of which it is possible to price money as whatever asset. Capital investment determines future production possibilities and thus future income which becomes therefore endogenous, in opposition to previous models *à la* Lucas in which quantities were given. The return on investment in capital equipment becomes also endogenous since it equalizes the marginal productivity of capital, and this improves models (Svensson (1985), Lucas and Stokey (1987)) in which assets' yields were given exogenously. The choice among capital and money is made by comparing the opportunity cost (the nominal interest rate) of holding money with the value of its liquidity services and by taking into account that capital proceedings must be re-invested in money before being transformed in consumption.

We have focused on the stationary economy as well as on the economy displaying long-run growth. We have shown that, provided consumption is not very substitutable, the unique steady state as well as the unique balanced growth rate may be indeterminate in the sense that there exists a continuum of equilibrium trajectories converging to them and indexed by the initial conditions of the non-predetermined variables. The mechanisms at the origin of indeterminacy and cycles relies on the expanding effect on capital accumulation of a high price level followed by an opposite contracting effect induced by a successive decline in prices. If consumption were highly intertemporally substitutable, this mechanism would lead to exploding dynamics. On the contrary, a low degree of intertemporal substitution may smooth consumption oscillations.

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