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Sunspots, Money and Capital

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Abstract

In this paper we characterize the indeterminacy of equilibria in a cash-in-advance economy with liquidity constraints and capital accumulation. In particular we show that, even though no extrinsic uncertainty a¤ects fundamentals, rational expectations equilibria exist in which prices and quantities exhibit repetitive and persistent ‡uctuations. A method of general applicability is proposed to construct sunspot equilibria in discrete-time three-dimensional models with one predetermined variable and lower dimensional stable manifold.

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0.1 Introduction

We consider an economy with productive capital: money and capital are treated as two competing assets in individuals' portfolios. Individuals enter each period with a given wealth distributed between money and capital. They observe the current state of the economy and form expectations upon prices, real and nominal interest rates in the following period. They then use money to …nance their consumption purchases and decide the holdings of money and capital to be carried into the following period. Although at an extremely aggregate level, our model permits to integrate monetary, …nancial and real aspects in a general equilibrium framework.

Money is a speci…c asset which, in addition to being a store of value, provides liquidity services. The demand for money is motivated by a cash-in-advance (or Clower) constraint: in every period consumption purchases must be …nanced out of money balances accumulated in the previous period. This transactional rule captures the fact that money and other assets di¤er in their degree of liquidity. The existence of a liquidity constraint drives a wedge between real returns over capital investment and money balances. Money entails a cost (roughly, the nominal interest rate) which is compensated by the evaluation of its liquidity services. Returns on capital exceed returns on the monetary asset in each period, but proceedings from capital accumulation can only be transformed in consumption (hence utility) through a reinvestment in money. In a cash-in-advance economy without accumulable production factors, the optimal allocation of resources can be achieved at equilibrium when money is withdrawn from circulation at exactly the rate of time preference (the "Chicago rule").

Introducing capital in a cash-in-advance economy entails important consequences on equilibrium determination of prices and quantities. Around the unique steady state, equilibrium is indeterminate under reasonable assumptions upon technology and preferences. Such an indeterminacy suggests that agents' expectations are likely to in‡uence the equilibrium trajectories though a mechanism of self-ful…lling prophecies, inducing sunspot equilibria. In a sunspot equilibrium, stochastic ‡uctuations result from spontaneous, self-ful…lling revisions of agents' expectations. The mechanics of this business cycle is related to oscillations in the cost of holding money induced by oscillations in prices. When prices raise, individuals are highly liquidity constrained and, consequently, production must expand. This leads to a decrease in prices in the following period. When prices decline, individuals' consumption purchases are higher and production contracts, thus generating an increase in prices in the next period. If consumption were highly intertemporally substitutable, this mechanism would lead to exploding dynamics. On the contrary, a low degree of intertemporal substitution smooths consumption oscillations.

To prove the existence of sunspot equilibria, we introduce a technique of general applicability in nonlinear macroeconomic models with predetermined variables. Our method is based on previous results by Woodford [17] and Grandmont, Pintus and de Vilder [9]. The sunspot equilibria we construct around the indeterminate steady state is a Markov process and do not depend on the whole history of sunspot shocks as in Woodford [17].

The remainder of the paper is organized as follows. In section 1, we describe the model and the notion of sunspot equilibrium. In section 2, we show that indeterminacy occurs under mild assumptions on preferences. In section 3, we present the main result concerning the existence of sunspot equilibria. Finally, some concluding remarks follow.

1 The Model

The model is formulated in discrete time and infinite horizon. There is a single commodity available in each period which can be either consumed or devoted to production, and two assets: money and capital. Production is carried out by means of a private, convex technology: a quantity k of commodity invested in a given period produces $f(k)$ units of commodity in the next period. We assume throughout the paper that this technology satisfies some standard hypotheses. The production function $f(k)$ is smooth, strictly increasing and strictly concave. The Inada conditions hold: $f(0) = 0$; $\lim_{k \to 0} f'(k) = \infty$; $\lim_{k \to \infty} f'(k) = 0$.

Sunspot shocks don't convey any relevant information about fundamentals. The prevailing stock of capital in the economy k_t, and the (realization of the) sunspot shock z_t in any period constitutes the state variable $s_t \equiv (k_t, z_t)$, which determines the distribution of sunspot shocks in the following period (Markovian process). The next period overall state of the economy is $s_{t+1} \equiv (k_{t+1}, z_{t+1})$, where $k_{t+1} = k(s_t)$ and the next period sunspot variable z_{t+1} is randomly distributed according to a transition map $Q = Q(s_{t+1} \mid s_t)$. Let Q be measurable. At the end of the article the existence of an equilibrium transition map Q which is measurable will be proved. Preferences are represented by $E\left[\sum_{t=0}^{\infty}\beta^{t}u(c_t)\right]$ where $0<\beta<1$ is the discount factor and the expectation is taken over realizations of the sunspot shocks. The period utility function $u(c_t)$ is assumed to be bounded, smooth, strictly increasing and strictly concave with $\lim_{c\to 0}u'(c_t)=\infty$.

The nominal monetary asset is denoted by a. Consumption is subject to a Clower (cash-in-advance) constraint. For sellers, consumption good sales result in currency units that simply accumulate during the period and are carried into the next period. We describe in details the decision problem facing a single agent, for whom the money price m_t as the inverse of the good price, is fixed and known. Suppose that his monetary asset (in nominal terms) is a (ma are the real balances) and his physical asset devoted to production is k_t. His knowledge about the system consists of the current state s_t. He purchases a quantity c_t of the commodity at the relative price m_t subject to the budget constraint $m_t a_{t+1}+k_{t+1}+c_t\le m_t a_t+f(k_t)$ and the cash-in-advance constraint $c_t\le m_t a_t$. The individual's portfolio at the beginning of the period is $y_t\equiv(a_t,k_t)$, while $\Gamma(y_t,s_t)$ denotes the set of positive values (y_{t+1},c_t) satisfying the constraints. It is easy to check that Γ is non-empty, compact and convex valued, and continuous (because the continuity and concavity of f and linearity of the constraints).

Let $v(y_t,s_t)$ be the value function for a consumer beginning with a portfolio y_t when the economy is in state s_t.

$$v(y_t,s_t)=\sup_{(y_{t+1},c_t)\in\Gamma(y_t,s_t)}\left\{u(c_t)+\beta\int v(y_{t+1},s_{t+1})\,dQ(s_{t+1}\mid s_t)\right\}.$$

Proposition 1 *The value function v exists and is unique. With respect to y_t the value function v is strictly increasing, strictly concave and continuously differentiable. The arg max correspondence g is a measurable function and it is continuous in y_t.*

Proof. We want to prove the existence and uniqueness of the value function by a usual fixed point argument. It is enough to show that the following operator T is a contraction in the function space: $(Tw)(y_t,s_t)\equiv\sup_{(y_{t+1},c_t)\in\Gamma(y_t,s_t)}\left\{u(c_t)+\beta\int w(y_{t+1},s_{t+1})\,dQ(s_{t+1}\mid s_t)\right\}$. We want to show that T maps the set of the bounded, continuous in y_t and measurable functions into itself. Note that Tw is bounded. The map $u(c_t)+\beta\int w(y_{t+1},s_{t+1})\,dQ(s_{t+1}\mid s_t)$ is continuous in (y_{t+1},c_t) since $\int w(y_{t+1},s_{t+1})\,dQ(s_{t+1}\mid s_t)$ is continuous in y_{t+1} by Lebesgue's Dominated Convergence Theorem. As Γ is compact-valued the Maximum Theorem implies that the supremum is attained, Tw

is a continuous function in y_t. Finally $u(c_t) + \beta \int w(y_{t+1}, s_{t+1})\, dQ(s_{t+1} \mid s_t)$ is a measurable function because of the continuity of u, the measurability of Q, w, and map $Q \to \int w\, dQ$, and from the Measurable Maximum Theorem (Aliprantis and Border [1, Theorem 14.91]), Tw is a measurable function. As $\beta < 1$, Blackwell's sufficient conditions for a contractions hold, i.e. T is a contraction of modulus β and the fixed point v exists and is unique.

Since u is strictly increasing and strictly concave and, for each fixed s_t, the correspondence $\Gamma(s_t)$ is convex, T maps functions that are, for each fixed s_t, strictly increasing and strictly concave in y_t into functions that are strictly increasing and strictly concave in y_t Hence $g(y_t, s_t)$ is unique, so that g turns out to be a measurable policy function, and continuous for each fixed s_t (Theorems 9.7, 9.8, 9.9; Stokey & Lucas (1989)). Finally the envelope theorem (Benveniste & Scheinkman (1977)) ensures that v is differentiable in y_t and provide a necessary condition (Theorems 9.10, 9.11; Stokey & Lucas (1989)), which jointly with the Kuhn-Tucker first order condition completes the demand side of the equilibrium conditions. ■

Equilibrium. The government's behavior in this economy reduces to keep constant the money supply. For the sake of simplicity the nominal money a is normalized to one. A stationary monetary equilibrium consists of a stationary sunspot process and a policy function such that all market clears: $(k(s_t), 1, c(s_t)) = g(k_t, 1, s_t)$. A stationary equilibrium is said to be a sunspot stationary equilibrium whenever the sunspot process is non-degenerate. Note that we require equilibrium stochastic process to be a Markovian process, i.e. to depend upon the history only through the quantity of physical capital inherited form the past and the last realization of the sunspot, rather than through any memory of the sunspot shocks as in Woodford (1986).

The Kuhn-Tucker first order conditions and the envelope theorem according to Benveniste and Scheinkman's (1977) formula provide the following system:

$$\begin{cases} f(k_t) - c(s_t) - k(s_t) = 0 \\ m(s_t)\,\mu(s_t) - \beta \int m(s_{t+1})\left(\mu(s_{t+1}) + \nu(s_{t+1})\right) dQ(s_{t+1} \mid s_t) = 0 \\ \mu(s_t) - \beta f'(k(s_t)) \int \mu(s_{t+1})\, dQ(s_{t+1} \mid s_t) = 0 \\ u'(c(s_t)) - \mu(s_t) - \nu(s_t) = 0 \\ c(s_t) - m(s_t) \leq 0 \\ \nu(s_t)\left[c(s_t) - m(s_t)\right] = 0 \end{cases}$$

where $\mu(s_t)$ and $\nu(s_t)$ are the non-negative Lagrange multipliers of the budget constraint and cash-in-advance. Note that the cash-in-advance is binding if and only if $\mu(s_t) - u'(c(s_t)) < 0$. In this case the system simplifies:

$$\begin{cases} f(k_t) - c(s_t) = k(s_t) \\ m(s_t)\,\mu(s_t) = \beta \int u'(m(s_{t+1}))\, m(s_{t+1})\, dQ(s_{t+1} \mid s_t) \\ \mu(s_t) = \beta f'(k(s_t)) \int \mu(s_{t+1})\, dQ(s_{t+1} \mid s_t) \end{cases} \quad (1)$$

An explicit sunspot process is introduced: $z_t \equiv (m_t, \mu_t)$. The reduced system can be rewritten in a compact form: $G_0(s_t) = \int G_1(k(s_t), z_{t+1})\, dQ(dz_{t+1} \mid s_t)$ where G_0 and G_1 are defined, respectively, by the left-hand and right-hand sides of the system. Note that the random component of s_{t+1} is just z_{t+1} in a very similar manner to Grandmont, Pintus and de Vilder (1998).

The underlying deterministic equilibrium dynamics is the following one:

$$\begin{cases} f(k_t) - m_t - k_{t+1} = 0 \\ m_t \mu_t - \beta u'(m_{t+1})\, m_{t+1} = 0 \\ \mu_t - \beta f'(k_{t+1})\, \mu_{t+1} = 0 \end{cases} \quad (2)$$

or in a more compact form: $G(x_t, x_{t+1}) = G_0(x_t) - G_1(x_{t+1}) = 0$, with $x_t \equiv (k_t, m_t, \mu_t)$.

The stationary state is given by $x^* = (k^*, m^*, \mu^*)$ s.t. $f(k^*) = k^* + m^*$, $\beta u'(m^*) = \mu^*$, $\beta f'(k^*) = 1$. Consumption is simply $c^* = m^*$. The inequality $\mu(s_t) - u'(c(s_t)) < 0$ is satisfied in any small, open neighborhood of the steady state, thus implying that the cash-in-advance constraint must be binding in the case of small enough fluctuations around the steady state.

2 Indeterminacy

For any initial amount of capital k_0, close to the steady state amount k^*, there will exist a continuum of perfect foresight equilibria all remaining within an arbitrarily small neighborhood of the steady state forever (Kehoe and Levine [11]). It is indeterminacy of deterministic equilibrium turning out to be a sufficient condition for sunspot stationary equilibria arbitrarily close to the steady state (in precise accord with Woodford's [18] conjecture). The study of indeterminacy requires the characterization of the linearized dynamics, i.e. the Jacobian matrix $J^* \equiv [DG_1(x^*)]^{-1} DG_0(x^*)$ regulating the tangent planar motion to G at the steady state:

$$J^* \equiv \begin{bmatrix} 1/\beta & -1 & 0 \\ 0 & 1/(1-\varepsilon) & m^*/[\beta(1-\varepsilon)u'] \\ \theta u' & -\theta\beta u' & 1 \end{bmatrix}$$

where the (convexity of) technology is captured by the new parameter: $\theta \equiv -f''/f'$; and preferences are summarized by the relative risk aversion: $\varepsilon \equiv -u''m^*/u' \neq 1$ (i.e. the inverse of the intertemporal substitution in consumption).

Proposition 2 The equilibrium is indeterminate if and only if

$$\varepsilon > 2 + m^*\theta\beta/[2(1+\beta)] \equiv \varepsilon^*.$$

In that case A displays two real stable eigenvalues and one unstable eigenvalue.

Proof. As there are two non-predetermined variables, the money price m_t and its shadow price μ_t; and one predetermined variable, the capital k_t; there is indeterminacy if and only if the stable manifold is at least two-dimensional. In force of the Stable Manifold Theorem the characteristic polynomial P of J^*; must display (at least) two roots inside the unit circle and no root outside. Note that $P(-1) = [(4-2\varepsilon)(1+1/\beta) + m^*\theta]/(1-\varepsilon)$; $P(0) = 1/[\beta(1-\varepsilon)]$; $P(1) = -m^*\theta/(1-\varepsilon)$; and then $\varepsilon > \varepsilon^*$ if and only if $P(-1) > 0$; $P(0) < 0$; $P(1) > 0$: As P is a continuous function two real eigenvalues lie inside the unit circle and one outside. ■

3 Sunspot equilibria

In this section we verify the celebrated conjecture of Woodford (1986): indeterminacy implies endogenous fluctuations. More precisely the following section is an adaptation of Woodford [18, Theorem 1] to a framework à la Grandmont, Pintus and de Vilder (1998). For the sake of simplicity we focus only on the existence of Markovian sunspot equilibria (as endogenous transition function) in a model where the realization of the random state can be decomposed in a deterministic step concerning the capital and a true randomization over prices (and related quantities) depending on the sunspot signal. Abstracting from the speci…c economic model which is examined in this paper, our analysis also clari…es the relation intercurrent between the two di¤erent approaches and allows to treat more general cases in which the stable manifold is not full dimensional.

Proposition 3 Indeterminacy implies the existence of sunspot equilibria.

The proof is relegated in the appendix. The idea is to …nd a non-explosive, non-degenerate randomization outside the stable manifold consisting in a measurable transition function. Note that in the above proof of existence of the value function and related policy function the transition function was required to be measurable. Thus the proof will be split in two parts. First of all the existence of a non-explosive, non-degenerate discrete randomization in the neighborhood of the steady state will be shown, then the existence of a measurable transition function as a direct application of the Measurable Selection Theorem.

Our method of constructing sunspot equilibria slightly improves upon the established body of existence results. In particular, apart from Woodford [18], we are not aware of any theoretical study which considers the case of an economic model with predetermined variables and an indeterminate steady state whose stable manifold is lower dimensional. It is worth emphasizing how Woodford's di¤ers from our work. A …rst di¤erence is purely methodological. We use an argument which is a slight variation of Woodford's [18] excellent theorem. However, Woodford introduces the technique to directly approach the problem of sunspot equilibrium existence; in our case, instead, Woodford's argument is used to show the existence of an invariant set and then a di¤erent argument is used to show the existence of sunspot equilibrium processes. A second di¤erence is inherent in the nature of equilibrium stochastic processes. Woodford's sunspot equilibria a priori depend on the whole past history of sunspot shocks. In our case, conversely, the current state of the economy provides the entire information about the sunspot shock in the next period. The history of sunspot shocks only a¤ects future sunspot shocks through the induced quantity of accumulated capital.

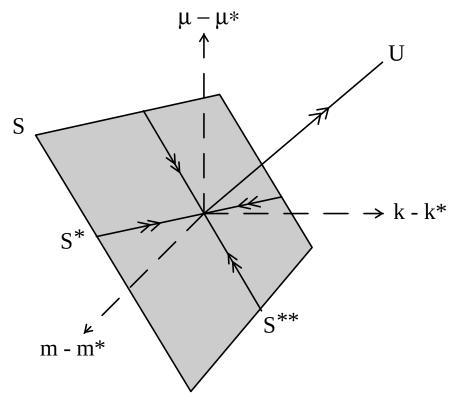
4 Conclusion

Our paper shows how indeterminacy arises in simple monetary economies represented by a cash-in-advance. In particular the high risk-aversion of the consumer or equivalently his low intertemporal substitution in consumption are recognized to be source of indeterminacy. To the contrary the intertemporal substitution in consumption frees the consumer from his monetary constraint and destroys the role of the cash-in-advance as source of indeterminacy and endogenous fluctuations.

The second most important result is just the proof of the Woodford's conjecture: indeterminacy implies sunspot equilibria. Abstracting a little from the specific model under study, our contribution should allow to extend previous existence results to cases in which the stable manifold is not full dimensional, i.e. its dimension, although greater than the number of predetermined variables, is less than the space dimension.

5 Appendix

Proof. We want to find a simple non-degenerate randomization concerning the sunspot z_{t+1}. First of all let L be the linear application associated to J^* and the space of variables $\{(k - k^*, m - m^*, \mu - \mu^*)\}$ be decomposed as direct sum: $S \oplus U$, where S is the stable plane of such a linear operator L and U the unstable line. Let L_s and L_u^{-1} be contractions on, respectively, S and U. We further decompose the subspace S in a direct sum $S_1 \oplus S_2$, where $S_1 \equiv S \cap (\{0\} \times \mathbb{R}^2)$.



The signaling space is now constructed. Each period a binary sunspot signal i_t is observed, which belongs to the space $\{0, 1\}$ endowed with the discrete metric. The signaling vector $\sigma_t \equiv (i_t; \sigma_{t-1})$ belongs to the infinite dimensional space $\Sigma \equiv \{0;1\}^{\infty}$; endowed with the metric: $d(\sigma_t;\varsigma_t) \equiv \sum_{\tau=t}^{-\infty} (1/2^{t-\tau})\, d(i_\tau; j_\tau) / [1 + d(j_\tau; j_\tau)]$ where $\varsigma_t \equiv (j_t; \varsigma_{t-1})$; $j_t \in \{0;1\}$. We define two operators α and δ in the function space as follows: $\alpha\varphi(\sigma_t) \equiv [\varphi(0;\sigma_t) + \varphi(1;\sigma_t)]/2$ and $\delta\varphi(\sigma_t) \equiv \varphi(0;\sigma_t) - \varphi(1;\sigma_t)$; where $\varphi(\sigma_t) = s_t$. As the discrete probability distribution in α is fixed, now the randomization problem consists in proving the existence of the function φ; which maps the sunspot signal σ_{t+1} in the state variable $s_{t+1} \equiv (k_{t+1}; z_{t+1})$. Such a function φ must solve the non-linear functional equation $G_0(\varphi(\sigma_t)) - (1/2)\sum_{i=0}^{1} G_1(\varphi(i;\sigma_t)) = 0$; i.e.

$$(G_0 - \alpha G_1)\varphi(\sigma_t) = 0 \tag{3}$$

for every σ_t and it must be non-degenerate, i.e. $\varphi(0;\sigma_t) \neq \varphi(1;\sigma_t)$ to implement a non-degenerate randomization. As $\varphi(i_{t+1};\sigma_t) = (k(s_t); z_{t+1})$ and s_t is given, a non-degenerate randomization implies $z_{t+1}(0;\sigma_t) \neq z_{t+1}(1;\sigma_t)$. Let $\varphi_1; \varphi_2; \varphi_u$ be the projections of φ on the sub-spaces S_1; S_2 and U respectively. It is possible to fix a nonzero vector $u \in S_1$; define a bounded scalar λ and constrain the choice of φ to avoid a degenerate randomization whenever $\lambda \neq 0$ as follows: $\varphi_1(0;\sigma_t) - \varphi_1(1;\sigma_t) = \varphi_1(0;\sigma_t) - s^* - [\varphi_1(1;\sigma_t) - s^*] = \lambda u$; i.e.

$$\delta\varphi_1(\sigma_t) - \lambda u = 0 \tag{4}$$

for every σ_t. If $\lambda = 0$; system (1) becomes nonstochastic, i.e. collapses in (2). Note that $\delta\varphi \in \{0\} \times R^2$ because $\varphi(0;\sigma_t) - \varphi(1;\sigma_t) = (k(s_t); z_{t+1}(0;\sigma_t)) - (k(s_t); z_{t+1}(1;\sigma_t)) = (0; z_{t+1}(0;\sigma_t) - z^*) - (0; z_{t+1}(1;\sigma_t) - z^*) \in \{0\} \times R^2$; i.e. we take into account the fact that the capital is predetermined in our construction of sunspot equilibria.

Remember that the Fréchet differential approximates a composition of functions $g(f)$ around the function $f^* = f^*(x)$ by $g(f^*) + Dg(f^*)(f - f^*) = [g(f^*) - Dg(f^*) f^*] + Dg(f^*) f$. Consider the system (3)-(4) in the hybrid unknown pair $(\varphi; \lambda)$ constituted by a function and a point. We linearize the system by taking its Fréchet differential around the pair $(\varphi^*; \lambda^*)$:

$$\begin{cases} (G_0 - \alpha G_1)(\varphi^*) + D(G_0 - \alpha G_1)(\varphi^*)(\varphi - \varphi^*) \approx 0 \\ \delta\varphi_1^* + \delta(\varphi_1 - \varphi_1^*) - [\lambda^* u + u(\lambda - \lambda^*)] = 0 \end{cases}$$

with $(\varphi; \lambda)$ in the neighborhood of $(\varphi^*; \lambda^*)$. Let $\psi^* \equiv (G_0 - \alpha G_1)(\varphi^*) - D(G_0 - \alpha G_1)(\varphi^*)\varphi^*$ and $\omega^* \equiv -\lambda u$. Note that if φ^* and λ are given, then $\psi^* = \psi^*(\sigma_t)$ and ω^* are two well defined fixed functions. As D; α;

δ and the projection are linear operators we obtain $D(G_0 - \alpha G_1)(\varphi^*) = (DG_0 - \alpha DG_1)(\varphi^*)$ and $D\delta(\varphi_1^*) = \delta(\varphi_1^*)$. Hence

$$\begin{cases} \chi^* + (DG_0 - \alpha DG_1)(\varphi^*)\,\varphi \approx 0 \\ \psi^* + \delta\varphi_1 = 0 \end{cases}$$

The Implicit Function Theorem is applied to the functional system to obtain

$$\begin{cases} (DG_1)^{-1}(\varphi^*)\,\chi^* + \left[(DG_1)^{-1}DG_0 - \alpha\right](\varphi^*)\,\varphi \approx 0 \\ \psi^* + \delta\varphi_1 = 0 \end{cases}$$

Let $\omega^* \equiv (DG_1)^{-1}(\varphi^*)\,\chi^*$, a fixed function of σ_t. As we consider a neighborhood of s^*, it is natural to consider $\varphi^*(\sigma_t) = s^*$, the constant function, as element in this function space around which the Fréchet differential is taken. In that case φ^* is the degenerate randomization ($\alpha(\varphi^*) = \varphi^*$, $\delta(\varphi^*) = 0$) and $\left[(DG_1)^{-1}DG_0\right](\varphi^*)$ becomes exactly the Jacobian J^* of the deterministic system (4) computed at steady state s^*, or in terms of linear operators, the application L above. Our system becomes

$$\begin{cases} \omega^* + (L - \alpha)\,\varphi \approx 0 \\ \psi^* + \delta\varphi_1 = 0 \end{cases} \qquad (5)$$

We want to prove that, given λ, the solution is unique. Take the projection of the first equation on the unstable line U: $\omega_u^* + (L_u - \alpha)\varphi_u \approx 0$, i.e. $\varphi_u \approx (L_u^{-1}\alpha)\varphi_u - L_u^{-1}\omega_u^*$, where L_u is the projection of the linear operator L defined by J^* on U. Define the operator T_u as follows:

$$T_u\varphi_u \equiv \left(L_u^{-1}\alpha\right)\varphi_u - L_u^{-1}\omega_u^*. \qquad (6)$$

By definition of U, L_u^{-1} is a contraction and $L_u^{-1}\alpha$ as well. As $L_u^{-1}\omega_u^*$ is a fixed function, T_u is a contraction too. So T_u has a unique fixed point φ_u in the function space (Contraction Mapping Theorem). The projection of the first equation in system (5) gives $\omega_s^* + (L_s - \alpha)\varphi_s = \omega_s^* + L_s\varphi_s - \frac{1}{2}\varphi_s(0;\sigma_t) - \frac{1}{2}\varphi_s(1;\sigma_t) \approx 0$, i.e. $\varphi_s(0;\sigma_t) \approx \omega_s^* + L_s\varphi_s + \frac{1}{2}[\varphi_s(0;\sigma_t) - \varphi_s(1;\sigma_t)] = \omega_s^* + L_s\varphi_s + \frac{1}{2}\delta\varphi_s$ and $\varphi_s(1;\sigma_t) \approx \omega_s^* + L_s\varphi_s - \frac{1}{2}[\varphi_s(0;\sigma_t) - \varphi_s(1;\sigma_t)] = \omega_s^* + L_s\varphi_s - \frac{1}{2}\delta\varphi_s$. The first restriction for the candidate solution φ is $\psi^* + \delta\varphi_1 = 0$ embodying the possibility of non-degenerate randomization. The second one is that $\delta(\varphi_2 + \varphi_u)$ must belong to $\{0\} \times R^2$ to capture the fact that the capital is a predetermined variable. Note that for each $\delta\varphi_u$ there is a unique $\delta\varphi_2$ s.t. $\delta\varphi_2 + \delta\varphi_u \in \{0\} \times R^2$ and the function $\delta\varphi_2 \equiv l(\delta\varphi_u)$ is linear. So

$\delta\varphi_s = \delta(\varphi_1 + \varphi_2) = \delta\varphi_2 + \delta\varphi_1 = I(\delta\varphi_u) - \psi^*$ and finally $\varphi_s(0;\sigma_t) \approx \omega_s^* + L_s\varphi_s + \frac{1}{2}[I(\delta\varphi_u) - \psi^*]$ and $\varphi_s(1;\sigma_t) \approx \omega_s^* + L_s\varphi_s - \frac{1}{2}[I(\delta\varphi_u) - \psi^*]$: where φ_u is now fixed and given by (6). So to show that a solution exists and is unique it suffices to prove that the following second operator T_s is just a contraction:

$$\left\{\begin{array}{l} T_s\varphi_s(0;\sigma_t) = L_s\varphi_s(\sigma_t) + \omega_s^+(\sigma_t) \\ T_s\varphi_s(1;\sigma_t) = L_s\varphi_s(\sigma_t) + \omega_s^-(\sigma_t) \end{array}\right.$$

where $\left\{\omega_s^\pm \equiv \omega_s^* \pm \frac{1}{2}[I(\delta\varphi_u) - \psi^*]\right\}$ are given known functions of σ_t: This is true because the linear operator L_s is a contraction along S: Hence we have found the unique projections φ_u and φ_s and thus the unique solution of the functional system (3)-(4): $\varphi = \varphi_s + \varphi_u$; a non-explosive, (maybe non-degenerate) randomization.

We must show now the existence of sunspot equilibria. The motion of the state variable can be decomposed into two steps. The first step is a deterministic move from s into $\bar{s}_{t+1} = (k_{t+1}; \bar{z}_{t+1}) = \left(G_1^{-1} \circ G_0\right)(s_t)$: The second step is a random perturbation of $\bar{z}_{t+1}$: conditions (3)-(4) allow to choose a non-degenerate probability measure over z which satisfies the stochastic Euler equations (1). Using a terminology analogous to Grandmont, Pintus and de Vilder's (1998), the support S of the state variables s is such that $G_0(S)$ belongs to the vertical convex hull of $G_1(S)$: Specifically, for all $(k_t; z_t)$ in S; there are two distinct points $\left(k_{t+1}; z^0_{t+1}\right)$ and $\left(k_{t+1}; z^1_{t+1}\right)$ in S such that $G_0(k_t; z_t)$ belongs to the (relative interior of the) vertical convex hull of $G_1\left(k_{t+1}; z^0_{t+1}\right)$ and $G_1\left(k_{t+1}; z^1_{t+1}\right)$: We want to find a random perturbation of $\bar{z}_{t+1}$; i.e. to show the existence of a measurable transition map Q such that $G_0(s_t) = \int G_1(k_{t+1}; z_{t+1})\, dQ(s_{t+1} | s_t)$:

The first part of the proof has shown that a non-degenerate discrete randomization exists, i.e. there exists a correspondence $\gamma_0(s_t)$ to which belong two points $\left(z^0_{t+1}; z^1_{t+1}\right)$ such that $G_0(s_t) = (1/2)\sum_{i=0}^{1} G_1\left(k(s_t); z^i_{t+1}\right)$ where $\left(k(s_t); z^i_{t+1}\right) = \varphi(s_t)$ and $\left\|z^1_{t+1} - z^0_{t+1}\right\| > 0$: This correspondence is non-empty valued and has a closed graph.

Define two non-empty valued, closed graph correspondences $\gamma_1; \gamma_2 : S \rightrightarrows M(Z)$ where M is the set of measures defined on the support Z of z; such that

$$\left\{\begin{array}{l} \gamma_1(s_t) \equiv \{\mu \in M(Z) : \mu(\gamma_0(s_t)) = 1\} \\ \gamma_2(s_t) \equiv \left\{\mu \in M(Z) : G_0(s_t) = \int_Z G_1(k(s_t); z_{t+1})\, d\mu(s_{t+1} | s_t)\right\} \end{array}\right. :$$

The intersection correspondence $\circ_1 \backslash \circ_2$ is non-empty valued. As intersection of two closed graph correspondences, it has a closed graph as well.

Finally the Measurable Selection Theorem applies: we can extract a measurable selection Q from the correspondence $\circ_1 \backslash \circ_2$ (Kuratowski-Ryll-Nardzewski Selection Theorem).

The existence of a measurable selection makes meaningful the proof of the existence of the policy function. Moreover it is the required transition function constituting the stochastic (sunspot) equilibrium. Hence when the steady state is indeterminate, sunspot stationary equilibria exist. ■

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